

OUTPUT TRACKING AND DISTURBANCE REJECTION FOR 1-D ANTI-STABLE WAVE EQUATION*

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Abstract. In this paper, we solve the output tracking and disturbance rejection problem for a system described by a one-dimensional anti-stable wave equation, with reference and disturbance signals that belong to $W^{1,\infty}[0, \infty)$ and $L^\infty[0, \infty)$, respectively. Generally, these signals cannot be generated from an exosystem. We explore an approach based on proportional control. It is shown that a proportional gain controller can achieve exponentially the output tracking while rejecting disturbance. Our method consists of three steps: first, we convert the original system without disturbance into two transport equations with an ordinary differential equation by using Riemann variables, then we propose a proportional control law by making use of the properties of transport systems and time delay systems. Second, based on our recent result on disturbance estimator, we apply the estimation/cancellation strategy to cancel to the external disturbance and to track the reference asymptotically. Third, we design a controller using a state observer. Since disturbance does not appear in the observer explicitly (the disturbance is exactly compensated), the controlled output signal is exponentially tracking the reference signal. As a byproduct, we obtain a new output feedback stabilizing control law by which the resulting closed-loop system is exponentially stable using only two displacement output signals.

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1. INTRODUCTION

The *output regulation problem* is to design a feedback control law to achieve asymptotic tracking of any reference signal from a prescribed class and asymptotic rejection of any disturbance signal from a prescribed class, for an uncertain system while maintaining closed-loop boundedness. The classical approach is to use the internal model principle (as in [4, 9]) for finite-dimensional systems and in [1, 24, 26] for infinite-dimensional systems. Internal model-based tracking and disturbance rejection for stable well-posed systems are considered in [27], where the reference and disturbance signals that are finite superpositions of sinusoids, and the frequencies are also assumed to be known. In all these references, the reference and disturbance signals are generated by a finite-dimensional exosystem. In [18, 19], the authors consider the case where the exosystem is infinite-dimensional. However, in many real control systems, the reference signal and the unknown disturbance belong to a very large class, such as $L^\infty[0, \infty)$, so that they cannot be generated by an exosystem. Interesting studies on the

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output regulation of parabolic partial differential equations (PDEs) are in [5, 6] that use the backstepping design for the (robust) output regulation of boundary controlled parabolic PDEs [35]; considers the backstepping design for the output regulation of a boundary controlled Schrödinger equation. In the recent study [21], backstepping approach is applied to the output tracking problem for a general 2×2 system of first order linear hyperbolic PDEs, but no disturbances are taken into consideration. Combining backstepping and differential flatness is adopted to achieve the tracking control for parabolic PDEs without disturbance in [23]. Another powerful method in dealing with output tracking is based on adaptive control in [11, 22], where the system is exponentially stable if no disturbance enters the system, and the disturbance signal is sinusoidal. The assumption used in [11] is removed in [13, 14], where the adopted method is the adaptive control approach. A recent result on output tracking on wave equation is in [7], where the disturbance signal and the reference signal are generated by an exosystem. Very recently, the output to be regulated to a constant reference signal for stable nonlinear system has been established in [31], where low gain integral control is proposed and the proof of asymptotically tracking is rather technical. For boundary proportional-integral controllers for the linearized Saint Venant model, we refer to [32], where the reference signal is also constant.

In this paper, motivated mainly by [7], [11], [13] and [14], we consider the following one-dimensional anti-stable wave equation with Neumann boundary control matched unknown disturbance:

$$\begin{cases} w_{tt}(x, t) = w_{xx}(x, t), & 0 < x < 1, t > 0, \\ w_x(0, t) = -qw_t(0, t), & t \geq 0, \\ w_x(1, t) = u(t) + d(t), & t \geq 0, \\ w(x, 0) = w_0(x), w_t(x, 0) = w_1(x), & 0 \leq x \leq 1, \\ y_o(t) = w(0, t), y_m(t) = w(1, t). \end{cases} \quad (1.1)$$

We denote by $w'(x, t)$ or $w_x(x, t)$ the derivative of $w(x, t)$ with respect to x and by $\dot{w}(x, t)$ or $w_t(x, t)$ the derivative of $w(x, t)$ with respect to t . (w, w_t) is the state, u the control input signal, and y_o is the output signal to be regulated. y_m is the measurement, parameter $q \neq 1$ is a real positive constant. d represents the unknown external disturbance which is only supposed to satisfy $d \in L^\infty[0, \infty)$. Let r be the known reference signal satisfying $r \in W^{1, \infty}[0, \infty)$, where $W^{1, \infty}[0, \infty) = \{\phi : \phi \in L^\infty[0, \infty), \phi' \in L^\infty[0, \infty)\}$.

In this paper, we solve the output tracking and disturbance rejection problem for the one-dimensional anti-stable wave equation with general reference and disturbance signals. Our control objective in this paper is *to design an output feedback regulator such that for all initial states of the systems (1.1), (i) all the internal signals should be bounded; (ii) the tracking error $e_y = y_o - r$ satisfies for some $M, \mu > 0$,*

$$|e_y(t)| = |y_o(t) - r(t)| \leq Me^{-\mu t} \quad \text{for all } t \geq 0. \quad (1.2)$$

This means that the tracking error is exponentially convergent to zero as time goes to infinity. Particularly, when $r \equiv 0$, the state of the system (1.1) converges to zero, exponentially, which provides a new output feedback exponentially stabilizing control law for system (1.1). These results have been announced (without proof) in the CDC conference paper [38].

We consider system (1.1) in the state Hilbert space $\mathcal{H} = H^1(0, 1) \times L^2(0, 1)$ with the inner product given by

$$\langle [\phi_1, \psi_1], [\phi_2, \psi_2] \rangle_{\mathcal{H}} = \int_0^1 [\phi_1'(x)\overline{\phi_2'(x)} + \psi_1(x)\overline{\psi_2(x)}] dx + \phi_1(0)\overline{\phi_2(0)}, \quad \forall [\phi_i, \psi_i] \in \mathcal{H}, i = 1, 2.$$

In comparison to the existing known results [2, 5–7, 10, 11, 13, 22, 28, 34], the major contributions of this paper are that:

- Our reference signal is much more general than what is normally considered in the literature [5–7]. The disturbance signal is only assumed to be essentially bounded. Generally speaking, these signals cannot be generated from an exosystem, hence the internal model principle cannot be applied for our problem.
- In contrast to the exponentially stable system in [11, 34] and the Lyapunov stable system in [13, 14], the paper deals with an anti-stable wave system where the real parts of all the eigenvalues (except for one) of the associated operator are positive. Note that the associated operator of [11, 13, 14, 34] has no unstable eigenvalues.
- Compared with the asymptotic tracking in [5, 7, 11, 13, 22], the novelty of the paper is that it is the first time to derive an exponential tracking controller, which makes the output to be regulated track the reference signal, exponentially, while rejecting the general disturbance.
- Different from the control law of [2, 28], the control law presented here is much simple (see Rem. 2.4). As a byproduct of this paper, a new output feedback exponentially stabilizing control law is proposed by utilizing only two displacement output signals. Compared with the results in [10, 15] where more measurements signals are needed, the result of this paper is very sharp.

We proceed as follows. In Section 2, we propose a tracking controller design for the system without disturbance. This controller is also a new exponentially stabilizing controller if the reference signal vanishes. In Section 3, an asymptotic tracking controller is developed. Section 4 is devoted to exponentially track the reference signal by using a state observer. In Section 5, we explore the robustness to the bounded errors in the estimation of disturbance. To simplify the notation, all obvious arguments and domains both for time and spatial variables will be omitted, when we feel that this will not lead to confusion.

2. PRELIMINARY: CASE STUDY OF $d(t) \equiv 0$.

To regulate system with the external disturbance, we should know how to regulate system without disturbance. In this section, we consider the output regulation problem for system (1.1) without disturbance (that is, $d \equiv 0$). We reformulate system (1.1) with $d \equiv 0$ by introducing the following intermediate Riemann variables:

$$\alpha(x, t) = \frac{1-q}{1+q} [w_t(x, t) - w_x(x, t)], \quad \beta(x, t) = w_t(x, t) + w_x(x, t). \quad (2.1)$$

This leads to the following equivalent new system:

$$\begin{cases} \alpha_t(x, t) = -\alpha_x(x, t), & \beta_t(x, t) = \beta_x(x, t), \\ \alpha(0, t) = \beta(0, t), & \beta(1, t) = u(t) + w_t(1, t), \\ w_t(0, t) = \frac{1}{1-q} \beta(0, t). \end{cases} \quad (2.2)$$

In this new framework, the wave equation is represented as the cascade of two transport PDEs, with one ODE being driven by the second of the two PDEs. The “ODE-part” of (2.2) with state $w(0, t)$ plays a central role and it has to be regulated to track the reference signal r asymptotically (exponentially) by feedback. It is seen that if the control input signal $u(t)$ and the velocity signal $w_t(1, t)$ are known, the solution α, β of the transport equation can be solved explicitly. Particularly, $\beta(0, t)$ takes the value $u(t) + w_t(1, t)$ delayed by 1 unit of time. Now, motivated by the “ODE-part” of (2.2) and the state $w(0, t)$ needed to be regulated to $r(t)$ asymptotically, we propose the following feedback controller

$$u(t) = -w_t(1, t) + (1-q)\dot{r}(t+1) - ke_y(t), \quad (2.3)$$

where $-ke_y(t)$ is the error proportional feedback, the gain k is a tuning parameter, and e_y is the tracking error. Set $\sigma(t) = w(0, t)$. Under feedback (2.3), the closed-loop of system (2.2) becomes

$$\begin{cases} \alpha_t(x, t) = -\alpha_x(x, t), & \beta_t(x, t) = \beta_x(x, t), \\ \alpha(0, t) = \beta(0, t), & \beta(1, t) = (1 - q)\dot{r}(t + 1) - k[\sigma(t) - r(t)], \\ \dot{\sigma}(t) = \frac{1}{1 - q}\beta(0, t). \end{cases} \quad (2.4)$$

We consider system (2.4) in the state space $\mathbb{H} = [L^2(0, 1)]^2 \times \mathbb{R}$ with the inner produce given by

$$\langle [\phi_1, \psi_1, h_1], [\phi_2, \psi_2, h_2] \rangle = \int_0^1 [\phi_1(x)\overline{\phi_2(x)} + \psi_1(x)\overline{\psi_2(x)}] dx + h_1\overline{h_2}, \quad \forall [\phi_i, \psi_i, h_i] \in \mathbb{H}, i = 1, 2. \quad (2.5)$$

System (2.4) can be re-formulated in operator form:

$$\dot{Z}(\cdot, t) = AZ(\cdot, t) + Bf(t), \quad (2.6)$$

where $Z(\cdot, t) = [\alpha(\cdot, t), \beta(\cdot, t), \sigma(t)]$, $f(t) := (1 - q)\dot{r}(t + 1) + kr(t)$, $B = [0, \delta_1, 0]$ (where δ_a is the Dirac pulse at $x = a$, with a suitable interpretation). A is a linear operator defined in \mathbb{H} by

$$\begin{cases} A[\phi, \psi, h] = \left[-\phi_x, \psi_x, \frac{1}{1 - q}\psi(0) \right], & \forall [\phi, \psi, h] \in D(A), \\ \mathcal{D}(A) = \{[\phi, \psi, h] \in [H^1(0, 1)]^2 \times \mathbb{R} : \phi(0) = \psi(0), \psi(1) = -kh\}. \end{cases} \quad (2.7)$$

A direct computation shows that A^* , the adjoint operator of A , is given by

$$\begin{cases} A^*[\phi, \psi, h] = [\phi_x, -\psi_x, -k\psi(1)], & \forall [\phi, \psi, h] \in D(A^*), \\ \mathcal{D}(A^*) = \left\{ [\phi, \psi, h] \in [H^1(0, 1)]^2 \times \mathbb{R} : \phi(1) = 0, \psi(0) = \phi(0) + \frac{1}{1 - q}h \right\}, \end{cases} \quad (2.8)$$

and B^* , the adjoint operator of B , is given by

$$B^*[\phi, \psi, h] = \psi(1), \quad D(B^*) = L^2(0, 1) \times H^1(0, 1) \times \mathbb{R}. \quad (2.9)$$

