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STABILITY OF ERROR BOUNDS FOR CONIC SUBSMOOTH INEQUALITIES*

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Abstract. Under either linearity or convexity assumption, several authors have studied the stability of error bounds for inequality systems when the concerned data undergo small perturbations. In this paper, we consider the corresponding issue for a more general conic inequality (most of the constraint systems in optimization can be described by an inequality of this type). In terms of coderivatives for vector-valued functions, we study perturbation analysis of error bounds for conic inequalities in the subsmooth setting. The main results of this paper are new even in the convex/smooth case.

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1. Introduction

Let X be a Banach space and $f: X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function with $\bar{x} \in \mathrm{dom}(f)$, the domain of f. Recall that f has a local error bound at $\bar{x}$ if there exist $\tau, \delta \in (0, +\infty)$ such that

$$d(x, S(f, \bar{x})) \leq \tau[f(x) - f(\bar{x})]_+ \quad \forall x \in B(\bar{x}, \delta), \tag{1.1}$$

where $S(f, \bar{x}) := \{x \in X : f(x) \leq f(\bar{x})\}$ is the sublevel set of f at $\bar{x}$ and $B(\bar{x}, \delta)$ is the open ball with center $\bar{x}$ and radius δ. Error bound theory has been recognized to be significant in sensitivity analysis and convergence analysis of some algorithms for solving optimization problems. Since Hoffman's pioneering work [8], a great deal of works have been reported in mathematical programming literature discussing the error bound issues (for details see [2, 6, 9, 14, 15, 19, 21, 22, 24–26] and references therein).

In practical problems, all data obtained are not perfectly ideal, one is forced to use "test" data, and the gap between the ideal and "test" data is unavoidable. Therefore, from the points of view of theoretical interest as well as for applications, it is important to study the issue of stability when the system data undergo small perturbations; error bound issue is of no exception. In 1994, Luo and Tseng [16] first studied the effect on error bounds when linear inequalities (in Euclidean spaces) are perturbed. In 2005, Zheng and Ng [28] established the stability results on error bounds for systems of conic linear inequalities in general Banach spaces. In 2010,

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relaxing the linearity restriction and in terms of the subdifferential, Ngai *et al.* [20] established the stability of error bounds for convex inequalities; this stability result was further extended in [31] to the subsmooth case from the convex case.

Let F be a proper vector-valued function from a Banach space X to another Banach space Y ordered by a closed convex cone C and consider the following conic inequality

$$F(x) \leq_C F(\bar{x}), \tag{CIE}$$

where $\leq_C$ is the pre-order induced by C. Many constraints in optimization can be described in terms of conic inequalities. In this paper, we study the stability of error bounds for (CIE). Recall that (CIE) is said to have an error bound at $\bar{x}$ if there exist $\tau, \delta \in (0, +\infty)$ such that

$$d(x, S(F, \bar{x}, C)) \leq \tau d(F(x) - F(\bar{x}), -C) \quad \forall x \in B(\bar{x}, \delta), \tag{1.2}$$

where $S(F, \bar{x}, C)$ denotes the solution set of (CIE), that is,

$$S(F, \bar{x}, C) := \{x \in X : F(x) \leq_C F(\bar{x})\}.$$

We say that (CIE) has a strong error bound at $\bar{x}$ if there exist $\tau, \delta \in (0, +\infty)$ such that (1.2) holds and $\bar{x}$ is an isolated point of $S(F, \bar{x}, C)$, equivalently, there exist $\tau, \delta \in (0, +\infty)$ such that

$$\|x - \bar{x}\| \leq \tau d(F(x) - F(\bar{x}), -C) \quad \forall x \in B(\bar{x}, \delta). \tag{1.3}$$

In the case when $Y = \mathbb{R}$ and $C = \mathbb{R}_+$, (1.2) reduces to (1.1), while (1.3) reduces to

$$\|x - \bar{x}\| \leq \tau [F(x) - F(\bar{x})]_+ \quad \forall x \in B(\bar{x}, \delta).$$

It is known and easy to see that (CIE) has an error bound (resp. strong error bound) at $\bar{x}$ if and only if the multifunction $\mathcal{F}_C$ is metrically subregular (resp. strongly metrically subregular) at $(\bar{x}, F(\bar{x}))$, where $\mathcal{F}_C(x) := F(x) + C$. The notion of metric subregularity is useful in variational analysis and optimization and has been well studied (*cf.* [4, 9] and the references therein).

