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CONVERGENCE ANALYSIS OF TIME-DISCRETISATION SCHEMES FOR RATE-INDEPENDENT SYSTEMS*

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Abstract. It is well known that rate-independent systems involving nonconvex energy functionals in general do not allow for time-continuous solutions even if the given data are smooth. In the last years, several solution concepts were proposed that include discontinuities in the notion of solution, among them the class of global energetic solutions and the class of BV-solutions. In general, these solution concepts are not equivalent and numerical schemes are needed that reliably approximate that type of solutions one is interested in. In this paper, we analyse the convergence of solutions of three time-discretisation schemes, namely an approach based on local minimisation, a relaxed version of it and an alternate minimisation scheme. For all three cases, we show that under suitable conditions on the discretisation parameters discrete solutions converge to limit functions that belong to the class of BV-solutions. The proofs rely on a reparametrisation argument. We illustrate the different schemes with a toy example.

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1. Introduction

In recent years, several discretisation schemes were proposed and applied in order to approximate solutions $z : [0,T] \to \mathcal{Z}$ of doubly nonlinear differential inclusions of the type

$$0 \in \partial\mathcal{R}(\partial_t z(t)) + \mathrm{D}_z\mathcal{I}(t, z(t)), \quad z(0) = z_0,\ t \in [0,T] \tag{1.1}$$

for the rate-independent case. In this case, the functional $\mathcal{R} : \mathcal{X} \to [0,\infty)$ (with $\mathcal{Z} \subset \mathcal{X}$) is convex and positively homogeneous of degree one. A variety of material models for complex solids rely on evolution laws of the type (1.1). There, $\mathcal{R}$ describes a dissipation (pseudo) potential while $\mathcal{I} : [0,T] \times \mathcal{Z} \to \mathbb{R}$ is a time or load dependent stored energy functional. In many cases, the mapping $z \mapsto \mathcal{I}(t,z)$ is not convex. It is a well known fact that (1.1) with a nonconvex (but smooth) functional $\mathcal{I}$ in general does not allow for solutions that are continuous on

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the whole time interval $[0, T]$. Several different solution concepts that include discontinuities in the notion of solution were proposed in the literature. Here, we mention the concepts of (global) energetic solutions (GESs), balanced viscosity and vanishing viscosity solutions (BV-solutions) and different types of local solutions. In the nonconvex case (*i.e.* $\mathcal{I}$ is not convex), GES and BV-solutions imply substantially different jump criteria and thus the solution concepts are not equivalent. It is a question of modelling to decide which type of solutions is most appropriate for a given problem. From a numerical point of view discretisation schemes are needed that approximate reliably the type of solutions one is interested in.

In literature, many different schemes are discussed for the approximation of (1.1). For simplifying the following presentation, we assume an equidistant partition of $[0, T]$ with $0 = t_0^N < \ldots < t_N^N$, $N \in \mathbb{N}$, $\tau^N = T/N$, $t_k^N = k\tau^N$. Moreover, let $\mathcal{X}$ be a Banach space and $\mathcal{Z}, \mathcal{V}$ Hilbert spaces such that $\mathcal{Z} \Subset \mathcal{V} \subset \mathcal{X}$ with compact and continuous embeddings, respectively.

Global energetic solutions $z : [0, T] \to \mathcal{Z}$ are characterised by the following global stability condition (S) and energy balance (E): for every $t \in [0, T]$

$$\mathcal{I}(t, z(t)) \leq \mathcal{I}(t, v) + \mathcal{R}(v - z(t)) \quad \text{for all } v \in \mathcal{Z}, \tag{S}$$

$$\mathcal{I}(t, z(t)) + \text{diss}_{\mathcal{R}}(z, [0, t]) = \mathcal{I}(0, z_0) + \int_0^t \partial_t \mathcal{I}(r, z(r))\,\mathrm{d}r\,. \tag{E}$$

Here, $\text{diss}_{\mathcal{R}}(z, [0, t]) := \sup_{\text{partitions } (t_i)_i \text{ of } [0,t]} \sum_{i=1}^N \mathcal{R}(z(t_i) - z(t_{i-1}))$ quantifies the dissipation with respect to $\mathcal{R}$ along the curve z. GES can be approximated by a time incremental global minimisation scheme (we omit the index N):

$$z(0) = z_0, \quad z_k \in \text{Argmin}\{\, \mathcal{I}(t_k, v) + \mathcal{R}(v - z_{k-1})\,;\, v \in \mathcal{Z}\,\}, \quad 1 \leq k \leq N\,, \tag{1.2}$$

for details we refer to [10, 15].

BV-solutions and vanishing viscosity solutions can be obtained starting from the viscously regularised minimisation problem

$$z^\mu(0) := z_0, \quad z_k^\mu \in \text{Argmin}\{\, \mathcal{I}(t_k, v) + \mathcal{R}(v - z_{k-1}^\mu) + \tfrac{\mu}{2\tau} \|v - z_{k-1}\|_{\mathcal{V}}^2\,;\, v \in \mathcal{Z}\,\}\,. \tag{1.3}$$

The parameter $\mu > 0$ plays the role of a viscosity parameter and the choice of the norm in the quadratic term is a question of modelling. For $N \to \infty$, $\mu \to 0$ and $\mu/\tau \to \infty$, suitable interpolants of $(z_k^\mu)_{0 \leq k \leq N}$ converge to vanishing viscosity solutions belonging to the class of BV-solutions, see *e.g.* ([14], Thm. 3.12). In Section 1.1, we give the definition of parametrised BV-solutions in the spirit of ([15], Def. 3.8.2). This will be the framework we are working in. Let us remark that for computations it is often difficult to find a good choice for μ in dependence of τ, see *e.g.* [8], where this approach is investigated analytically and computationally for a crack propagation model.

Several alternatives to (1.2) and (1.3) were proposed and applied in literature. However, a detailed convergence analysis is missing in many cases, and it often is even not clear which type of solutions might be approximated in the limit. In some cases, it has been shown that the limit function is a local solution which means that $z : [0, T] \to \mathcal{Z}$ satisfies $0 \in \partial\mathcal{R}(0) + \mathrm{D}_z\mathcal{I}(t, z(t))$ for almost every t together with the energy dissipation estimate

$$\mathcal{I}(t_1, z(t_1)) + \text{diss}_{\mathcal{R}}(z; [t_0, t_1]) \leq \mathcal{I}(t_0, z(t_0)) + \int_{t_0}^{t_1} \partial_t \mathcal{I}(r, z(r))\,\mathrm{d}r$$

that is valid for every $0 \leq t_0 < t_1 \leq T$, see for instance ([15], Chap. 1.8). The class of local solutions comprises both, GES and BV-solutions, and it is the most general and weakest notion of solutions for rate-independent systems of the type (1.1).

In this paper, we focus on three discretisation schemes: a local minimisation approach originally proposed in [4], a relaxed version of it that is closely related to a scheme discussed in [1] and a modified alternating minimisation scheme including a penalty term. These schemes will be analysed in an abstract infinite-dimensional framework for a semilinear model equation. This framework is general enough to be applied to basic models from ferroelectrics, see Section 5.3. However, for more complex models with stronger nonlinearities, like for instance damage models, the analysis has to be adapted accordingly. In all three cases, it turns out that the limit functions belong to the class of BV-solutions and thus are different from GESs.

Let us discuss the results in more detail. The precise assumptions on the functionals $\mathcal{R}$ and $\mathcal{I}$ are collected in Section 1.1.

Local minimisation: The following scheme was first proposed in [4]. Let $h > 0$. For $k \geq 1$, the quantities z_k^h and t_k^h are iteratively defined as

$$z_k^h \in \operatorname{Argmin}\{\, \mathcal{I}(t_{k-1}^h, v) + \mathcal{R}(v - z_{k-1}^h)\,;\, v \in \mathcal{Z},\ \left\| v - z_{k-1}^h \right\|_{\mathbb{V}} \leq h \,\} \tag{1.4}$$

$$t_k^h = \min\left\{ t_{k-1}^h + h - \left\| z_k^h - z_{k-1}^h \right\|_{\mathbb{V}},\, T \right\} . \tag{1.5}$$

Observe that the time increment is not fixed *a priori* but it is a result of the minimisation procedure. Thus, the scheme has a time adaptive character with finer time steps at those points where the solution might develop a discontinuity. It is proved in [4] in finite dimensions that for $h \to 0$ suitable interpolants converge to (parametrised) BV-solutions. However, it is not shown that the desired final time T is reached after a finite number of minimisation steps and that the interpolating curves have finite arc-length. Thus, in that paper it was not clarified whether in the limit $h \to 0$ the original problem (1.1) is solved on the whole time interval $[0, T]$. A version of this approach was investigated in [20] in the infinite dimensional setting. Also here, it was not clarified whether the time T is reached after a finite number of steps. A further variant of (1.4)–(1.5) was investigated in [21] in the context of a cohesive fracture model. However, in contrast to (1.4)–(1.5) the time increment in [21] is fixed *a priori* and scaled with a (non-explicit) constant c. Thus, this version does not have the time adaptive character of the original scheme (1.4)–(1.5). In Section 2, we provide a full convergence analysis for (1.4)–(1.5) in the infinite-dimensional setting. In particular, we prove that T is reached after a finite number of steps and we derive a uniform (with respect to h) estimate for the piecewise constant interpolants of the incremental values $(z_k^h)_k$ in $BV([0, T]; \mathcal{Z})$, see Proposition 2.3. This estimate allows us to apply a reparametrisation technique and to identify a limit system that is satisfied by limits of the incremental solutions, see Theorem 2.5. It turns out that the limits are parametrised BV-solutions.

Relaxed local minimisation: The scheme discussed in Section 3.1 can be interpreted as a relaxed version of (1.4)–(1.5). Given $N \in \mathbb{N}$, a time-step size $\tau = T/N$ and a parameter $\eta > 0$ we define for $1 \leq k \leq N$ and $i \in \mathbb{N}_0$: $t_k = k\tau$, $z_{k,0} := z_{k-1}$ and for $i \geq 1$

$$z_{k,i} \in \operatorname{Argmin}\{\, \mathcal{I}(t_k, v) + \frac{\eta}{2} \left\| v - z_{k,i-1} \right\|_{\mathbb{V}}^2 + \mathcal{R}(v - z_{k,i-1})\,;\, v \in \mathcal{Z} \,\}, \tag{1.6}$$

$$z_k := z_{k,\infty} := \lim_{i \to \infty} z_{k,i} \quad \text{(weak limit in } \mathcal{Z}). \tag{1.7}$$

Here, the constraint $\left\| z_k^h - z_{k-1}^h \right\|_{\mathbb{V}} \leq h$ from (1.4) is replaced with the term $\frac{\eta}{2} \left\| v - z_{k,i-1} \right\|_{\mathbb{V}}^2$. The parameter η plays the role of a penalty parameter that should be sent to infinity. It can also be interpreted as a generalized viscosity parameter, *i.e.* in comparison with (1.3) it plays the role of μ/τ. In Section 3.1, we show the convergence of discrete solutions under the assumption $\eta_N \to \infty$ for $N \to \infty$ and obtain again BV-solutions in the limit. Observe that (1.6)–(1.7) is a modified version of an algorithm studied in [1]: instead of the term $\mathcal{R}(v - z_{k,i-1})$ the authors in [1] use the term $\mathcal{R}(v - z_k)$ in (1.6) and they study the convergence of the scheme for fixed $\eta > 0$ and $N \to \infty$. For the version (1.6) we show that the sequence $(z_{k,i})_{i \in \mathbb{N}}$ itself converges to a critical point (*i.e.* $-\mathrm{D}_z \mathcal{I}(t_k, z_k) \in \partial \mathcal{R}(0)$), while such a result is derived in [1] for a subsequence, only. Moreover, the authors from [1] show that the limit function belongs to the class of local solutions, the weakest notion of solutions for rate-independent systems of the type (1.1), while we are able to classify the limit function as a (parametrised) BV-solution. In order to have a closer comparison with the results from [1], in Section 3.2, we also study the limit behaviour of (1.6)–(1.7) for $N \to \infty$ but with fixed $\eta > 0$. In this case, it is not clear whether the limits of piecewise constant and piecewise affine interpolating curves coincide. Hence, we characterise the limiting system by using the individual limits explicitly. In this way, we obtain an energy dissipation balance that slightly differs from the one for the original limit to (1.6)–(1.7). In comparison to the energy dissipation estimate obtained in [1], it contains more information and the behaviour at jump points can be characterised more precisely. In this context, let us finally mention [19, 22]. There, the authors consider (1.6) for fixed $\eta > 0$ and at each t_k they carry out only one minimisation step (*i.e.* $z_k := z_{k,1}$). They prove the convergence of suitable interpolants to so-called visco-energetic solutions. Depending on the size of η these solutions may behave more like GES or like BV-solutions or they show some intermediate behaviour.

Alternate minimisation with penalty term: Finally, in Section 4, we discuss an alternate minimisation scheme with a penalty term and a stopping criterion. The underlying energy $\mathcal{E} = \mathcal{E}(t, u, z)$ contains an additional variable u that in the context of material models plays for instance the role of the displacement field. The scheme is defined as follows: let $z_{k,0} := z_{k-1}$, $u_{k,0} := u_{k-1}$. Then for $i \geq 1$

$$u_{k,i} = \operatorname{argmin}\{\, \mathcal{E}(t_k, v, z_{k,i-1})\,;\, v \in \mathcal{U}\,\}, \tag{1.8}$$

$$z_{k,i} \in \operatorname{Argmin}\{\, \mathcal{E}(t_k, u_{k,i}, \xi) + \frac{\eta}{2}\,\|\xi - z_{k,i-1}\|_{\mathbb{V}}^2 + \mathcal{R}(\xi - z_{k,i-1})\,;\, \xi \in \mathcal{Z}\,\}, \tag{1.9}$$

$$\text{stop if } \|z_{k,i} - z_{k,i-1}\|_{\mathcal{V}} \leq \delta; \quad (u_k, z_k) := (u_{k,i}, z_{k,i})\,. \tag{1.10}$$

We show that the criterion (1.10) is satisfied after a finite number of minimisation steps. We further prove that for $\eta_N \to \infty$, $\delta_N \to 0$ and $\eta_N \delta_N \to 0$ the interpolants converge to a (parametrised) solution triple $(\hat{t}, \hat{u}, \hat{z}) : [0, S] \to [0, T] \times \mathcal{U} \times \mathcal{Z}$ with $\mathrm{D}_u\mathcal{E}(\hat{t}(s), \hat{u}(s), \hat{z}(s)) = 0$ for every s and $\hat{z}$ is again a BV-type solution. This result is different from the alternate minimisation scheme analysed in [6] (for a damage model and with $\eta = 0$), where in the limit also visco-elastic dissipation is present. By defining $\mathcal{I}(t, z) := \min_{u \in \mathcal{U}} \mathcal{E}(t, u, z)$ we are back in the setting of Section 3.1, and hence the results of Section 4 can be interpreted as a convergence result for (1.6) with an additional stopping criterion. Alternate minimisation schemes are frequently applied in simulations since they split the problem into subproblems that usually are easier to solve. In the context of rate-independent systems we refer to [2, 6, 23] for first results.

The key estimate for all convergence proofs is a BV-type bound for the incremental solutions

$$\sum_{k=1}^{N} \sum_{i} \|z_{k,i} - z_{k,i-1}\|_{\mathcal{Z}} \leq C$$

that is uniform with respect to the discretisation parameters. Estimates of this type lie at the heart of any vanishing viscosity result. After deriving this estimate for the different schemes we define interpolating curves $(t^N, z^N) : [0, S_N] \to [0, T] \times \mathcal{Z}$ by introducing an artificial arclength parameter and formulate discrete energy dissipation identities that are satisfied by these curves. Passing to the limit in these identities yields the desired results. This general approach is frequently applied in the context of vanishing viscosity analysis for rate-independent systems, see for example [4, 11, 17] for abstract settings and [7] for a damage model.

Finally, Section 5 contains a finite-dimensional example for which solutions can be constructed explicitly. It turns out that even within the same class of solutions (BV-solutions in this case) the limits related to the local minimisation scheme and those related to the schemes with a penalisation parameter may differ. We further illustrate the predictions of the different schemes with the help of a finite-dimensional toy example for which exact solutions can be constructed explicitly. Finally, we show that a (simplified) rate-independent version of the ferroelectric model introduced in [24] falls into the abstract framework of this paper. Thus, the analysis in this paper in particular guarantees the convergence of the alternate minimisation scheme (1.8)–(1.10) to solutions of BV-type for the ferroelectric model.

1.1. Basic assumptions and estimates

The analysis will be carried out for the semilinear system introduced in [17] and ([15], Example 3.8.4). Let $\mathcal{X}$ be a Banach space and $\mathcal{Z}, \mathcal{V}$ separable Hilbert spaces that are densely and compactly resp. continuously embedded in the following way:

$$\mathcal{Z} \Subset \mathcal{V} \subset \mathcal{X}. \tag{1.11}$$

Let further $A \in \mathrm{Lin}(\mathcal{Z}, \mathcal{Z}^*)$ and $\mathbb{V} \in \mathrm{Lin}(\mathcal{V}, \mathcal{V}^*)$ be linear symmetric, bounded $\mathcal{Z}$- and $\mathcal{V}$-elliptic operators, *i.e.* there exist constants $\alpha, \gamma > 0$ such that

$$\forall z \in \mathcal{Z}, \forall v \in \mathcal{V}: \quad \langle Az, z\rangle \geq \alpha \|z\|_{\mathcal{Z}}^2 , \quad \langle \mathbb{V}v, v\rangle \geq \gamma \|v\|_{\mathcal{V}}^2 , \tag{1.12}$$

and $\langle Az_1, z_2\rangle = \langle Az_2, z_1\rangle$ for all $z_1, z_2 \in \mathcal{Z}$ (and similar for $\mathbb{V}$). Here, $\langle\cdot,\cdot\rangle$ denotes the duality pairings in $\mathcal{Z}$ and $\mathcal{V}$, respectively. We define $\|v\|_{\mathbb{V}} := (\langle \mathbb{V}v, v\rangle)^{\frac{1}{2}}$, which is a norm that is equivalent to the Hilbert space norm $\|\cdot\|_{\mathcal{V}}$. By rescaling the inner product on $\mathcal{Z}$ we may assume that $\|z\|_{\mathbb{V}} \leq \|z\|_{\mathcal{Z}}$ for all $z \in \mathcal{Z}$. Let further

$$\ell \in C^1([0,T]; \mathcal{V}^*) \text{ and } \mathcal{F} \in C^2(\mathcal{Z}; \mathbb{R}) \text{ with } \mathcal{F} \geq 0. \tag{1.13}$$

The energy functional $\mathcal{I}$ is of the form

$$\mathcal{I} : [0,T] \times \mathcal{Z} \to \mathbb{R}, \quad \mathcal{I}(t,z) := \frac{1}{2}\langle Az, z\rangle + \mathcal{F}(z) - \langle \ell(t), z\rangle . \tag{1.14}$$

Clearly, $\mathcal{I} \in C^1([0,T] \times \mathcal{Z}; \mathbb{R})$, $\mathcal{I}$ is bounded from below and

$$\exists \mu, c > 0\, \forall t \in [0,T],\, z \in \mathcal{Z}: \qquad |\partial_t \mathcal{I}(t,z)| \leq \mu\big(\mathcal{I}(t,z) + c\big). \tag{1.15}$$

Referring to ([15], Sect. 2.1.1), these conditions imply that for all $t, s \in [0,T], z \in \mathcal{Z}$ the estimates

$$\mathcal{I}(t,z) + c \leq (\mathcal{I}(s,z) + c)e^{\mu|t-s|}, \qquad |\partial_t \mathcal{I}(t,z)| \leq \mu(\mathcal{I}(s,z) + c)e^{\mu|t-s|} \tag{1.16}$$

are valid. The dissipation functional $\mathcal{R} : \mathcal{X} \to [0,\infty)$ is assumed to be convex, lower semicontinuous, positively homogeneous of degree one and

$$\exists c, C > 0\, \forall x \in \mathcal{X}: \quad c\|x\|_{\mathcal{X}} \leq \mathcal{R}(x) \leq C\|x\|_{\mathcal{X}} . \tag{1.17}$$

The functional $\mathcal{F}$ shall play the role of a possibly nonconvex lower order term (*cf.* [15], Sect. 3.8). By lower order we mean that if we reinterpret the rate independent system in the context of partial differential equations, then the term $\mathrm{D}_z\mathcal{F}$ is a lower order term with respect to the differential operator defined by $\mathrm{D}_z\mathcal{I}$, compare for instance the example in Section 5.3. Hence, we assume that

$$\mathrm{D}_z\mathcal{F} \in C^1(\mathcal{Z}; \mathcal{V}^*), \quad \left\|\mathrm{D}_z^2\mathcal{F}(z)v\right\|_{\mathcal{V}^*} \leq C(1 + \|z\|_{\mathcal{Z}}^q)\|v\|_{\mathcal{Z}} \tag{1.18}$$

for some $q \geq 1$. From (1.18) and (1.17) we deduce the following interpolation estimate:

Lemma 1.1. *Assume* (1.11)*,* (1.13)*,* (1.17) *and* (1.18)*. For every $\rho > 0$ and $\varepsilon > 0$ there exists $C_{\rho,\varepsilon} > 0$ such that for all $z_1, z_2 \in \mathcal{Z}$ with $\|z_i\|_{\mathcal{Z}} \leq \rho$ we have*

$$|\langle \mathrm{D}\mathcal{F}(z_1) - \mathrm{D}\mathcal{F}(z_2), z_1 - z_2\rangle| \leq \varepsilon\|z_1 - z_2\|_{\mathcal{Z}}^2 + C_{\rho,\varepsilon}\min\{\mathcal{R}(z_1 - z_2), \mathcal{R}(z_2 - z_1)\}\|z_1 - z_2\|_{\mathbb{V}} . \tag{1.19}$$

Proof. The proof relies on an abstract Ehrling Lemma [26], which adapted to our situation reads: for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that for all $z \in \mathcal{Z}$

$$\|z\|_{\mathcal{V}} \le \varepsilon \|z\|_{\mathcal{Z}} + C_\varepsilon \|z\|_{\mathcal{X}} . \tag{1.20}$$

Let now $\varepsilon, \rho > 0$, $z_i \in \mathcal{Z}$ with $\|z_i\|_{\mathcal{Z}} \le \rho$. Then for $\varepsilon_1 := \varepsilon/(2C(1+\rho^q))$ and $\varepsilon_2 = \varepsilon/2$

$$\begin{aligned}|\langle \mathrm{D}\mathcal{F}(z_1) - \mathrm{D}\mathcal{F}(z_2), z_1 - z_2\rangle| &\le \|\mathrm{D}_z\mathcal{F}(z_1) - \mathrm{D}_z\mathcal{F}(z_2)\|_{\mathcal{V}^*} \|z_1 - z_2\|_{\mathcal{V}} \\ &\le C(1+\rho^q) \|z_1 - z_2\|_{\mathcal{Z}} (\varepsilon_1 \|z_1 - z_2\|_{\mathcal{Z}} + C_{\varepsilon_1} \|z_1 - z_2\|_{\mathcal{X}}) \\ &\le \left(\frac{\varepsilon}{2} + \varepsilon_2\right) \|z_1 - z_2\|_{\mathcal{Z}}^2 + C_{\varepsilon_2}(C_{\varepsilon_1} C(1+\rho^q))^2 \|z_1 - z_2\|_{\mathcal{X}}^2 .\end{aligned}$$

The proof is complete since by (1.17) and (1.11) we have $\|z_1 - z_2\|_{\mathcal{X}}^2 \le C \min\{\mathcal{R}(z_1 - z_2), \mathcal{R}(z_2 - z_1)\} \|z_1 - z_2\|_{\mathbb{V}}$. □

For the proof of the convergence theorems we need a further assumption on $\mathcal{F}$:

$$\mathcal{F} : \mathcal{Z} \to \mathbb{R} \text{ and } \mathrm{D}_z\mathcal{F} : \mathcal{Z} \to \mathcal{Z}^* \text{ are weak–weak continuous.} \tag{1.21}$$

Finally, we give here the definition of parametrised BV-solutions following ([15], Def. 3.8.2).

Definition 1.2. A pair $(\hat{t}, \hat{z}) : [0, S] \to [0, T] \times \mathcal{Z}$ is a (normalized) $\mathcal{V}$-parametrised solution associated with $(\mathcal{I}, \mathcal{R}, \mathbb{V})$ if $(\hat{t}, \hat{z}) \in W^{1,\infty}([0, S]; \mathbb{R} \times \mathcal{V})$ and if there exists a measurable function $\lambda : [0, S] \to [0, \infty)$ such that for almost all $s \in [0, S]$

$$\hat{t}(0) = 0,\ \hat{t}(S) = T,\ \hat{z}(0) = z_0,\ \hat{t}'(s) \ge 0,\ \hat{t}'(s) + \|z'(s)\|_{\mathbb{V}} = 1, \tag{1.22a}$$

$$\lambda(s) \ge 0,\ \lambda(s)\hat{t}'(s) = 0, \tag{1.22b}$$

$$0 \in \partial\mathcal{R}(\hat{z}'(s)) + \lambda(s)\mathbb{V}\hat{z}'(s) + \mathrm{D}_z\mathcal{I}(\hat{t}(s), \hat{z}(s)) . \tag{1.22c}$$

The pair $(\hat{t}, \hat{z})$ is a degenerate $\mathcal{V}$-parametrised solution associated with $(\mathcal{I}, \mathcal{R}, \mathbb{V})$ if all of the above conditions but the last one in (1.22a) are satisfied.