Lemma 2.1. *Let k be a constant satisfying $\frac{k}{1 - q} \in (0, \pi/2)$, $r \in W^{1, \infty}[0, \infty)$. The operator A defined by (2.7) generates a C_0 -semigroup e^{At} on \mathbb{H} and B is admissible for e^{At} . Therefore, for any initial value $(\alpha(\cdot, 0), \beta(\cdot, 0), \sigma(0)) \in \mathbb{H}$, system (2.4) admits a unique solution $(\alpha, \beta, \sigma) \in C(0, \infty; \mathbb{H})$ that is bounded, i.e., $\|(\alpha(\cdot, t), \beta(\cdot, t), \sigma(t))\|_{\mathbb{H}} < +\infty$. Moreover, there exist two constants $M, \mu > 0$ such that $|\sigma(t) - r(t)| \leq Me^{-\mu t}$ for all $t \geq 0$.*

Proof. We first claim that A generates a C_0 -semigroup e^{At} . To this end, we show that A^{-1} exists. Indeed, a simply computation shows

$$A^{-1}[\phi, \psi, h] = \left[(1 - q)h - \int_0^x \phi(\xi) d\xi, (1 - q)h + \int_0^x \psi(\xi) d\xi, -\frac{1}{k} \left((1 - q)h + \int_0^1 \psi(\xi) d\xi \right) \right].$$

Hence, A^{-1} exists and is also compact on \mathbb{H} by the Sobolev embedding theorem. Define a new inner product in \mathbb{H} as follows: for all $[\phi_i, \psi_i, h_i] \in \mathbb{H}$, $i = 1, 2$,

$$\left\langle [\phi_1, \psi_1, h_1], [\phi_2, \psi_2, h_2] \right\rangle_1 = \int_0^1 \left[\phi_1(x) \overline{\phi_2(x)} + 2\psi_1(x) \overline{\psi_2(x)} \right] dx + h_1 \overline{h_2}. \quad (2.10)$$

It is obvious that the norm induced by (2.10) is equivalent to the one induced by (2.5). For any $Z = [\phi, \psi, h] \in D(A)$, we have

$$\langle AZ, Z \rangle_1 = \int_0^1 \left[-\phi_x(x) \overline{\phi(x)} + 2\psi_x(x) \overline{\psi(x)} \right] dx + \frac{1}{1-q} \psi(0) \overline{h}$$

and hence

$$\begin{aligned} \operatorname{Re} \langle AZ, Z \rangle_1 &= \frac{1}{2} |\phi(0)|^2 - \frac{1}{2} |\phi(1)|^2 + |\psi(1)|^2 - |\psi(0)|^2 + \frac{1}{1-q} \operatorname{Re} \psi(0) \overline{h} \\ &\leq k^2 |h|^2 + \frac{1}{2(1-q)^2} |h|^2 \\ &\leq \left(k^2 + \frac{1}{2(1-q)^2} \right) \|\phi, \psi, h\|_1^2. \end{aligned}$$

Hence, $A - (k^2 + 1/2(1-q)^2)I$ is dissipative. Since A is surjective (A^{-1} exists), by the Lumer–Phillips theorem ([25], Thm. 1.4.3), $A - (k^2 + 1/2(1-q)^2)I$ generates a C_0 -semigroup of contractions on \mathbb{H} , and so A and A^* generate a C_0 -semigroup.

Now we show that B is admissible for e^{At} ([30]). Actually, a direct computation gives

$$A^{*-1}[\phi, \psi, h] = \left[-\int_x^1 \phi(\xi) d\xi, -\frac{h}{k} + \int_x^1 \psi(\xi) d\xi, (1-q) \left(-\frac{h}{k} + \int_0^1 [\psi(\xi) + \phi(\xi)] d\xi \right) \right],$$

and

$$B^* A^{*-1}[\phi, \psi, h] = -\frac{h}{k}, \quad (2.11)$$

which is bounded from \mathbb{H} to \mathbb{C} . Now, we consider the dual system associated with A^* as follows:

$$\dot{Z}^*(\cdot, t) = A^* Z^*(\cdot, t), \quad y(t) = B^* Z^*(\cdot, t), \quad (2.12)$$

that is,

$$\begin{cases} \alpha_t^*(x, t) = \alpha_x^*(x, t), & \beta_t^*(x, t) = -\beta_x^*(x, t), \\ \alpha^*(1, t) = 0, & \beta^*(0, t) = \alpha^*(0, t) + \frac{\sigma^*(t)}{1-q}, \\ \dot{\sigma}^*(t) = -k\beta^*(1, t), & y(t) = \beta^*(1, t). \end{cases} \quad (2.13)$$

Define the energy function for (2.13) as

$$E(t) = \int_0^1 \left[(\alpha^*(x, t))^2 + (\beta^*(x, t))^2 \right] dx + (\sigma^*(t))^2.$$

Since A^* generates a C_0 -semigroup e^{A^*t} , we know that there exists two constant $M_{A^*}, \omega_{A^*} > 0$ such that $E(t) \leq M_{A^*} e^{\omega_{A^*}t} E(0)$. Set $\rho(t) = \int_0^1 x(\beta^*(x, t))^2 dx$. Differentiating $\rho(t)$ with respect to t along the solution to (2.13) yields

$$\dot{\rho}(t) = -(\beta^*(1, t))^2 + \int_0^1 (\beta^*(x, t))^2 dx. \quad (2.14)$$

For any $T \geq 0$, integrating from 0 to T with respect to t for (2.14) and noting that $|\rho(t)| \leq E(t)$ for all $t \geq 0$, we have

$$\begin{aligned} \int_0^T (\beta^*(1, s))^2 ds &\leq \int_0^T \int_0^1 (\beta^*(x, s))^2 dx ds + \rho(0) - \rho(T) \\ &\leq \int_0^T E(s) ds + E(0) + E(T) \leq M_{A^*} e^{\omega_{A^*}T} (T + 2) E(0). \end{aligned}$$

This fact, together with the boundedness of B^*A^{*-1} , shows the admissibility of B . By ([29], Prop. 4.2.5), for any initial value $(\alpha(\cdot, 0), \beta(\cdot, 0), \sigma(0)) \in \mathbb{H}$, system (2.4) admits a unique solution $(\alpha, \beta, \sigma) \in C(0, \infty; \mathbb{H})$. Next, we claim $|e_y(t)| \leq M e^{-\mu t}$ for some $M, \mu > 0$. First, note that

$$\beta(x, t) = \begin{cases} (1-q)\dot{r}(t+x) - k[\sigma(t+x-1) - r(t+x-1)], & t+x \geq 1, \\ \beta_0(t+x), & t+x < 1, \end{cases} \quad (2.15)$$

solve the “ β -part” of (2.4) provided that $\sigma(t)$ is known. Thus, in this case, $\beta(0, t) = (1-q)\dot{r}(t) - k[\sigma(t-1) - r(t-1)]$ for $t \geq 1$. Substituting $\beta(0, t)$ into the last equation of (2.4), and noting $e_y(t) = \sigma(t) - r(t)$, we obtain

$$\dot{e}_y(t) = -\frac{k}{1-q} e_y(t-1), \quad \forall t > 1, \quad (2.16)$$

which is classical time delay system. It is well-known, by the frequency domain analysis ([8], p. 40), that system (2.16) is exponentially stable if and only if $0 < \frac{k}{1-q} < \frac{\pi}{2}$. Therefore, $|\sigma(t) - r(t)| \leq M e^{-\mu t}$ for all $t \geq 0$ holds with some $M, \mu > 0$.

Finally, we claim that the solution of (2.4) is bounded. Indeed, since $r \in W^{1,\infty}[0, \infty)$, we derive

$$|\sigma(t)| \leq \|r\|_{L^\infty[0, \infty)} + M, \quad \forall t \geq 0, \quad (2.17)$$

i.e., $\sigma(t)$ is bounded. By (2.15), (2.17), we obtain

$$\begin{aligned} \int_0^1 \beta^2(x, t) dx &\leq 3 \int_0^1 \left[(1-q)^2 \dot{r}^2(t+x) + k^2 [\sigma^2(t+x-1) + r^2(t+x-1)] \right] dx \\ &\leq 3 \left[(1-q)^2 \|\dot{r}\|_{L^\infty[0, \infty)}^2 + k^2 (\|r\|_{L^\infty[0, \infty)} + M)^2 + k^2 \|r\|_{L^\infty[0, \infty)}^2 \right] < +\infty, \quad \forall t > 1. \end{aligned} \quad (2.18)$$

On the other hand, note that

$$\alpha(x, t) = \begin{cases} (1-q)\dot{r}(t-x) - k[\sigma(t-x-1) - r(t-x-1)], & t-x > 1, \\ \beta_0(t-x), & 0 \leq t-x \leq 1 \\ \alpha_0(x-t), & t < x, \end{cases} \quad (2.19)$$

solve the “ α -part” of (2.4) provided that $\sigma(t)$ is known. Similar to (2.18), we obtain that for $t > 2$,

$$\int_0^1 \alpha^2(x, t) dx \leq 3 \left[(1-q)^2 \|\dot{r}\|_{L^\infty[0, \infty)} + k^2 (\|r\|_{L^\infty[0, \infty)} + M)^2 + k^2 \|r\|_{L^\infty[0, \infty)}^2 \right] < +\infty. \quad (2.20)$$

It follows from (2.17), (2.18) and (2.20) that $\sup_{t \geq 0} \|(\alpha(\cdot, t), \beta(\cdot, t), \sigma(t))\|_{\mathbb{H}} < +\infty$. \square

Remark 2.2. The proportional gain k in (2.3) cannot be large. From the proof of Lemma 2.1 and ([8], p. 40), if $|k| > \frac{\pi}{2}|1-q|$, the equation (2.16) is unstable and $|e_y(t)| = |w(0, t) - r(t)|$ maybe tends to infinity. In this aspect, our controller (2.3) is a special low-gain controller.

Corollary 2.3. *Suppose that the gain k satisfies $\frac{k}{1-q} \in (0, \pi/2)$. The operator A given by (2.7) generates an exponentially stable C_0 -semigroup e^{At} .*

Proof. Obviously, it suffices to prove the solution of (2.4) is exponentially stable whenever the reference signal $r(t) \equiv 0$. Actually, by Lemma 2.1, $|\sigma(t)| \leq M e^{-\mu t}$ for all $t \geq 0$ holds with some $M, \mu > 0$. By (2.15) and (2.19), we have

$$\int_0^1 \left[\alpha^2(x, t) + \beta^2(x, t) \right] dx \leq k^2 M^2 e^{-2\mu(t-2)} + k^2 M^2 e^{-2\mu(t-1)}, \quad \forall t > 2, \quad (2.21)$$

which yields that the solution of (2.4) is exponentially stable. \square

Remark 2.4. By Corollary 2.3, we actually have obtained a new stabilizing feedback control law for anti-stable wave equation, *i.e.*, the control $u(t) = -w_t(1, t) - kw(0, t)$ with $\frac{k}{1-q} \in (0, \pi/2)$ can stabilize exponentially system (1.1) with $d \equiv 0$, that is,

$$\begin{cases} w_{tt}(x, t) = w_{xx}(x, t), & 0 < x < 1, t > 0, \\ w_x(0, t) = -qw_t(0, t), & t \geq 0, \\ w_x(1, t) = -w_t(1, t) - kw(0, t), & t \geq 0, \end{cases} \quad (2.22)$$

is exponentially stable if $\frac{k}{1-q} \in (0, \pi/2)$. This controller is very simpler than those in [2, 28], where the controller is much complicated, and backstepping approach and the state observer are used. It is worth to note that using our controller, the anti-damping constant q in (1.1) is allowed to be unknown but with known bounds \underline{q} and \bar{q} such that that $q \in [\underline{q}, \bar{q}]$ and either $\underline{q} > 1$ or $\bar{q} < 1$. The proportional gain k is taken as the following ways: if $\underline{q} > 1$, we take k such that $\frac{k}{1-\underline{q}} \in (0, \pi/2)$; if $\bar{q} < 1$, we take k such that $\frac{k}{1-\bar{q}} \in (0, \pi/2)$.