Let $\tau(F, \bar{x}, C)$ denote the modulus of the error bound of conic inequality (CIE), that is,

$$\tau(F, \bar{x}, C) := \inf\{\tau > 0 : \text{(1.2) holds for some } \delta > 0\}. \tag{1.4}$$

Clearly, F has an error bound at $\bar{x}$ if and only if $\tau(F, \bar{x}, C) < +\infty$. Our main concern of this paper is to consider the stability of error bound for conic inequality (CIE) when the objective function F undergoes small perturbations: (CIE) *is said to have a stable error bound at* $\bar{x}$ *if there exist* $\eta, r \in (0, +\infty)$ *such that* $\tau(G, \bar{x}, C) \leq \eta$ *for each function* $G : X \to Y^\bullet := Y \cup \{\infty_Y\}$ *with*

$$\|G - F\|_{\bar{x}} := \limsup_{x \to \bar{x}} \frac{\|G(x) - F(x) - (G(\bar{x}) - F(\bar{x}))\|}{\|x - \bar{x}\|} < r.$$

For convenience of presentation as well as printing, let $\partial_C F(\bar{x})$ denote $D^*\mathcal{F}_C(\bar{x}, F(\bar{x}))(\mathcal{I}_{C^+})$, where $D^*\mathcal{F}_C(\bar{x}, F(\bar{x}))$ is the coderivative of the multifunction $\mathcal{F}_C$ at $(\bar{x}, F(\bar{x}))$ (see Sect. 2 for the detail) and

$$\mathcal{I}_{C^+} := \{v^* \in C^+ : \|v^*\| = 1\} \text{ and } C^+ := \{v^* \in Y^* : \langle v^*, y \rangle \geq 0 \ \forall y \in C\}. \tag{1.5}$$

In Section 3, as a natural extension of the convexity and smoothness, we adopt the notion of subsmooth vector-valued functions, and provide examples and results on such subsmooth functions. In Section 4, two types of stability properties for error bounds of F at $\bar{x} \in \mathrm{dom}(F)$ are considered: one is that F has a stable strong error bound at $\bar{x}$, while the other is a weaker one that F has a stable error bound at $\bar{x}$ (the exact definitions are given in Sect. 4). When F is assumed to be subsmooth at $\bar{x}$, they are shown to be implied by the conditions $0 \in \mathrm{int}(\partial_C F(\bar{x}))$ and $0 \notin \mathrm{bd}(\partial_C F(\bar{x}))$, respectively. The converses are also established in Theorems 4.2, 4.7 and 4.8 under the additional assumption that F is locally Lipschitz at $\bar{x}$.

2. Preliminaries

Let X be a Banach space with topological dual X^* and let B_X denote the closed unit ball of X. We denote by $B(x, r)$ and $B[x, r]$ the open and closed balls with center x and radius r, respectively. For a subset A of X, we define the boundary $\mathrm{bd}(A)$ of A as $\mathrm{bd}(A) := \mathrm{cl}(A) \setminus \mathrm{int}(A)$, where $\mathrm{cl}(A)$ and $\mathrm{int}(A)$ denote the closure and interior of A, respectively. For $a \in A$, we use $T(A, a)$ and $N(A, a)$ to denote the Clarke tangent cone and the Clarke normal cone of A to a, respectively, defined by

$$T(A,a) := \Big\{ v \in X : \ \forall a_n \xrightarrow{A} a \text{ and } \forall t_n \to 0^+ \ \exists v_n \to v \text{ s.t. } a_n + t_n v_n \in A \ \forall n \in \mathbb{N} \Big\}$$

and

$$N(A,a) := \Big\{ x^* \in X^* : \ \langle x^*, h \rangle \le 0 \ \text{ for all } h \in T(A,a) \Big\}.$$

For $\varepsilon \ge 0$, let $\hat{N}_\varepsilon(A, a)$ denote the set of Fréchet ε-normals of A to a, that is,

$$\hat{N}_\varepsilon(A,a) := \left\{ x^* \in X^* : \ \limsup_{x \xrightarrow{A} a} \frac{\langle x^*, x - a \rangle}{\|x - a\|} \le \varepsilon \right\}.$$