Normalized parametrised BV-solutions can equivalently be characterised by an energy dissipation identity. The proof of the next proposition is identical to the one of ([12], Cor. 5.4).

Proposition 1.3. *Let the pair $(\hat{t}, \hat{z}) \in W^{1,\infty}([0, S]; \mathbb{R} \times \mathcal{Z})$ satisfy (1.22a). Then it is a $\mathcal{V}$-parametrised solution associated with $(\mathcal{I}, \mathcal{R}, \mathbb{V})$ (i.e. there exists a function $\lambda : [0, S] \to [0, \infty)$ such that $(\lambda, \hat{t}, \hat{z})$ satisfies (1.22b)–(1.22c)) if and only if the following complementarity relation and energy dissipation identity are satisfied:*

$$\textit{for almost all } s \in [0, S]\!: \qquad \hat{t}'(s) \operatorname{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{t}(s), \hat{z}(s)), \partial\mathcal{R}(0)) = 0, \tag{1.23}$$

$$\begin{aligned}\textit{for all } s \in [0, S]\!: \quad \mathcal{I}(\hat{t}(s), \hat{z}(s)) + \int_0^s \mathcal{R}(\hat{z}'(r)) + \|\hat{z}'(r)\|_{\mathbb{V}} \operatorname{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{t}(r), \hat{z}(r)), \partial\mathcal{R}(0))\,\mathrm{d}r \\ = \mathcal{I}(0, z_0) + \int_0^s \partial_t\mathcal{I}(\hat{t}(r), \hat{z}(r))t'(r)\,\mathrm{d}r .\end{aligned} \tag{1.24}$$

2. An approximation scheme relying on local minimisation

In this section, we analyse the scheme proposed in [4] for approximating solutions to the rate-independent model (1.1). It was already shown in [4] (for the finite-dimensional case) that suitable interpolants generated by this scheme converge to solutions that belong to the class of BV-solutions. However, in [4] it is not shown that a finite number of minimisation steps are sufficient to reach the desired final time T, and that the interpolating curves have a finite length that is uniformly bounded with respect to the discretisation parameter. The aim of this section is to fill this gap for the infinite- dimensional setting introduced in the previous section, see Proposition 2.3 ahead.

Let us describe the local minimisation algorithm from [4]. Fix $h > 0$. Given initial values $t_0 = 0$ and $z_0 \in \mathcal{Z}$, for $k \geq 1$ the quantities z_k^h and t_k^h are iteratively defined as

$$z_k^h \in \operatorname{Argmin}\{\, \mathcal{I}(t_{k-1}^h, z) + \mathcal{R}(z - z_{k-1}^h)\,;\, z \in \mathcal{Z},\ \|z - z_{k-1}^h\|_{\mathbb{V}} \leq h\,\} \tag{2.1}$$

$$t_k^h = \min\left\{t_{k-1}^h + h - \|z_k^h - z_{k-1}^h\|_{\mathbb{V}}\,,\, T\right\}\,. \tag{2.2}$$

The existence of minimizers follows by the direct method in the calculus of variations.

Proposition 2.1 (Basic estimates)**.** *Under the above assumptions on $\mathcal{I}$ and $\mathcal{R}$, for all $h > 0$, $k \in \mathbb{N}$ and with c, μ from (1.15) we have*

$$\mathcal{I}\Big(t_k^h, z_k^h\Big) + \mathcal{R}\Big(z_k^h - z_{k-1}^h\Big) \leq \mathcal{I}\Big(t_{k-1}^h, z_{k-1}^h\Big) + \int_{t_{k-1}^h}^{t_k^h} \partial_t \mathcal{I}\Big(\tau, z_k^h\Big)\mathrm{d}\tau\,, \tag{2.3}$$

$$\mathcal{I}\Big(t_k^h, z_k^h\Big) + \sum_{i=1}^{k} \mathcal{R}\Big(z_k^h - z_{k-1}^h\Big) \leq \Big(c + \mathcal{I}(0, z_0)\Big)e^{\mu T}\,, \tag{2.4}$$

$$\sup_{h>0, k\in\mathbb{N}} \|z_k^h\|_{\mathcal{Z}} < \infty\,. \tag{2.5}$$

Proof. Estimate (2.3) follows as in ([4], Prop. 4.2), estimate (2.4) follows from ([15], Thm. 2.1.5) and (2.5) is a consequence of (2.4) and the coercivity of $\mathcal{I}$ (uniformly in t). ☐

Proposition 2.2 (Optimality properties)**.** *The pairs $(z_k^h, t_k^h)_{k\geq 1}$ satisfy the following optimality properties: There exist Lagrange multipliers $\lambda_k^h \geq 0$ with*

$$\lambda_k^h\Big(\,\|z_k^h - z_{k-1}^h\|_{\mathbb{V}} - h\Big) = 0\,, \tag{2.6}$$

$$h\,\mathrm{dist}_{\mathcal{V}^*}\left(-\mathrm{D}_z\mathcal{I}(t_{k-1}^h, z_k^h), \partial\mathcal{R}(0)\right) = \lambda_k^h\,\|z_k^h - z_{k-1}^h\|_{\mathbb{V}}^2\,, \tag{2.7}$$

$$\mathcal{R}\Big(z_k^h - z_{k-1}^h\Big) + h\,\mathrm{dist}_{\mathcal{V}^*}\left(-\mathrm{D}_z\mathcal{I}(t_{k-1}^h, z_k^h), \partial\mathcal{R}(0)\right) = \Big\langle -\mathrm{D}_z\mathcal{I}(t_{k-1}^h, z_k^h), z_k^h - z_{k-1}^h\Big\rangle\,, \tag{2.8}$$

$$\mathcal{R}\Big(z_k^h - z_{k-1}^h\Big) + \|z_k^h - z_{k-1}^h\|_{\mathbb{V}}\,\mathrm{dist}_{\mathcal{V}^*}\left(-\mathrm{D}_z\mathcal{I}(t_{k-1}^h, z_k^h), \partial\mathcal{R}(0)\right) = \Big\langle -\mathrm{D}_z\mathcal{I}(t_{k-1}^h, z_k^h), z_k^h - z_{k-1}^h\Big\rangle\,, \tag{2.9}$$

$$\forall v \in \mathcal{Z} \quad \mathcal{R}(v) \geq -\Big\langle \lambda_k^h\mathbb{V}(z_k^h - z_{k-1}^h) + \mathrm{D}_z\mathcal{I}(t_{k-1}^h, z_k^h), v\Big\rangle\,. \tag{2.10}$$

We refer to Lemma A.1 for identities relying on convex analysis and for the definition of the distance function $\mathrm{dist}_{\mathcal{V}^*}(\cdot,\cdot)$.

Proof. Let $\Psi_h = \mathcal{R} + I_h$ be given as in (A.1), where I_h is the characteristic function of the set $\{\, v \in \mathcal{V}\,;\, \|v\|_{\mathbb{V}} \leq h\,\}$. Observe that z_k^h minimizes $\mathcal{I}(t_{k-1}^h, \cdot) + \Psi_h(\cdot - z_{k-1}^h)$. Hence,

$$0 \in \partial\Psi_h\Big(z_k^h - z_{k-1}^h\Big) + \mathrm{D}_z\mathcal{I}\Big(t_{k-1}^h, z_k^h\Big)\,, \tag{2.11}$$

which is equivalent to

$$\Psi_h\Big(z_k^h - z_{k-1}^h\Big) + \Psi_h^*\Big(-\mathrm{D}_z\mathcal{I}\Big(t_{k-1}^h, z_k^h\Big)\Big) = \Big\langle -\mathrm{D}_z\mathcal{I}\Big(t_{k-1}^h, z_k^h\Big), z_k^h - z_{k-1}^h\Big\rangle\,.$$

Taking into account Lemma A.1 we arrive at (2.8). Furthermore, there exists $\xi_k^h \in \partial I_h(z_k^h - z_{k-1}^h)$ such that $0 \in \partial\mathcal{R}(z_k^h - z_{k-1}^h) + \xi_k^h + \mathrm{D}_z\mathcal{I}(t_{k-1}^h, z_k^h)$ and

$$\mathcal{R}\Big(z_k^h - z_{k-1}^h\Big) + \mathcal{R}^*\Big(- \xi_k^h - \mathrm{D}_z\mathcal{I}\Big(t_{k-1}^h, z_k^h\Big)\Big) = -\Big\langle \xi_k^h + \mathrm{D}_z\mathcal{I}\Big(t_{k-1}^h, z_k^h\Big), z_k^h - z_{k-1}^h\Big\rangle .$$

Subtracting (2.8) from this relation yields

$$\mathcal{R}^*\Big(- \Big(\xi_k^h + \mathrm{D}_z\mathcal{I}(t_{k-1}^h, z_k^h)\Big)\Big) - h\,\mathrm{dist}_{\mathcal{V}^*}\Big(- \mathrm{D}_z\mathcal{I}\Big(t_{k-1}^h, z_k^h\Big), \partial\mathcal{R}(0)\Big) = -\Big\langle \xi_k^h, z_k^h - z_{k-1}^h\Big\rangle.$$

We next exploit the one-homogeneity of $\mathcal{R}$. Since $-((\xi_k^h + \mathrm{D}_z\mathcal{I}(t_{k-1}^h, z_k^h)) \in \partial\mathcal{R}(z_k^h - z_{k-1}^h) \subset \partial\mathcal{R}(0)$, the term $\mathcal{R}^*(-(\xi_k^h + \mathrm{D}_z\mathcal{I}(t_{k-1}^h, z_k^h)))$ equals to zero. Furthermore, by Lemma A.1, relation (A.4), there exists $\lambda_k^h \geq 0$ such that $\xi_k^h = \lambda_k^h \mathbb{V}(z_k^h - z_{k-1}^h)$ which ultimately implies (2.7). Relation (2.6) is again a consequence of Lemma A.1, relation (A.4). Relation (2.10) is a consequence of the one-homogeneity of $\mathcal{R}$ implying that $\partial\mathcal{R}(z_k^h - z_{k-1}^h) \subset \partial\mathcal{R}(0)$. Finally, relation (2.9) follows from (2.8) combined with (2.6) and (2.7). □

Observe that

$$h\lambda_k^h = \begin{cases} 0 & \text{if } \left\|z_k^h - z_{k-1}^h\right\|_{\mathbb{V}} < h \\ \mathrm{dist}_{\mathcal{V}^*}\big(- \mathrm{D}_z\mathcal{I}(t_{k-1}^h, z_k^h), \partial\mathcal{R}(0)\big) & \text{if } \left\|z_k^h - z_{k-1}^h\right\|_{\mathbb{V}} = h \end{cases} . \tag{2.12}$$

The next proposition is the main result of this section and guarantees that the procedure in (2.1)–(2.2) leads to $t_{N(h)}^h = T$ after a finite number of iteration steps $N(h)$.

Proposition 2.3. *Let $z_0 \in \mathcal{Z}$ satisfy $\mathrm{D}_z\mathcal{I}(0, z_0) \in \mathcal{V}^*$. For every $h > 0$ there exists $N(h) \in \mathbb{N}$ such that $t_{N(h)}^h = T$. Moreover, there exist constants $c_1, c_2, c_3 > 0$ such that for all $h > 0, k \leq N(h)$ we have*

$$\lambda_{k+1}^h \left\|z_{k+1}^h - z_k^h\right\|_{\mathbb{V}} + c_1 \sum_{i=0}^k \left\|z_{i+1}^h - z_i^h\right\|_{\mathcal{Z}} \leq c_2\Big(t_k^h + \|\mathrm{D}_z\mathcal{I}(t_0, z_0)\|_{\mathcal{V}^*} + \sum_{i=0}^k \mathcal{R}\Big(z_{i+1}^h - z_i^h\Big)\Big), \tag{2.13}$$

$$\Big\|\mathrm{D}_z\mathcal{I}\Big(t_{k-1}^h, z_k^h\Big)\Big\|_{\mathcal{V}^*} \leq c_3. \tag{2.14}$$

Proof. Inserting (2.7) into (2.8), rewriting this identity for the index $k+1$ (instead of k) and subtracting the resulting equation from (2.10) with $v = z_{k+1}^h - z_k^h$ yields

$$0 \geq \lambda_{k+1}^h \left\|z_{k+1}^h - z_k^h\right\|_{\mathbb{V}}^2 - \lambda_k^h\Big\langle \mathbb{V}\Big(z_k^h - z_{k-1}^h\Big), z_{k+1}^h - z_k^h\Big\rangle + \Big\langle \mathrm{D}_z\mathcal{I}\Big(t_k^h, z_{k+1}^h\Big) - \mathrm{D}_z\mathcal{I}\Big(t_{k-1}^h, z_k^h\Big), z_{k+1}^h - z_k^h\Big\rangle . \tag{2.15}$$

Substituting $\mathcal{I}$ and rearranging the terms yields

$$\begin{aligned} \lambda_{k+1}^h \left\|z_{k+1}^h - z_k^h\right\|_{\mathbb{V}}^2 - \lambda_k^h\Big\langle \mathbb{V}\Big(z_k^h - z_{k-1}^h\Big), z_{k+1}^h - z_k^h\Big\rangle + \Big\langle A\Big(z_{k+1}^h - z_k^h\Big), \Big(z_{k+1}^h - z_k^h\Big)\Big\rangle \\ \leq \Big\langle \mathrm{D}_z\mathcal{F}\Big(z_k^h\Big) - \mathrm{D}_z\mathcal{F}\Big(z_{k+1}^h\Big), z_{k+1}^h - z_k^h\Big\rangle + \Big\langle \ell\Big(t_{k-1}^h\Big) - \ell\Big(t_k^h\Big), z_{k+1}^h - z_k^h\Big\rangle . \end{aligned} \tag{2.16}$$

With Lemma 1.1, the assumptions on ℓ and (2.5), the right hand side is estimated by

$$r.h.s. \leq \frac{\alpha}{2}\left\|z_{k+1}^h - z_k^h\right\|_{\mathcal{Z}}^2 + C\left\|z_{k+1}^h - z_k^h\right\|_{\mathcal{V}}\Big(\Big(t_k^h - t_{k-1}^h\Big) + \mathcal{R}\Big(z_{k+1}^h - z_k^h\Big)\Big), \tag{2.17}$$

where $\alpha > 0$ is the constant from (1.12). The left hand side of (2.16) can be estimated as follows:

$$l.h.s \geq \lambda_{k+1}^h \left\|z_{k+1}^h - z_k^h\right\|_{\mathbb{V}}^2 - \lambda_k^h \left\|z_k^h - z_{k-1}^h\right\|_{\mathbb{V}} \left\|z_{k+1}^h - z_k^h\right\|_{\mathbb{V}} + \alpha \left\|z_{k+1}^h - z_k^h\right\|_{\mathcal{Z}}^2 . \tag{2.18}$$

Joining (2.17) with (2.18) and rearranging the terms, we arrive at

$$\begin{aligned}\lambda_{k+1}^h \left\|z_{k+1}^h - z_k^h\right\|_{\mathbb{V}}^2 - \lambda_k^h \left\|z_k^h - z_{k-1}^h\right\|_{\mathbb{V}} \left\|z_{k+1}^h - z_k^h\right\|_{\mathbb{V}} + c_1 \left\|z_{k+1}^h - z_k^h\right\|_{\mathcal{Z}} \left\|z_{k+1}^h - z_k^h\right\|_{\mathbb{V}} \\ \leq c_2\big(\mathcal{R}(z_{k+1}^h - z_k^h) + t_k^h - t_{k-1}^h\big) \left\|z_{k+1}^h - z_k^h\right\|_{\mathbb{V}} ,\end{aligned} \tag{2.19}$$

which implies

$$\lambda_{k+1}^h \left\|z_{k+1}^h - z_k^h\right\|_{\mathbb{V}} - \lambda_k^h \left\|z_k^h - z_{k-1}^h\right\|_{\mathbb{V}} + c_1 \left\|z_{k+1}^h - z_k^h\right\|_{\mathcal{Z}} \leq c_2\big(\mathcal{R}(z_{k+1}^h - z_k^h) + t_k^h - t_{k-1}^h\big). \tag{2.20}$$

Summing up this estimate with respect to k finally yields

$$\lambda_{k+1}^h \left\|z_{k+1}^h - z_k^h\right\|_{\mathbb{V}} + c_1 \sum_{i=1}^{k} \left\|z_{i+1}^h - z_i^h\right\|_{\mathcal{Z}} \leq \lambda_1^h \left\|z_1^h - z_0^h\right\|_{\mathbb{V}} + c_2\big(t_k^h + \sum_{i=1}^{k} \mathcal{R}(z_{i+1}^h - z_i^h)\big) . \tag{2.21}$$

It remains to estimate the term $\left\|z_1^h - z_0^h\right\|_{\mathcal{Z}}$. Starting again from (2.8) for $k = 1$ in combination with (2.7) and inserting a zero we obtain after rearranging the terms

$$\langle \mathrm{D}_z\mathcal{I}(t_0, z_1^h) - \mathrm{D}_z\mathcal{I}(t_0, z_0^h), z_1^h - z_0^h\rangle + \mathcal{R}(z_1^h - z_0^h) + \lambda_1^h \left\|z_1^h - z_0^h\right\|_{\mathbb{V}}^2 = -\langle \mathrm{D}_z\mathcal{I}(t_0, z_0^h), z_1^h - z_0^h\rangle .$$

The first term on the left hand side is treated as above, so that finally

$$c \left\|z_1^h - z_0^h\right\|_{\mathcal{Z}}^2 + \lambda_1^h \left\|z_1^h - z_0^h\right\|_{\mathbb{V}}^2 \leq c\big(\mathcal{R}(z_1^h - z_0^h) + \left\|\mathrm{D}_z\mathcal{I}(t_0, z_0^h)\right\|_{\mathcal{V}^*}\big) \left\|z_1^h - z_0^h\right\|_{\mathbb{V}} \tag{2.22}$$

which is the analogue to (2.20). Adding this estimate to (2.21) we arrive at (2.13).

Since $\left\|z_k^h - z_{k-1}^h\right\|_{\mathbb{V}} \leq h$, the identity (2.7) implies that

$$\mathrm{dist}_{\mathcal{V}^*}\Big(-\mathrm{D}_z\mathcal{I}\Big(t_{k-1}^h, z_k^h\Big), \partial\mathcal{R}(0)\Big) \leq \lambda_k^h \left\|z_k^h - z_{k-1}^h\right\|_{\mathbb{V}} ,$$

which together with (2.13) and (2.4) leads to (2.14).

Thanks to (2.4) and the assumption on z_0, the right hand side of (2.13) is uniformly bounded with respect to k. Hence, if T is not reached after a finite number of steps, then the series $\sum_{k=0}^{\infty} \left\|z_{k+1}^h - z_k^h\right\|_{\mathbb{V}}$ converges and there exists $t_* \leq T$ such that $\lim_{k\to\infty} t_k^h = t_*$. In particular, the sequence $(t_{k+1}^h - t_k^h)_{k\in\mathbb{N}}$ tends to zero. But this implies that $(\left\|z_{k+1}^h - z_k^h\right\|_{\mathbb{V}})_{k\in\mathbb{N}}$ tends to h for $k \to \infty$, a contradiction to the convergence of the series $\sum_{k=0}^{\infty} \left\|z_{k+1}^h - z_k^h\right\|_{\mathbb{V}}$. □

Similarly to [4] we introduce the piecewise affine and the left and right continuous piecewise constant interpolants: Let $S_h := T + \sum_{i=1}^{N(h)} \left\|z_i^h - z_{i-1}^h\right\|_{\mathbb{V}}$ and $s_k^h = kh$. For $s \in [s_{k-1}^h, s_k^h) \subset [0, S_h]$

$$\hat{z}_h(s) = z_{k-1}^h + (s - s_{k-1}^h)h^{-1}(z_k^h - z_{k-1}^h) , \qquad \hat{t}_h(s) = t_{k-1}^h + (s - s_{k-1}^h)h^{-1}(t_k^h - t_{k-1}^h) , \tag{2.23}$$

$$\overline{z}_h(s) := z_k^h, \qquad \overline{t}_h(s) := t_k^h , \qquad \underline{z}_h(s) := z_{k-1}^h, \qquad \underline{t}_h(s) := t_{k-1}^h . \tag{2.24}$$

Observe that by construction we have $\left\|z_k^h - z_{k-1}^h\right\|_{\mathbb{V}} \le h$, see (2.1), and hence, $\hat{z}_h$ is Lipschitz continuous with respect to $\mathcal{V}$ with a Lipschitz constant that is independent of h. Furthermore, as a consequence of Proposition 2.1 and Proposition 2.3 we deduce

$$\hat{t}_h(S_h) = \bar{t}_h(S_h) = T, \quad (\hat{t}_h, \hat{z}_h) \in W^{1,\infty}([0,S_h], \mathbb{R}\times\mathcal{V}), \tag{2.25}$$

$$\sup_h \Big(\|\hat{t}_h\|_{W^{1,\infty}([0,S_h],\mathbb{R})} + \|\hat{z}_h\|_{W^{1,\infty}([0,S_h];\mathcal{V})} + \|\hat{z}_h\|_{L^\infty([0,S_h];\mathcal{Z})} + S_h \Big) < \infty \tag{2.26}$$

$$\text{for a.a. } s \in [0,S_h]: \quad \hat{t}_h'(s) \ge 0,\ \hat{t}_h'(s) + \|\hat{z}_h'(s)\|_{\mathbb{V}} = 1\,. \tag{2.27}$$

Proposition 2.4 (Discrete energy-dissipation identity)**.** *For all $\sigma_1 \le \sigma_2 \in [0,S_h]$ we have*

$$\begin{aligned} \mathcal{I}(\hat{t}_h(\sigma_2), \hat{z}_h(\sigma_2)) + \int_{\sigma_1}^{\sigma_2} \mathcal{R}(\hat{z}_h'(s)) + \|\hat{z}_h'(s)\|_{\mathbb{V}} \operatorname{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\underline{t}_h(s), \bar{z}_h(s)), \partial\mathcal{R}(0))\,\mathrm{d}s \\ = \mathcal{I}(\hat{t}_h(\sigma_1), \hat{z}_h(\sigma_1)) + \int_{\sigma_1}^{\sigma_2} \partial_t\mathcal{I}(\hat{t}_h(s), \hat{z}_h(s))\hat{t}_h'(s)\,\mathrm{d}s + \int_{\sigma_1}^{\sigma_2} r_h(s)\,\mathrm{d}s\,, \end{aligned} \tag{2.28}$$

where $r_h(s) = \langle \mathrm{D}_z\mathcal{I}(\hat{t}_h(s), \hat{z}_h(s)) - \mathrm{D}_z\mathcal{I}(\underline{t}_h(s), \bar{z}_h(s)), \hat{z}_h'(s)\rangle$. Moreover the complementarity condition

$$\textit{for a.a. } s \in [0,S_h]: \quad \hat{t}_h'(s) \operatorname{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\underline{t}_h(s), \bar{z}_h(s), \partial\mathcal{R}(0)) = 0 \tag{2.29}$$

is fulfilled. There exists a constant $C > 0$ such that the remainder r_h satisfies for all $h > 0$ and all $\sigma_1 < \sigma_2 \in [0,S_h]$

$$\int_{\sigma_1}^{\sigma_2} r_h(s)\,\mathrm{d}s \le Ch\,. \tag{2.30}$$

Proof. Relation (2.29) is an immediate consequence of (2.6)–(2.7). The energy identity follows from (2.9) by applying the chain rule and integrating with respect to s. In order to estimate r_h we proceed as follows: observe first that $\hat{z}_h(s) - \bar{z}_h(s) = (s - s_{k+1}^h)\hat{z}_h'(s)$ for $s \in [s_k^h, s_{k+1}^h)$. Taking into account the definition of $\mathcal{I}$ and of the interpolants, we find by applying Lemma 1.1 with $\epsilon = \alpha/2$ (α as the ellipticity constant of A)

$$\begin{aligned} r_h(s) \le \alpha(s - s_{k+1}^h)\,\|\hat{z}_h'(s)\|_{\mathcal{Z}}^2 + (s_{k+1}^h - s)\left(\epsilon\,\|\hat{z}_h'(s)\|_{\mathcal{Z}}^2 + C_\epsilon\mathcal{R}(\hat{z}_h'(s))\,\|\hat{z}_h'(s)\|_{\mathcal{V}}\right) \\ + (s_{k+1}^h - s)\hat{t}'(s)\,\|\hat{z}'(s)\|_{\mathcal{V}}\,\|\ell\|_{C^1([0,T];\mathcal{V}^*)}\,. \end{aligned}$$

Integration with respect to s yields

$$\int_{\sigma_0}^{\sigma_1} r_h(s)\,\mathrm{d}s \le ch\left(T + \sum_{i=1}^{N(h)} \mathcal{R}(z_i^h - z_{i-1}^h)\right)\,.$$

Taking into account (2.4) we finally arrive at (2.30). □

Thanks to (2.26), $(S_h)_h$ is uniformly bounded and hence there exists a subsequence $h_n \searrow 0$ and $S \ge T$ such that $S_{h_n} \to S$. In the following, if $S_h < S$ the corresponding discrete functions are extended to the interval (S_h, S) by their value at S_h.