Now, we go back to the closed-loop system of (1.1) without disturbance:

$$\begin{cases} w_{tt}(x, t) = w_{xx}(x, t), & 0 < x < 1, t > 0, \\ w_x(0, t) = -qw_t(0, t), & t \geq 0, \\ w_x(1, t) = -w_t(1, t) + (1-q)\dot{r}(t+1) - k[w(0, t) - r(t)], & t \geq 0. \end{cases} \quad (2.23)$$

Theorem 2.5. *Let k be a constant satisfying $\frac{k}{1-q} \in (0, \pi/2)$, $r \in W^{1, \infty}[0, \infty)$. For any initial value $(w(\cdot, 0), w_t(\cdot, 0)) \in \mathcal{H}$, system (2.23) admits a unique solution $(w, w_t) \in C(0, \infty; \mathcal{H})$ that is bounded, *i.e.*, $\sup_{t \geq 0} \|(w(\cdot, t), w_t(\cdot, t))\|_{\mathcal{H}} < +\infty$. Moreover, the output regulation is exponentially achieved, *i.e.*, there exist two constants $M, \mu > 0$ such that $|y_o(t) - r(t)| \leq M e^{-\mu t}$ for all $t \geq 0$.*

Proof. Using Riemann variables (α, β) given by (2.1), we convert system (2.23) into system (2.4). The initial value of system (2.4) is given by $\alpha(x, 0) = \frac{1-q}{1+q}(w_t(x, 0) - w_x(x, 0)) \in L^2(0, 1)$, $\beta(x, 0) = w_t(x, 0) + w_x(x, 0) \in$

$L^2(0, 1)$ and $\sigma(0) = w(0, 0) - r(0)$. It follows from Lemma 2.1 that (i) system (2.4) has a unique solution and thus system (2.23) has a unique solution; (ii) $|y_o(t) - r(t)| \leq M e^{-\mu t}$ for all $t \geq 0$ with some $M, \mu > 0$. Moreover, by (2.1), we obtain

$$w_t(x, t) = \frac{1}{2} \left[\frac{1+q}{1-q} \alpha(x, t) + \beta(x, t) \right], \quad w_x(x, t) = \frac{1}{2} \left[-\frac{1+q}{1-q} \alpha(x, t) + \beta(x, t) \right], \quad (2.24)$$

which, together with $\|(\alpha(\cdot, t), \beta(\cdot, t))\|_{[L^2(0,1)]^2} < +\infty$, implies that

$$\sup_{t \geq 0} \|w_t(\cdot, t)\|_{L^2(0,1)}, \sup_{t \geq 0} \|w_x(\cdot, t)\|_{L^2(0,1)} < +\infty. \quad (2.25)$$

On the other hand, by Lemma 2.1 again, $\sigma(t) = w(0, t)$ is bounded for all $t \geq 0$, it follows from (2.25) that $\|(w(\cdot, t), w_t(\cdot, t))\|_{\mathcal{H}} < +\infty$. \square

Remark 2.6. If the objective is to design u such that $w_t(0, t) \rightarrow \dot{r}(t)$ without requirement $w(0, t) \rightarrow r(t)$, we can simply use the way in [21] to find a control law $u(t) = (1-q)\dot{r}(t+1)$. In general, the control $u(t) = (1-q)\dot{r}(t+1)$ cannot achieve $\lim_{t \rightarrow \infty} |w(0, t) - r(t)| = 0$.

Remark 2.7. If the reference signal is a constant signal r , then the control law (2.3) becomes $u(t) = -w_t(1, t) - k[w(0, t) - r]$ with k satisfying $\frac{k}{1-q} \in (0, \pi/2)$, which is different from [31] where the control is low gain integral control. Noting that the anti-damping constant q does not appear in the control law, it is seen from Remark 2.4 that for the output $w(0, t)$ to be regulated to a constant signal r , the anti-damping constant q in (1.1) can be unknown but with known bounds \underline{q} and \bar{q} such that that $q \in [\underline{q}, \bar{q}]$ and either $\underline{q} > 1$ or $\bar{q} < 1$.

3. TRACKING CONTROLLER DESIGN

In this section, like the previous section, we assume provisionally that the signal $w_t(1, t)$ is available. This assumption will be removed in the next section. In [37] (also see [36]), the following disturbance estimator of system (1.1):

$$\left\{ \begin{array}{l} v_{tt}(x, t) = v_{xx}(x, t), \quad 0 < x < 1, \quad t > 0, \\ v_x(0, t) = -q v_t(0, t) + c_1 [v(0, t) - w(0, t)], \quad t \geq 0, \\ v_x(1, t) = u(t) - W_x(1, t), \quad t \geq 0, \\ v(x, 0) = v_0(x), \quad v_t(x, 0) = v_1(x), \quad 0 \leq x \leq 1, \\ z_{tt}(x, t) = z_{xx}(x, t), \quad 0 < x < 1, \quad t > 0, \\ z_x(0, t) = \frac{c_1}{1-c_0} z(0, t) + \frac{c_0 - q}{1-c_0} z_t(0, t), \quad t \geq 0, \\ z(1, t) = v(1, t) + W(1, t) - w(1, t), \quad t \geq 0, \\ z(x, 0) = z_0(x), \quad z_t(x, 0) = z_1(x), \quad 0 \leq x \leq 1, \end{array} \right. \quad (3.1)$$

$$\left\{ \begin{array}{l} W_t(x, t) = -W_x(x, t), \quad 0 < x < 1, \\ W(0, t) = -c_0 [v(0, t) - w(0, t)], \quad t > 0, \\ W(x, 0) = W_0(x), \quad 0 \leq x \leq 1, \end{array} \right. \quad (3.2)$$

was used for output feedback exponential stabilization, where c_0 and c_1 are two design parameters so that $\frac{c_1}{1-c_0} > 0$ and $\frac{c_0-q}{1-c_0} > 0$; $(v_0, v_1, z_0, z_1, W_0) \in \mathcal{H}^2 \times H^1(0, 1)$ is the arbitrary initial state. Let

$$\tilde{z}(x, t) = z(x, t) - v(x, t) + w(x, t) - W(x, t). \quad (3.3)$$

Then we can see that $\tilde{z}(x, t)$ is governed by

$$\begin{cases} \tilde{z}_{tt}(x, t) = \tilde{z}_{xx}(x, t), \\ \tilde{z}_x(0, t) = \frac{c_1}{1-c_0}\tilde{z}(0, t) + \frac{c_0-q}{1-c_0}\tilde{z}_t(0, t), \quad \tilde{z}(1, t) = 0, \end{cases} \quad (3.4)$$

which is exponentially stable and serves as a target system for the disturbance estimator. It is seen from [37] that $\tilde{z}_x(1, t) = z_x(1, t) + d(t) \in L^2[0, \infty)$ and $-z_x(1, t)$ can be regarded as an approximated value of $d(t)$.

Let $\alpha(x, t), \beta(x, t)$ be the Riemann variables given by (2.1). Then, system (1.1) is reformulated the new dynamics, that is, $(\alpha(x, t), \beta(x, t), w(0, t))$ is governed by

$$\begin{cases} \alpha_t(x, t) = -\alpha_x(x, t), \quad \beta_t(x, t) = \beta_x(x, t), \\ \alpha(0, t) = \beta(0, t), \quad \beta(1, t) = u(t) + d(t) + w_t(1, t), \\ w_t(0, t) = \frac{1}{1-q}\beta(0, t). \end{cases} \quad (3.5)$$

Since the control law (2.3) regulates system (1.1) without disturbance, and we have estimated the unknown disturbance $d(t)$ and its estimated value is $-z_x(1, t)$, naturally, we propose an disturbance-estimator-based feedback controller as follows:

$$u(t) = z_x(1, t) - w_t(1, t) + (1-q)\dot{r}(t+1) - k[w(0, t) - r(t)]. \quad (3.6)$$

It is seen that the first term in the right side of (3.6) is used to cancel the effect of the disturbance which is just the estimation/cancellation strategy, the last term $-k[w(0, t) - r(t)]$ is the error proportional feedback. Let $\sigma(t) = w(0, t)$. Under feedback (3.6), the closed-loop system of (3.5) becomes

$$\begin{cases} \alpha_t(x, t) = -\alpha_x(x, t), \quad \beta_t(x, t) = \beta_x(x, t), \\ \alpha(0, t) = \beta(0, t), \\ \beta(1, t) = z_x(1, t) + d(t) + (1-q)\dot{r}(t+1) - ke_y(t) \\ \quad = \tilde{z}_x(1, t) + (1-q)\dot{r}(t+1) - k[\sigma(t) - r(t)], \\ \dot{\sigma}(t) = \frac{1}{1-q}\beta(0, t). \end{cases} \quad (3.7)$$

Lemma 3.1. *Suppose that the signal $\tilde{z}_x(1, t)$ is generated by system (3.4), and $d \in L^\infty[0, \infty)$, $r \in W^{1,\infty}[0, \infty)$. Let k be a constant satisfying $\frac{k}{1-q} \in (0, \pi/2)$. For any initial value $(\alpha(\cdot, 0), \beta(\cdot, 0), \sigma(0)) \in \mathbb{H}$, system (3.7) admits a unique solution $(\alpha, \beta, \sigma) \in C(0, \infty; \mathbb{H})$ that is bounded, i.e., there exists a constant $M > 0$ such that $\|(\alpha(\cdot, t), \beta(\cdot, t), \sigma(t))\|_{\mathbb{H}} \leq M$. Moreover, $\lim_{t \rightarrow \infty} |\sigma(t) - r(t)| = 0$.*

Proof. We first claim that system (3.7) has a unique solution that is bounded. To this end, we re-write system (3.7) as

$$\frac{d}{dt}[\alpha(\cdot, t), \beta(\cdot, t), \sigma(t)] = A[\alpha(\cdot, t), \beta(\cdot, t), \sigma(t)] + Bg(t), \quad (3.8)$$

where the operators A, B are given by (2.7), and $g(t) := (1 - q)\dot{r}(t + 1) + kr(t) + \tilde{z}_x(1, t) \in L^2_{loc}(0, \infty)$. Since A generates a C_0 -semigroup e^{At} and B is admissible for e^{At} that are justified in Lemma 2.1, it follows that (3.8) admits a unique solution $(\alpha(\cdot, t), \beta(\cdot, t), \sigma(t)) \in C(0, \infty; \mathbb{H})$. Noting that $(1 - q)\dot{r}(t + 1) + kr(t) \in L^\infty[0, \infty)$, $\tilde{z}_x(1, t) \in L^2[0, \infty)$, e^{At} is exponentially stable due to Corollary 2.3, it follows from ([37], Lem. 2.1) that the solution of (3.8) is bounded.