When $\varepsilon = 0$, $\hat{N}_\varepsilon(A, a)$ is a convex cone which is called the Fréchet normal cone of A to a and is denoted by $\hat{N}(A, a)$. Let $\bar{N}(A, a)$ denote Mordukhovich's limiting normal cone of A to a which is defined by

$$\bar{N}(A,a) := \limsup_{x \xrightarrow{A} a, \varepsilon \to 0^+} \hat{N}_\varepsilon(A, x).$$

That is, $x^* \in \bar{N}(A, a)$ if and only if there exists a sequence $\{(x_n, \varepsilon_n, x_n^*)\}$ in $A \times R_+ \times X^*$ such that $(x_n, \varepsilon_n) \to (a, 0)$, $x_n^* \xrightarrow{w^*} x^*$ and $x_n^* \in \hat{N}_{\varepsilon_n}(A, x_n)$ for each n.

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function. We use $\mathrm{dom}(f)$ and $\mathrm{epi}(f)$ to denote the domain and the epigraph of f, respectively. The Clarke subdifferential $\partial f(x)$ of f at $x \in \mathrm{dom}(f)$ is defined as

$$\partial f(x) := \Big\{ x^* \in X^* : \ (x^*, -1) \in N(\mathrm{epi}(f), (x, f(x))) \Big\}. \tag{2.1}$$

If f is locally Lipschitz at x, then

$$\partial f(x) := \Big\{ x^* \in X^* : \ \langle x^*, h \rangle \le f^\circ(x, h) := \limsup_{z \to x, \, t \downarrow 0} \frac{f(z + th) - f(z)}{t} \ \ \forall h \in X \Big\}.$$

The Fréchet subdifferential $\hat{\partial} f(x)$ of f at x is defined by

$$\hat{\partial} f(x) := \left\{ x^* \in X^* : \ \liminf_{u \to x} \frac{f(u) - f(x) - \langle x^*, u - x \rangle}{\|u - x\|} \ge 0 \right\},$$

or equivalently

$$\hat{\partial} f(x) = \Big\{ x^* \in X^* :\ (x^*, -1) \in \hat{N}(\mathrm{epi}(f), (x, f(x))) \Big\}.$$

Let $\bar{\partial} f(x)$ denote Mordukhovich's limiting subdifferential of f at x, that is,

$$\bar{\partial} f(x) = \Big\{ x^* \in X^* :\ (x^*, -1) \in \bar{N}(\mathrm{epi}(f), (x, f(x))) \Big\}.$$

Recall that $\hat{\partial} f(x) \subset \bar{\partial} f(x) \subset \partial f(x)$ and that if f is convex then

$$\partial f(x) = \bar{\partial} f(x) = \hat{\partial} f(x) = \Big\{ x^* \in X^* :\ \langle x^*, h\rangle \le f(x+h) - f(x) \text{ for all } h \in X \Big\}.$$

It is known that if X is a separable Banach space such that its dual X^* is nonseparable (*e.g.*, $X = l^1$), then there exists a Lipschitz function f on X such that $\hat{\partial} f(x) = \bar{\partial} f(x) = \emptyset$ for all $x \in X$. Thus, it cannot be expected to develop Fréchet/limiting subdifferential theory in the general Banach space framework. Recall that a Banach space X is an Asplund space if every continuous convex function on X is Fréchet differentiable at every point of a dense subset of X. It is well known that X is an Asplund space if and only if every separable subspace of X has a separable dual space. With regard to these notions, the following two lemmas collect some useful results on the Clarke and Fréchet subdifferentials (and normal cones) (*cf.* [3, 17]).

Lemma 2.1. *Let X be a Banach space, $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function, and let $g : X \to \mathbb{R} \cup \{+\infty\}$ be locally Lipschitz at $x \in \mathrm{dom}(f)$. Then $\partial g(x) \neq \emptyset$ and $\partial(f + g)(x) \subset \partial f(x) + \partial g(x)$.*