Theorem 2.5. *Let $z_0 \in \mathcal{Z}$ satisfy $\mathrm{D}_z\mathcal{I}(0,z_0) \in \mathcal{V}^*$ and assume that $\mathcal{F}$ satisfies (1.13), (1.18) and (1.21). There exists a sequence $(h_n)_{n\in\mathbb{N}}$ with $h_n \to 0$ as $n \to \infty$, $S \in (0,\infty)$ and functions $\hat{t} \in W^{1,\infty}((0,S);\mathbb{R})$ and $\hat{z} \in W^{1,\infty}((0,S);\mathcal{V}) \cap L^\infty((0,S);\mathcal{Z})$ such that for $n \to \infty$*

$$S_{h_n} \to S, \tag{2.31}$$

$$\hat{t}_{h_n} \overset{*}{\rightharpoonup} \hat{t} \text{ in } W^{1,\infty}((0,S);\mathbb{R}), \quad \hat{t}_{h_n}(s) \to \hat{t}(s) \text{ for every } s \in [0,S], \tag{2.32}$$

$$\hat{z}_{h_n} \overset{*}{\rightharpoonup} \hat{z} \text{ weakly* in } W^{1,\infty}((0,S);\mathcal{V}) \cap L^\infty((0,S);\mathcal{Z}) \tag{2.33}$$

$$\hat{z}_{h_n}(s) \rightharpoonup \hat{z}(s) \text{ weakly in } \mathcal{Z} \text{ for every } s \in [0,S]\,. \tag{2.34}$$

Moreover, the limit pair $(\hat{t},\hat{z})$ satisfies

$$\hat{t}(0) = 0,\ \hat{t}(S) = T,\ \hat{z}(0) = z_0\,, \tag{2.35}$$

$$\text{for a.a. } s \in [0,S]: \quad \hat{t}'(s) \geq 0,\ \hat{t}'(s) + \|\hat{z}'(s)\|_{\mathbb{V}} \leq 1,\ \hat{t}'(s)\,\mathrm{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{t}(s),\hat{z}(s)),\partial\mathcal{R}(0)) = 0 \tag{2.36}$$

together with the energy identity

$$\mathcal{I}(\hat{t}(s_1),\hat{z}(s_1)) + \int_0^{s_1} \mathcal{R}(\hat{z}'(s)) + \|\hat{z}'(s)\|_{\mathbb{V}}\,\mathrm{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{t}(s),\hat{z}(s)),\partial\mathcal{R}(0))\,\mathrm{d}s$$
$$= \mathcal{I}(\hat{t}(0),\hat{z}(0)) + \int_0^{s_1} \partial_t\mathcal{I}(\hat{t}(s),\hat{z}(s))\hat{t}'(s)\,\mathrm{d}s \tag{2.37}$$

that is valid for all $s_1 \in [0,S]$. Every accumulation point $(\hat{t},\hat{z})$ (in the sense of (2.31)–(2.34)) of time incremental sequences $(\hat{t}_h,\hat{z}_h)_{h>0}$ satisfies (2.35)–(2.37).

Proof. The convergence results in (2.31)–(2.34) are an immediate consequence of the uniform estimates formulated in (2.26) and in Proposition 2.1. Let us discuss in more detail assertion (2.34). Since the sequence $(\hat{z}_h)_h$ is uniformly bounded in $W^{1,\infty}((0,S);\mathcal{V}) \cap L^\infty((0,S);\mathcal{Z})$, the Arzéla–Ascoli theorem (*cf.* [3], Sect. 7.5), implies the uniform convergence in $C([0,S];\mathcal{V})$ to $\hat{z}$. Since $\sup_{h>0,s\in[0,S_h]}\|\hat{z}(s)\|_{\mathcal{Z}} < \infty$, for each s there exists a subsequence such that $\hat{z}_{h_{\tilde{n}}}(s) \rightharpoonup \hat{z}(s)$ weakly in $\mathcal{Z}$. A proof by contradiction ultimately shows the weak convergence of the sequence $(\hat{z}_{h_n}(s))_n$ itself, whence (2.34).

Clearly, the limit pair $(\hat{t},\hat{z})$ satisfies the first two relations in (2.36). In the following, we omit the index n. Observe further that for all $s \in [0,S]$

$$\underline{t}_h(s), \bar{t}_h(s) \to \hat{t}(s), \quad \underline{z}_h(s), \bar{z}_h(s) \rightharpoonup \hat{z}(s) \text{ weakly in } \mathcal{Z}.$$

Indeed, from the definitions it follows that $\|\hat{z}_h(s) - \underline{z}_h(s)\|_{\mathbb{V}} = (s - s_{k-1}^h)h^{-1}\left\|z_k^h - z_{k-1}^h\right\|_{\mathbb{V}} \leq h$ for $s \in (s_{k-1}^h, s_k^h)$, and hence, $\underline{z}_h \to \hat{z}$ uniformly in $\mathcal{V}$. By the very same argument as before the pointwise weak convergence in $\mathcal{Z}$ ensues. Together with the uniform bound (2.14), this implies that for all s

$$\mathrm{D}_z\mathcal{I}(\underline{t}_h(s),\bar{z}_h(s)) \rightharpoonup \mathrm{D}_z\mathcal{I}(\hat{t}(s),\hat{z}(s)) \text{ weakly in } \mathcal{Z}^* \text{ and in } \mathcal{V}^*.$$

Hence, the following lower semicontinuity estimate is valid for all s:

$$\liminf_{h\to 0} \mathrm{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\underline{t}_h(s),\bar{z}_h(s)),\partial\mathcal{R}(0)) \geq \mathrm{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{t}(s),\hat{z}(s)),\partial\mathcal{R}(0))\,.$$

For arbitrary $\alpha < \beta$ we therefore obtain from (2.29) with Lemma B.2

$$0 \geq \liminf_{h\to 0} \int_\alpha^\beta \hat{t}_h'(s)\,\mathrm{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\underline{t}_h(s),\bar{z}_h(s)),\partial\mathcal{R}(0))\,\mathrm{d}s \geq \int_\alpha^\beta \hat{t}'(s)\,\mathrm{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(t(s),z(s)),\partial\mathcal{R}(0))\,\mathrm{d}s \geq 0,$$

whence the last relation in (2.36). Moreover, by the generalized version of Ioffe's theorem ([25], Thm. 21) applied to $\int_0^{s_1}\mathcal{R}(\hat{z}_h')\,\mathrm{d}s$ and by Proposition B.1 we arrive at the following estimate: $\forall s_1 \in [0,S]$

$$\begin{aligned}
&\mathcal{I}(\hat{t}(s_1),\hat{z}(s_1)) + \int_0^{s_1} \mathcal{R}(\hat{z}'(s)) + \|\hat{z}'(s)\|_{\mathbb{V}}\,\mathrm{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{t}(s),\hat{z}(s)),\partial\mathcal{R}(0))\,\mathrm{d}s\\
&\quad\leq \liminf_{h\to 0}\Big(\mathcal{I}(\hat{t}_h(s_1),\hat{z}_h(s_1)) + \int_0^{s_1}\mathcal{R}(\hat{z}_h'(s)) + \|\hat{z}_h'(s)\|_{\mathbb{V}}\,\mathrm{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\underline{t}_h(s),\bar{z}_h(s)),\partial\mathcal{R}(0))\,\mathrm{d}s\Big)\\
&\quad\overset{(2.28)}{\leq} \limsup_{h\to 0}\Big(\mathcal{I}(\hat{t}_h(0),\hat{z}_h(0)) + \int_0^{s_1}\partial_t\mathcal{I}(\hat{t}_h(s),\hat{z}_h(s))\hat{t}_h'(s)\,\mathrm{d}s + \int_0^{s_1} r_h(s)\,\mathrm{d}s\Big)\\
&\quad\leq \mathcal{I}(\hat{t}(0),\hat{z}(0)) + \int_0^{s_1}\partial_t\mathcal{I}(\hat{t}(s),\hat{z}(s))\hat{t}'(s)\,\mathrm{d}s\,,
\end{aligned}$$

which is (2.37) with $\leq$ instead of an equality. Here, we also used that $\partial_t\mathcal{I}(\hat{t}_h(s),\hat{z}_h(s))$ converges pointwise to $\partial_t\mathcal{I}(\hat{t}(s),\hat{z}(s))$ and is uniformly bounded with respect to h and s, which implies strong $L^1(0,s_1)$-convergence of $s\mapsto\partial_t\mathcal{I}(\hat{t}_h(s),\hat{z}_h(s))$. Arguing in exactly the same way as for instance in the proof of ([7], Lem. 5.2) the inequality can be replaced by an equality, and (2.37) is shown. □

Remark 2.6. The above proof does not guarantee that the limit pair $(\hat{t},\hat{z})$ is nondegenerate meaning that $\hat{t}'(s) + \|\hat{z}'(s)\|_{\mathbb{V}} > 0$ for almost all $s \in [0,S]$. We refer to [4, 17] for a discussion of nondegeneracy conditions in an abstract setting and to [6] for a discussion in the context of a damage model.

The solution obtained by the local minimisation algorithm belongs to the class of parametrised BV-solutions, see Proposition 1.3 and Definition 1.2. The example in Section 5 reveals that parametrised BV-solutions obtained by vanishing viscosity approximations may differ from those obtained by the local minimisation algorithm.

3. An approximation scheme relying on relaxed local minimisation

3.1. Convergence with an unbounded sequence of penalty parameters

We briefly recall the setting of Section 1.1:

The spaces $\mathcal{X},\mathcal{V},\mathcal{Z}$ satisfy (1.11), (3.1a)

the functional $\mathcal{I} : [0,T]\times\mathcal{Z}\to\mathbb{R}$ is given by (1.14) with operators $A,\mathbb{V}$ as in (1.12), (3.1b)

$\ell,\mathcal{F}$ satisfy (1.13), (1.18) and (1.21), (3.1c)

$\mathcal{R} : \mathcal{X}\to[0,\infty)$ is convex, lower semicontinuous, pos. one-homogeneous and satisfies (1.17), (3.1d)

$z_0 \in \mathcal{Z}$ satisfies $\mathrm{D}_z\mathcal{I}(0,z_0)\in\mathcal{V}^*$. (3.1e)

The following variant of a procedure proposed in [1] will be analysed:

Given $N\in\mathbb{N}$, a time-step size $\tau = T/N$, a parameter $\eta > 0$ and an initial datum $z_0\in\mathcal{Z}$ we define for $1\leq k\leq N$ and $i\in\mathbb{N}_0$: $t_k = k\tau$, $z_{k,0} := z_{k-1}$ and for $i\geq 1$

$$z_{k,i} \in \mathrm{Argmin}\{\,\mathcal{I}(t_k,v) + \frac{\eta}{2}\|v - z_{k,i-1}\|_{\mathbb{V}}^2 + \mathcal{R}(v - z_{k,i-1})\,;\,v\in\mathcal{Z}\,\}, \tag{3.2}$$

$$z_k := z_{k,\infty} := \lim_{i\to\infty} z_{k,i} \quad \text{(weak limit in } \mathcal{Z}). \tag{3.3}$$

Remark 3.1. This approximation scheme can be interpreted as a relaxed version of the scheme discussed in Section 1.1 where the constraint $\left\|v - z_{k-1}^h\right\|_{\mathbb{V}} \leq h$ is replaced with the additional term $\frac{\eta}{2}\left\|v - z_{k,i-1}\right\|_{\mathbb{V}}^2$, where η plays the role of a penalty parameter. This scheme is a variation of a procedure suggested in ([1], Sect. 3.1). There, instead of $\mathcal{R}(v - z_{k,i-1})$ the term $\mathcal{R}(v - z_{k-1})$ is used in (3.2). Different from ([1], Sect. 3.1) we can prove that the sequence $(z_{k,i})_{i\in\mathbb{N}}$ itself converges, see Proposition 3.2 here below. For a more detailed comparison with the results from [1], we refer to Section 3.2.

In a first step, we discuss the behaviour of (3.2)–(3.3) for fixed k. Let $\mathcal{H}(t, v, w) := \mathcal{I}(t, v) + \frac{\eta}{2}\left\|v - w\right\|_{\mathbb{V}}^2 + \mathcal{R}(v - w)$. For $t \in [0, T]$, $z_0 \in \mathcal{Z}$ and $i \geq 1$ let

$$z_i \in \operatorname{Argmin}\{\, \mathcal{H}(t, v, z_{i-1})\,;\, v \in \mathcal{Z}\,\}. \tag{3.4}$$

Clearly, minimizers exist and we have the following estimates for all $i \geq 1$:

$$\mathcal{H}(t, z_{i+1}, z_i) \leq \mathcal{H}(t, z_i, z_i) = \mathcal{I}(t, z_i) \leq \mathcal{H}(t, z_i, z_{i-1}) \leq \mathcal{I}(t, z_{i-1}), \tag{3.5}$$

i.e., the sequences $(\mathcal{H}(t, z_i, z_{i-1}))_{i\geq 1}$ and $(\mathcal{I}(t, z_i))_{i\geq 0}$ are non-increasing. Due to the coercivity of $\mathcal{I}$, they are bounded from below. Hence, there exists $I_\infty \in \mathbb{R}$ such that

$$\lim_{i\to\infty} \mathcal{I}(t, z_i) = I_\infty = \lim_{i\to\infty} \mathcal{H}(t, z_i, z_{i-1}), \tag{3.6}$$

which implies that

$$\lim_{i\to\infty} \mathcal{R}(z_i - z_{i-1}) = 0, \quad \lim_{i\to\infty} \left\|z_i - z_{i-1}\right\|_{\mathbb{V}} = 0. \tag{3.7}$$

Moreover, summing up the left part of estimate (3.5) with respect to i one arrives at

$$\mathcal{I}(t, z_i) + \sum_{j=s}^{i-1} \Big(\mathcal{R}(z_{j+1} - z_j) + \frac{\eta}{2}\left\|z_{j+1} - z_j\right\|_{\mathbb{V}}^2\Big) \leq \mathcal{I}(t, z_s), \tag{3.8}$$

which is valid for all $0 \leq s \leq i$.

Proposition 3.2. *Assume* (3.1a)–(3.1d).

There exists a constant $C > 0$ (possibly depending on t and z_0 but independent of η) such that

$$\sup_{i\in\mathbb{N}} \left\|z_i\right\|_{\mathcal{Z}} \leq C, \tag{3.9}$$

$$\sum_{j=0}^{\infty} \Big(\mathcal{R}(z_{j+1} - z_j) + \frac{\eta}{2}\left\|z_{j+1} - z_j\right\|_{\mathbb{V}}^2\Big) \leq C. \tag{3.10}$$

Moreover, there exists $z_\infty \in \mathcal{Z}$ such that the sequence $(z_i)_{i\in\mathbb{N}}$ converges to z_∞ weakly in $\mathcal{Z}$. The limit z_∞ satisfies $\mathrm{D}_z\mathcal{I}(t, z_\infty) \in \mathcal{V}^*$ and

$$0 \in \partial\mathcal{R}(0) + \mathrm{D}_z\mathcal{I}(t, z_\infty). \tag{3.11}$$

Proof. Estimates (3.9)–(3.10) follow from (3.8) (with $s = 0$) and the coercivity of $\mathcal{I}$. Since $\mathcal{R}$ is convex and positively homogeneous of degree one and hence satisfies a triangle inequality, together with (1.17) and (3.8) it follows that for $s \le i$

$$c\,\|z_i - z_s\|_{\mathcal{X}} \le \mathcal{R}(z_i - z_s) \le \sum_{j=s}^{i-1} \mathcal{R}(z_{j+1} - z_j) \le \mathcal{I}(t, z_s) - \mathcal{I}(t, z_i).$$

Since the sequence $(\mathcal{I}(t, z_j))_{j\in\mathbb{N}}$ is converging, this estimate shows that $(z_j)_{j\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $\mathcal{X}$. Together with (3.9) we obtain the convergence of the sequence $(z_j)_{j\in\mathbb{N}}$ to some z_∞ weakly in $\mathcal{Z}$.

In order to obtain (3.11) observe that for every $i \ge 1$ we have

$$-\mathrm{D}_z\mathcal{I}(t, z_i) - \eta\mathbb{V}(z_i - z_{i-1}) \in \partial\mathcal{R}(z_i - z_{i-1}) \subset \partial\mathcal{R}(0), \tag{3.12}$$

where the last inclusion again follows from the one-homogeneity of $\mathcal{R}$. This inclusion is valid in both spaces, in $\mathcal{Z}^*$ and in $\mathcal{V}^*$, thanks to the upper estimate for $\mathcal{R}$ in (1.17). Since by the assumptions the operator A and $\mathrm{D}_z\mathcal{F} : \mathcal{Z} \to \mathcal{Z}^*$ are weak–weak-continuous, it follows that

$$\mathrm{D}_z\mathcal{I}(t, z_i) + \eta\mathbb{V}(z_i - z_{i-1}) \rightharpoonup \mathrm{D}_z\mathcal{I}(t, z_\infty) \quad \text{weakly in } \mathcal{Z}^*. \tag{3.13}$$

Moreover, thanks to (1.18) and since $\partial\mathcal{R}(0)$ is a bounded subset of $\mathcal{V}^*$, the sequences $(\mathrm{D}_z\mathcal{F}(z_i))_{i\in\mathbb{N}}$ and $(\mathrm{D}_z\mathcal{I}(t, z_i) + \eta\mathbb{V}(z_i - z_{i-1}))_{i\in\mathbb{N}}$ are bounded in $\mathcal{V}^*$. Together with (1.21) and (3.13) this implies that $\mathrm{D}_z\mathcal{F}(z_i) \rightharpoonup \mathrm{D}_z\mathcal{F}(z_\infty)$ weakly in $\mathcal{V}^*$ and ultimately $Az_i \rightharpoonup Az_\infty$ weakly in $\mathcal{V}^*$. Since $\partial\mathcal{R}(0)$ is weakly closed in $\mathcal{V}^*$ one finally obtains (3.11). □

The next aim is to derive uniform estimates for the sequences $(z^\tau_{k,i})_{0\le k\le N, i\in\mathbb{N}\cup\{0,\infty\}}$ generated by the full scheme (3.2)–(3.3). Observe first that with $z^\tau_{k,0} = z^\tau_{k-1} = z^\tau_{k-1,\infty}$ we have

$$\mathcal{I}\Big(t_k, z^\tau_{k,0}\Big) = \mathcal{I}\Big(t_k, z^\tau_{k-1}\Big) = \mathcal{I}\Big(t_{k-1}, z^\tau_{k-1}\Big) - \int_{t_{k-1}}^{t_k} \Big\langle \dot{\ell}(t), z^\tau_{k-1}\Big\rangle\,\mathrm{d}t.$$

Hence, summing up (3.8) with respect to i yields

$$\mathcal{I}\Big(t_k, z^\tau_k\Big) + \sum_{j=0}^{\infty} \mathcal{R}_\eta\Big(z^\tau_{k,j+1} - z^\tau_{k,j}\Big) \le \mathcal{I}\Big(t_{k-1}, z^\tau_{k-1}\Big) + \int_{t_{k-1}}^{t_k} \partial_t\mathcal{I}(t, z_{k-1})\,\mathrm{d}t, \tag{3.14}$$

where we use the short-hand notation $\mathcal{R}_\eta(v) = \mathcal{R}(v) + \frac{\eta}{2}\|v\|^2_{\mathbb{V}}$. Additional summation with respect to k gives

$$\mathcal{I}\Big(t_k, z^\tau_{k,i+1}\Big) + \sum_{j=0}^{i} \mathcal{R}_\eta\Big(z^\tau_{k,j+1} - z^\tau_{k,j}\Big) + \sum_{s=1}^{k-1}\sum_{j=0}^{\infty} \mathcal{R}_\eta\Big(z^\tau_{s,j+1} - z^\tau_{s,j}\Big) \le \mathcal{I}(t_0, z_0) + \sum_{l=1}^{k} \int_{t_{l-1}}^{t_l} \partial_t\mathcal{I}\Big(t, z^\tau_{l-1}\Big)\,\mathrm{d}t, \tag{3.15}$$

which is valid for all $\tau = T/N > 0$, $\eta > 0$, $1 \le k \le N$, $i \in \mathbb{N}\cup\{0,\infty\}$.

Proposition 3.3. *Assume* (3.1a)–(3.1d).

There exists a constant $C > 0$ such that for all $\eta > 0$, $N \in \mathbb{N}$, $\tau = T/N$, $0 \le k \le N$, $i \in \mathbb{N}\cup\{0,\infty\}$ we have

$$\left\|z^\tau_{k,i}\right\|_{\mathcal{Z}} \le C, \tag{3.16}$$

$$\sum_{s=1}^{N}\sum_{j=0}^{\infty} \Big(\mathcal{R}(z^\tau_{s,j+1} - z^\tau_{s,j}) + \frac{\eta}{2}\left\|z^\tau_{s,j+1} - z^\tau_{s,j}\right\|^2_{\mathbb{V}}\Big) \le C. \tag{3.17}$$

Proof. From (3.14) and (1.16) one obtains similar to ([15], Sect. 2.1.2, estimates (2.1.10)) the estimate

$$\mathcal{I}\Big(t_k, z_k^\tau\Big) + \sum_{s=1}^{k}\sum_{j=0}^{\infty}\mathcal{R}_\eta\Big(z_{k,j+1}^\tau - z_{k,j}^\tau\Big) \le \Big(c + \mathcal{I}(0,z_0)\Big)e^{\mu T}$$

with $\mu, c \ge 0$ independently of τ, k, η. Together with the coercivity of $\mathcal{I}$ this yields (3.16) for $i = \infty$ and (3.17). Exploiting now (3.15) for arbitrary i leads to (3.16) for every $i \in \mathbb{N} \cup \{0, \infty\}$. □

Like in Section 1.1, from the data generated by (3.2)–(3.3) we construct interpolating curves in an arc-length parametrised setting. However, due to slight differences in the estimates that we find for the $(z_{k,i}^\tau)$, the interpolating curves will be constructed in the spirit of [6]. For that purpose we first derive an analogue of Proposition 2.3 guaranteeing that the lengths of the interpolating curves will be uniformly bounded.