Next, we show $\lim_{t \rightarrow \infty} |\sigma(t) - r(t)| = 0$. Observing that

$$\beta(x, t) = \begin{cases} (1 - q)\dot{r}(t + x) - k[\sigma(t + x - 1) - r(t + x - 1)] + \tilde{z}_x(1, t + x - 1), & t + x \geq 1, \\ \beta_0(t + x), & t + x < 1, \end{cases}$$

solve the “ β -part” of (3.7). Thus, $\beta(0, t) = (1 - q)\dot{r}(t) - k[\sigma(t - 1) - r(t - 1)] + \tilde{z}_x(1, t - 1)$ for $t \geq 1$. Substituting $\beta(0, t)$ into the last equation of (3.7), and noting $e_y(t) = \sigma(t) - r(t)$, we obtain that $e_y(t)$ satisfies

$$\dot{e}_y(t) = -\frac{k}{1 - q}e_y(t - 1) + \frac{1}{1 - q}\tilde{z}_x(1, t - 1), \quad t > 1. \quad (3.9)$$

To prove $\lim_{t \rightarrow \infty} |e_y(t)| = 0$, we consider (3.9) in the state space $\mathcal{H}_0 = \mathbb{R} \times L^2(-1, 0)$, and reformulate (3.9) as

$$\dot{\chi}(t) = A_0\chi(t) + B_0\tilde{z}_x(1, t - 1), \quad (3.10)$$

where $\chi(t) = (e_y(t), e_y(t + \cdot))$, the operators A_0, B_0 are given by

$$\begin{cases} A_0[h, \phi(\cdot)] = \left[-\frac{k}{1 - q}\phi(-1), \frac{d\phi}{d\theta} \right], \\ \mathcal{D}(A_0) = \{ [h, \phi(\cdot)] \in \mathcal{H}_0 : \phi \in H^1(-1, 0), \phi(0) = h \} \end{cases} \quad (3.11)$$

and $B_0 = [1/(1 - q), 0]$. It follows from ([3], Chap. 2, Thm. 2.4.6) that the operator A_0 generates a C_0 -semigroup $e^{A_0 t}$. Since $\frac{k}{1 - q} \in (0, \pi/2)$, the solution of $\dot{e}_y(t) = -\frac{k}{1 - q}e_y(t - 1)$ is exponentially stable. It follows that $e^{A_0 t}$ is also exponentially stable. By the boundedness of B_0 , B_0 is admissible for $e^{A_0 t}$. Since $\tilde{z}_x(1, t) \in L^2[0, \infty)$ and thus $\tilde{z}_x(1, t - 1) \in L^2[0, \infty)$, it follows from ([37], Lem. 2.1 or [36], Lem. 2.1) that system (3.10) admits a unique solution $(e_y(t), e_y(t + \cdot)) \in C(0, \infty; \mathcal{H}_0)$ satisfying $\lim_{t \rightarrow \infty} \|(e_y(t), e_y(t + \cdot))\|_{\mathcal{H}_0} = 0$, which yields $\lim_{t \rightarrow \infty} |e_y(t)| = 0$. \square

We go back to the closed-loop system (1.1) under the feedback (3.6):

$$\left\{ \begin{array}{l} w_{tt}(x, t) = w_{xx}(x, t), \\ w_x(0, t) = -qw_t(0, t), \\ w_x(1, t) = z_x(1, t) - w_t(1, t) + (1-q)\dot{r}(t+1) - k[w(0, t) - r(t)] + d(t), \\ v_{tt}(x, t) = v_{xx}(x, t), \\ v_x(0, t) = -qv_t(0, t) + c_1[v(0, t) - w(0, t)], \\ v_x(1, t) = z_x(1, t) - w_t(1, t) + (1-q)\dot{r}(t+1) - k[w(0, t) - r(t)] - W_x(1, t), \\ z_{tt}(x, t) = z_{xx}(x, t), \\ z_x(0, t) = \frac{c_1}{1-c_0}z(0, t) + \frac{c_0-q}{1-c_0}z_t(0, t), \\ z(1, t) = v(1, t) + W(1, t) - w(1, t), \\ W_t(x, t) = -W_x(x, t), \quad W(0, t) = -c_0[v(0, t) - w(0, t)]. \end{array} \right. \quad (3.12)$$

Theorem 3.2. *Suppose that $\frac{c_0-q}{1-c_0} > 0$, $\frac{c_1}{1-c_0} > 0$, $d \in L^\infty[0, \infty)$, $r \in W^{1,\infty}[0, \infty)$, and k is a constant satisfying $\frac{k}{1-q} \in (0, \pi/2)$. For any initial value $(w(\cdot, 0), w_t(\cdot, 0), v(\cdot, 0), v_t(\cdot, 0), z(\cdot, 0), z_t(\cdot, 0), W(\cdot, 0)) \in \mathcal{H}^3 \times H^1(0, 1)$ with the compatibility conditions*

$$z(1, 0) - v(1, 0) - W(1, 0) + w(1, 0) = 0, \quad W(0, 0) + c_0[v(0, 0) - w(0, 0)] = 0,$$

system (3.12) admits a unique solution $(w, w_t, v, v_t, z, z_t, W) \in C(0, \infty; \mathcal{H}^3 \times H^1(0, 1))$ that is bounded, i.e., $\|(w(\cdot, t), w_t(\cdot, t), v(\cdot, t), v_t(\cdot, t), z(\cdot, t), z_t(\cdot, t), W(\cdot, t))\|_{\mathcal{H}^3 \times H^1(0, 1)} \leq M$ with some $M > 0$ for all $t \geq 0$. Moreover, the output regulation is achieved, i.e., $\lim_{t \rightarrow \infty} |y_o(t) - r(t)| = 0$.

Proof. Introduce variables

$$\hat{v}(x, t) = v(x, t) - w(x, t), \quad \tilde{z}(x, t) = z(x, t) - \hat{v}(x, t) - W(x, t). \quad (3.13)$$

By [37], we know that $(\hat{v}(\cdot, t), \hat{v}_t(\cdot, t), W(\cdot, t))$ and $(\tilde{z}(\cdot, t), \tilde{z}_t(\cdot, t))$ are well-defined, and satisfy

$$\sup_{t \geq 0} \|(\hat{v}(\cdot, t), \hat{v}_t(\cdot, t), W(\cdot, t))\|_{\mathcal{H} \times H^1(0, 1)} < +\infty, \quad \|(\tilde{z}(\cdot, t), \tilde{z}_t(\cdot, t))\|_{\mathcal{H}} \leq M_1 e^{-\mu_1 t}, \quad (3.14)$$

with $M_1, \mu_1 > 0$. On the other hand, by Lemma 3.1, $\sup_{t \geq 0} \|(\alpha(\cdot, t), \beta(\cdot, t), \sigma(t))\|_{\mathbb{H}} < +\infty$ and $\lim_{t \rightarrow \infty} |\sigma(t) - r(t)| = 0$, which gives $\lim_{t \rightarrow \infty} |y_o(t) - r(t)| = 0$. Noting that $\sigma(t) = w(0, t)$ and

$$w_t(x, t) = \frac{1}{2} \left[\frac{1+q}{1-q} \alpha(x, t) + \beta(x, t) \right], \quad w_x(x, t) = \frac{1}{2} \left[-\frac{1+q}{1-q} \alpha(x, t) + \beta(x, t) \right],$$

which, together with $\sup_{t \geq 0} \|(\alpha(\cdot, t), \beta(\cdot, t), \sigma(t))\|_{\mathbb{H}} < +\infty$, implies that

$$\sup_{t \geq 0} \|w_t(\cdot, t)\|_{L^2(0, 1)}, \quad \sup_{t \geq 0} \left[|w(0, t)|^2 + \|w_x(\cdot, t)\|_{L^2(0, 1)}^2 \right] < +\infty. \quad (3.15)$$

Thus, $(w(\cdot, t), w_t(\cdot, t))$ is well-defined and $\sup_{t \geq 0} \|(w(\cdot, t), w_t(\cdot, t))\|_{\mathcal{H}} < +\infty$. It follows from (3.13) and (3.14) that $(v(\cdot, t), v_t(\cdot, t))$ and $(z(\cdot, t), z_t(\cdot, t))$ are well-defined and $\sup_{t \geq 0} \|(v(\cdot, t), v_t(\cdot, t))\|_{\mathcal{H}} < +\infty$, $\sup_{t \geq 0} \|(z(\cdot, t), z_t(\cdot, t))\|_{\mathcal{H}} < +\infty$. \square

From Theorem 3.2, the output signal to be controlled is asymptotically tracking the reference signal, because $\tilde{z}_x \in L^2[0, \infty)$ and the disturbance are not completely rejected. To completely reject the disturbance, we need the state observer to overcome this difficulty, which is the subject of the next section.

4. A STATE OBSERVER BASED TRACKING CONTROLLER DESIGN

In this section, we employ the following state observer ([37]) for system (1.1)

$$\begin{cases} \hat{w}_{tt}(x, t) = \hat{w}_{xx}(x, t), & 0 < x < 1, t > 0, \\ \hat{w}_x(0, t) = -q\hat{w}_t(0, t) + c_1[\hat{w}(0, t) - w(0, t)], & t \geq 0, \\ \hat{w}_x(1, t) = u(t) - z_x(1, t) - Y_x(1, t), & t \geq 0, \\ \hat{w}(x, 0) = \hat{w}_0(x), \hat{w}_t(x, 0) = \hat{w}_1(x), & 0 \leq x \leq 1, \\ Y_t(x, t) = -Y_x(x, t), & 0 < x < 1, t > 0, \\ Y(0, t) = -c_0[\hat{w}(0, t) - w(0, t)], & t \geq 0, \\ Y(x, 0) = Y_0(x), & 0 \leq x \leq 1, \end{cases} \quad (4.1)$$

where c_1 and c_2 are two design parameters that are the same as in (3.1) and (3.2). $-z_x(1, t)$ plays the role of disturbance. Let $\varepsilon(x, t) = \hat{w}(x, t) - w(x, t)$. Then, it is easy to see that $(\varepsilon(x, t), Y(x, t))$ is governed by

$$\begin{cases} \varepsilon_{tt}(x, t) = \varepsilon_{xx}(x, t), \\ \varepsilon_x(0, t) = -q\varepsilon_t(0, t) + c_1\varepsilon(0, t), \quad \varepsilon_x(1, t) = -\tilde{z}_x(1, t) - Y_x(1, t), \\ Y_t(x, t) = -Y_x(x, t), \quad Y(0, t) = -c_0\varepsilon(0, t). \end{cases} \quad (4.2)$$

The following lemma follows from [37].

Lemma 4.1. *Suppose that $\frac{c_1}{1-c_0} > 0$, $\frac{c_0-q}{1-c_0} > 0$. Let $\tilde{z}_x(1, t)$ be generated by system (3.4). For any initial value $(\varepsilon(\cdot, 0), \varepsilon_t(\cdot, 0), Y(\cdot, 0)) \in \mathcal{H} \times H^1(0, 1)$, with compatibility condition $Y_0(0) = -c_0\varepsilon_0(0)$, then system (4.2) admits a unique solution $(\varepsilon, \varepsilon_t, Y) \in C(0, \infty; \mathcal{H} \times H^1(0, 1))$ satisfying $\|(\varepsilon(\cdot, t), \varepsilon_t(\cdot, t), Y(\cdot, t))\|_{\mathcal{H} \times H^1(0, 1)} \leq Me^{-\mu t}$, with some $M, \mu > 0$.*

By Lemma 4.1, to regulate system (1.1), it suffices to find a control law for system (4.1) such that (i) all the internal signals of (4.1) should be bounded; (ii) the tracking error $\hat{e}_y = \hat{w}(0, t) - r(t)$ satisfies for some $M, \mu > 0$,

$$|\hat{e}_y(t)| = |\hat{w}(0, t) - r(t)| \leq Me^{-\mu t}, \quad \text{for all } t \geq 0. \quad (4.3)$$

Similar to Section 2, we introduce the following Riemann variables

$$\zeta(x, t) = \frac{1-q}{1+q}[\hat{w}_t(x, t) - \hat{w}_x(x, t)], \quad \eta(x, t) = \hat{w}_t(x, t) + \hat{w}_x(x, t). \quad (4.4)$$

Then, system (4.1) is reformulated the new dynamics, that is, $(\zeta(x, t), \eta(x, t), \widehat{w}(0, t))$ satisfies

$$\begin{cases} \zeta_t(x, t) = -\zeta_x(x, t), & \eta_t(x, t) = \eta_x(x, t), \\ \zeta(0, t) = \eta(0, t) - [2 - q]c_1\varepsilon(0, t), \\ \eta(1, t) = u(t) - z_x(1, t) - Y_x(1, t) + \widehat{w}_t(1, t), \\ \widehat{w}_t(0, t) = \frac{1}{1 - q}\eta(0, t) - \frac{c_1}{1 - q}\varepsilon(0, t). \end{cases} \quad (4.5)$$

Now, we propose the following feedback controller

$$u(t) = z_x(1, t) + Y_x(1, t) - \widehat{w}_t(1, t) + (1 - q)\dot{r}(t + 1) - k[\widehat{w}(0, t) - r(t)]. \quad (4.6)$$

It should be noted that the control $u(t)$ is implemented based on the boundary value $\widehat{w}_t(1, t), Y_x(1, t)$ of the solution $\widehat{w}(x, t), Y(x, t)$ of the state observer system (4.1) and on the boundary value $z_x(1, t)$ of the solution $z(x, t)$ of the disturbance observer system (3.1)–(3.2). Thus, this controller is implementable. Set $\gamma(t) = \widehat{w}(0, t)$. Under feedback (4.6), the closed-loop of observer (4.1) becomes

$$\begin{cases} \zeta_t(x, t) = -\zeta_x(x, t), & \eta_t(x, t) = \eta_x(x, t), \\ \zeta(0, t) = \eta(0, t) - [2 - q]c_1\varepsilon(0, t), \\ \eta(1, t) = (1 - q)\dot{r}(t + 1) - k[\gamma(t) - r(t)], \\ \dot{\gamma}(t) = \frac{1}{1 - q}\eta(0, t) - \frac{c_1}{1 - q}\varepsilon(0, t). \end{cases} \quad (4.7)$$

We consider system (4.7) in the state space $\mathbb{H} = [L^2(0, 1)]^2 \times \mathbb{R}$.