Proposition 3.4. *Assume* (3.1a)–(3.1e) *and let*

$$\gamma_k^\tau := \sum_{i=0}^{\infty}\left\|z_{k,i+1}^\tau - z_{k,i}^\tau\right\|_{\mathcal{Z}}.$$

There exists a constant $C > 0$ such that for all $N \in \mathbb{N}$, $\tau = T/N$ and $\eta > 0$ we have

$$\sum_{k=1}^{N}\gamma_k^\tau \le C\left(T\,\|\ell\|_{C^1([0,T],\mathcal{V}^*)} + \|\mathrm{D}_z\mathcal{I}(0,z_0)\|_{\mathbb{V}} + \sum_{k=1}^{N}\sum_{i=0}^{\infty}\mathcal{R}\left(z_{k,i+1}^\tau - z_{k,i}^\tau\right)\right). \tag{3.18}$$

Thanks to Proposition 3.3, the right hand side is uniformly bounded. Moreover, there exists a constant $C > 0$ such that for all $N \in \mathbb{N}$, $\eta > 0$, $1 \le k \le N$,$i \ge 0$:

$$\eta\left\|z_{k,i+1}^\tau - z_{k,i}^\tau\right\|_{\mathbb{V}} \le C, \tag{3.19}$$

$$\left\|\mathrm{D}_z\mathcal{I}(t_k^\tau, z_{k,i}^\tau)\right\|_{\mathcal{V}^*} \le C\,. \tag{3.20}$$

Proof. The arguments to prove Proposition 3.4 are similar to those in the proof of Proposition 2.3. Let $1 \le k \le N$ and $1 \le i < \infty$. In the following we omit the index τ. Exploiting the one-homogeneity of $\mathcal{R}$, the inclusion (3.12) implies that

$$0 \ge \Big\langle -(\mathrm{D}_z\mathcal{I}(t_k, z_{k,i}) + \eta\mathbb{V}(z_{k,i} - z_{k,i-1})) - (-(\mathrm{D}_z\mathcal{I}(t_k, z_{k,i+1}) + \eta\mathbb{V}(z_{k,i+1} - z_{k,i}))), z_{k,i+1} - z_{k,i}\Big\rangle, \tag{3.21}$$

which can be rewritten as

$$\eta\,\|z_{k,i+1} - z_{k,i}\|_{\mathbb{V}}^2 - \eta\Big\langle\mathbb{V}(z_{k,i} - z_{k,i-1}), z_{k,i+1} - z_{k,i}\Big\rangle + \Big\langle A(z_{k,i+1} - z_{k,i}), (z_{k,i+1} - z_{k,i})\Big\rangle \le \Big\langle \mathrm{D}_z\mathcal{F}(z_{k,i}) - \mathrm{D}_z\mathcal{F}(z_{k,i+1}), z_{k,i+1} - z_{k,i}\Big\rangle. \tag{3.22}$$

This is exactly (2.16) if one identifies η with λ_k^h and λ_{k+1}^h. Here, i plays the role of k in (2.16). Transferring the arguments leading to (2.21) to the present setting results in ($k \ge 1$, $i \ge 1$)

$$\eta\,\|z_{k,i+1} - z_{k,i}\|_{\mathbb{V}} + \frac{\alpha}{2}\sum_{j=1}^{i}\|z_{k,j+1} - z_{k,j}\|_{\mathcal{Z}} \le \eta\,\|z_{k,1} - z_{k,0}\|_{\mathbb{V}} + c_\alpha\sum_{j=1}^{i}\mathcal{R}(z_{k,j+1} - z_{k,j}). \tag{3.23}$$

It remains to estimate the term $\|z_{k,1}-z_{k,0}\|_{\mathbb{V}}$. Exploiting the convex analysis identity $\xi\in\partial\mathcal{R}(\eta)\Leftrightarrow\mathcal{R}(\eta)+\mathcal{R}^*(\xi)=\langle\xi,\eta\rangle$ and taking into account that $\mathcal{R}$ is positively homogeneous of degree one, the first inclusion of (3.12) written for $i=1$ yields:

$$\mathcal{R}(z_{k,1}-z_{k,0})=-\Big(\Big\langle \mathrm{D}_z\mathcal{I}(t_k,z_{k,1}),z_{k,1}-z_{k,0}\Big\rangle+\eta\,\|z_{k,1}-z_{k,0}\|_{\mathbb{V}}^2\Big). \tag{3.24}$$

If $k\geq 2$, then by (3.11) we have (since $z_{k-1,\infty}=z_{k,0}$) $0\in\partial\mathcal{R}(0)+\mathrm{D}_z\mathcal{I}(t_{k-1},z_{k,0})$, and thus

$$\mathcal{R}\Big(z_{k,1}-z_{k,0}\Big)\geq\Big\langle -\mathrm{D}_z\mathcal{I}(t_{k-1},z_{k,0}),z_{k,1}-z_{k,0}\Big\rangle.$$

Subtracting (3.24) from this estimate leads to (3.21) and we finally obtain (3.23) with the additional term $+c\tau\,\|\ell\|_{C^1([0,T],\mathcal{V}^*)}$ on the right hand side and starting from $j=0$. The constants are independent of η,N,k. If $k=1$, then adding $-\langle\mathrm{D}_z\mathcal{I}(t_1,z_0),z_{1,1}-z_{1,0}\rangle$ to both sides of (3.24) and rearranging the terms results in

$$\begin{aligned}\mathcal{R}(z_{1,1}-z_0)+\Big\langle \mathrm{D}_z\mathcal{I}(t_1,z_{1,1})-\mathrm{D}_z\mathcal{I}(t_1,z_0),z_{1,1}-z_0\Big\rangle+\eta\,\|z_{1,1}-z_0\|_{\mathbb{V}}^2\\ =-\Big\langle \mathrm{D}_z\mathcal{I}(t_1,z_0),z_{1,1}-z_0\Big\rangle=-\Big\langle \mathrm{D}_z\mathcal{I}(t_0,z_0),z_{1,1}-z_0\Big\rangle-\Big\langle \ell(t_0)-\ell(t_1),z_0\Big\rangle.\end{aligned}$$

Observe that in the last identity we switched from t_1 to t_0 in $\mathrm{D}_z\mathcal{I}$. Hence, similar arguments as those leading to (2.22) can be applied. We finally obtain

$$\begin{aligned}\eta\,\|z_{k,i+1}-z_{k,i}\|_{\mathbb{V}}+\sum_{j=0}^{i}\|z_{k,j+1}-z_{k,j}\|_{\mathcal{Z}}\\ \leq C\left(\delta_{k,1}\,\|\mathrm{D}_z\mathcal{I}(0,z_0)\|_{\mathbb{V}}+\tau\,\|\ell\|_{C^1([0,T],\mathcal{V}^*)}+\sum_{j=0}^{i}\mathcal{R}(z_{k,j+1}-z_{k,j})\right),\end{aligned}$$

which is valid for all $k\geq1$, $i\geq0$. Here, $\delta_{k,j}$ denotes the Kronecker symbol, and the constant C is independent of η,N,k,i. Summing up with respect to k gives (3.18) and (3.19). From the inclusion (3.12), the uniform estimate (3.19) and assumption (1.18) on $\mathcal{F}$ we deduce that $Az_{k,i}^\tau$ is uniformly bounded in $\mathcal{V}^*$, which implies (3.20). □

Next we construct interpolating curves generated by the data $(z_{k,i}^\tau)_{k,i}$ following the ideas in [6], see also Figure 1. Let $N\in\mathbb{N}$, $\tau=T/N$, $t_k^\tau=k\tau$, $z_{0,-1}:=z_0$ and $s_0^\tau:=t_0^\tau=0$. For each $k\geq1$, given s_{k-1}^τ and $i\geq0$ we define

$$s_{k,-1}^\tau:=s_{k-1}^\tau,\quad s_{k,0}^\tau:=s_{k,-1}^\tau+\tau=s_{k-1}^\tau+\tau, \tag{3.25}$$

$$\sigma_{k,i+1}^\tau:=\left\|z_{k,i+1}^\tau-z_{k,i}^\tau\right\|_{\mathbb{V}},\quad s_{k,i+1}^\tau:=s_{k,i}^\tau+\sigma_{k,i+1}^\tau. \tag{3.26}$$

Furthermore, $s_k^\tau:=\lim_{i\to\infty}s_{k,i}^\tau$. Proposition 3.4 guarantees that this limit exists and that the quantities s_N^τ are finite and uniformly bounded with respect to N and η. In the time update interval we set

$$t_\tau(s):=t_{k-1}^\tau+(s-s_{k-1}^\tau)\qquad\text{for } s\in[s_{k-1}^\tau,s_{k,0}^\tau], \tag{3.27}$$

$$z_\tau(s):=z_{k,0}^\tau\qquad\text{for } s\in[s_{k-1}^\tau,s_{k,0}^\tau]. \tag{3.28}$$

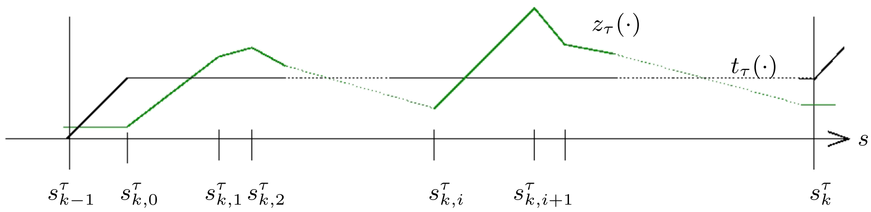


FIGURE 1. Notation and interpolating curves for (3.25)–(3.29).

Observe that $t_\tau(s^\tau_{k,0}) = t^\tau_k$. Next, for $i \geq 0$ and $s \in [s^\tau_{k,i}, s^\tau_{k,i+1}]$ we define the interpolants as follows:

$$t_\tau(s) := t^\tau_k, \qquad z_\tau(s) := \begin{cases} z^\tau_{k,i} + \frac{(s-s^\tau_{k,i})}{\sigma^\tau_{k,i+1}}(z^\tau_{k,i+1} - z^\tau_{k,i}) & \text{if } z^\tau_{k,i+1} \neq z^\tau_{k,i} \\ z^\tau_{k,i} & \text{if } z^\tau_{k,i+1} = z^\tau_{k,i} \end{cases}. \tag{3.29}$$

By definition, for almost all s we have

$$t'_\tau(s) + \|z'_\tau(s)\|_{\mathbb{V}} = 1.$$

Furthermore, we introduce the piecewise constant, left or right continuous interpolants

$$\overline{z}_\tau(s) := z_\tau(s^\tau_{k,i+1}),\ \overline{t}_\tau(s) := t_\tau(s^\tau_{k,i+1}), \qquad \text{if } s \in (s^\tau_{k,i}, s^\tau_{k,i+1}] \text{ for some } k \geq 1, i \geq -1, \tag{3.30}$$

$$\underline{z}_\tau(s) := z_\tau(s^\tau_{k,i}),\ \underline{t}_\tau(s) := t_\tau(s^\tau_{k,i}) \qquad \text{if } s \in [s^\tau_{k,i}, s^\tau_{k,i+1}) \text{ for some } k \geq 1, i \geq -1, \tag{3.31}$$

and the increment

$$\overline{\sigma}_\tau(s) := s^\tau_{k,i+1} - s^\tau_{k,i} \quad \text{if } s \in (s^\tau_{k,i}, s^\tau_{k,i+1}] \text{ for some } k \geq 1, i \geq -1. \tag{3.32}$$

Observe that $\overline{\sigma}_\tau(s) = \tau$ for $s \in (s_{k,-1}, s_{k,0}]$ and that $\overline{\sigma}_\tau(s) > 0$ for almost all $s \in [0, s^\tau_N]$.

Proposition 3.5. *Assume (3.1a)–(3.1e) and $-\mathrm{D}_z\mathcal{I}(0, z_0) \in \partial\mathcal{R}(0)$.*
Then $t_\tau(s^\tau_N) = T$, and for all $\tau = T/N$ we have $z_\tau \in W^{1,\infty}((0, s^\tau_N); \mathcal{V}) \cap L^\infty((0, s^\tau_N); \mathcal{Z})$ with $\|z'_\tau(s)\|_{\mathbb{V}} \leq 1$, and

$$\sup_N \Big(s^\tau_N + \|z_\tau\|_{W^{1,\infty}((0,s^\tau_N);\mathcal{V})} + \|z_\tau\|_{L^\infty((0,s^\tau_N);\mathcal{Z})} + \|t_\tau\|_{W^{1,\infty}((0,s^\tau_N);\mathbb{R})} \Big) < \infty. \tag{3.33}$$

Moreover, the interpolating curves satisfy the following energy-dissipation relation for all $\alpha < \beta \in [0, s^\tau_N]$:

$$\mathcal{I}(t_\tau(\beta), z_\tau(\beta)) + \int_\alpha^\beta \mathcal{R}_{\overline{\sigma}_\tau(s)\eta}(z'_\tau(s)) + \mathcal{R}^*_{\overline{\sigma}_\tau(s)\eta}(-\mathrm{D}_z\mathcal{I}(\underline{t}_\tau(s), \overline{z}_\tau(s)))\,\mathrm{d}s$$
$$= \mathcal{I}(t_\tau(\alpha), z_\tau(\alpha)) - \int_\alpha^\beta \Big\langle \dot{\ell}(t_\tau(s)), z_\tau(s) \Big\rangle t'_\tau(s)\,\mathrm{d}s + \int_\alpha^\beta r_\tau(s)\,\mathrm{d}s. \tag{3.34}$$

where we use the notation $\mathcal{R}_\mu(v) = \mathcal{R}(v) + \frac{\mu}{2}\|v\|^2_{\mathbb{V}}$. The remainder r_τ is given by

$$r_\tau(s) = \Big\langle \mathrm{D}_z\mathcal{I}(t_\tau(s), z_\tau(s)) - \mathrm{D}_z\mathcal{I}(\underline{t}_\tau(s), \overline{z}_\tau(s)), z'_\tau(s) \Big\rangle. \tag{3.35}$$

There exist constants $c, C > 0$ (independently of N, η) such that for all $\beta \in [0, s_N^\tau]$

$$\int_0^\beta r_\tau(s)\,\mathrm{d}s \leq C\eta^{-1}, \tag{3.36}$$

$$\|\overline{\sigma}_\tau\|_{L^\infty(0,s_N^\tau)} + \|\overline{z}_\tau - z_\tau\|_{L^\infty((0,s_N^\tau);\mathcal{V})} \leq C(\tau + \eta^{-1})\,. \tag{3.37}$$

Proof. Estimate (3.33) is an immediate consequence of Proposition 3.4.

In order to derive (3.34) assume first that $i \geq 0$, $k \geq 1$. Let $s \in (s_{k,1}^\tau, s_{k,i+1}^\tau)$ such that $\overline{\sigma}_\tau(s) = \sigma_{k,i+1}^\tau \neq 0$. From (3.12) (left inclusion) we deduce that

$$-\mathrm{D}_z\mathcal{I}(t_k^\tau, z_{k,i+1}^\tau) - \sigma_{k,i+1}^\tau \eta \mathbb{V}\left(\frac{z_{k,i+1}^\tau - z_{k,i}^\tau}{\sigma_{k,i+1}^\tau}\right) \in \partial\mathcal{R}\Big((z_{k,i+1}^\tau - z_{k,i}^\tau)\Big/\sigma_{k,i+1}^\tau\Big), \tag{3.38}$$

where on the right hand side we have used the one-homogeneity of $\mathcal{R}$. This is equivalent to

$$-\mathrm{D}_z\mathcal{I}(\underline{t}_\tau(s), \overline{z}_\tau(s)) \in \partial\mathcal{R}_{\overline{\sigma}_\tau(s)\eta}(z_\tau'(s)), \tag{3.39}$$

which in fact is valid for all $s \in (s_{k,0}^\tau, s_k^\tau)\backslash(\cup_{i=1}^\infty\{s_{k,i}^\tau\})$ (and not only for those with $\overline{\sigma}_\tau(s) \neq 0$). For $s \in (s_{k,-1}, s_{k,0})$, relation (3.11) yields $-\mathrm{D}_z\mathcal{I}(t_{k-1}, z_{k-1}) \in \partial\mathcal{R}(0)$, which is equivalent to

$$-\left(\mathrm{D}_z\mathcal{I}(\underline{t}_\tau(s), \overline{z}_\tau(s)) + \overline{\sigma}_\tau(s)\eta\mathbb{V}(z_\tau'(s))\right) \in \partial\mathcal{R}(z_\tau'(s)).$$

Here, we used that $z_\tau'(s) = 0$ for $s \in (s_{k,-1}, s_{k,0})$. This shows that (3.39) is valid for all $s \in (0, s_N^\tau)\backslash\{\, s_{k,i}^\tau\,;\, 1 \leq k \leq N, i \geq -1\,\}$. By convex analysis, (3.39) can be rewritten as

$$\begin{aligned}\mathcal{R}_{\overline{\sigma}_\tau(s)\eta}(z_\tau'(s)) + \mathcal{R}^*_{\overline{\sigma}_\tau(s)\eta}(-\mathrm{D}_z\mathcal{I}(\underline{t}_\tau(s), \overline{z}_\tau(s)) &= \langle -\mathrm{D}_z\mathcal{I}(\underline{t}_\tau(s), \overline{z}_\tau(s)), z_\tau'(s)\rangle \\ &= \langle -\mathrm{D}_z\mathcal{I}(t_\tau(s), z_\tau(s)), z_\tau'(s)\rangle + r_\tau(s)\end{aligned} \tag{3.40}$$

with $r_\tau(s)$ as in (3.35). Combining (3.40) with the integrated chain rule identity

$$\mathcal{I}(t_\tau(\beta), z_\tau(\beta)) - \mathcal{I}(t_\tau(\alpha), z_\tau(\alpha)) = \int_\alpha^\beta \partial_t\mathcal{I}(t_\tau(s), z_\tau(s))t_\tau'(s) + \langle \mathrm{D}_z\mathcal{I}(t_\tau(s), z_\tau(s)), z_\tau'(s)\rangle\,\mathrm{d}s \tag{3.41}$$

yields (3.34).

It remains to estimate the term r_τ. Observe first that $\langle \ell(t_\tau(s)) - \ell(\underline{t}_\tau(s)), z_\tau'(s)\rangle = 0$ for almost all $s \in (0, s_N)$ since $(t_\tau(s) - \underline{t}_\tau(s))\,\|z_\tau'(s)\|_\mathcal{V} = 0$ for almost all s. Let $s \in (s_{k,i}^\tau, s_{k,i+1}^\tau)$ for some $k \geq 1$ and $i \geq -1$. Then $z_\tau(s) - \overline{z}_\tau(s) = (s - s_{k,i+1}^\tau)z_\tau'(s)$ and

$$\begin{aligned} r_\tau(s) &= \Big\langle A(z_\tau(s) - \overline{z}_\tau(s)), z_\tau'(s)\Big\rangle + \Big\langle \mathrm{D}_z\mathcal{F}(z_\tau(s)) - \mathrm{D}_z\mathcal{F}(\overline{z}_\tau(s)), z_\tau'(s)\Big\rangle \\ &\overset{(1)}{\leq} (s - s_{k,i+1}^\tau)\alpha\,\|z_\tau'(s)\|_\mathcal{Z}^2 + c\,\|z_\tau(s) - \overline{z}_\tau(s)\|_\mathcal{Z}\,\|z_\tau'(s)\|_\mathbb{V} \\ &= (s - s_{k,i+1}^\tau)\alpha\,\|z_\tau'(s)\|_\mathcal{Z}^2 + c(s_{k,i+1}^\tau - s)\,\|z_\tau'(s)\|_\mathcal{Z}\,\|z_\tau'(s)\|_\mathbb{V} \\ &\leq (s - s_{k,i+1}^\tau)\alpha\,\|z_\tau'(s)\|_\mathcal{Z}^2 + \frac{\alpha}{2}(s_{k,i+1}^\tau - s)\,\|z_\tau'(s)\|_\mathcal{Z}^2 + c_\alpha(s_{k,i+1}^\tau - s)\,\|z_\tau'(s)\|_\mathbb{V}^2 \\ &\overset{(2)}{\leq} c_\alpha(s_{k,i+1}^\tau - s)\,\|z_\tau'(s)\|_\mathbb{V}^2 \overset{(3)}{\leq} c_\alpha C\eta^{-1}. \end{aligned} \tag{3.42}$$

Estimate (1) is due to the ellipticity of A, assumption (1.18) and the uniform bound for the $(z^\tau_{k,i})_{k,i}$ (see (3.16)); (2) follows since the sum of the first two terms in the previous line is nonpositive and (3) follows from (3.19) and $\|z'_\tau(s)\|_{\mathbb{V}} \in \{0,1\}$. Together with (3.33) this proves (3.36). For the last estimate observe that for almost all $s \in (0, s^\tau_N)$ we have

$$\|\overline{z}_\tau(s) - z_\tau(s)\|_{\mathbb{V}} \le \overline{\sigma}_\tau(s)\, \|z'_\tau(s)\|_{\mathbb{V}}$$

and we conclude using again (3.19). □

Remark 3.6. In Section 1.1, it follows by construction that $\sup_k \left\|z^h_k - z^h_{k-1}\right\|_{\mathbb{V}} \le h$, and hence, for $h \to 0$ these differences converge to zero uniformly. In the present setting from Proposition 3.4 we obtain the uniform estimate $\left\|z^\tau_{k,i+1} - z^\tau_{k,i}\right\|_{\mathcal{V}} \le C\eta^{-1}$, which, for constant η, does not imply that these differences converge. If this (uniform) convergence is not available, it is not clear whether piecewise linear and piecewise constant interpolants of the $(z^\tau_{k,i})_{k,i}$ converge to the same limit function for $\tau \to 0$. In order to enforce this convergence, we will require $\eta_N \to \infty$, *cf.* Theorem 3.7. In Section 3.2, we will discuss the case with $\eta > 0$ fixed.

The main result in this Section is the following theorem, which is the analogue to Theorem 2.5.

Theorem 3.7. *Assume* (3.1a)–(3.1e) *and that* $-\mathrm{D}_z\mathcal{I}(0,z_0) \in \partial\mathcal{R}(0)$.
For every sequence $\tau \searrow 0$ *and* $\eta \nearrow \infty$ *there exists a subsequence* $(\tau_n,\eta_n)_{n\in\mathbb{N}}$, $S \in (0,\infty)$ *and functions* $\hat{t} \in W^{1,\infty}((0,S);\mathbb{R})$ *and* $\hat{z} \in W^{1,\infty}((0,S);\mathcal{V}) \cap L^\infty((0,S);\mathcal{Z})$ *such that for* $n \to \infty$ *(we omit the index* n *in the following)*

$$s^\tau_N \to S, \tag{3.43}$$

$$t_\tau \overset{*}{\rightharpoonup} \hat{t} \text{ in } W^{1,\infty}((0,S);\mathbb{R}), \quad t_\tau(s) \to \hat{t}(s) \text{ for every } s \in [0,S], \tag{3.44}$$

$$z_\tau \overset{*}{\rightharpoonup} \hat{z} \text{ weakly* in } W^{1,\infty}((0,S);\mathcal{V}) \cap L^\infty((0,S);\mathcal{Z}) \tag{3.45}$$

$$\overline{z}_\tau(s), z_\tau(s) \rightharpoonup \hat{z}(s) \text{ weakly in } \mathcal{Z} \text{ for every } s \in [0,S]\,. \tag{3.46}$$

Moreover, the limit pair $(\hat{t},\hat{z})$ *satisfies* (2.35)–(2.37).

Proof. The proof is nearly identical to the proof of Theorem 2.5, and we highlight the differences, only. In the following, we omit the index n. The convergence results in (3.43)–(3.46) follow from the uniform estimates formulated in Proposition 3.5. Clearly, the limit pair $(\hat{t},\hat{z})$ satisfies the first two relations in (2.36). We will next discuss the complementarity relation in (2.36). Observe first that thanks to (3.20) the term $-\mathrm{D}_z\mathcal{I}(\underline{t}_\tau(s),\overline{z}_\tau(s))$ is bounded in $\mathcal{V}^*$ uniformly in τ and s. Hence, by the boundedness of $\partial\mathcal{R}(0)$ in $\mathcal{V}^*$ we obtain

$$\sup_{\tau>0,0\le s\le s^\tau_N} \operatorname{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\underline{t}_\tau(s),\overline{z}_\tau(s)),\partial\mathcal{R}(0)) < \infty. \tag{3.47}$$

By (3.46) and the weak $\mathcal{Z}$–$\mathcal{Z}^*$-continuity of $\mathrm{D}_z\mathcal{I}(t,\cdot)$ it follows that for every s we have $\mathrm{D}_z\mathcal{I}(\underline{t}_\tau(s),\overline{z}_\tau(s)) \rightharpoonup \mathrm{D}_z\mathcal{I}(\hat{t}(s),\hat{z}(s))$ weakly in $\mathcal{Z}^*$ and in $\mathcal{V}^*$. The latter is a consequence of the uniform $\mathcal{V}^*$-bound. By lower semicontinuity, we therefore obtain for all s:

$$\liminf_{\tau\to 0} \operatorname{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\underline{t}_\tau(s),\overline{z}_\tau(s)),\partial\mathcal{R}(0)) \ge \operatorname{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{t}(s),\hat{z}(s)),\partial\mathcal{R}(0))\,,$$

which in particular shows that $\mathrm{D}_z\mathcal{I}(\hat{t},\hat{z}) \in L^\infty((0,S);\mathcal{V}^*)$. The following discrete complementarity relation is satisfied for almost all $s \in [0, s^\tau_N]$:

$$t'_\tau(s)\operatorname{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\underline{t}_\tau(s),\overline{z}_\tau(s)),\partial\mathcal{R}(0)) = 0\,.$$

Indeed, this identity is trivial for $s \in [0, s_N^\tau] \setminus \cup_{k=1}^N [s_{k-1}, s_{k,0}]$ since then $t_\tau'(s) = 0$ (together with (3.47)). Thanks to (3.11), for $s \in (s_{k-1}, s_{k,0})$ we have $\operatorname{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\underline{t}_\tau(s), \overline{z}_\tau(s)), \partial\mathcal{R}(0)) = 0$. The same arguments as in the proof of Theorem 2.5 now lead to the last relation in (2.36).

By Young's inequality, for $\mu > 0$, $v \in \mathcal{V}$ and $\zeta \in \mathcal{V}^*$ we have

$$\mathcal{R}_\mu(v) + \mathcal{R}_\mu^*(\zeta) = \mathcal{R}(v) + \frac{\mu}{2}\|v\|_{\mathbb{V}}^2 + \frac{1}{2\mu}\left(\operatorname{dist}_{\mathcal{V}^*}(\zeta, \partial\mathcal{R}(0))\right)^2 \geq \mathcal{R}(v) + \|v\|_{\mathbb{V}} \operatorname{dist}_{\mathcal{V}^*}(\zeta, \partial\mathcal{R}(0))$$

which implies

$$\mathcal{R}_{\overline{\sigma}_\tau(s)\eta}(z_\tau'(s)) \; + \; \mathcal{R}^*_{\overline{\sigma}_\tau(s)\eta}(-\mathrm{D}_z\mathcal{I}(\underline{t}_\tau(s), \overline{z}_\tau(s))) \;\; \geq \;\; \mathcal{R}(z_\tau'(s)) \; + \; \|z_\tau'(s)\|_{\mathbb{V}} \operatorname{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\underline{t}_\tau(s), \overline{z}_\tau(s)), \partial\mathcal{R}(0))$$

for almost all s. The arguments from the proof of Theorem 2.5 in combination with Proposition B.1 applied to the energy-dissipation estimate (3.34) finally complete the proof. □

3.2. Convergence for fixed penalty parameter

Let us finally discuss the convergence of the incremental solutions for $N \to \infty$ but with fixed penalty parameter $\eta > 0$. Again, we will start from the discrete energy dissipation identity in a parametrised framework. However, as already mentioned in Remark 3.6, with $\eta > 0$ fixed we cannot show that the piecewise affine and the piecewise constant interpolating functions $z_\tau, \overline{z}_\tau, \underline{z}_\tau$ converge to the same limit. Hence, we have to carry out a more detailed analysis for the remainder term r_τ in the energy dissipation balance. In order to be able to identify the limits of the quadratic part of r_τ which involves $\langle A z_\tau', z_\tau' \rangle$ we use an arclength parametrisation in terms of the $\mathcal{Z}$-norm instead of the $\mathcal{V}$-norm.

The analysis of this section refines the results from [1] as we can characterise more precisely the behaviour of the solution at jump points by deriving a more detailed energy dissipation estimate.

For $\eta > 0$ fixed, $N \in \mathbb{N}$ and $\tau = T/N$ let the sequence $(z_{k,i}^\tau)_{k,i}$ with $0 \leq k \leq N$, $i \in \mathbb{N} \cup \{0, \infty\}$ be generated by (3.2)–(3.3). The piecewise linear and piecewise constant interpolating functions are constructed as in (3.25)–(3.32) with the difference that now we define the z-increment with respect to the $\mathcal{Z}$-norm, *i.e.* for each $k \geq 1$ and $i \geq 0$, given s_{k-1}^τ,

$$s_{k,-1}^\tau := s_{k-1}^\tau, \quad s_{k,0}^\tau := s_{k,-1}^\tau + \tau = s_{k-1}^\tau + \tau, \tag{3.48}$$

$$\sigma_{k,i+1}^\tau := \left\| z_{k,i+1}^\tau - z_{k,i}^\tau \right\|_{\mathcal{Z}}, \quad s_{k,i+1}^\tau := s_{k,i}^\tau + \sigma_{k,i+1}^\tau. \tag{3.49}$$

Observe that the $\mathcal{Z}$-parametrised interpolants satisfy the discrete energy dissipation identity (3.34). The proof is identical to the one of Proposition 3.5. Thanks to (3.17) the BV-type estimates

$$\operatorname{diss}_{\mathcal{R}}(\overline{z}_\tau; [0, s_N^\tau]),\; \operatorname{diss}_{\mathcal{R}}(\underline{z}_\tau; [0, s_N^\tau]) \leq C$$

are valid uniformly in N. Here, for a function $v : [0, S] \to \mathcal{X}$ the $\mathcal{R}$-dissipation is defined in the usual way (*cf.* [15], Sect. 2.1.1) as

$$\operatorname{diss}_{\mathcal{R}}(v; [0, S]) := \sup_{\substack{\text{partitions} \\ 0 = s_0 < \ldots < s_K = S}} \sum_{k=1}^K \mathcal{R}(v(s_k) - v(s_{k-1}))\,.$$

Moreover, Propositions 3.3 and 3.4 provide the uniform (with respect to N) bounds

$$s_N^\tau \leq C,$$

$$\|\bar{\sigma}_\tau\|_{L^\infty((0,s_N^\tau);\mathbb{R}))}, \|\bar{z}_\tau\|_{L^\infty((0,s_N^\tau);\mathcal{Z})}, \|\underline{z}_\tau\|_{L^\infty((0,s_N^\tau);\mathcal{Z})} \leq C,$$
$$\|z_\tau\|_{W^{1,\infty}((0,s_N^\tau);\mathcal{Z})} \leq C.$$

Hence, there exist $S > 0$, functions $\hat{z}, \bar{z}, \underline{z} : [0, S] \to \mathcal{Z}$ and a (not relabelled) subsequence of $(z_\tau, \bar{z}_\tau, \underline{z}_\tau)_{N\in\mathbb{N}}$ such that for $N \to \infty$ (and $\eta > 0$ fixed) the convergences stated in (3.43)–(3.44) are valid and moreover

$$z_\tau \overset{*}{\rightharpoonup} \hat{z} \text{ weakly}* \text{ in } W^{1,\infty}((0, S); \mathcal{Z}), \tag{3.50}$$

$$z_\tau(s) \rightharpoonup \hat{z}(s),\ \bar{z}_\tau(s) \rightharpoonup \bar{z}(s),\ \underline{z}_\tau(s) \rightharpoonup \underline{z}(s) \text{ weakly in } \mathcal{Z} \text{ and strongly in } \mathcal{V} \text{ for all } s. \tag{3.51}$$

Here, we applied the generalized Helly selection principle to the sequences $(\bar{z}_\tau)_\tau$ and $(\underline{z}_\tau)_\tau$, see *e.g.* ([15], Thm. B.5.13) or [9]. It is not clear whether the limit functions $\hat{z}, \bar{z}, \underline{z}$ coincide. However, the following relation is satisfied: Let $\bar{s}_\tau(s) := \inf\{\, s_{k,i}\, ;\, s \leq s_{k,i},\, 1 \leq k \leq N, i \geq 0\,\}$, $\underline{s}_\tau(s) := \sup\{\, s_{k,i}\, ;\, s \geq s_{k,i},\, 1 \leq k \leq N, i \in \mathbb{N} \cup \{-1, 0\}\,\}$. Clearly, $\bar{s}_\tau$ is left continuous, while $\underline{s}_\tau$ is right continuous. Both functions are nondecreasing and uniformly bounded from above, hence uniformly bounded in $BV([0, s_N^\tau])$. Again by Helly's principle, they contain a subsequence that converges pointwise (for all s) to the nondecreasing functions $\bar{s}, \underline{s} : [0, S] \to [0, \infty]$, respectively (w.l.o.g. the same subsequence as the one for $(z_\tau)_\tau$). Moreover, for all $s \in [0, S]$ we have $\underline{s}(s) \leq s \leq \bar{s}(s)$ and $\underline{s}$ is right continuous, $\bar{s}$ is left continuous. For almost all $s \in [0, s_N^\tau]$ the identities

$$z_\tau(s) - \bar{z}_\tau(s) = (s - \bar{s}_\tau(s))z_\tau'(s), \quad z_\tau(s) - \underline{z}_\tau(s) = (s - \underline{s}_\tau(s))z_\tau'(s) \tag{3.52}$$

are valid. Passing to the limit $N \to \infty$ we obtain

$$\hat{z}(s) - \bar{z}(s) = (s - \bar{s}(s))\hat{z}'(s), \quad \hat{z}(s) - \underline{z}(s) = (s - \underline{s}(s))\hat{z}'(s) \tag{3.53}$$

that is valid for almost all $s \in (0, S)$. This can be verified as follows: the sequence $(z_\tau - \bar{z}_\tau)_\tau$ converges weakly$*$ in $L^\infty((0, S); \mathcal{Z})$ to the function $\hat{z} - \bar{z}$. Moreover, for every $\phi \in L^1((0, S); \mathcal{Z}^*)$ the sequence $(\bar{s}_\tau(\cdot) - \cdot)\phi(\cdot))_\tau$ converges to $(\bar{s}(\cdot) - \cdot)\phi(\cdot)$ strongly in $L^1((0, S); \mathcal{Z}^*)$. Due to the weak$*$ convergence of $(z_\tau')_\tau$ in $L^\infty((0, S); \mathcal{Z})$ we ultimately obtain

$$\int_0^S \Big\langle \phi(s), (\bar{s}_\tau(s) - s)z_\tau'(s) \Big\rangle_{\mathcal{Z}^*,\mathcal{Z}} \mathrm{d}s \to \int_0^S \Big\langle \phi(s), (\bar{s}(s) - s)\hat{z}'(s) \Big\rangle_{\mathcal{Z}^*,\mathcal{Z}} \mathrm{d}s$$

for all $\phi \in L^1((0, S); \mathcal{Z}^*)$ and thus weak$*$ convergence in $L^\infty((0, S); \mathcal{Z})$ of the sequence $(\cdot - \bar{s}_\tau(\cdot))z_\tau'(\cdot))_\tau$ to $(\cdot - \bar{s}(\cdot))\hat{z}'(\cdot))_\tau$. This proves (3.53).

Theorem 3.8. *Assume* (3.1a)–(3.1e) *and that* $\mathrm{D}_z\mathcal{F} : \mathcal{Z} \to \mathcal{Z}^*$ *is weakly-strongly continuous.*

The limit functions $\hat{t}$ *and* $(\hat{z}, \bar{z}, \underline{z})$ *defined above satisfy* (2.35) *with* $\bar{z}(0) = z_0 = \underline{z}(0)$, *the first two relations in* (2.36), *the complementarity condition*

$$\textit{for almost all } s \in (0, S) \quad \hat{t}'(s)\, \mathrm{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{t}(s), \bar{z}(s)), \partial\mathcal{R}(0)) = 0 \tag{3.54}$$

and the energy dissipation identity

$$\mathcal{I}(\hat t(\beta),\hat z(\beta)) + \int_0^\beta \mathcal{R}(\hat z'(s)) + \|\hat z'(s)\|_{\mathbb{V}} \operatorname{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat t(s),\bar z(s)),\partial\mathcal{R}(0))\,\mathrm{d}s \\ + \int_0^\beta \Big\langle \mathrm{D}_z\mathcal{I}(\hat t(s),\bar z(s)) - \mathrm{D}_z\mathcal{I}(\hat t(s),\hat z(s)),\hat z'(s)\Big\rangle\,\mathrm{d}s \\ = \mathcal{I}(0,z_0) + \int_0^\beta \partial_t\mathcal{I}(\hat t(s),\hat z(s))\hat t'(s)\,\mathrm{d}s \quad (3.55)$$

that is valid for all $\beta\in[0,S]$. *If* $\overline{s}(s)\neq\underline{s}(s)$, *then* $\hat t$ *is constant on* $(\underline{s}(s),\overline{s}(s))$. *Moreover, by lower semicontinuity,* $\overline{s}(s)-\underline{s}(s)\geq\|\bar z(s)-\underline{z}(s)\|_{\mathcal{Z}}$ *for all* s. *Finally, the following relation is valid for almost all* s*:*

$$\hat t'(s)\big((\overline{s}(s)-\underline{s}(s)) + \|\bar z(s)-\underline{z}(s)\|_{\mathcal{Z}} + \|\bar z(s)-\hat z(s)\|_{\mathcal{Z}} + \|\hat z(s)-\underline{z}(s)\|_{\mathcal{Z}}\big) = 0. \quad (3.56)$$

Remark 3.9. Observe that in Theorem 3.8 the assumption on $\mathrm{D}_z\mathcal{F}$ is slightly stronger than what is required in (1.21).

Proof. The complementarity relation (3.54) follows with the same arguments as in the proof of Theorem 3.7. Starting again from the discrete energy dissipation identity (3.34) with $\alpha=0$ and $\beta>0$ on the left hand side we may pass to the limit inferior using the same arguments as in the proof of Theorem 3.7 and obtain

$$\liminf_{N\to\infty}\left(\mathcal{I}(t_\tau(\beta),z_\tau(\beta)) + \int_0^\beta \mathcal{R}_{\overline{\sigma}_\tau(s)\eta}(z_\tau'(s)) + \mathcal{R}^*_{\overline{\sigma}_\tau(s)\eta}(-\mathrm{D}_z\mathcal{I}(\underline{t}_\tau(s),\overline{z}_\tau(s)))\,\mathrm{d}s\right) \\ \geq \mathcal{I}(\hat t(\beta),\hat z(\beta)) + \int_0^\beta \mathcal{R}(\hat z'(s)) + \|\hat z'(s)\|_{\mathbb{V}}\operatorname{dist}_{\mathcal{V}^*}\big(-\mathrm{D}_z\mathcal{I}(\hat t(s),\bar z(s)),\partial\mathcal{R}(0)\big)\,\mathrm{d}s\,.$$

On the right hand side of (3.34) we have to be more careful with the remainder term $\int_0^\beta r_\tau(s)\,\mathrm{d}s$. From (3.42) we obtain

$$\int_0^\beta r_\tau(s)\,\mathrm{d}s = \int_0^\beta -(\overline{s}_\tau(s)-s)\Big\langle Az_\tau'(s),z_\tau'(s)\Big\rangle\,\mathrm{d}s + \int_0^\beta \Big\langle \mathrm{D}_z\mathcal{F}(z_\tau(s)) - \mathrm{D}_z\mathcal{F}(\overline{z}_\tau(s)),z_\tau'(s)\Big\rangle\,\mathrm{d}s \\ =: I_1^\tau + I_2^\tau\,.$$

Thanks to (3.51) and the continuity assumption on $\mathrm{D}_z\mathcal{F}$, for all $s\in(0,S)$ the terms $\mathrm{D}_z\mathcal{F}(\overline{z}_\tau(s))$ and $\mathrm{D}_z\mathcal{F}(z_\tau(s))$ converge strongly in $\mathcal{Z}^*$ to the limits $\mathrm{D}_z\mathcal{F}(\overline{z}(s))$ and $\mathrm{D}_z\mathcal{F}(\hat z(s))$, respectively. Since these terms are uniformly bounded in $\mathcal{Z}^*$ (uniformly with respect to s and τ) they also converge strongly in $L^1((0,S);\mathcal{Z}^*)$. Together with the weak$*$ convergence of $(z_\tau')_\tau$ in $L^\infty((0,S);\mathcal{Z}^*)$ it follows that

$$\lim_{N\to\infty} I_2^\tau = \int_0^\beta \Big\langle \mathrm{D}_z\mathcal{F}(\hat z(s)) - \mathrm{D}_z\mathcal{F}(\overline{z}(s)),\hat z'(s)\Big\rangle\,\mathrm{d}s\,.$$

As for I_1^τ thanks to the non-negativity and the pointwise and strong convergence in $L^1((0,S);\mathbb{R})$ of the sequence $(\overline{s}_\tau(\cdot)-\cdot)_\tau$, the weak$*$ convergence of $(z_\tau')_\tau$ in $L^\infty((0,S);\mathcal{Z})$ and the convexity of the mapping

$v \to \int_0^\beta (\overline{s}(r)-r)\langle Av(r), v(r)\rangle \,\mathrm{d}r$ with ([25], Thm. 21) we conclude that

$$\limsup_{N\to\infty} \int_0^\beta -(\overline{s}_\tau(s)-s)\Big\langle Az_\tau'(s), z_\tau'(s)\Big\rangle \,\mathrm{d}s = -\liminf_{N\to\infty}\int_0^\beta (\overline{s}_\tau(s)-s)\Big\langle Az_\tau'(s), z_\tau'(s)\Big\rangle\,\mathrm{d}s$$
$$\le \int_0^\beta -(\overline{s}(s)-s)\Big\langle A\hat{z}'(s), \hat{z}'(s)\Big\rangle\,\mathrm{d}s = \int_0^\beta \Big\langle A(\hat{z}(s)-\overline{z}(s)), \hat{z}'(s)\Big\rangle\,\mathrm{d}r\,. \tag{3.57}$$

This yields (3.55) with $\le$ instead of an equality. By the very same arguments as in Section 2, we finally obtain (3.55) with an equality.

Relation (3.56) can be verified as follows:

By the definition of the interpolating curves, we have $\hat{t}_\tau'(s)\,\|\overline{z}_\tau(s)-\underline{z}_\tau(s)\|_{\mathcal{Z}} = 0$ for almost all s. Moreover, for every s we obtain

$$\liminf_{N\to\infty} \|\overline{z}_\tau(s)-\underline{z}_\tau(s)\|_{\mathcal{Z}} \ge \|\overline{z}(s)-\underline{z}(s)\|_{\mathcal{Z}}\,.$$

With Lemma B.2 applied to $\int_\alpha^\beta t_\tau'(s)\,\|\overline{z}_\tau(s)-\underline{z}_\tau(s)\|_{\mathcal{Z}}\,\mathrm{d}s$ with arbitrary $\alpha<\beta\in[0,S]$ we conclude. The other terms involving $\hat{z},\overline{z},\underline{z}$ can be treated similarly.

Assume that $\overline{s}(s)\neq\underline{s}(s)$ for some $s\in[0,S]$. Let further $(\epsilon_0,\epsilon_1)\subset[\underline{s}(s),\overline{s}(s)]$ be an arbitrary nonempty interval. Then there exists $N_0\in\mathbb{N}$ such that for all $N\ge N_0$ we have $(\epsilon_0,\epsilon_1)\subset[\underline{s}_\tau(s),\overline{s}_\tau(s)]$. It follows that t_τ is constant on $[\underline{s}_\tau(s),\overline{s}_\tau(s)]$ for all $N\ge N_0$ since otherwise these intervals coincide with the time-update interval and have the width τ tending to zero for $N\to\infty$. Altogether it follows that the limit function $\hat{t}$ is constant on (ϵ_0,ϵ_1), as well. ☐

Analogously to Proposition 1.3 we finally obtain the following characterisation of the limit curves $(\hat{t},\hat{z},\overline{z})$ in terms of a differential inclusion: assume that the limit curve $(\hat{t},\hat{z},\overline{z})$ is nondegenerate, *i.e.* $\hat{t}'(s)+\|\hat{z}'(s)\|_{\mathcal{Z}}>0$ for almost all s. Then there exists a measurable function $\lambda:[0,S]\to[0,\infty)$ such that

$$\text{for almost all } s\in[0,S]: \qquad \lambda(s)\hat{t}'(s)=0, \quad 0\in\partial\mathcal{R}(\hat{z}'(s))+\lambda(s)\mathbb{V}\hat{z}'(s)+\mathrm{D}_z\mathcal{I}(\hat{t}(s),\overline{z}(s))\,.$$

Under the above assumptions, we have $\lambda(s)=\mathrm{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{I}(\hat{t}(s),\overline{z}(s)),\partial\mathcal{R}(0))/\|\hat{z}'(s)\|_{\mathbb{V}}$ if $\hat{z}'(s)\neq 0$ and $\lambda(s)=0$ otherwise.

4. An alternate minimisation scheme with penalty term

Let $\mathcal{U}$ be a further Hilbert space and $\mathcal{Q}:=\mathcal{U}\times\mathcal{Z}$. Let $\mathcal{Z},\mathcal{V},\mathcal{X}$ satisfy (1.11). With $\mathbb{C}\in\mathrm{Lin}(\mathcal{U},\mathcal{U}^*)$, $\mathbb{B}\in\mathrm{Lin}(\mathcal{V},\mathcal{U}^*)$, $\mathbb{A}\in\mathrm{Lin}(\mathcal{Z},\mathcal{Z}^*)$ we define $\mathcal{A}\in\mathrm{Lin}(\mathcal{Q},\mathcal{Q}^*)$ via

$$\mathcal{A}(u,z) := \begin{pmatrix}\mathbb{C} & \mathbb{B}\\ \mathbb{B}^* & \mathbb{A}\end{pmatrix}\begin{pmatrix}u\\ z\end{pmatrix} = \begin{pmatrix}\mathbb{C}u+\mathbb{B}z\\ \mathbb{B}^*u+\mathbb{A}z\end{pmatrix}\,. \tag{4.1}$$

It is assumed that $\mathcal{A}$ is self-adjoint and positive definite with

$$\forall q\in\mathcal{Q}: \quad \langle\mathcal{A}q,q\rangle \ge \alpha\,\|q\|_{\mathcal{Q}}^2 \tag{4.2}$$

for some positive constant α. For $\ell=(\ell_u,\ell_z)\in C^1([0,T],(\mathcal{U}^*\times\mathcal{V}^*))$, $\mathcal{F}\in C^2(\mathcal{Z},\mathbb{R})$ and $q=(u,z)\in\mathcal{Q}$ we define the energy

$$\mathcal{E}(t,q) := \frac{1}{2}\langle\mathcal{A}q,q\rangle + \mathcal{F}(z) - \langle\ell(t),q\rangle\,, \tag{4.3}$$

and use the same dissipation potential $\mathcal{R} : \mathcal{X} \to [0, \infty)$ as before, *i.e.* $\mathcal{R}$ is convex, lower semicontinuous, positively homogeneous of degree one and satisfies (1.17). Given $z_0 \in \mathcal{Z}$ the aim is to find solutions $q = (u, z) : [0, T] \to \mathcal{Q}$ of the system

$$0 = \mathbb{C}u + \mathbb{B}z - \ell_u(t), \tag{4.4}$$

$$0 \in \partial\mathcal{R}(\dot{z}(t)) + (\mathbb{B}^*u(t) + \mathbb{A}z(t)) + \mathrm{D}_z\mathcal{F}(z(t)) - \ell_z(t) \tag{4.5}$$

with $z(0) = z_0$. In applications, the first equation typically represents the (stationary) balance of linear momentum while the second inclusion describes the evolution of the internal variable z. Solving the first equation for u in dependence of z the system can be reduced to a sole evolution law in z and we are back in the situation discussed in the previous sections. Hence, if in each incremental step one looks for minimizers simultaneously in (u, z), the analysis of the previous sections guarantees the convergence of suitable interpolants of the incremental solutions of (3.2)–(3.3) to a limit function as described in Theorem 3.7. However, from a practical point of view the iteration in (3.2)–(3.3) will be stopped after a finite number of steps and in addition it is sometimes more convenient to follow an operator splitting ansatz.

The aim of this section is to analyse the following alternate minimisation scheme combined with iterated viscous minimisation (relaxed local minimisation):

Given $N \in \mathbb{N}$, time-step size $\tau = T/N$, $\eta, \delta > 0$, an initial datum $z_0 \in \mathcal{Z}$ and $u_0 \in \mathcal{U}$ with $\mathrm{D}_u\mathcal{E}(0, u_0, z_0) = 0$ determine recursively $u_{k,i}$ and $z_{k,i}$ for $1 \le k \le N$ and $i \ge 1$ by the following procedure: Let $z_{k,0} := z_{k-1}$, $u_{k,0} := u_{k-1}$. Then for $i \ge 1$

$$u_{k,i} = \operatorname{argmin}\{\, \mathcal{E}(t_k, v, z_{k,i-1})\, ;\, v \in \mathcal{U}\,\}, \tag{4.6}$$

$$z_{k,i} \in \operatorname{Argmin}\{\, \mathcal{E}(t_k, u_{k,i}, \xi) + \frac{\eta}{2}\,\|\xi - z_{k,i-1}\|_{\mathbb{V}}^2 + \mathcal{R}(\xi - z_{k,i-1})\, ;\, \xi \in \mathcal{Z}\,\}, \tag{4.7}$$

$$\text{stop if } \|z_{k,i} - z_{k,i-1}\|_{\mathcal{V}} \le \delta; \quad (u_k, z_k) := (u_{k,i}, z_{k,i})\,. \tag{4.8}$$

Remark 4.1. Observe that for $\mathbb{B} = 0$ (which is an admissible choice) this approach coincides with (3.2)–(3.3) with a stopping criterion instead of (3.3).

Clearly, minimizers exist in (4.6)–(4.7). A straightforward adaption of the arguments leading to Proposition 3.2 results in

Proposition 4.2. *Assume* (3.1a)–(3.1e) *and* (4.2).

For every $N \in \mathbb{N}$, $\eta, \delta > 0$ and $1 \le k \le N$ there exists $M_k^N \in \mathbb{N}$ such that the stopping criterion (4.8) *is satisfied after M_k^N minimisation steps. Moreover, there exists a constant $C > 0$ such that for all $N \in \mathbb{N}$, $\eta, \delta > 0$, $1 \le k \le N$, $1 \le i \le M_k^N$ the corresponding minimizers satisfy the bounds*

$$\|u_{k,i}\|_{\mathcal{U}} + \|z_{k,i}\|_{\mathcal{Z}} \le C, \tag{4.9}$$

$$\sum_{s=1}^{N} \sum_{j=0}^{M_s^N - 1} \Big(\mathcal{R}(z_{s,j+1} - z_{s,j}) + \frac{\eta}{2}\,\|z_{s,j+1} - z_{s,j}\|_{\mathbb{V}}^2 \Big) \le C. \tag{4.10}$$

Proof. Let $\delta > 0$. Note first that as a consequence of coercivity and the assumptions on ℓ the energy functional $\mathcal{E}$ is uniformly bounded from below. This implies that the sequences $(e_i)_{i\ge0} := (\mathcal{E}(t_k, u_{k,i}, z_{k,i}))_{i\ge0}$ and $(e_i + \delta_i)_{i\ge0}$ with $\delta_i := \mathcal{R}(z_{k,i} - z_{k,i-1}) + \frac{\eta}{2}\,\|z_{k,i} - z_{k,i-1}\|_{\mathbb{V}}^2$ are uniformly bounded from below, as well. Moreover, arguing as subsequent to (3.4) we see that these sequences are nested (*i.e.* $e_i + \delta_i \le e_{i-1} \le e_{i-1} + \delta_{i-1}$ for all $i \ge 1$) and nonincreasing. Hence both sequences converge to the same limit. This in turn implies that $\|z_{k,i} - z_{k,i-1}\|_{\mathcal{V}}$ tends to zero for $i \to \infty$. Hence, $M_k^N := \inf\{\, i \in \mathbb{N}\, ;\, \|z_{k,i} - z_{k,i-1}\|_{\mathcal{V}} \le \delta\,\}$ is finite, which proves the first statement of the Proposition.