Lemma 4.2. *Suppose that k is a constant satisfying $\frac{k}{1 - q} \in (0, \pi/2)$, and $\varepsilon(0, t)$ is generated by system (4.2). For any initial value $(\zeta(\cdot, 0), \eta(\cdot, 0), \gamma(0)) \in \mathbb{H}$, system (4.7) admits a unique solution $(\zeta, \eta, \gamma) \in C(0, \infty; \mathbb{H})$ that is bounded, i.e., $\|(\zeta(\cdot, t), \eta(\cdot, t), \gamma(t))\|_{\mathbb{H}} < +\infty$. Moreover, there exist two constants $M, \mu > 0$ such that $|\gamma(t) - r(t)| \leq Me^{-\mu t}$ for all $t \geq 0$. Further, we assume that $r \equiv 0$, then $\|(\zeta(\cdot, t), \eta(\cdot, t), \gamma(t))\|_{\mathbb{H}} \leq M'e^{-\mu' t}$ with some $M', \mu' > 0$.*

Proof. We first claim that system (4.7) has a unique solution and its solution is bounded. For this purpose, we re-write system (4.7) as

$$\frac{d}{dt}(\zeta(\cdot, t), \eta(\cdot, t), \gamma(t)) = A(\zeta(\cdot, t), \eta(\cdot, t), \gamma(t)) + Bg(t) + B_1\varepsilon(0, t) + B_2\varepsilon(0, t), \quad (4.8)$$

where the operators A, B are given by (2.7), the operators B_1, B_2 are defined by $B_1 = [(2 - q)c_1\delta(x), 0, 0]$, $B_2 = -\frac{c_1}{1 - q}[0, 0, 1]$, and $g(t) := (1 - q)\dot{r}(t + 1) + kr(t) \in L^\infty[0, \infty)$. Since B_2 is bounded operator, B_2 is admissible for e^{At} . By Lemma 2.1, B is admissible for e^{At} . Similar to the proof of Lemma 2.1, it is easy to verify that B_1 is also admissible for e^{At} . In view of Lemma 4.1 and Sobolev trace theorem, $|\varepsilon(0, t)| \leq M_0e^{-\mu_0 t}$ with some $M_0, \mu_0 > 0$ for all $t \geq 0$. Since e^{At} is exponentially stable, it follows from ([37], Lem. 2.1, or [36], Lem. 2.1) that system (4.8) admits a unique solution that is bounded.

Next, we claim the last assertion. Note that

$$\eta(x, t) = \begin{cases} (1 - q)\dot{r}(t + x) - k[\gamma(t + x - 1) - r(t + x - 1)], & t + x \geq 1, \\ \eta_0(t + x), & t + x < 1, \end{cases} \quad (4.9)$$

solve the “ η -part” of (4.7) provided that $\gamma(t)$ is known. Thus, in this case, $\eta(0, t) = (1 - q)\dot{r}(t) - k(\gamma(t - 1) - r(t - 1))$ for $t \geq 1$. Substituting $\eta(0, t)$ into the last equation of (4.7), and noting $\widehat{e}_y(t) = \gamma(t) - r(t)$, we obtain

$$\dot{\widehat{e}}_y(t) = -\frac{k}{1-q}\widehat{e}_y(t-1) - \frac{c_1}{1-q}\varepsilon(0, t), \quad t > 1. \quad (4.10)$$

Similar to the way in Lemma 3.1, we reformulate (4.10) as

$$\dot{\chi}(t) = A_0\chi(t) + B_0\varepsilon(0, t), \quad (4.11)$$

where $\chi(t) = (\widehat{e}_y(t), \widehat{e}_y(t + \cdot))$, the operators A_0, B_0 are given by (3.11). Since $e^{A_0 t}$ is also exponentially stable and B_0 is admissible for $e^{A_0 t}$, it follows from $|\varepsilon(0, t)| \leq M_0 e^{-\mu_0 t}$ with some $M_0, \mu_0 > 0$ and ([37], Lem. 2.1) that system (4.11) admits a unique solution $(\widehat{e}_y(t), \widehat{e}_y(t + \cdot)) \in C(0, \infty; \mathcal{H}_0)$ satisfying $\|(\widehat{e}_y(t), \widehat{e}_y(t + \cdot))\|_{\mathcal{H}_0} \leq M e^{-\mu t}$ with some $M, \mu > 0$, which yields $|\widehat{e}_y(t)| \leq M e^{-\mu t}$.

Finally, suppose that $r \equiv 0$, then $g \equiv 0$ in (4.8). Since $e^{A t}$ is exponentially stable and $|\varepsilon(0, t)| \leq M_0 e^{-\mu_0 t}$, it follows from Lemma ([37], Lem. 2.1) that the solution of system (4.8) is exponentially stable. \square

We go back to the closed-loop system (1.1) under the feedback (4.6)

$$\left\{ \begin{array}{l} w_{tt}(x, t) = w_{xx}(x, t), \\ w_x(0, t) = -q w_t(0, t), \\ w_x(1, t) = z_x(1, t) + Y_x(1, t) - \widehat{w}_t(1, t) + (1 - q)\dot{r}(t + 1) - k[\widehat{w}(0, t) - r(t)] + d(t), \\ v_{tt}(x, t) = v_{xx}(x, t), \\ v_x(0, t) = -q v_t(0, t) + c_1[v(0, t) - w(0, t)], \\ v_x(1, t) = z_x(1, t) + Y_x(1, t) - \widehat{w}_t(1, t) + (1 - q)\dot{r}(t + 1) - k[\widehat{w}(0, t) - r(t)] - W_x(1, t), \\ z_{tt}(x, t) = z_{xx}(x, t), \\ z_x(0, t) = \frac{c_1}{1 - c_0} z(0, t) + \frac{c_0 - q}{1 - c_0} z_t(0, t), \\ z(1, t) = v(1, t) + W(1, t) - w(1, t), \\ W_t(x, t) = -W_x(x, t), \quad W(0, t) = -c_0[v(0, t) - w(0, t)], \\ \widehat{w}_{tt}(x, t) = \widehat{w}_{xx}(x, t), \\ \widehat{w}_x(0, t) = -q \widehat{w}_t(0, t) + c_1[\widehat{w}(0, t) - w(0, t)], \\ \widehat{w}_x(1, t) = -\widehat{w}_t(1, t) + (1 - q)\dot{r}(t + 1) - k[\widehat{w}(0, t) - r(t)], \\ Y_t(x, t) = -Y_x(x, t), \quad Y(0, t) = -c_0[\widehat{w}(0, t) - w(0, t)]. \end{array} \right. \quad (4.12)$$

We consider system (4.12) in the state space $\mathcal{H}^4 \times [H^1(0, 1)]^2$.

Theorem 4.3. *Suppose that $\frac{c_1}{1 - c_0} > 0$, $\frac{c_0 - q}{1 - c_0} > 0$, and $d \in L^\infty[0, \infty)$, $r \in W^{1, \infty}[0, \infty)$. Suppose that k is a constant satisfying $\frac{k}{1 - q} \in (0, \pi/2)$. For any initial value $(w(\cdot, 0), w_t(\cdot, 0), \widehat{w}(\cdot, 0), \widehat{w}_t(\cdot, 0), v(\cdot, 0), v_t(\cdot, 0), z(\cdot, 0), z_t(\cdot, 0), W(\cdot, 0), Y(\cdot, 0)) \in \mathcal{H}^4 \times [H^1(0, 1)]^2$ with the compatibility conditions*

$$z(1, 0) - v(1, 0) - W(1, 0) + w(1, 0) = 0, \quad W(0, 0) + c_0[v(0, 0) - w(0, 0)] = 0,$$

$$Y(0, 0) + c_0[\widehat{w}(0, 0) - w(0, 0)] = 0,$$

system (4.12) admits a unique solution $(w, w_t, \widehat{w}, \widehat{w}_t, v, v_t, z, z_t, W, Y) \in C(0, \infty; \mathcal{H}^4 \times [H^1(0, 1)]^2)$ that is bounded, i.e., $\sup_{t \geq 0} \|(w(\cdot, t), w_t(\cdot, t), \widehat{w}(\cdot, t), \widehat{w}_t(\cdot, t), v(\cdot, t), v_t(\cdot, t), z(\cdot, t), z_t(\cdot, t), W(\cdot, t), Y(\cdot, t))\|_{\mathcal{H}^4 \times [H^1(0, 1)]^2} \leq M_0$ with some $M_0 > 0$. Moreover, the output regulation is achieved, i.e., there exist two constants $M, \mu > 0$ such that $|y_o(t) - r(t)| \leq Me^{-\mu t}$ for all $t \geq 0$.

Proof. Let $\varepsilon(x, t) = \widehat{w}(x, t) - w(x, t)$ and let $\widehat{v}(x, t)$ and $\widehat{z}(x, t)$ be given by (3.13). By [37], we know that $(\varepsilon(\cdot, t), \varepsilon_t(\cdot, t), Y(\cdot, t), (\widehat{v}(\cdot, t), \widehat{v}_t(\cdot, t), W(\cdot, t)))$ and $(\widehat{z}(\cdot, t), \widehat{z}_t(\cdot, t))$ are well-defined, and satisfy

$$\begin{aligned} \sup_{t \geq 0} \|(\widehat{v}(\cdot, t), \widehat{v}_t(\cdot, t), W(\cdot, t))\|_{\mathcal{H} \times H^1(0, 1)} &< +\infty, \\ \|(\varepsilon(\cdot, t), \varepsilon_t(\cdot, t), Y(\cdot, t), \widehat{z}(\cdot, t), \widehat{z}_t(\cdot, t))\|_{\mathcal{H}} &\leq M_1 e^{-\mu_1 t}. \end{aligned} \quad (4.13)$$

With $M_1, \mu_1 > 0$. By Lemma 4.2, $\sup_{t \geq 0} \|(\zeta(\cdot, t), \eta(\cdot, t), \gamma(t))\|_{\mathbb{H}} < +\infty$ and $|\gamma(t) - r(t)| \leq M_1 e^{-\mu_1 t}$ with some $M_1, \mu_1 > 0$, which gives $|\widehat{w}(0, t) - r(t)| \leq M_1 e^{-\mu_1 t}$. On the other hand, by the second equation of (4.13) and Sobolev trace theorem, $|\varepsilon(0, t)| \leq M_2 e^{-\mu_2 t}$ with some $M_2, \mu_2 > 0$ for all $t \geq 0$. Thus, we have

$$|y_o(t) - r(t)| \leq |\varepsilon(0, t)| + |\widehat{w}(0, t) - r(t)| \leq M e^{-\mu t}, \quad \forall t \geq 0,$$

with $M = M_1 + M_2$, $\mu = \min\{\mu_1, \mu_2\}$. Noting that $\gamma(t) = \widehat{w}(0, t)$ and

$$\widehat{w}_t(x, t) = \frac{1}{2} \left[\frac{1+q}{1-q} \zeta(x, t) + \eta(x, t) \right], \quad \widehat{w}_x(x, t) = \frac{1}{2} \left[-\frac{1+q}{1-q} \zeta(x, t) + \eta(x, t) \right], \quad (4.14)$$

which, together with $\sup_{t \geq 0} \|(\zeta(\cdot, t), \eta(\cdot, t), \gamma(t))\|_{\mathbb{H}} < +\infty$, implies that

$$\sup_{t \geq 0} \|\widehat{w}_t(\cdot, t)\|_{L^2(0, 1)}, \sup_{t \geq 0} \left[|\widehat{w}(0, t)|^2 + \|\widehat{w}_x(\cdot, t)\|_{L^2(0, 1)}^2 \right] < +\infty. \quad (4.15)$$

Thus, $\sup_{t \geq 0} \|(\widehat{w}(\cdot, t), \widehat{w}_t(\cdot, t))\|_{\mathcal{H}} < +\infty$, which, jointly with (3.13) and (4.13), yields

$$\sup_{t \geq 0} \|(w(\cdot, t), w_t(\cdot, t), v(\cdot, t), v_t(\cdot, t), z(\cdot, t), z_t(\cdot, t))\|_{\mathcal{H}^3} < +\infty.$$

□

Remark 4.4. Compared with the result in [12] where the adopted method is the Lyapunov functional method based variable structure design method and the initial state is required to be smooth, proportional gain on smooth initial state is required.