Observe further that relations (4.6)–(4.7) imply that $e_i + \delta_i \le e_{i-1}$ for all $i \ge 2$ and that

$$e_1 + \delta_1 \le \mathcal{E}(t_k, u_{k,1}, z_{k,0}) \le \mathcal{E}(t_{k-1}, u_{k,0}, z_{k,0}) = \mathcal{E}(t_{k-1}, u_{k-1}, z_{k-1}) + \int_{t_{k-1}}^{t_k} \partial_t \mathcal{E}(r, u_{k-1}, z_{k-1}) \mathrm{d}r\,.$$

Taking the sum with respect to i yields (with $\mathcal{R}_\eta(v) = \mathcal{R}(v) + \frac{\eta}{2}\|v\|_{\mathbb{V}}^2$)

$$\mathcal{E}(t_k, u_{k,i}, z_{k,i}) + \sum_{j=1}^{i} \mathcal{R}_\eta(z_{k,j} - z_{k,j-1}) \le \mathcal{E}(t_{k-1}, u_{k-1}, z_{k-1}) + \int_{t_{k-1}}^{t_k} \partial_t \mathcal{E}(r, u_{k-1}, z_{k-1}) \mathrm{d}r\,.$$

Now, the uniform bounds (4.9)–(4.10) follow by similar arguments as in ([15], Chap. 2.1.2). □

As in the previous sections, the arc length of the linear interpolation curves of the minimizers generated by (4.6)–(4.8) is uniformly bounded:

Proposition 4.3. *Assume* (3.1a)–(3.1e), (4.2) *and* $\mathrm{D}_z\mathcal{E}(0, z_0, u_0) \in \mathcal{V}^*$. *Let*

$$\gamma_k^\tau := \sum_{i=0}^{M_k^N - 1} \left\| z_{k,i+1}^\tau - z_{k,i}^\tau \right\|_{\mathcal{Z}}\,, \qquad \mu_k^\tau := \sum_{i=0}^{M_k^N - 1} \left\| u_{k,i+1}^\tau - u_{k,i}^\tau \right\|_{\mathcal{U}}\,. \tag{4.11}$$

Then there exists a constant $C > 0$ *such that for all* $N \in \mathbb{N}$ *and* $\eta, \delta > 0$

$$\sum_{k=1}^{N} \gamma_k^\tau \le C \left(T \|\ell\|_{C^1([0,T],\mathcal{Q}^*)} + \|\mathrm{D}_z\mathcal{E}(0, u_0, z_0)\|_{\mathcal{V}^*} + \sum_{k=1}^{N} \sum_{i=0}^{M_k^N - 1} \mathcal{R}(z_{k,i+1}^\tau - z_{k,i}^\tau) \right)\,, \tag{4.12}$$

$$\sum_{k=1}^{N} \mu_k^\tau \le C \left(T + \sum_{k=1}^{N} \gamma_k^\tau \right). \tag{4.13}$$

Moreover, for all $0 \le i \le M_k^N - 1$

$$\eta \left\| z_{k,i+1}^\tau - z_{k,i}^\tau \right\|_{\mathbb{V}} \le C, \tag{4.14}$$

$$\|u_{k,i+1} - u_{k,i}\|_{\mathcal{U}} \le C(\tau + \eta^{-1}). \tag{4.15}$$

Proof. Let $u_{k,0} := u_{k-1}$ and $z_{k,-1} := z_{k-1, M_{k-1}^N - 1}$. Then from the minimality of $u_{k,i}$ in (4.6) we deduce for $0 \le i \le M_k^N - 1$ that

$$\|u_{k,i+1} - u_{k,i}\|_{\mathcal{U}} \le C \left(\|z_{k,i} - z_{k,i-1}\|_{\mathcal{V}} + \delta_{i,0}\tau \|\ell_u\|_{C^1([0,T];\mathcal{U}^*)} \right), \tag{4.16}$$

where $\delta_{i,0}$ is the Kronecker-symbol. Summation with respect to k and i yields (4.13). In order to prove (4.12) we proceed as follows: For $i \ge 1$ let $\xi_{k,i} := -\mathrm{D}_z\mathcal{E}(t_k, u_{k,i}, z_{k,i}) - \eta\mathbb{V}(z_{k,i} - z_{k,i-1})$. Since $\xi_{k,i} \in \partial\mathcal{R}(z_{k,i} - z_{k,i-1})$, by the convexity and one-homogeneity of $\mathcal{R}$ we deduce that $0 \ge \langle \xi_{k,i} - \xi_{k,i+1}, z_{k,i+1} - z_{k,i} \rangle$, which can be rewritten as

$$\begin{aligned} \eta \|z_{k,i+1} - z_{k,i}\|_{\mathbb{V}}^2 - \eta\langle \mathbb{V}(z_{k,i} - z_{k,i-1}), z_{k,i+1} - z_{k,i} \rangle + \langle \mathbb{A}(z_{k,i+1} - z_{k,i}), (z_{k,i+1} - z_{k,i}) \rangle \\ \le \langle \mathbb{B}^*(u_{k,i} - u_{k,i+1}), z_{k,i+1} - z_{k,i} \rangle + \langle \mathrm{D}_z\mathcal{F}(z_{k,i}) - \mathrm{D}_z\mathcal{F}(z_{k,i+1}), z_{k,i+1} - z_{k,i} \rangle. \end{aligned} \tag{4.17}$$

This is the analogue of (3.22). Taking into account estimate (4.16) and applying Ehrling's Lemma (*cf.* (1.20)) with $\varepsilon = \alpha/4$ the first term on the right hand side of (4.17) can be estimated as

$$|\langle \mathbb{B}^*(u_{k,i} - u_{k,i+1}), z_{k,i+1} - z_{k,i}\rangle| \leq \Big(\frac{\alpha}{4}\left\|z_{k,i} - z_{k,i-1}\right\|_{\mathcal{Z}} + C\mathcal{R}(z_{k,i} - z_{k,i-1})\Big)\left\|z_{k,i+1} - z_{k,i}\right\|_{\mathbb{V}},$$

while the second term on the right hand side is estimated with Lemma 1.1, again with $\varepsilon = \alpha/4$, so that in total we arrive at

$$\begin{aligned} &\eta\left\|z_{k,i+1} - z_{k,i}\right\|_{\mathbb{V}}^2 - \eta\left\|z_{k,i} - z_{k,i-1}\right\|_{\mathbb{V}}\left\|z_{k,i+1} - z_{k,i}\right\|_{\mathbb{V}} + \tfrac{3\alpha}{4}\left\|z_{k,i+1} - z_{k,i}\right\|_{\mathcal{Z}}^2 \\ &\qquad \leq \big(\tfrac{\alpha}{4}\left\|z_{k,i} - z_{k,i-1}\right\|_{\mathcal{Z}} + C(\mathcal{R}(z_{k,i} - z_{k,i-1}) + \mathcal{R}(z_{k,i+1} - z_{k,1}))\big)\left\|z_{k,i+1} - z_{k,i}\right\|_{\mathbb{V}}, \end{aligned} \tag{4.18}$$

and hence

$$\begin{aligned} &\eta\left\|z_{k,i+1} - z_{k,i}\right\|_{\mathbb{V}} + \tfrac{3\alpha}{4}\left\|z_{k,i+1} - z_{k,i}\right\|_{\mathcal{Z}} \\ &\qquad \leq \eta\left\|z_{k,i} - z_{k,i-1}\right\|_{\mathbb{V}} + \tfrac{\alpha}{4}\left\|z_{k,i} - z_{k,i-1}\right\|_{\mathcal{Z}} + C(\mathcal{R}(z_{k,i} - z_{k,i-1}) + \mathcal{R}(z_{k,i+1} - z_{k,i})), \end{aligned} \tag{4.19}$$

which is valid for $1 \leq i \leq M_k^N - 1$. If $k \geq 2$ and $i = 0$, then arguing as above we find (4.17) with the additional term $\langle \ell_z(t_k) - \ell_z(t_{k-1}), z_{k,1} - z_{k,0}\rangle$ on the right hand side. This leads to (4.19) for $i = 0$ and with the additional term $\tau\left\|\ell\right\|_{C^1([0,T];(\mathcal{U}\times\mathcal{V})^*)}$ on the right hand side.

Fix $k \geq 2$. Taking the sum of (4.19) with respect to $1 \leq i \leq M_k^N - 1$ and adding the inequality for $i = 0$ we obtain after exploiting several cancellations:

$$\begin{aligned} &\eta\left\|z_{k,i+1} - z_{k,i}\right\|_{\mathbb{V}} + \tfrac{\alpha}{4}\left\|z_{k,i+1} - z_{k,i}\right\|_{\mathcal{Z}} + \tfrac{\alpha}{2}\sum_{j=0}^{i}\left\|z_{k,j+1} - z_{k,j}\right\|_{\mathcal{Z}} \\ &\leq \eta\left\|z_{k,0} - z_{k,-1}\right\|_{\mathbb{V}} + \tfrac{\alpha}{4}\left\|z_{k,0} - z_{k,-1}\right\|_{\mathcal{Z}} + C\big(\tau\left\|\ell\right\|_{C^1([0,T];(\mathcal{U}\times\mathcal{Z})^*)} + \sum_{j=0}^{i+1}\mathcal{R}(z_{k,j} - z_{k,j-1})\big), \end{aligned} \tag{4.20}$$

which is valid for $0 \leq i \leq M_k^N - 1$.

Let us now discuss the case $k = 1$ and $i = 0$. Again we have $\xi_{1,1} \in \partial\mathcal{R}(z_{1,1} - z_{1,0})$ and thus $\mathcal{R}(z_{1,1} - z_{1,0}) = \langle \xi_{1,1}, z_{1,1} - z_{1,0}\rangle$. Adding $\langle -\mathrm{D}_z\mathcal{E}(0, u_0, z_0), z_{1,1} - z_{1,0}\rangle$ on both sides yields after rearranging the terms and exploiting the positivity of $\mathbb{A}$

$$\begin{aligned} &\mathcal{R}(z_{1,1} - z_{1,0}) + \eta\left\|z_{1,1} - z_{1,0}\right\|_{\mathbb{V}}^2 + \alpha\left\|z_{1,1} - z_{1,0}\right\|_{\mathcal{Z}}^2 \\ &\quad \leq \langle -\mathrm{D}_z\mathcal{E}(0, u_0, z_0), z_{1,1} - z_{1,0}\rangle + \langle \ell(0) - \ell(\tau), z_{1,1} - z_{1,0}\rangle + \langle \mathbb{B}^*(u_0 - u_{1,1}), z_{1,1} - z_{1,0}\rangle \\ &\quad \leq C\big(\left\|\mathrm{D}_z\mathcal{E}(0, u_0, z_0)\right\|_{\mathcal{V}^*} + \tau\left\|\ell\right\|_{C^1([0,T];(\mathcal{U}\times\mathcal{V})^*)}\big)\left\|z_{1,1} - z_{1,0}\right\|_{\mathbb{V}}. \end{aligned} \tag{4.21}$$

From this inequality we deduce that

$$\eta\left\|z_{1,1} - z_{1,0}\right\|_{\mathbb{V}} + \alpha\left\|z_{1,1} - z_{1,0}\right\|_{\mathcal{Z}} \leq C\big(\left\|\mathrm{D}_z\mathcal{E}(0, u_0, z_0)\right\|_{\mathcal{V}^*} + \tau\left\|\ell\right\|_{C^1([0,T];(\mathcal{U}\times\mathcal{V})^*)}\big). \tag{4.22}$$

For $k = 1$, taking the sum of (4.19) with respect to i and adding (4.22) we obtain

$$\eta\left\|z_{1,i+1} - z_{1,i}\right\|_{\mathbb{V}} + \tfrac{\alpha}{4}\left\|z_{1,1} - z_{1,0}\right\|_{\mathcal{Z}} + \tfrac{\alpha}{4}\left\|z_{1,i+1} - z_{1,i}\right\|_{\mathcal{Z}} + \tfrac{\alpha}{2}\sum_{j=0}^{i}\left\|z_{1,j+1} - z_{1,j}\right\|_{\mathcal{Z}}$$

$$\leq C\big(\tau \left\|\ell\right\|_{C^1([0,T];(\mathcal{U}\times\mathcal{V})^*)} + \left\|\mathrm{D}_z\mathcal{E}(0,u_0,z_0)\right\|_{\mathcal{V}^*} + \sum_{j=0}^{i}\mathcal{R}(z_{1,j+1}-z_{1,j})\big)\,, \tag{4.23}$$

which is valid for $0 \leq i \leq M_1^N - 1$.

For arbitrary $k \in \{1,\dots,N\}$ and $i \in \{0,\dots,M_k^N - 1\}$ the summation of (4.20) and (4.23) up to (k,i) yields (with $z_{1,-1} := z_0$ and omitting the term $\frac{\alpha}{4}\left\|z_{1,1}-z_{1,0}\right\|_{\mathbb{Z}}$ on the left hand side)

$$\begin{aligned}
\eta\left\|z_{k,i+1}-z_{k,i}\right\|_{\mathbb{V}} + \tfrac{\alpha}{4}\left\|z_{k,i+1}-z_{k,i}\right\|_{\mathbb{Z}} + \sum_{s=1}^{k-1}\big(\eta\left\|z_{s,M_s^N}-z_{s,M_s^N-1}\right\|_{\mathbb{V}} + \tfrac{\alpha}{4}\left\|z_{s,M_s^N}-z_{s,M_s^N-1}\right\|_{\mathbb{Z}}\big)&\\
+ \tfrac{\alpha}{2}\sum_{s=1}^{k-1}\sum_{j=0}^{M_{k-1}^N-1}\left\|z_{s,j+1}-z_{s,j}\right\|_{\mathbb{Z}} + \tfrac{\alpha}{2}\sum_{j=0}^{i}\left\|z_{k,j+1}-z_{k,j}\right\|_{\mathbb{Z}}&\\
\leq \sum_{s=1}^{k}\big(\eta\left\|z_{s,0}-z_{s,-1}\right\|_{\mathbb{V}} + \tfrac{\alpha}{4}\left\|z_{s,0}-z_{s,-1}\right\|_{\mathbb{Z}}\big)&\\
+ C\Big(k\tau\left\|\ell\right\|_{C^1([0,T];(\mathcal{U}\times\mathcal{V})^*)} + \left\|\mathrm{D}_z\mathcal{E}(0,u_0,z_0)\right\|_{\mathcal{V}^*} + \sum_{s=1}^{k-1}\sum_{j=0}^{M_{k-1}^N-1}\mathcal{R}(z_{s,j}-z_{s,j-1}) + \sum_{j=0}^{i+1}\mathcal{R}(z_{k,j}-z_{k,j-1})\Big)\,.&
\end{aligned} \tag{4.24}$$

Observe that $z_{s,0} = z_{s-1,M_{s-1}^N}$ and $z_{s,-1} = z_{s-1,M_{s-1}^N-1}$ and hence

$$\sum_{s=1}^{k-1}\left\|z_{s,M_s^N}-z_{s,M_s^N-1}\right\|_{\mathbb{V}} = \sum_{\sigma=2}^{k}\left\|z_{\sigma,0}-z_{\sigma,-1}\right\|_{\mathbb{V}}\,.$$

Thus, the previous estimate reduces to

$$\begin{aligned}
\eta\left\|z_{k,i+1}-z_{k,i}\right\|_{\mathbb{V}} + \tfrac{\alpha}{4}\left\|z_{k,i+1}-z_{k,i}\right\|_{\mathbb{Z}} + \tfrac{\alpha}{2}\sum_{s=1}^{k-1}\gamma_s + \tfrac{\alpha}{2}\sum_{j=0}^{i}\left\|z_{k,j+1}-z_{k,j}\right\|_{\mathbb{Z}}&\\
\leq C\Big(k\tau\left\|\ell\right\|_{C^1([0,T];(\mathcal{U}\times\mathcal{V})^*)} + \left\|\mathrm{D}_z\mathcal{E}(0,u_0,z_0)\right\|_{\mathcal{V}^*}&\\
+ \sum_{s=1}^{k-1}\sum_{j=0}^{M_{k-1}^N-1}\mathcal{R}(z_{s,j}-z_{s,j-1}) + \sum_{j=0}^{i+1}\mathcal{R}(z_{k,j}-z_{k,j-1})\Big)\,,&
\end{aligned} \tag{4.25}$$

which implies (4.12) and (4.14). Finally, (4.15) is a consequence of (4.16) and (4.14). □

Like in the previous section interpolating curves will be defined with respect to an artificial arclength parameter. For $k \geq 1$ let

$$s_0 := 0, \quad s_k := s_{k-1} + \tau + \sum_{i=1}^{M_k^N}\Big(\left\|u_{k,i}-u_{k,i-1}\right\|_{\mathcal{U}} + \left\|z_{k,i}-z_{k,i-1}\right\|_{\mathcal{V}}\Big)\,.$$

Thanks to Proposition 4.3 we have $\sup_{N\in\mathbb{N},\delta>0,\eta>0} s_N < \infty$. Let furthermore

$$s_{k,-1} := s_{k-1},\ \ s_{k,0} = s_{k-1} + \tau \tag{4.26}$$

and for $i \geq 1$

$$s_{k,i} := s_{k,i-1} + \|u_{k,i} - u_{k,i-1}\|_{\mathcal{U}} + \|z_{k,i} - z_{k,i-1}\|_{\mathcal{V}}$$

with $s_{k,M_k^N} = s_k$. The piecewise affine interpolations are given by ($1 \leq k \leq N$, $1 \leq i \leq M_k^N$)

$$\text{for } s \in [s_{k-1}, s_{k,0}): \qquad t_\tau(s) = t_{k-1} + (s - s_{k-1}),\ u_\tau(s) = u_{k-1},\ z_\tau(s) = z_{k-1}, \tag{4.27a}$$

$$\text{for } s \in [s_{k,i-1}, s_{k,i}): \qquad u_\tau(s) := u_{k,i-1} + \frac{s - s_{k,i-1}}{\sigma_\tau(s)}(u_{k,i} - u_{k,i-1}), \tag{4.27b}$$

$$z_\tau(s) := z_{k,i-1} + \frac{s - s_{k,i-1}}{\sigma_\tau(s)}(z_{k,i} - z_{k,i-1}) \tag{4.27c}$$

$$t_\tau(s) := t_k, \tag{4.27d}$$

where the increment σ_τ is defined as

$$\sigma_\tau(s) := \begin{cases} \tau & \text{for } s \in (s_{k-1}, s_{k,0}) \\ \|u_{k,i} - u_{k,i-1}\|_{\mathcal{U}} + \|z_{k,i} - z_{k,i-1}\|_{\mathcal{V}} & \text{for } s \in (s_{k,i-1}, s_{k,i}) \\ 0 & \text{otherwise} \end{cases}. \tag{4.28}$$

Observe that in (4.27) we do not divide by zero. The piecewise left or right continuous interpolants are defined as follows for $g \in \{t, u, z\}$: Let $1 \leq k \leq N$, $0 \leq i \leq M_k^N$. Then

$$\overline{g}_\tau(s) = g_\tau(s_{k,i}) \qquad \text{for } s \in (s_{k,i-1}, s_{k,i}],$$
$$\underline{g}_\tau(s) = g_\tau(s_{k,i-1}) \quad \text{for } s \in [s_{k,i-1}, s_{k,i}).$$

With this, $\sigma_\tau(s) = \left|\overline{t}_\tau(s) - \underline{t}_\tau(s)\right| + \|\overline{u}_\tau(s) - \underline{u}_\tau(s)\|_{\mathcal{U}} + \|\overline{z}_\tau(s) - \underline{z}_\tau(s)\|_{\mathcal{V}}$. By definition, for almost all s we have $t_\tau'(s) + \|u_\tau'(s)\|_{\mathcal{U}} + \|z_\tau'(s)\|_{\mathcal{V}} = 1$, $t_\tau(s_N) = T$ and z_τ is uniformly bounded in $L^\infty((0, s_N); \mathcal{Z}) \cap W^{1,\infty}((0, s_N); \mathcal{V})$ while u_τ is uniformly bounded in $W^{1,\infty}((0, s_N); \mathcal{U})$. Moreover, due to Proposition 4.3 for all $s \in [0, s_N]$ and uniformly in N, η and δ we have

$$\|u_\tau(s) - \overline{u}_\tau(s)\|_{\mathcal{U}} + \|\underline{u}_\tau(s) - \overline{u}_\tau(s)\|_{\mathcal{U}} \leq C(\tau + \eta^{-1}), \tag{4.29}$$

$$\|z_\tau(s) - \overline{z}_\tau(s)\|_{\mathcal{V}} + \|\underline{z}_\tau(s) - \overline{z}_\tau(s)\|_{\mathcal{V}} \leq C\eta^{-1}, \tag{4.30}$$

$$|\sigma_\tau(s)| \leq C(\tau + \eta^{-1}). \tag{4.31}$$

Let us finally define

$$J_\tau(s) := \begin{cases} \eta \mathbb{V}(z_{k-1,M_{k-1}^N} - z_{k-1,M_{k-1}^N - 1}) & \text{for } s \in (s_{k-1}, s_{k,0}) \text{ and } k \geq 2 \\ 0 & \text{otherwise} \end{cases}.$$

Thanks to the stopping criterion (4.8) we have

$$\|J_\tau\|_{L^\infty((0,s_N);\mathcal{V}^*)} \leq C\eta\delta.$$

Moreover, $\|J_\tau(s)\|_{\mathcal{V}^*} \|z_\tau'(s)\|_{\mathcal{V}} = 0$ for almost all $s \in [0, s_N]$. The next proposition is the analogue to Proposition 3.5. We recall the notation $\mathcal{R}_\mu(v) = \mathcal{R}(v) + \frac{\mu}{2}\|v\|_{\mathbb{V}}^2$.

Proposition 4.4 (Discrete energy dissipation estimate)**.** *Assume* (3.1a)–(3.1e), (4.2) *and that* $-\mathrm{D}_z\mathcal{E}(0,u_0,z_0)\in\partial\mathcal{R}(0)$*. The interpolating curves satisfy the following relation for every* $\alpha<\beta\in[0,s_N]$ *and with* $q_\tau(s):=(u_\tau(s),z_\tau(s))$*:*

$$\begin{aligned}\mathcal{E}(t_\tau(\beta),q_\tau(\beta))-\mathcal{E}(t_\tau(\alpha),q_\tau(\alpha))&\\ +\int_\alpha^\beta \mathcal{R}_{\sigma_\tau(s)\eta}(z_\tau'(s))+\mathcal{R}^*_{\sigma_\tau(s)\eta}\big(-\mathrm{D}_z\mathcal{E}(\underline{t}_\tau(s),\overline{u}_\tau(s),\overline{z}_\tau(s))-J_\tau(s)\big)\,\mathrm{d}s&\\ =\int_\alpha^\beta \partial_t\mathcal{E}(t_\tau(s),q_\tau(s))t_\tau'(s)\,\mathrm{d}s+\int_\alpha^\beta\Big\langle \mathrm{D}_u\mathcal{E}(t_\tau(s),q_\tau(s)),u_\tau'(s)\Big\rangle\,\mathrm{d}s+\int_\alpha^\beta r_\tau(s)\,\mathrm{d}s,&\end{aligned}\tag{4.32}$$

where $r_\tau=\langle \mathrm{D}_z\mathcal{E}(t_\tau,q_\tau)-\mathrm{D}_z\mathcal{E}(\underline{t}_\tau,\overline{u}_\tau,\overline{z}_\tau),z_\tau'\rangle$*. Moreover,*

$$\int_\alpha^\beta r_\tau(s)\,\mathrm{d}s\le C(\beta-\alpha)(\tau+\eta^{-1}),\tag{4.33}$$

$$\|\mathrm{D}_u\mathcal{E}(t_\tau,q_\tau)\|_{L^\infty((0,s_N);\mathcal{U}^*)}\le C(\tau+\eta^{-1}),\tag{4.34}$$

and the constant C *independent of* η*,* N *and* δ*.*

Proof. By the chain rule, for every $\alpha<\beta\in[0,s_N]$ we deduce

$$\begin{aligned}&\mathcal{E}(t_\tau(\beta),q_\tau(\beta))-\mathcal{E}(t_\tau(\alpha),q_\tau(\alpha))\\ &=\int_\alpha^\beta \partial_t\mathcal{E}(t_\tau(s),q_\tau(s))t_\tau'(s)\,\mathrm{d}s+\int_\alpha^\beta\Big\langle \mathrm{D}_u\mathcal{E}(t_\tau(s),q_\tau(s)),u_\tau'(s)\Big\rangle\,\mathrm{d}s+\int_\alpha^\beta\Big\langle \mathrm{D}_z\mathcal{E}(t_\tau(s),q_\tau(s)),z_\tau'(s)\Big\rangle\,\mathrm{d}s\,.\end{aligned}\tag{4.35}$$

For the term involving $\mathrm{D}_z\mathcal{E}$ we proceed as follows: Let first $k,i\ge1$ and $s\in(s_{k,i-1},s_{k,i})$. Then from the minimality of $z_{k,i}$ we obtain $-\mathrm{D}_z\mathcal{E}(t_k,u_{k,i},z_{k,i})-\eta\mathbb{V}(z_{k,i}-z_{k,i-1})\in\partial\mathcal{R}(z_{k,i}-z_{k,i-1})$, which can be rewritten as

$$-\mathrm{D}_z\mathcal{E}(\underline{t}_\tau(s),\overline{u}_\tau(s),\overline{z}_\tau(s))-J_\tau(s)\in\partial\mathcal{R}_{\sigma_\tau(s)\eta}(z_\tau'(s))\,.\tag{4.36}$$

Next, for $k\ge2$ and $s\in(s_{k,-1},s_{k,0})$ we have

$$\begin{aligned}-\mathrm{D}_z\mathcal{E}\Big(t_{k-1},u_{k-1,M^N_{k-1}},z_{k-1,M^N_{k-1}}\Big)-\eta\mathbb{V}\Big(z_{k-1,M^N_{k-1}}-z_{k-1,M^N_{k-1}-1}\Big)&\\ \in\partial\mathcal{R}\Big(z_{k-1,M^N_{k-1}}-z_{k-1,M^N_{k-1}-1}\Big)\subset\partial\mathcal{R}(0).&\end{aligned}$$

Since $\sigma_\tau(s)\eta\mathbb{V}z_\tau'(s)=0$, the previous relation can be rewritten in the form (4.36), as well. Finally for $k=1$ and $s\in(s_0,s_{1,0})$ thanks to the assumptions we have $-\mathrm{D}_z\mathcal{E}(0,u_0,z_0)\in\partial\mathcal{R}(0)$, which again can be rewritten in the form (4.36). Thus, (4.36) is valid for almost all $s\in(0,s_N)$. By convex analysis, relation (4.36) is equivalent to

$$\begin{aligned}&\mathcal{R}_{\sigma_\tau(s)\eta}(z_\tau'(s))+\mathcal{R}^*_{\sigma_\tau(s)\eta}(-\mathrm{D}_z\mathcal{E}(\underline{t}_\tau(s),\overline{u}_\tau(s),\overline{z}_\tau(s))-J_\tau(s))\\ &\quad=\langle-\mathrm{D}_z\mathcal{E}(\underline{t}_\tau(s),\overline{u}_\tau(s),\overline{z}_\tau(s)),z_\tau'(s)\rangle-\langle J_\tau(s),z_\tau'(s)\rangle=-\langle \mathrm{D}_z\mathcal{E}(t_\tau(s),q_\tau(s)),z_\tau'(s)\rangle+r_\tau(s)\end{aligned}$$

with $r_\tau(s)$ as in the proposition and taking into account that $\langle J_\tau(s),z_\tau'(s)\rangle=0$. Inserting this identity into (4.35) results in (4.32).