Remark 4.5. From the closed-loop system (4.12), the output measurement signals only are $w(1, t)$ and the tracking error signal $e_y(t)$. We do not need to measure the output to be controlled $w(0, t)$. This is because the reference signal $r(t)$ is known, and $v(0, t) - w(0, t) = v(0, t) - r(t) - (w(0, t) - r(t)) = v(0, t) - r(t) - e_y(t)$, $\widehat{w}(0, t) - w(0, t) = \widehat{w}(0, t) - r(t) - (w(0, t) - r(t)) = \widehat{w}(0, t) - r(t) - e_y(t)$.

Output feedback stabilization for wave equation is a challenging issues (see [15, 28]). By coincidence, Theorem 4.3 gives a byproduct which elegantly solves this output feedback exponential stabilization problem. We have the following corollary.

Corollary 4.6. Suppose that the reference signal $r \equiv 0$. Suppose that $\frac{c_1}{1-c_0} > 0$, $\frac{c_0-q}{1-c_0} > 0$ and $d \in L^\infty[0, \infty)$. Suppose that k is a constant satisfying $\frac{k}{1-q} \in (0, \pi/2)$. For any initial value $(w(\cdot, 0), w_t(\cdot, 0), \widehat{w}(\cdot, 0), \widehat{w}_t(\cdot, 0), v(\cdot, 0), v_t(\cdot, 0), z(\cdot, 0), z_t(\cdot, 0), W(\cdot, 0), Y(\cdot, 0)) \in \mathcal{H}^4 \times H^2(0, 1)$, the solution of system (4.12) satisfies $\|(w(\cdot, t), w_t(\cdot, t), \widehat{w}(\cdot, t), \widehat{w}_t(\cdot, t), Y(\cdot, t))\|_{\mathcal{H}^2 \times H^1(0, 1)} \leq M e^{-\mu t}$ with some $M, \mu > 0$ for all $t \geq 0$, and $\|(v(\cdot, t), v_t(\cdot, t), z(\cdot, t), z_t(\cdot, t), W(\cdot, t))\|_{\mathcal{H}^2 \times H^1(0, 1)} < +\infty$.

Proof. When $r \equiv 0$, Lemma 4.2 and (4.14) imply that $\|(\widehat{w}(\cdot, t), \widehat{w}_t(\cdot, t))\|_{\mathcal{H}} \leq M_1 e^{-\mu_1 t}$ for some $M_1, \mu_1 > 0$. With this, the conclusions follow from (3.13) and (4.13). \square

Remark 4.7. In the light of Corollary 4.6, we actually have designed a new output feedback exponentially stabilizing control law for system (1.1) using only two displacement signals $\{w(0, t), w(1, t)\}$. The new controller is given by

$$u(t) = z_x(1, t) + Y_x(1, t) - \widehat{w}_t(1, t) - k\widehat{w}(0, t), \quad (4.16)$$

which is different from the controller in [37], where an extra dynamics “Z system” is needed to stabilize system (1.1). The controller (4.16) is also different the one in [15], where the controller is not only discontinuous but also need more measurements signals than ours, and the initial state is required to be smooth. Thus, the controller (4.16) is more simple and easily implementable.

Remark 4.8. If the disturbance $d(t)$ in (1.1) is replaced by the total disturbance $f(w(\cdot, t), w_t(\cdot, t)) + d(t)$ where $f \in C(H^1(0, 1) \times L^2(0, 1); \mathbb{R})$ is an unknown possibly nonlinear mapping that represents the *internal uncertainty* in the model, and d represents the unknown *external disturbance*, the main results (Thm. 4.3 and Cor. 4.6) of this paper are still true.

5. ROBUSTNESS TO THE BOUNDED ERRORS IN THE ESTIMATION OF $d(t)$

In Sections 3 and 4, we have regarded $-z_x(1, t)$ as the estimation of $d(t)$, *i.e.*, $-z_x(1, t) \approx d(t)$. Here, $z_x(1, t)$ is the output of the disturbance estimator (3.1)–(3.2). In this section, we study the robustness of the feedback law (4.6) to a small bounded errors in the estimation $d(t)$. More precisely, we consider the disturbance estimation $-z_x(1, t)$ being varying into $-z_x(1, t) + \delta(t)$, where $\delta(t)$ is a function with small value, and then we verify whether tracking error $e_y = y_o - r$ is small when the computation error $\|\delta\|_{L^\infty(0, \infty)}$ is small. Replacing $-z_x(1, t)$ by $-z_x(1, t) + \delta(t)$ in (4.12), the closed-loop system (4.12) becomes

$$\left\{ \begin{array}{l} w_{tt}(x, t) = w_{xx}(x, t), \\ w_x(0, t) = -qw_t(0, t), \\ w_x(1, t) = z_x(1, t) - \delta(t) + Y_x(1, t) - \widehat{w}_t(1, t) + (1 - q)\dot{r}(t + 1) - k[\widehat{w}(0, t) - r(t)] + d(t), \\ v_{tt}(x, t) = v_{xx}(x, t), \\ v_x(0, t) = -qv_t(0, t) + c_1[v(0, t) - w(0, t)], \\ v_x(1, t) = z_x(1, t) - \delta(t) + Y_x(1, t) - \widehat{w}_t(1, t) + (1 - q)\dot{r}(t + 1) - k[\widehat{w}(0, t) - r(t)] - W_x(1, t), \\ z_{tt}(x, t) = z_{xx}(x, t), \\ z_x(0, t) = \frac{c_1}{1 - c_0}z(0, t) + \frac{c_0 - q}{1 - c_0}z_t(0, t), \\ z(1, t) = v(1, t) + W(1, t) - w(1, t), \\ W_t(x, t) = -W_x(x, t), \quad W(0, t) = -c_0[v(0, t) - w(0, t)], \\ \widehat{w}_{tt}(x, t) = \widehat{w}_{xx}(x, t), \\ \widehat{w}_x(0, t) = -q\widehat{w}_t(0, t) + c_1[\widehat{w}(0, t) - w(0, t)], \\ \widehat{w}_x(1, t) = -\widehat{w}_t(1, t) + (1 - q)\dot{r}(t + 1) - k[\widehat{w}(0, t) - r(t)], \\ Y_t(x, t) = -Y_x(x, t), \quad Y(0, t) = -c_0[\widehat{w}(0, t) - w(0, t)]. \end{array} \right. \quad (5.1)$$

The following robustness result with respect to a small δ holds.

Theorem 5.1. *Suppose that $\frac{c_1}{1-c_0} > 0$, $\frac{c_0-q}{1-c_0} > 0$, and $d, \delta \in L^\infty[0, \infty)$, $r \in W^{1,\infty}[0, \infty)$. Suppose that k is a constant satisfying $\frac{k}{1-q} \in (0, \pi/2)$. For any initial value $(w(\cdot, 0), w_t(\cdot, 0), \widehat{w}(\cdot, 0), \widehat{w}_t(\cdot, 0), v(\cdot, 0), v_t(\cdot, 0), z(\cdot, 0), z_t(\cdot, 0), W(\cdot, 0), Y(\cdot, 0)) \in \mathcal{H}^4 \times [H^1(0, 1)]^2$ with the compatibility conditions*

$$z(1, 0) - v(1, 0) - W(1, 0) + w(1, 0) = 0, \quad W(0, 0) + c_0[v(0, 0) - w(0, 0)] = 0,$$

$$Y(0, 0) + c_0[\widehat{w}(0, 0) - w(0, 0)] = 0,$$

system (5.1) admits a unique solution $(w, w_t, \widehat{w}, \widehat{w}_t, v, v_t, z, z_t, W, Y) \in C(0, \infty; \mathcal{H}^4 \times [H^1(0, 1)]^2)$ that is bounded, i.e., $\sup_{t \geq 0} \|(w(\cdot, t), w_t(\cdot, t), \widehat{w}(\cdot, t), \widehat{w}_t(\cdot, t), v(\cdot, t), v_t(\cdot, t), z(\cdot, t), z_t(\cdot, t), W(\cdot, t), Y(\cdot, t))\|_{\mathcal{H}^4 \times [H^1(0, 1)]^2} \leq M_0$ with some $M_0 > 0$. Moreover, the output regulation is robust with respect to δ in the sense that there exist three constants $M_1, M_2, \mu > 0$ such that

$$|y_o(t) - r(t)| \leq M_1 e^{-\mu t} + M_2 \|\delta\|_{L^\infty(0, \infty)}, \quad \text{for all } t \geq 0. \quad (5.2)$$

Proof. Following the proof of Theorem 4.3, we only need to prove the last assertion that output regulation is robust with respect to δ . Let $\varepsilon(x, t) = \widehat{w}(x, t) - w(x, t)$. Then, it is easy to see that $(\varepsilon(x, t), Y(x, t))$ is governed by

$$\begin{cases} \varepsilon_{tt}(x, t) = \varepsilon_{xx}(x, t), \\ \varepsilon_x(0, t) = -q\varepsilon_t(0, t) + c_1\varepsilon(0, t), \quad \varepsilon_x(1, t) = -\tilde{z}_x(1, t) + \delta(t) - Y_x(1, t), \\ Y_t(x, t) = -Y_x(x, t), \quad Y(0, t) = -c_0\varepsilon(0, t). \end{cases}$$

Now, we claim that there exist three constants $C_1, C_2, \mu_0 > 0$ such that

$$|\varepsilon(0, t)| \leq C_1 e^{-\mu_0 t} + C_2 \|\delta\|_{L^\infty(0, \infty)}, \quad \text{for all } t \geq 0. \quad (5.3)$$

Indeed, let $\tilde{\varepsilon}(x, t) = \varepsilon(x, t) + \tilde{z}(x, t) + Y(x, t)$. Then it is seen that $\tilde{\varepsilon}(x, t)$ satisfies

$$\begin{cases} \tilde{\varepsilon}_{tt}(x, t) = \tilde{\varepsilon}_{xx}(x, t), \\ \tilde{\varepsilon}_x(0, t) = \frac{c_1}{1-c_0} \tilde{\varepsilon}(0, t) + \frac{c_0-q}{1-c_0} \tilde{\varepsilon}_t(0, t), \quad \tilde{\varepsilon}_x(1, t) = \delta(t). \end{cases} \quad (5.4)$$

The system (5.4) can be rewritten in abstract form as

$$\frac{d}{dt} \begin{pmatrix} \tilde{\varepsilon}(\cdot, t) \\ \tilde{\varepsilon}_t(\cdot, t) \end{pmatrix} = \mathcal{A} \begin{pmatrix} \tilde{\varepsilon}(\cdot, t) \\ \tilde{\varepsilon}_t(\cdot, t) \end{pmatrix} + \mathcal{B} \delta(t),$$

where the operators \mathcal{A} and \mathcal{B} are given by

$$\begin{cases} \mathcal{A}(\phi, \psi) = (\psi, \phi'') & \forall (\phi, \psi) \in \mathcal{D}(\mathcal{A}), \quad \mathcal{B} = (0, \delta_1), \\ \mathcal{D}(\mathcal{A}) = \left\{ (\phi, \psi) \in H^2(0, 1) \times H^1(0, 1) \mid \phi'(0) = \frac{c_1}{1-c_0} \phi(0) + \frac{c_0-q}{1-c_0} \psi(0), \phi'(1) = 0 \right\}. \end{cases}$$

It is well-known ([16], Thm. 2.1) that \mathcal{A} generates an exponentially stable operator semigroup $e^{\mathcal{A}t}$ on \mathbb{H} and \mathcal{B} is admissible for $e^{\mathcal{A}t}$. It follows from ([37], Lem. 2.1) that there exist three constants $C_3, C_4, \mu_1 > 0$ such that

$$\|(\tilde{\varepsilon}(\cdot, t), \tilde{\varepsilon}_t(\cdot, t))\|_{\mathcal{H}} \leq C_3 e^{-\mu_1 t} + C_4 \|\delta\|_{L^\infty(0, \infty)}, \quad \text{for all } t \geq 0.$$

This, together with the Sobolev trace theorem, implies that

$$|\tilde{\varepsilon}(0, t)| \leq C_5 \|(\tilde{\varepsilon}(\cdot, t), \tilde{\varepsilon}_t(\cdot, t))\|_{\mathcal{H}} \leq C_3 C_5 e^{-\mu_1 t} + C_4 C_5 \|\delta\|_{L^\infty(0, \infty)}, \text{ for all } t \geq 0 \quad (5.5)$$

for some $C_5 > 0$. On the other hand, due to the fact that system (3.4) is exponentially stable, we get $|\tilde{z}(0, t)| \leq C_6 e^{-\mu_2 t}$ for some $C_6, \mu_2 > 0$. Since

$$\varepsilon(0, t) = \frac{1}{1 - c_0} \tilde{\varepsilon}(0, t) - \frac{1}{1 - c_0} \tilde{z}(0, t),$$

by (5.5), we know that (5.3) holds and C_1, C_2, μ_0 can be taken as $C_1 = \max\{C_3 C_5 / (1 - c_0), C_6\}$, $C_2 = C_4 C_5 / (1 - c_0)$ and $\mu_0 = \min\{\mu_1, \mu_2\}$.

Next, we claim that there exist three constants $C_7, C_8, \mu_3 > 0$ such that

$$|\hat{e}_y(t)| \leq C_7 e^{-\mu_3 t} + C_8 \|\delta\|_{L^\infty(0, \infty)}, \text{ for all } t \geq 0,$$

where $\hat{e}_y = \hat{w}(0, t) - r(t)$. Following the proof of Lemma 4.2, we have that $\hat{e}_y(t)$ satisfies

$$\dot{\hat{e}}_y(t) = -\frac{k}{1 - q} \hat{e}_y(t - 1) - \frac{c_1}{1 - q} \varepsilon(0, t), \quad t > 1, \quad (5.6)$$

which can be reformulated (5.6) as

$$\dot{\chi}(t) = A_0 \chi(t) + B_0 \varepsilon(0, t),$$

where $\chi(t) = (\hat{e}_y(t), \hat{e}_y(t + \cdot))$, the operators A_0, B_0 are given by (3.11). Since $e^{A_0 t}$ is also exponentially stable and B_0 is admissible for $e^{A_0 t}$, it follows from (5.3) and ([37], Lem. 2.1) that the solution $(\hat{e}_y(t), \hat{e}_y(t + \cdot)) \in C(0, \infty; \mathcal{H}_0)$ satisfies

$$\|(\hat{e}_y(t), \hat{e}_y(t + \cdot))\|_{\mathcal{H}_0} \leq C_7 e^{-\mu_3 t} + C_8 \|\delta\|_{L^\infty(0, \infty)}$$

with some $C_7, C_8 > 0$ and $\mu_3 > 0$. Hence we get $|\hat{e}_y(t)| \leq C_7 e^{-\mu_3 t} + C_8 \|\delta\|_{L^\infty(0, \infty)}$, which, jointly with (5.3) and the inequality $|y_o(t) - r(t)| \leq |\varepsilon(0, t)| + |\hat{w}(0, t) - r(t)|$, implies that (5.2) holds. \square

From Theorem 5.1, we can say that if the computation error $\|\delta\|_{L^\infty(0, \infty)}$ is small, the tracking error $e_y(t)$ is also small. This can be considered as robustness of approximation of $-z_x(1, t)$ to $d(t)$.

Remark 5.2. Although Theorem 5.1 shows the robustness of the feedback law (4.6) to a small bounded errors in the estimation $d(t)$, due to the fact that we do not take the robustness into account in the design process, we do not know if the designed controller is robust against a small perturbation of the propagation speed meaning that in the closed-loop (4.12), the governing equation is changed from $w_{tt}(x, t) = w_{xx}(x, t)$, into $w_{tt}(x, t) = (1 + \delta)w_{xx}(x, t)$ for small perturbation parameter δ . By robustness, we need to verify that (i) when $d(t) = 0$, $r(t) = 0$, the closed-loop (4.12) is robustly asymptotically stable for all small perturbation parameter δ or δ in a compact set; (ii) for all small perturbation parameter δ or δ in a compact set, the output is again tracking: $|w(0, t) - r(t)| \rightarrow 0$ as $t \rightarrow \infty$. In this paper, we are not able to answer any of these two questions, which need a separate paper to discuss it systematically. For an interesting example on the robustness of the feedback to a small perturbation of the propagation speed, we refer to ([20], Chap. 16, p. 283) where a robust state stabilizing feedback control for ODEs coupled with wave equation with small error in wave propagation speed was designed.

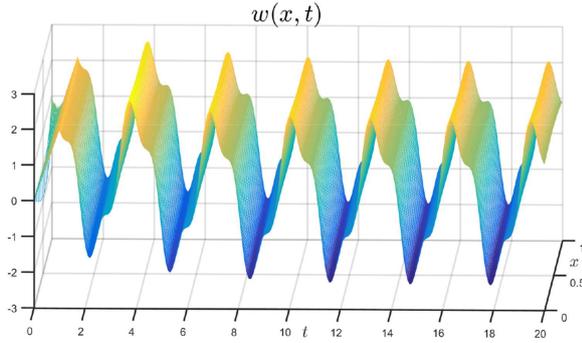
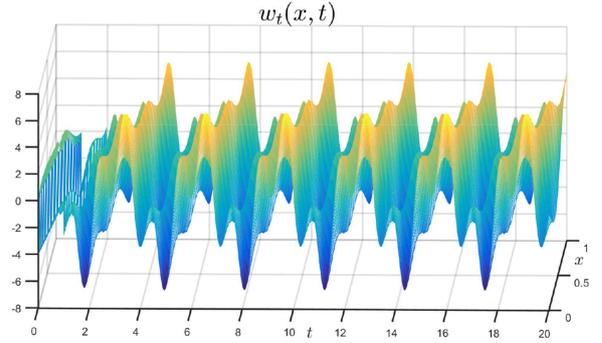

 (a) The displacement $w(x, t)$

 (b) The velocity $w_t(x, t)$

FIGURE 1. The displacement $w(x, t)$ and the velocity $w_t(x, t)$ of w -system (4.12) (for interpretation of the references to color of the figure's legend in this section, we refer to the PDF version of this article).

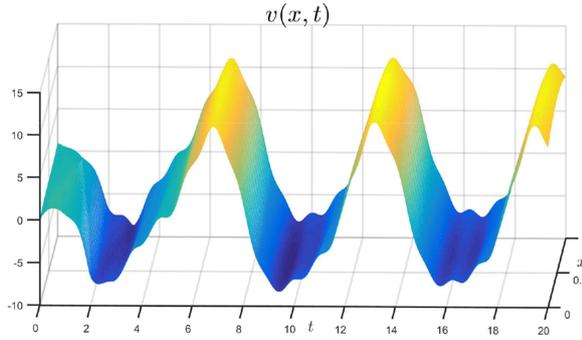
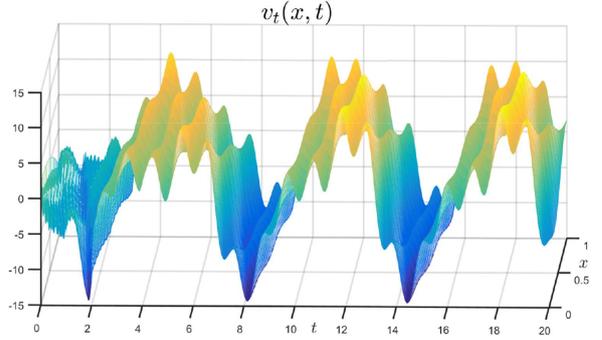

 (a) The displacement $v(x, t)$

 (b) The velocity $v_t(x, t)$

FIGURE 2. The displacement $v(x, t)$ and the velocity $v_t(x, t)$ of v -system (4.12) (for interpretation of the references to color of the figure's legend in this section, we refer to the PDF version of this article).

6. NUMERICAL SIMULATION

In this section, we present some numerical simulations for illustration to system (4.12). For numerical computations, the disturbance is taken as $d(t) = 2\sin(t) + 0.5\cos(5t)$ and the reference signal $r(t) = 2\sin(t) - 0.5\cos(6t)$. The parameters are taken as $q = 0.5$, $c_0 = 0.7$, $c_1 = 0.3$, $k = 0.1$ and the initial values are

$$\begin{aligned} w(x, 0) &= 2x - x^2, & w_t(x, 0) &= 2x + x^2, \\ v(x, 0) &= x - 2x^2, & v_t(x, 0) &= -x + 2x^2, \\ z(x, 0) &= 2x - x^3, & z_t(x, 0) &= 0, \\ \hat{w}(x, 0) &= -2x + x^2, & \hat{w}_t(x, 0) &= 0, \\ W(x, 0) &= 2x - x^2, & Y(x, 0) &= -2x^2 + 3x^3. \end{aligned}$$

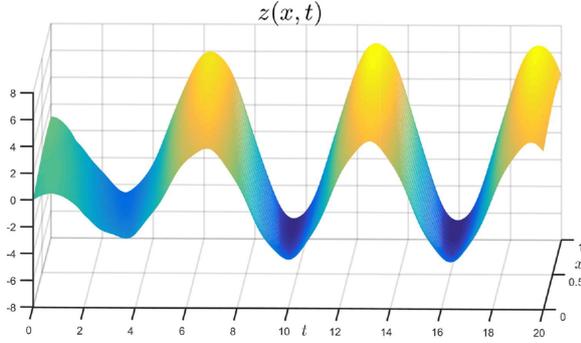
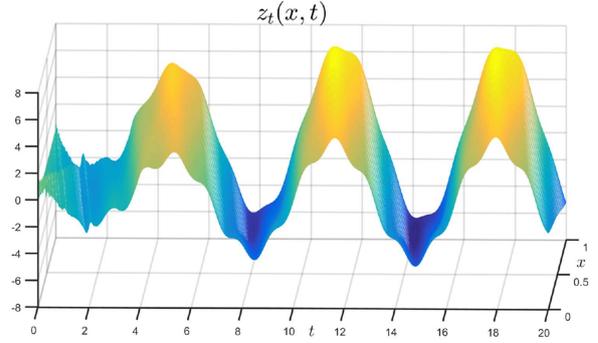
(a) The displacement $z(x,t)$ (b) The velocity $z_t(x,t)$

FIGURE 3. The displacement $z(x,t)$ and the velocity $z_t(x,t)$ of z -system (4.12) (for interpretation of the references to color of the figure's legend in this section, we refer to the PDF version of this article).

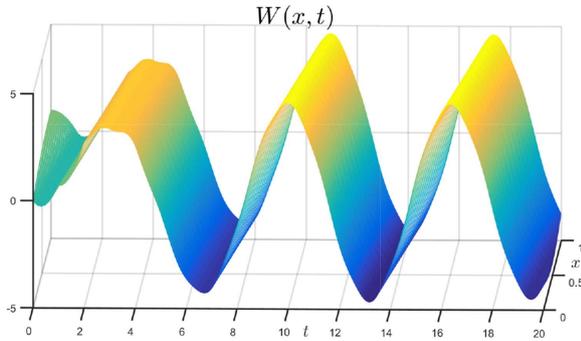
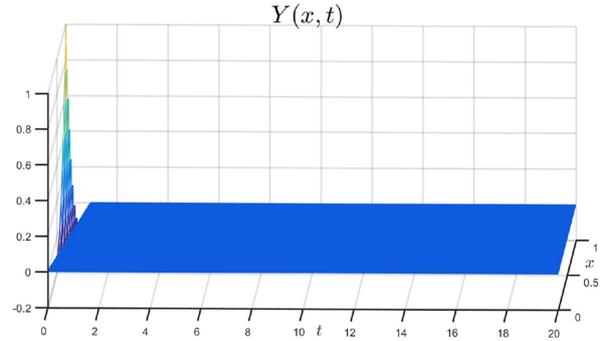
(a) The displacement $W(x,t)$ (b) The velocity $Y(x,t)$

FIGURE 4. The displacement $Y(x,t)$ and $W(x,t)$ of system (4.12) (for interpretation of the references to color of the figure's legend in this section, we refer to the PDF version of this article).