For proving (4.34) observe first that due to the minimality of the $u_{k,i}$ and the assumption on u_0 we have

$$
\mathrm{D}_u\mathcal{E}(t_\tau(s),q_\tau(s)) = \begin{cases} \mathrm{D}_u\mathcal{E}(t_\tau(s),q_\tau(s)) - \mathrm{D}_u\mathcal{E}(t_\tau(s),\overline{u}_\tau(s),\underline{z}_\tau(s)) & \text{for } k,i\geq 1 \text{ and } s\in(s_{k,i-1},s_{k,i}) \\ \mathrm{D}_u\mathcal{E}(t_\tau(s),q_\tau(s)) - \mathrm{D}_u\mathcal{E}(\underline{t}_\tau(s),\underline{u}_\tau(s),z_{k-1,M_{k-1}^N-1}) & \text{for } k\geq 2 \text{ and } s\in(s_{k,-1},s_{k,0}) \\ \mathrm{D}_u\mathcal{E}(t_\tau(s),q_\tau(s)) - \mathrm{D}_u\mathcal{E}(\underline{t}_\tau(s),\underline{u}_\tau(s),\underline{z}_\tau(s)) & \text{for } s\in(s_0,s_{1,0}) \end{cases} .
$$

Taking into account estimates (4.29)–(4.30) we arrive at (4.34). By similar arguments as those in the proof of Proposition 3.5 we finally obtain (4.33) applying again Lemma 1.1 and the estimates (4.29)–(4.30). □

We are now ready to pass to the limit.

Theorem 4.5. *Assume (3.1a)–(3.1e), (4.2) and that $-\mathrm{D}_z\mathcal{E}(0,u_0,z_0)\in\partial\mathcal{R}(0)$. Let the sequences $(u_{k,i}^\tau)_{k,i}\subset\mathcal{U}$ and $(z_{k,i}^\tau)_{k,i}$ with $\tau=T/N$ be generated by (4.6)–(4.8).*
For every sequence $N_n\to\infty,\eta_n\to\infty$, $\tau_{N_n}\to 0$ and $\delta_n\to 0$ with $\eta_n\delta_n\to 0$ there exists a (not relabelled) subsequence $(s_{N_n},t_{\tau_n},u_{\tau_n},z_{\tau_n})_{n\in\mathbb{N}}$ of the interpolating curves, a number $S>0$ and functions $\hat t\in W^{1,\infty}((0,S);\mathbb{R})$, $\hat u\in W^{1,\infty}((0,S);\mathcal{U})$, $\hat z\in W^{1,\infty}((0,S);\mathcal{V})\cap L^\infty((0,S);\mathcal{Z})$ (with $\hat q:=(\hat u,\hat z)$) such that for $n\to\infty$ (we omit the index n)

$$ s_N\to S, \tag{4.37} $$
$$ t_\tau \overset{*}{\rightharpoonup} \hat t \text{ in } W^{1,\infty}((0,S);\mathbb{R}),\quad \underline{t}_\tau, t_\tau(s)\to\hat t(s) \text{ for every } s\in[0,S], \tag{4.38} $$
$$ u_\tau \overset{*}{\rightharpoonup} \hat u \text{ weakly* in } W^{1,\infty}((0,S);\mathcal{U}), \tag{4.39} $$
$$ z_\tau \overset{*}{\rightharpoonup} \hat z \text{ weakly* in } W^{1,\infty}((0,S);\mathcal{V})\cap L^\infty((0,S);\mathcal{Z}), \tag{4.40} $$
$$ \overline{u}_\tau(s),\underline{u}_\tau(s),u_\tau(s)\rightharpoonup\hat u(s) \text{ weakly in } \mathcal{U} \text{ for every } s\in[0,S]\,, \tag{4.41} $$
$$ \overline{z}_\tau(s),\underline{z}_\tau(s),z_\tau(s)\rightharpoonup\hat z(s) \text{ weakly in } \mathcal{Z} \text{ for every } s\in[0,S]\,. \tag{4.42} $$

Moreover, the limit functions satisfy $\hat t(0)=0$, $\hat t(S)=T$, $\hat z(0)=z_0$, $\hat u(0)=u_0$ and for a.a. $s\in[0,S]$

$$ \hat t'(s)\geq 0,\ \hat t'(s)+\|\hat u'(s)\|_{\mathcal{U}}+\|\hat z'(s)\|_{\mathcal{V}}\leq 1\,, \tag{4.43} $$
$$ \hat t'(s)\operatorname{dist}_{\mathcal{V}^*}\big(-\mathrm{D}_z\mathcal{E}(\hat t(s),\hat q(s)),\partial\mathcal{R}(0)\big)=0 \tag{4.44} $$

together with the energy dissipation identity

$$ \mathcal{E}(\hat t(s),\hat q(s))+\int_0^s \mathcal{R}(\hat z'(r))+\|\hat z'(r)\|_{\mathcal{V}}\operatorname{dist}(-\mathrm{D}_z\mathcal{E}(\hat t(r),\hat q(r)),\partial\mathcal{R}(0))\,\mathrm{d}r = \mathcal{E}(0,q(0))+\int_0^s\partial_t\mathcal{E}(\hat t(r),\hat q(r))\hat t'(r)\,\mathrm{d}r \tag{4.45} $$

for all $s\in[0,S]$. Finally, for all $s\in[0,S]$ we have $\mathrm{D}_u\mathcal{E}(\hat t(s),\hat u(s),\hat z(s))=0$.

Remark 4.6. Observe that the solutions generated by the combined alternate minimisation scheme with viscous regularisation are of the same type as the solutions generated by the schemes discussed in Sections 2 and 3.1 and hence belong to the class of BV-solutions, as well.

Proof. The proof is similar to the proof of Theorem 3.7 and 2.5, and we highlight here the differences, only. The convergences in (4.37)–(4.42) follow from the bounds provided in Propositions 4.2 and 4.3, compare also (4.29)–(4.30). Moreover, for every sequence $(\sigma_n)_n\subset[0,s_N]$ with $\sigma_n\to\sigma$ in $[0,S]$ we have

$$ t_{\tau_n}(\sigma_n)\to\hat t(\sigma),\ u_{\tau_n}(\sigma_n)\rightharpoonup\hat u(\sigma) \text{ weakly in } \mathcal{U},\ z_{\tau_n}(\sigma_n)\rightharpoonup\hat z(\sigma) \text{ weakly in } \mathcal{Z}. \tag{4.46} $$

This is an immediate consequence of the uniform Lipschitz bounds for the sequences $(t_\tau)_\tau$, $(u_\tau)_\tau$ and $(z_\tau)_\tau$ and the pointwise (weak) convergences in (4.39)–(4.42).

Let $(\widetilde{t}_\tau, \widetilde{u}_\tau, \widetilde{z}_\tau)_\tau$ be any triple of interpolating curves. Thanks to the assumptions on $\mathcal{E}$ and the convergences (4.41)–(4.42), for every s we have

$$\mathrm{D}_u\mathcal{E}(\widetilde{t}_\tau(s), \widetilde{u}_\tau(s), \widetilde{z}_\tau(s)) \rightharpoonup \mathrm{D}_u\mathcal{E}(\hat{t}(s), \hat{u}(s), \hat{z}(s)) \text{ weakly in } \mathcal{U}^*, \tag{4.47}$$

$$\mathrm{D}_z\mathcal{E}(\widetilde{t}_\tau(s), \widetilde{u}_\tau(s), \widetilde{z}_\tau(s)) \rightharpoonup \mathrm{D}_z\mathcal{E}(\hat{t}(s), \hat{u}(s), \hat{z}(s)) \text{ weakly in } \mathcal{Z}^*. \tag{4.48}$$

The relations in (4.43) follow by simple lower semicontinuity arguments. Let us next discuss the complementarity relation (4.44): observe first that

$$\sup_{s\in[0,S]} \mathrm{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{E}(\hat{t}(s), \hat{u}(s), \hat{z}(s)), \partial\mathcal{R}(0)) < \infty. \tag{4.49}$$

Indeed, from (4.36) it follows that for almost all $s \in [0, s_N]$ we have

$$-\big(\mathrm{D}_z\mathcal{E}(\underline{t}_\tau(s), \overline{u}_\tau(s), \overline{z}_\tau(s)) + \sigma_\tau(s)\eta z_\tau'(s) + J_\tau(s)\big) \in \partial\mathcal{R}(0). \tag{4.50}$$

Since $\partial\mathcal{R}(0) \subset \mathcal{V}^*$ is bounded (due to assumption (1.17)) and since the functions J_τ and $\sigma_\tau(s)\eta\hat{z}_\tau'(s)$ are uniformly bounded in $\mathcal{V}^*$ with respect to s, η, τ (see (4.14) and (4.30)–(4.31)) we obtain

$$\sup_{\tau,\eta} \|\mathrm{D}_z\mathcal{E}(\underline{t}_\tau, \overline{u}_\tau, \overline{z}_\tau)\|_{L^\infty((0,s_N);\mathcal{V}^*)} < \infty,$$

Thus, for this choice of the interpolants in (4.48) we actually have weak convergence in $\mathcal{V}^*$ and ultimately $\mathrm{D}_z\mathcal{E}(\hat{t}, \hat{u}, \hat{z}) \in L^\infty((0, S); \mathcal{V}^*)$. Due to the boundedness of $\partial\mathcal{R}(0)$ in $\mathcal{V}^*$ we finally obtain (4.49). Starting again from (4.50) for almost all s we have

$$t_\tau' \,\mathrm{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{E}(\underline{t}_\tau, \overline{u}_\tau, \overline{z}_\tau), \partial\mathcal{R}(0)) \le t_\tau'(\sigma_\tau\eta \|z_\tau'\|_{\mathcal{V}^*} + \|J_\tau\|_{\mathcal{V}^*}) = t_\tau' \|J_\tau\|_{\mathcal{V}^*},$$

where for the latter identity we have used that $t_\tau'(s) \|z_\tau'(s)\|_{\mathcal{V}^*} = 0$. Thus, taking into account (4.8) for all $\alpha < \beta \in [0, s_N]$ we have

$$0 \le \int_\alpha^\beta t_\tau' \,\mathrm{dist}_{\mathcal{V}^*}(-\mathrm{D}_z\mathcal{E}(\underline{t}_\tau, \overline{u}_\tau, \overline{z}_\tau), \partial\mathcal{R}(0))\,\mathrm{d}s \le \eta\delta s_N.$$

Proposition B.2 now yields (4.44).

From (4.33) and the convergences in (4.47) it follows that $\mathrm{D}_u\mathcal{E}(\hat{t}(s), \hat{u}(s), \hat{z}(s)) = 0$ for all $s \in [0, S]$ and moreover, $\int_\alpha^\beta \langle \mathrm{D}_u\mathcal{E}(t_\tau(s), q_\tau(s)), u_\tau'(s)\rangle\,\mathrm{d}s$ tends to zero for $N \to \infty$. We recall that $\mathcal{R}_\mu(v) + \mathcal{R}_\mu^*(\xi) \ge \mathcal{R}(v) + \|v\|_{\mathbb{V}} \mathrm{dist}_{\mathcal{V}^*}(\xi, \partial\mathcal{R}(0))$ for all $\mu > 0$, $v \in \mathcal{V}$ and $\xi \in \mathcal{V}^*$. Thus, for $\alpha = 0$ and arbitrary $\beta \in [0, S]$ from the discrete energy dissipation identity (4.32) we obtain in the limit $N \to \infty$ the energy dissipation inequality (4.45) with $\le$ instead of an equality. Here, we exploit the lower semicontinuity of $\mathcal{E}$ and Proposition B.1. With the same argument as in the proof of ([7], Lem. 5.2) one obtains

$$\Big\langle -\mathrm{D}_z\mathcal{E}(\hat{t}(s), \hat{q}(s)), \hat{z}'(s)\Big\rangle \le \mathcal{R}\Big(\hat{z}'(s)\Big) + \mathrm{dist}_{\mathcal{V}^*}\Big(-\mathrm{D}_z\mathcal{E}(\hat{t}(s), \hat{q}(s)), \partial\mathcal{R}(0)\Big).$$

Hence, applying again the chain rule to the right hand side of (4.45) we finally obtain the energy dissipation identity (4.45) with equality. This finishes the proof of Theorem 4.5. □

5. Examples

5.1. A finite dimensional example

The following finite-dimensional example illustrates that functions satisfying (2.35)–(2.37) with the same data z_0 and ℓ need not be unique. Moreover, the approaches discussed in Section 2 (local minimisation) and Section 3.1 (relaxed local minimisation) might converge to different solutions of (2.35)–(2.37).

Let $\mathcal{Z} = \mathcal{V} = \mathcal{X} = \mathbb{R}$, $\kappa > 0$, $z_0 = 2$, $\ell(t) \equiv 0$ for $t \in [0, T]$ and define

$$\mathcal{I}(t, z) := -\kappa z + \tfrac{1}{4}z^4 - \tfrac{8}{3}z^3 + 10z^2 - 16z, \quad \mathcal{R}(z) := \kappa\,|z|\,.$$

Observe that $\mathrm{D}_z\mathcal{I}(t, z) = -\kappa + (z-2)^2(z-4)$ and $\partial\mathcal{R}(0) = [-\kappa, \kappa]$. Clearly, $-\mathrm{D}_z\mathcal{I}(t, z) \in \partial\mathcal{R}(0)$ if and only if $z \in \{2\} \cup [4, z_*]$, where $z_* > 4$ is the (unique) solution of $(z-2)^2(z-4) = 2\kappa$. The pair $(\hat{t}_\infty, \hat{z}_\infty) : [0, S] \to [0, T] \times \mathbb{R}$ with $\hat{t}_\infty(t) = t$ and $\hat{z}_\infty(t) = 2$ is a solution of (2.35)–(2.37). Moreover, let $\alpha \in [0, T]$ be arbitrary. It is straightforward to verify that the pairs $(\hat{t}_\alpha, \hat{z}_\alpha) : [0, S] \to [0, T] \times \mathbb{R}$ with $S = 2 + T$ and

$$\hat{t}_\alpha(s) = \begin{cases} s & \text{if } s \le \alpha \\ \alpha & \text{if } \alpha < s \le \alpha + 2 \\ s - 2 & \text{if } s \ge \alpha + 2 \end{cases}, \qquad \hat{z}_\alpha(s) = \begin{cases} 2 & \text{if } s \le \alpha \\ 2 + s - \alpha & \text{if } \alpha < s \le \alpha + 2 \\ 4 & \text{if } s \ge \alpha + 2 \end{cases}$$

satisfy (2.35)–(2.37), as well. Starting with $z_0 = 2$ the algorithm (3.2)–(3.3) for every $\eta > 1$ and arbitrary $\tau > 0$ generates the constant values $z_{k,\infty} = 2$, hence approximating in the limit the solution $(\hat{t}_\infty, \hat{z}_\infty)$ from above. On the other hand, the local minimisation algorithm (2.1)–(2.2) generates the points $(t_k^h, z_k^h) = (0, 2 + kh)$ if $kh \le 2$ and $(t_k^h, z_k^h) = (kh + ((k_* + 1)h - 2), 4)$ if $k > k_*$, where $k_* = \lfloor 2/h \rfloor$. In the limit $(h \to 0)$ these curves converge to the solution $(\hat{t}_\alpha, \hat{z}_\alpha)$ with $\alpha = 0$. A similar example was presented in ([18], Sect. 5.3).

5.2. Comparison of the schemes for a finite-dimensional toy example

In order to illustrate the similarities and also differences of the above discussed schemes let us consider the following finite dimensional example with $\mathcal{Z} = \mathcal{V} = \mathcal{X} = \mathbb{R}$ and

$$\mathcal{I}(t, z) := 5z^2 - \frac{t^2}{2(0.1 + z^2)}, \quad \mathcal{R}(v) := 10\,|v|\,, \quad z_0 = 1 \quad \text{and } T = 1.5\,. \tag{5.1}$$

Note that the energy $\mathcal{I}$ is not exactly of the structure (1.14). Clearly, $\mathrm{D}_z\mathcal{I}(t, z) = (10 + \frac{t^2}{(0.1+z^2)^2})z$ and hence $\mathrm{D}_z\mathcal{I}(t, z)$ is positive if and only if z is positive. Hence, $z(t) > 0$ implies $\dot{z}(t) \le 0$. Moreover, $\inf\{\, \mathrm{D}_z^2\mathcal{I}(t, z)\,;\, 0 \le t \le T, 0 \le z \le 1 \,\} = \inf\{\, \mathrm{D}_z^2\mathcal{I}(T, z)\,;\, 0 \le z \le 1 \,\} \ge -46.3$.

Figure 2a shows the GES (dark red) and the BV-solution (blue) associated with (5.1) on the time interval $[0, T]$. In this particular example, these solutions are unique. The grey set in Figure 2a refers to points (t, z) with $-\mathrm{D}_z\mathcal{I}(t, z) \in \partial\mathcal{R}(0)$. The following tests were carried out:

Vanishing viscosity: Figure 2b shows the results obtained with the vanishing viscosity approach (1.3), where the discretisation parameters are chosen as in Table 1. Observe the rather slow convergence towards the BV-solution of the discrete solutions for the choice $\mu = 0.1\sqrt{\tau}$.

Local minimisation: The purple curve in Figure 3a is obtained by the local minimisation algorithm (2.1)–(2.2) with $h = T/90 = 0.01\overline{6}$. The total number of minimisation steps to reach the final time T is 150. Figure 3b shows the corresponding time increments.

Relaxed local minimisation with stopping criterion: Figure 4a shows the discrete solution obtained with the scheme (3.2) combined with the stopping criterion (4.8) for $N = 100$, $\eta = 100$ and $\delta = 10^{-3}$. The total number of minimisation steps is 400. Figure 4b displays the number of iterations in each time step $t_k = kT/N$ (with a

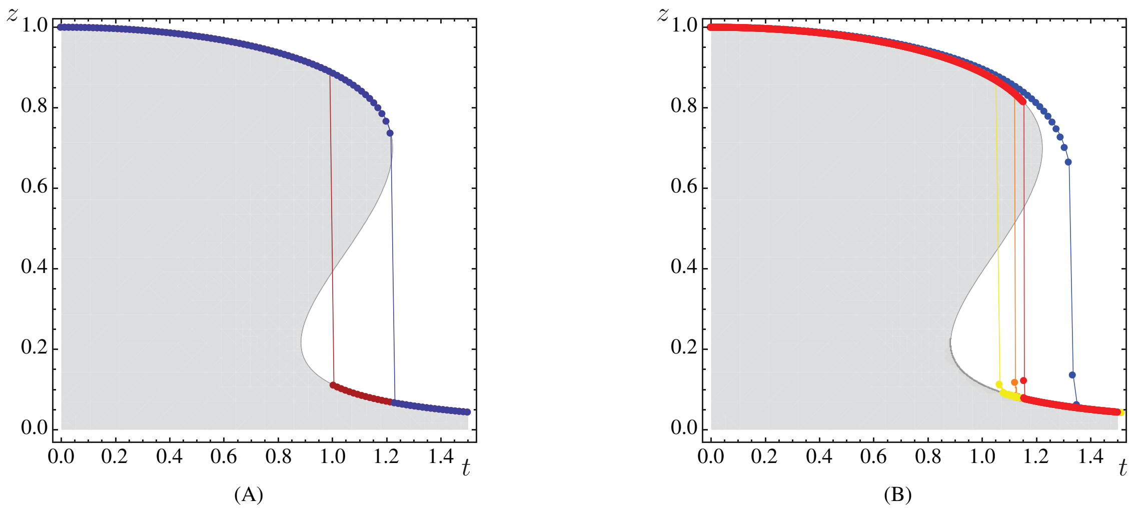


FIGURE 2. *Left*: global energetic (dark red) and BV-solution (blue); *Right*: solutions generated by the viscosity scheme (1.3) with parameters from Table 1.

TABLE 1. Discretisation parameters for the scheme (1.3) with $\tau = T/N$.

	N	μ
Blue	100	$\mu = \sqrt{\tau}$
Yellow	100	$\mu = 0.1\sqrt{\tau}$
Orange	500	$\mu = 0.1\sqrt{\tau}$
Red	1000	$\mu = 0.1\sqrt{\tau}$

maximum of 127 minimisation steps for $k = 86$). Observe that for this choice of η for all $t \in [0, T]$ the function $z \mapsto \mathcal{I}(t, z) + \mathcal{R}(z - v) + \frac{\eta}{2} |z - v|^2$ is uniformly convex on $[0, 1]$ (for arbitrary v).

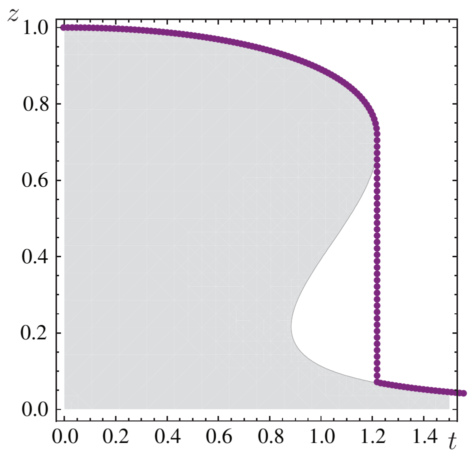
Visco-energetic solutions: Visco-energetic solutions are obtained as limit $\tau \to 0$ in (1.3) for fixed ratio μ/τ, [22]. Figure 5a shows the convergence for $\mu/\tau = 0.5$, while Figure 5b shows the convergence for $\mu/\tau = 10$. In both cases $\tau = T/N$ with $N \in \{100, 500, 1500, 3000\}$.

5.3. Application to a rate-independent ferroelectric model

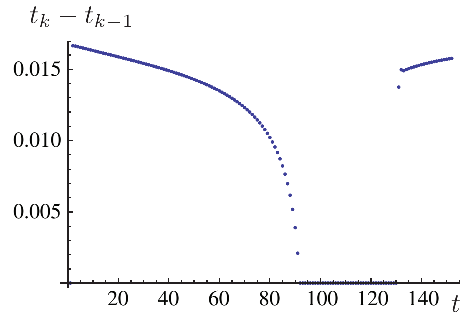
Ferroelectric ceramics exhibit coupled electrical and mechanical responses: mechanical deformations of such a material induce an electric field and vice versa. Furthermore, they show a hysteretic behaviour since the polarisation and the spontaneous eigenstrains might change provided the applied electrical or mechanical loads are large enough. The model discussed in this section is a rate-independent version of the phase field model from [24]. General rate-independent models for ferroelectric material behaviour were analysed in [16] in the global energetic framework.

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be a bounded domain with Lipschitz boundary. The following variables are used for the modelling: The displacement field $u : \Omega \to \mathbb{R}^d$, the strain field $e(u) = \text{sym}(\nabla u)$, the electric potential $\phi : \Omega \to \mathbb{R}$, the electric field $E = -\nabla\phi$, the electric displacement $D : \Omega \to \mathbb{R}^d$ and the spontaneous polarisation $P : \Omega \to \mathbb{R}^d$. In [24], the free energy density associated with this system is given as

$$\Psi(e(u), D, P, \nabla P) := \Psi_{\text{bulk}}(e(u), D, P) + \Psi_{\text{sep}}(P) + \Psi_{\text{grad}}(\nabla P), \tag{5.2}$$

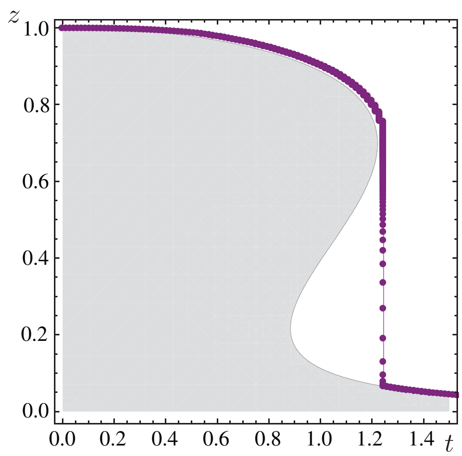


(A)

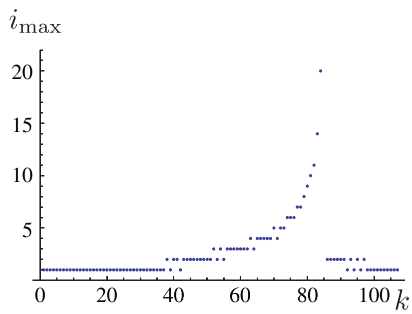


(B)

FIGURE 3. *Left*: solution generated with the local minimisation schemes (2.1)–(2.2) for $h = T/90$; *Right*: time-increment in each minimisation step.



(A)



(B)

FIGURE 4. *Left*: solution generated with the relaxed local minimisation scheme (3.2) combined with (4.8) for $N = 100$, $\eta = 100$, $\delta = 10^{-3}$; *Right*: number of iterations $i_{\max}$ in each time step.

where

$$\Psi_{\text{bulk}}(e(u), D, P) = \frac{1}{2}\left\langle \begin{pmatrix} \mathbb{C} + e^T\epsilon^{-1}e & -e^T\epsilon^{-1} \\ -\epsilon^{-T}e & \epsilon^{-1} \end{pmatrix} \begin{pmatrix} e(u) - \varepsilon^0 \\ D - P \end{pmatrix}, \begin{pmatrix} e(u) - \varepsilon^0 \\ D - P \end{pmatrix} \right\rangle , \tag{5.3}$$

$$\Psi_{\text{grad}}(\nabla P) = \frac{\kappa}{2}\left|\nabla P\right|^2 . \tag{5.4}$$

In [24], Ψ_{sep} is a nonconvex sixth order polynomial in P that is bounded from below. In the case $d = 2$ this polynomial fits to our assumptions. However, in the three-dimensional case, we will formulate more restrictive assumptions on Ψ_{sep}, see (5.5b) here below. In general, the material parameters $\mathbb{C}$ (elasticity tensor), e (piezo-electric tensor), ϵ (dielectric tensor) and the eigenstrain ε^0 depend on the polarisation P and explicit expressions can again be found in [24]. However, in order to apply the results from the previous sections directly, we here make the simplifying assumption that these quantities do not depend on P. It is the topic of a forthcoming

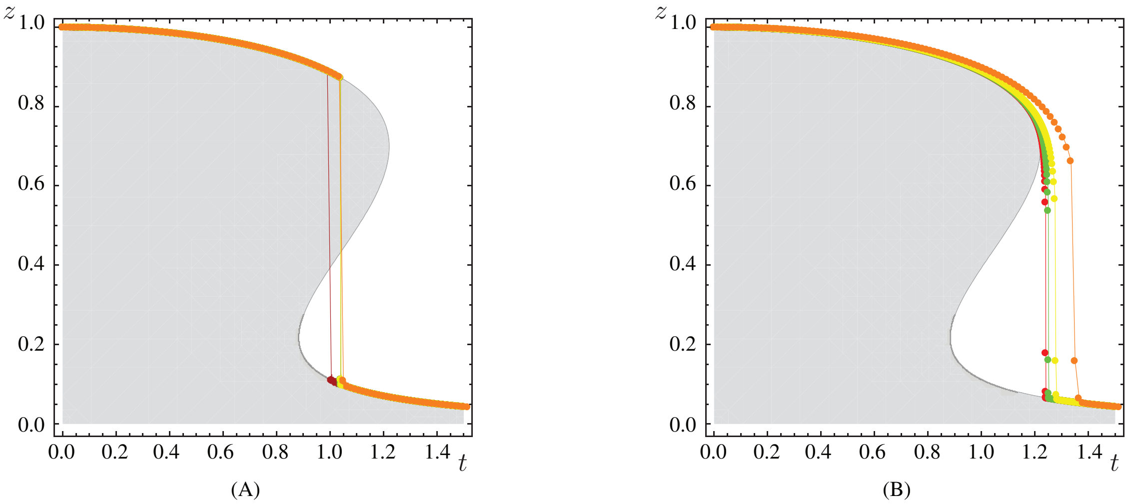


FIGURE 5. *Left*: visco-energetic solutions for $\mu/\tau = 0.5$ and different values for τ (for comparison the global energetic solution is plotted in dark red); *Right*: visco-energetic solutions for $\mu/\tau = 10$.

paper to include also P-dependent coefficients. To be more precise, we assume that

$$\mathbb{C} \in L^\infty\Big(\Omega; \mathrm{Lin}\Big(\mathbb{R}^{d\times d}_{\mathrm{sym}}, \mathbb{R}^{d\times d}_{\mathrm{sym}}\Big)\Big),\ e \in L^\infty\Big(\Omega; \mathrm{Lin}\Big(\mathbb{R}^{d\times d}, \mathbb{R}^d\Big)\Big),$$
$$\epsilon \in L^\infty\Big(\Omega; \mathrm{Lin}\Big(\mathbb{R}^{d}, \mathbb{R}^d\Big)\Big),\ \varepsilon^0 \in L^2\Big(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}}\Big),\ \kappa \in L^\infty\Big(\Omega; \mathbb{R}\Big)$$

with $\kappa(x) \ge \kappa_0 > 0$ almost everywhere in Ω. Moreover, the tensor fields $\mathbb{C}$ and ϵ shall be uniformly positive definite, *i.e.* there exists a constant $\alpha > 0$ such that for almost all $x \in \Omega$ we have

$$\Big\langle \mathbb{C}(x)\xi, \xi \Big\rangle \ge \alpha\,|\xi|^2 \text{ for all } \xi \in \mathbb{R}^{d\times d}_{\mathrm{sym}}, \qquad (\epsilon(x)v)\cdot v \ge \alpha\,|v|^2 \text{ for all } v \in \mathbb{R}^d.$$

Finally, we assume that $\Psi_{\mathrm{sep}} \in C^2(\mathbb{R}^d, \mathbb{R})$ has the following coercivity and growth properties

$$\exists \delta > 0, c_0 \in \mathbb{R}\, \forall P \in \mathbb{R}^d: \qquad \Psi_{\mathrm{sep}}(P) \ge \delta\,|P|^2 - c_0, \tag{5.5a}$$

$$\exists c_1 > 0\, \forall P \in \mathbb{R}^d: \qquad \left|\mathrm{D}^2_P \Psi_{\mathrm{sep}}(P)\right| \le c_1(1 + |P|^{r-2}) \quad \text{with } r \in \begin{cases} [1,\infty) & \text{if } d = 2 \\ [1,4] & \text{if } d = 3 \end{cases}. \tag{5.5b}$$

For simplicity we assume vanishing Dirichlet boundary conditions on $\partial\Omega$ for u and ϕ. This leads to the following choice for the function spaces:

$$\mathcal{U} := H^1_0(\Omega; \mathbb{R}^d) \times L^2_D(\Omega, \mathbb{R}^d), \quad \mathcal{Z} := H^1(\Omega, \mathbb{R}^d), \quad \mathcal{V} := L^2(\Omega, \mathbb{R}^d), \quad \mathcal{X} = L^1(\Omega; \mathbb{R}^d)$$

where $L^2_D(\Omega, \mathbb{R}^d) := \{\, D \in L^2(\Omega, \mathbb{R}^d)\,;\, \forall \phi \in H^1_0(\Omega, \mathbb{R})\ \int_\Omega D \cdot \nabla\phi \,\mathrm{d}x = 0\,\}$, equipped with the L^2-norm. For $(u, D) \in \mathcal{U}$, $P \in \mathcal{Z}$ and $\ell \in C^1([0,T], (\mathcal{U}^* \times \mathcal{V}^*))$ the energy functional $\mathcal{E} : [0,T] \times \mathcal{U} \times \mathcal{Z} \to \mathbb{R}$ takes the form

$$\mathcal{E}(t, u, D, P) := \int_\Omega \Psi_{\mathrm{bulk}}(e(u(x)), D(x), P(x)) + \Psi_{\mathrm{sep}}(P(x)) + \Psi_{\mathrm{grad}}(\nabla P(x))\,\mathrm{d}x - \langle \ell(t), (u, D, P)^T \rangle,$$

while the dissipation potential $\mathcal{R} : \mathcal{X} \to [0, \infty)$ is given by

$$\mathcal{R}(v) = \gamma \|v\|_{L^1(\Omega)} ,$$

with a constant $\gamma > 0$. The ferroelectric model reads: Find $(u, D) : [0, T] \to \mathcal{U}$ and $P : [0, T] \to \mathcal{Z}$ with $P(0) = P_0 \in \mathcal{Z}$ and

$$\begin{aligned} &0 = \mathrm{D}_u\mathcal{E}(t, u(t), D(t), P(t)), \quad 0 = \mathrm{D}_D\mathcal{E}(t, u(t), D(t), P(t)), \\ &0 \in \partial\mathcal{R}(\dot{P}(t)) + \mathrm{D}_P\mathcal{E}(t, u(t), D(t), P(t)) . \end{aligned}$$

Clearly, the assumptions (3.1a) and (3.1d) are satisfied. Moreover, for all $(u, D) \in \mathcal{U}$, $P \in \mathcal{Z}$, the quadratic part of $\mathcal{E}$ satisfies

$$\int_\Omega \Psi_{\mathrm{bulk}}(e(u)), D, P) + \Psi_{\mathrm{grad}}(\nabla P)\,\mathrm{d}x + \delta \|P\|^2_{L^2(\Omega)} \geq \beta\Big(\|(u, D)\|^2_{\mathcal{U}} + \|P\|^2_{\mathcal{Z}} - \left\|\varepsilon^0\right\|^2_{L^2(\Omega)}\Big) \tag{5.6}$$

with $\delta > 0$ from (5.5a) and $\beta > 0$ is a constant that is independent of (u, D, P). This estimate follows from the positivity assumption on the material tensors, Korn's inequality and after applying Young's inequality several times. This implies (4.2) (to be more precise, one should interpret terms involving ε^0 as parts of the loads ℓ). Let $\mathcal{F}(P) := \int_\Omega \Psi_{\mathrm{sep}}(P) - \delta |P|^2\,\mathrm{d}x$. Thanks to (5.5a)–(5.5b) and the embedding theorems for Sobolev spaces, $\mathcal{F}$ satisfies (1.13), (1.18), (1.21) and (3.1c). Hence, discrete solutions of this ferroelectric model generated by any of the schemes presented in the previous sections converge to solutions of BV-type. In particular, the alternate minimisation scheme discussed in Section 4 can be applied to approximate BV-solutions of the ferroelectric model.

Appendix A. Identities relying on convex analysis

With the assumptions and definitions introduced in Section 1.1 for $h > 0$, $v \in \mathcal{V}$ we define

$$\Psi_h(v) := \mathcal{R}(v) + I_h(v), \tag{A.1}$$

where $I_h(v) = 0$ if $\langle \mathbb{V}v, v\rangle \leq h^2$ and $I_h(v) = \infty$ otherwise. We denote by $\partial^{\mathcal{Z}}\Psi_h$ and Ψ_h^{*z} the subdifferential and the conjugate functional of Ψ_h with respect to the $\mathcal{Z} - \mathcal{Z}^*$-duality and by $\partial\Psi_h$ and Ψ_h^* the subdifferential and the conjugate functional with respect to the $\mathcal{V} - \mathcal{V}^*$-duality.

Lemma A.1. *Assume (1.11), (1.12) and (1.17). For every $z \in \mathcal{Z}$, $\eta \in \mathcal{V}^*$ we have*

$$\partial^{\mathcal{Z}}\Psi_h(z) \subset \mathcal{V}^*, \quad \partial^{\mathcal{Z}}\Psi_h(z) = \partial\Psi_h(z) , \tag{A.2}$$

$$\Psi_h^{*z}(\eta) = \Psi_h^*(\eta) = h\Psi_1^*(\eta) = h\,\mathrm{dist}_{\mathcal{V}^*}\big(\eta, \partial\mathcal{R}(0)\big) , \tag{A.3}$$

where $\mathrm{dist}_{\mathcal{V}^*}(\eta, \partial\mathcal{R}(0)\big) = \inf\{\, \|\eta - \sigma\|_{\mathbb{V}^{-1}} \,;\, \sigma \in \partial\mathcal{R}(0)\,\}$ and $\|\eta\|^2_{\mathbb{V}^{-1}} = \langle \mathbb{V}^{-1}\eta, \eta\rangle$. Furthermore, for $h > 0$ $\partial\Psi_h(0) = \partial\mathcal{R}(0)$ and $\partial\mathcal{R}(0)$ is bounded in $\mathcal{V}^*$. Moreover, for $v \in \mathcal{V}$, $\xi \in \mathcal{V}^*$ we have

$$\xi \in \partial I_h(v) \quad\Leftrightarrow\quad \|v\|_{\mathbb{V}} \leq h \text{ and } \exists\mu \geq 0 \text{ with } \mu(\|v\|_{\mathbb{V}} - h) = 0 \text{ and } \xi = \mu\mathbb{V}v . \tag{A.4}$$

Proof. In order to verify (A.2) observe first that by the sum rule for subdifferentials, [5], for all $z \in \mathcal{Z}$ we have $\partial^{\mathcal{Z}}\Psi_h(z) = \partial^{\mathcal{Z}}\mathcal{R}(z) + \partial^{\mathcal{Z}}I_h(z)$, and we discuss the terms on the right hand side separately. Since $\mathcal{R}$ is positively homogeneous of degree one we have $\partial^{\mathcal{Z}}\mathcal{R}(z) \subset \partial^{\mathcal{Z}}\mathcal{R}(0)$ for all $z \in \mathcal{Z}$. The upper bound (1.17) implies the estimate

$\langle \eta, z\rangle \leq \mathcal{R}(z) \leq C\,\|z\|_{\mathcal{X}} \leq \widetilde{C}\,\|z\|_{\mathcal{V}}$ that is valid for all $\eta \in \partial^{\mathcal{Z}}\mathcal{R}(0)$ and $z \in \mathcal{Z}$. Since $\mathcal{Z}$ is dense in $\mathcal{V}$ this estimate shows that η can be extended in a unique way to an element from $\mathcal{V}^*$ and thus $\partial^{\mathcal{Z}}\mathcal{R}(0) \subset \mathcal{V}^*$. Observe that

$$I_h^{*z}(\xi) = \begin{cases} I_h^*(\xi) & \text{if } \xi \in \mathcal{V}^* \\ \infty & \text{if } v \in \mathcal{Z}^* \backslash \mathcal{V}^* \end{cases} \tag{A.5}$$

with $I_h^*(\xi) = h\sqrt{\langle \xi, \mathbb{V}^{-1}\xi\rangle}$ for $\xi \in \mathcal{V}^*$. This can be seen as follows: The expression for I_h^* (conjugate functional of I_h with respect to $\mathcal{V} - \mathcal{V}^*$) follows by direct calculations. In order to determine $I_h^{*z}(\xi)$ let first $\xi \in \mathcal{Z}^*$ with $I_h^{*z}(\xi) = \sup\{\, \langle \xi, z\rangle\,;\, z \in \mathcal{Z},\, \langle \mathbb{V}z, z\rangle_{\mathcal{V}^*,\mathcal{V}} \leq h^2 \,\} =: c < \infty$. Then for all $z \in B_{\mathcal{Z}} := \{\, z \in \mathcal{Z}\,;\, \langle \mathbb{V}z, z\rangle_{\mathcal{V}^*,\mathcal{V}} \leq h^2 \,\}$ we have $|\langle \xi, z\rangle| \leq c$ which due to the density of $B_{\mathcal{Z}}$ in $B_{\mathcal{V}} := \{\, v \in \mathcal{V}\,;\, \langle \mathbb{V}v, v\rangle_{\mathcal{V}^*,\mathcal{V}} \leq h^2 \,\}$ implies that $\xi \in \mathcal{V}^*$ and $I_h^*(\xi) = I_h^{*z}(\xi)$. With the same argument we obtain that $I_h^{*z}(\xi) = I_h^*(\xi)$ for arbitrary $\xi \in \mathcal{V}^*$ and (A.5) is proved. Since $\operatorname{dom}(I_h^{*z}) \subset \mathcal{V}^*$, from the generalized Young inequality we conclude that $\partial^{\mathcal{Z}} I_h(z) \subset \mathcal{V}^*$ for all $z \in \mathcal{Z}$. This proves the first claim in (A.2). The second claim in (A.2) now is an immediate consequence. In a similar way the first identity in (A.3) follows. The last identity in (A.3) is a consequence of the inf-convolution formula and general properties of one-homogeneous functionals, *cf.* [5]. □

Appendix B. Lower semicontinuity properties

The following Proposition is a slight variant of ([11], Lem. 3.1).

Proposition B.1. *Let $v_n, v \in L^\infty(0,S;\mathcal{V})$ with $v_n \overset{*}{\rightharpoonup} v$ in $L^\infty(0,S;\mathcal{V})$ and $\delta_n, \delta \in L^1(0,S;[0,\infty))$ with $\liminf_{n\to\infty} \delta_n(s) \geq \delta(s)$ for almost all s. Then*

$$\liminf_{n\to\infty} \int_0^S \|v_n(s)\|_{\mathcal{V}}\, \delta_n(s)\,\mathrm{d}s \geq \int_0^S \|v(s)\|_{\mathcal{V}}\, \delta(s)\,\mathrm{d}s. \tag{B.1}$$

Proof. The proposition can be proved in exactly the same way as ([11], Lem. 3.1). Indeed, assume first that $\delta_n \to \delta$ strongly in $L^1(0,S)$. Since for every fixed $\widetilde{\delta} \in L^1(0,S;[0,\infty))$ the mapping $v \mapsto \int_0^S \|v\|_{\mathcal{V}}\, \widetilde{\delta}\,\mathrm{d}s$ is convex and lower semicontinuous on $L^\infty(0,S;\mathcal{V})$ a generalized version of Ioffe's theorem (see [25], Thm. 21) yields (B.1) for this case. For the general case fix $k > 0$ and define $\delta_{n,k}(s) := \min\{\delta_n(s), \delta(s), k\}$. Observe that $\delta_{n,k} \to \delta_k := \min\{\delta, k\}$ strongly in $L^1(0,S)$. Hence,

$$\liminf_{n\to\infty} \int_0^S \|v_n(s)\|_{\mathcal{V}}\, \delta_n(s)\,\mathrm{d}s \geq \liminf_{n\to\infty} \int_0^S \|v_n(s)\|_{\mathcal{V}}\, \delta_{n,k}(s)\,\mathrm{d}s \geq \int_0^S \|v(s)\|_{\mathcal{V}}\, \delta_k(s)\,\mathrm{d}s$$

by the first step. The limit $k \to \infty$ finally implies (B.1). □

The next lemma that we cite from ([13], Lem. 4.3) is closely related to the previous proposition:

Lemma B.2. *Let $I \subset \mathbb{R}$ be a bounded interval and $f, g, f_n, g_n : I \to [0,\infty)$, $n \in \mathbb{N}$, measurable functions satisfying $\liminf_{n\to\infty} f_n(s) \geq f(s)$ for a.a. $s \in I$ and $g_n \rightharpoonup g$ weakly in $L^1(I)$. Then*

$$\liminf_{n\to\infty} \int_I f_n(s) g_n(s)\,\mathrm{d}s \geq \int_I f(s) g(s)\,\mathrm{d}s\,.$$

References

[1] M. Artina, F. Cagnetti, M. Fornasier and F. Solombrino, Linearly constrained evolutions of critical points and an application to cohesive fractures. *Math. Models Methods Appl. Sci.* **27** (2017) 231–290.

[2] B. Bourdin, G. Francfort and J.-J. Marigo, Numerical experiments in revisited brittle fracture. *Mech. Phys. Solids* **48** (2000) 797–826.

[3] J. Dieudonné, Foundations of Modern Analysis. Enlarged and Corrected Printing. Academic Press, New York-London (1969), 387.

[4] M.A. Efendiev and A. Mielke, On the rate-independent limit of systems with dry friction and small viscosity. *J. Convex Anal.* **13** (2006) 151–167.

[5] A. Ioffe and V. Tikhomirov, Theorie der Extremalaufgaben. Übersetzung aus dem Russischen von Bernd Luderer. VEB Deutscher Verlag der Wissenschaften, Berlin (1979).

[6] D. Knees and M. Negri, Convergence of alternate minimization schemes for phase-field fracture and damage. *Math. Models Methods Appl. Sci.* **27** (2017) 1743–1794.

[7] D. Knees, R. Rossi and C. Zanini, A vanishing viscosity approach to a rate-independent damage model. *Math. Models Methods Appl. Sci.* **23** (2013) 565–616.

[8] D. Knees and A. Schröder, Computational aspects of quasi-static crack propagation. *Discrete Contin. Dyn. Syst. Ser. S* **6** (2013) 63–99.

[9] A. Mainik and A. Mielke, Existence results for energetic models for rate-independent systems. *Calc. Var. Partial Differ. Equ.* **22** (2005) 73–99.

[10] A. Mielke, Evolution of rate-independent systems. Vol II of *Handbook of Differential Equations: Evolutionary Equations.* Elsevier/North-Holland, Amsterdam (2005) 461–559.

[11] A. Mielke, R. Rossi and G. Savaré, Modeling solutions with jumps for rate-independent systems on metric spaces. *Discrete Contin. Dyn. Syst.* **25** (2009) 585–615.

[12] A. Mielke, R. Rossi and G. Savaré, BV solutions and viscosity approximations of rate-independent systems. *ESAIM: COCV* **18** (2012) 36–80.

[13] A. Mielke, R. Rossi and G. Savaré, Variational convergence of gradient flows and rate-independent evolutions in metric spaces. *Milan J. Math.* **80** (2012) 381–410.

[14] A. Mielke, R. Rossi and G. Savaré, Balanced viscosity (BV) solutions to infinite-dimensional rate-independent systems. *J. Eur. Math. Soc. (JEMS)* **18** (2016) 2107–2165.

[15] A. Mielke and T. Roubíček, Rate-Independent Systems: Theory and Application. Springer, New York, NY (2015).

[16] A. Mielke and A.M. Timofte, An energetic material model for time-dependent ferroelectric behaviour: existence and uniqueness. *Math. Methods Appl. Sci.* **29** (2006) 1393–1410.

[17] A. Mielke and S. Zelik, On the vanishing-viscosity limit in parabolic systems with rate-independent dissipation terms. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **13** (2014) 67–135.

[18] M.N. Minh, *Weak solutions to rate-independent systems: existence and regularity.* Ph.D. thesis, Università di Pisa (2012).

[19] L. Minotti and G. Savaré, Viscous corrections of the time incremental minimization scheme and visco-energetic solutions to rate-independent evolution problems. *Arch. Ration. Mech. Anal.* **227** (2018) 477–543.

[20] M. Negri, Quasi-static rate-independent evolutions: characterization, existence, approximation and application to fracture mechanics. *ESAIM: COCV* **20** (2014) 983–1008.

[21] M. Negri and R. Scala, A quasi-static evolution generated by local energy minimizers for an elastic material with a cohesive interface. *Nonlinear Anal. Real World Appl.* **38** (2017) 271–305.

[22] R. Rossi and G. Savaré, From visco-energetic to energetic and balanced viscosity solutions of rate-independent systems, in Solvability, Regularity, and Optimal Control of Boundary Value Problems for PDEs: In Honour of Prof. Gianni Gilardi. Springer, Cham (2017) 489–531.

[23] T. Roubíček, M. Thomas and C. Panagiotopoulos, Stress-driven local-solution approach to quasistatic brittle delamination. *Nonlinear Anal. Real World Appl.* **22** (2015) 645–663.

[24] D. Schrade, M.-A. Keip, H. Thai, J. Schröder, B. Svendsen, R. Müller and D. Gross, Coordinate-invariant phase field modeling of ferroelectrics, part I: model formulation and single-crystal simulations. *GAMM-Mitt.* **38** (2015) 102–114.

[25] M. Valadier, Young measures, in Methods of Nonconvex Analysis (Varenna, 1989.) Vol. 1446 of *Lecture Notes in Mathematics.* Springer, Berlin (1990) 152–188.

[26] J. Wloka, Partial Differential Equations. Cambridge University Press, Cambridge (1987).