It is clear that the above initial state satisfies the compatibility conditions of Theorem 4.3. The backward Euler method in time and the Chebyshev spectral method for space variable are used to discretize system (4.12). Here, we take the grid size $N = 20$ for x and the time step $dt = 5 \times 0.001$. The solution of system (4.12) is plotted in Figures 1–5, which shows that $(w, w_t, v, v_t, z, z_t, \hat{w}, \hat{w}_t, W, Y)$ is bounded satisfactorily. Particularly, it is seen that Y decay exponentially, which is shown to be in [37], theoretically. Figure 5 shows that the (\hat{w}, \hat{w}_t) is an observed state of (w, w_t) . Figure 6a shows that $w(0,t)$ tracks fast the reference signal $r(t)$. Figure 6b displays the feedback control in time.

7. CONCLUDING REMARKS

In this paper, we have studied the output tracking problem for one-dimensional anti-stable wave equation with Neumann boundary control with the general disturbance and general reference signal. The low proportional gain control law is designed by making full use of the property of time delay system and transport system, the disturbance estimator developed in [37]. The estimation/cancellation strategy is used to compensate the

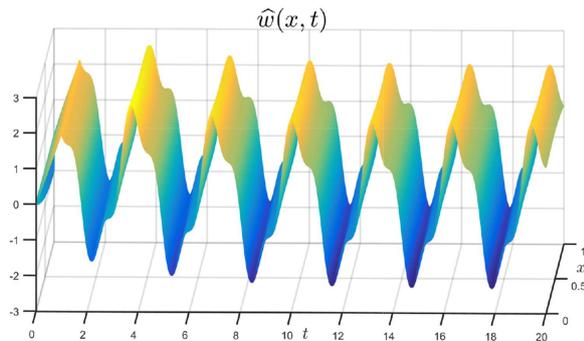
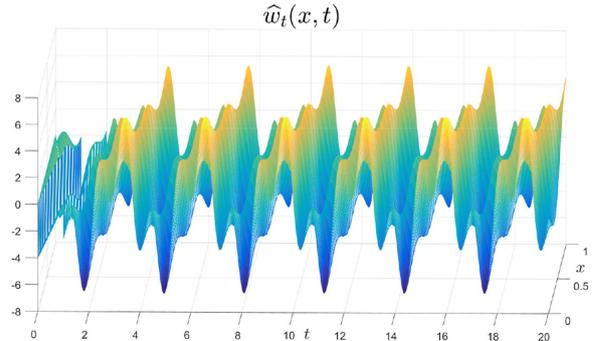

 (a) The displacement $\hat{w}(x, t)$

 (b) The velocity $\hat{w}_t(x, t)$

FIGURE 5. The displacement $\hat{w}(x, t)$ and the velocity $\hat{w}_t(x, t)$ of \hat{w} -system (4.12) (for interpretation of the references to color of the figure's legend in this section, we refer to the PDF version of this article).

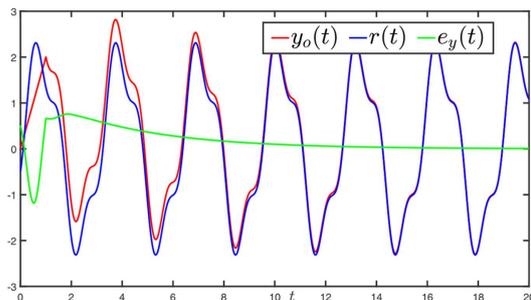
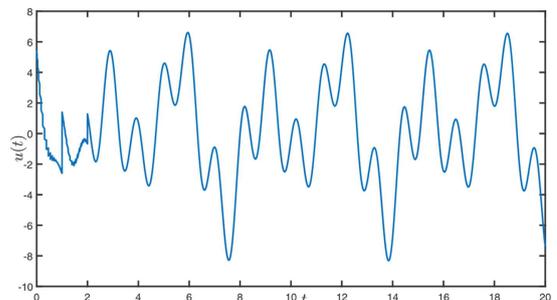

 (a) The reference signal $r(t)$ and the output $w(0, t)$

 (b) The control law $u(t)$

FIGURE 6. The reference signal $r(t)$, the output $w(0, t)$ and the control law $u(t)$ (4.12) (for interpretation of the references to color of the figure's legend in this section, we refer to the PDF version of this article).

disturbance. The novelty of the paper is that it is the first time to derive an exponential tracking controller, which makes the output to be controlled track the reference signal, exponentially, while rejecting the general disturbance. We also derive a byproduct, that is, a new output feedback stabilizing control law is found. This control law not only rejects the disturbance but also makes the closed-loop system exponentially stable by using two displacement output signals, which, compared with the results in [10, 15], is very sharp. In future works, applying our approach to one-dimensional wave equation with variable coefficients seems interesting, and considering the other boundary conditions to wave equation is also interesting problem. At addition, a future research direction may be to use the method used in the paper or internal model principle to solve output tracking problem for multi-dimensional PDEs, like multi-dimensional heat equation and wave equation. For the stabilization problem for variable coefficient wave equation and multi-dimensional wave equation, we refer to [17, 33].

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REFERENCES

- [1] C.I. Byrnes, I.G. Laukó, D.S. Gilliam and V.I. Shubov, Output regulation for linear distributed parameter systems. *IEEE Trans. Automat. Control* **45** (2000) 2236–2252.
- [2] D. Bresch-Pietri and M. Krstic, Output-feedback adaptive control of a wave PDE with boundary anti-damping. *Automatica* **50** (2014) 1407–1415.
- [3] R. Curtain and H. Zwart, An Introduction to Infinite-Dimensional Linear Systems Theory. Springer-Verlag, New York (1995).
- [4] E.J. Davison, The robust control of a servomechanism problem for linear time-invariant multivariable systems. *IEEE Trans. Automat. Control* **21** (1976) 25–34.
- [5] J. Deutscher, A backstepping approach to the output regulation of boundary controlled parabolic PDEs. *Automatica* **57** (2015) 56–64.
- [6] J. Deutscher, Backstepping design of robust output feedback regulators for boundary controlled parabolic PDEs. *IEEE Trans. Automat. Control* **61** (2016) 2288–2294.
- [7] J. Deutscher and S. Kerschbaum, Backstepping design of robust state feedback regulators for second order hyperbolic PIDEs. *IFAC-PapersOnLine* **49** (2016) 80–85.
- [8] E. Fridman, Introduction to Time-Delay Systems. Birkhäuser/Springer, Cham (2014).
- [9] B.A. Francis and W.M. Wonham, The internal model principle of control theory. *Automatica* **12** (1976) 457–465.
- [10] H. Feng and B.Z. Guo, A new active disturbance rejection control to output feedback stabilization for a one-dimensional anti-stable wave equation with disturbance. *IEEE Trans. Automat. Control* **62** (2017) 3774–3787.
- [11] W. Guo and B.Z. Guo, Performance output tracking for a wave equation subject to unmatched general boundary harmonic disturbance. *Automatica* **68** (2016) 194–202.
- [12] W. Guo, B.Z. Guo and F.F. Jin, Performance output tracking and disturbance rejection for one-dimensional wave equation with boundary disturbance. *IEEE, 54th, Annual Conference on Decision and Control, Osaka, Japan* (2015).
- [13] W. Guo, Z.C. Shao and M. Krstic, Adaptive rejection of harmonic disturbance anticollocated with control in 1D wave equation. *Automatica* **79** (2017) 17–26.
- [14] W. Guo, H.C. Zhou and M. Krstic, Adaptive error feedback output regulation for 1d wave equation. *Int. J. Robust Nonlinear Control* **28** (2018) 4309–4329.
- [15] B.Z. Guo and F.F. Jin, Output feedback stabilization for one-dimensional wave equation subject to boundary disturbance. *IEEE Trans. Automat. Control* **60** (2015) 824–830.
- [16] B.Z. Guo and C.Z. Xu, The stabilization of a one-dimensional wave equation by boundary feedback with noncollocated observation. *IEEE Trans. Automat. Control* **52** (2007) 371–377.
- [17] B.Z. Guo and H.C. Zhou, The active disturbance rejection control to stabilization for multi-dimensional wave equation with boundary control matched disturbance. *IEEE Trans. Automat. Control* **60** (2015) 143–157.
- [18] E. Immonen and S. Pohjolainen, Output regulation of periodic signals for DPS: an infinite-dimensional signal generator. *IEEE Trans. Automat. Control* **50** (2005) 1799–1804.
- [19] E. Immonen and S. Pohjolainen, Feedback and feedforward output regulation of bounded uniformly continuous signals for infinite-dimensional systems. *SIAM J. Control Optim.* **45** (2006) 1714–1735.
- [20] M. Krstic, Delay Compensation for Nonlinear, Adaptive, and PDE Systems. Birkhäuser, Boston (2009).
- [21] P.O. Lamare and N. Bekiaris-Liberis, Control of 2×2 linear hyperbolic systems: backstepping-based trajectory generation and PI-based tracking. *Syst. Control Lett.* **86** (2015) 24–33.
- [22] J.J. Liu, J.M. Wang and Y.P. Guo, Output tracking for one-dimensional Schrödinger equation subject to boundary disturbance. *Asian J. Control* **20** (2018) 659–668.
- [23] T. Meurer and A. Kugi, Tracking control for boundary controlled parabolic PDEs with varying parameters: combining backstepping and differential flatness. *Automatica* **45** (2009) 1182–1194.
- [24] V. Natarajan, D. Gilliam and G. Weiss, The state feedback regulator problem for regular linear systems. *IEEE Trans. Automat. Control* **59** (2014) 2708–2723.
- [25] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations. Springer-Verlag, New York (1983).
- [26] L. Paunonen and S. Pohjolainen, The internal model principle for systems with unbounded control and observation. *SIAM J. Control Optim.* **52** (2014) 3967–4000.
- [27] R. Rebarber and G. Weiss, Internal model based tracking and disturbance rejection for stable well-posed systems. *Automatica* **39** (2003) 1555–1569.
- [28] A. Smyshlyaev and M. Krstic, Boundary control of an anti-stable wave equation with antidamping on the uncontrolled boundary. *Syst. Control Lett.* **58** (2009) 617–623.
- [29] M. Tucsnak and G. Weiss, Observation and Control for Operator Semigroups, *Birkhäuser Advanced Texts Basler Lehrbücher*, Birkhäuser, Basel (2009).
- [30] G. Weiss, Admissibility of unbounded control operators. *SIAM J. Control Optim.* **27** (1989) 527–545.
- [31] G. Weiss and V. Natarajan, Integral control of stable nonlinear systems. Preprint [arXiv:1610.04868](https://arxiv.org/abs/1610.04868) (2016).

- [32] C.Z. Xu and G. Sallet, Multivariable boundary PI control and regulation of a fluid flow system. *Math. Control Relat. Fields* **4** (2014) 501–520.
- [33] Z.H. Xu, Y.G. Liu and J. Li, Adaptive stabilization for a class of PDE-ODE cascade systems with uncertain harmonic disturbances. *ESAIM: COCV* **23** (2017) 497–515.
- [34] H.C. Zhou and B.Z. Guo, Performance output tracking for one-dimensional wave equation subject to unmatched general disturbance and non-collocated control. *Eur. J. Control* **39** (2018) 39–52.
- [35] H.C. Zhou and G. Weiss, The regulation problem for the one-dimensional Schrödinger equation via the backstepping approach. *Proc. of the International Conference on the Science of Electrical Engineering (ICSEE), Eilat, Israel* (2016).
- [36] H.C. Zhou and G. Weiss, Output feedback exponential stabilization for one-dimensional unstable wave equations with boundary control matched disturbance. *SIAM J. Control Optim.* **56** (2018) 4098–4129.
- [37] H.C. Zhou and G. Weiss, Output feedback exponential stabilization of a nonlinear 1-D wave equation with boundary input. *Proc. of the IFAC World Congress, Toulouse, France* (2017).
- [38] H.C. Zhou and G. Weiss, Output tracking and disturbance rejection for a one-dimensional anti-stable wave equation. *IEEE 56th Annual Conference on Decision and Control, Melbourne, Australia* (2017).