Lemma 2.2. *Let X be an Asplund space, A be a closed subset of X, and let $f, g : X \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous functions. Then the following statements hold:*
(i) $\bar{N}(A, a) = \limsup\limits_{x \xrightarrow{A} a} \hat{N}(A, x)$ for all $a \in A$ and $N(A, a)$ is the weak* closed convex hull of $\bar{N}(A, a)$.
(ii) $\bar{\partial} f(x) = \limsup\limits_{z \xrightarrow{f} x} \hat{\partial} f(z)$ for all $x \in \mathrm{dom}(f)$.
(iii) If f is locally Lipschitz at x, then $\partial f(x)$ is the weak* closed convex hull of $\bar{\partial} f(x)$, that is, $\partial f(x) = \overline{\mathrm{co}}^{w^*}(\bar{\partial} f(x))$.
(iv) If f is locally Lipschitz at $x \in \mathrm{dom}(g)$ and $x^* \in \hat{\partial}(f + g)(x)$, then for any $\varepsilon > 0$ there exist $x_1, x_2 \in B(x, \varepsilon)$ with $|g(x_2) - g(x)| < \varepsilon$ such that $x^* \in \hat{\partial} f(x_1) + \hat{\partial} g(x_2) + \varepsilon B_{X^*}$.

Let X, Y be Banach spaces, and $C \subset Y$ be a closed convex cone, which defines a pre-order $\le_C$ in Y as follows: $y_1 \le_C y_2 \Leftrightarrow y_2 - y_1 \in C$. It is known that $\le_C$ is a partial order in Y if and only if the closed convex cone C is pointed (*i.e.*, $C \cap -C = \{0\}$). In the remainder of this paper, we always assume that C is a closed convex pointed cone in Y. Let C^+ and $\mathcal{I}_{C^+}$ be defined by (1.5). Note that, in our consideration, $\mathcal{I}_{C^+}$ will play a role of an "abstract one"; indeed, in the special case that $Y = \mathbb{R}$ and $C = \mathbb{R}_+$, one clearly has $\mathcal{I}_{C^+} = \{1\}$. Let ∞_Y denote an abstract infinity and $Y^\bullet := Y \cup \{\infty_Y\}$. For a vector-valued function $F : X \to Y^\bullet$, the epigraph of F with respect to the ordering cone C is defined by

$$\mathrm{epi}_C(F) := \Big\{ (x, y) \in X \times Y :\ F(x) \le_C y \Big\}.$$

Recall that F is C-convex if $\mathrm{epi}_C(F)$ is a convex subset of $X \times Y$; thus, F is C-convex if and only if

$$F(\lambda x_1 + (1-\lambda)x_2) \le_C \lambda F(x_1) + (1-\lambda) F(x_2) \quad \forall x_1, x_2 \in \mathrm{dom}(F) \text{ and } \lambda \in [0,\ 1],$$

where $\mathrm{dom}(F) := \{x \in X : \ F(x) \neq \infty_Y\}$. The following lemma regarding the epigraph $\mathrm{epi}_C(F)$ is useful for us.

Lemma 2.3. *Let X, Y be Banach spaces and $F : X \to Y^\bullet$ be a function such that $\mathrm{epi}_C(F)$ is closed. Let $x \in \mathrm{dom}(F)$ and $c \in C$. The following statements hold:*
(i) $\hat{N}(\mathrm{epi}_C(F), (x, F(x) + c)) \subset X^* \times -C^+$.
(ii) If, in addition, F is continuous at x, then

$$T(\mathrm{epi}_C(F), (x, F(x))) + \{0\} \times C = T(\mathrm{epi}_C(F), (x, F(x))) \subset T(\mathrm{epi}_C(F), (x, F(x) + c)) \tag{2.2}$$

and

$$N(\mathrm{epi}_C(F), (x, F(x) + c)) \subset N(\mathrm{epi}_C(F), (x, F(x))) \subset X^* \times -C^+. \tag{2.3}$$

Proof. Let $(x^*, y^*) \in \hat{N}(\mathrm{epi}_C(F), (x, F(x) + c))$. Then, for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\Big\langle x^*, x' - x \Big\rangle + \Big\langle y^*, y' - F(x) - c \Big\rangle \leq \varepsilon \Big(\|x' - x\| + \|y' - F(x) - c\| \Big).$$

for all $(x', y') \in \mathrm{epi}_C(F)$ with $\|x' - x\| + \|y' - F(x) - c\| < \delta$. Noting that $\{x\} \times (F(x) + c + C) \subset \mathrm{epi}_C(F)$, it follows that $\langle y^*, h \rangle \leq \varepsilon \|h\|$ for all $h \in C \cap B_Y(0, \delta)$. Since C is a closed convex set and ε is an arbitrary positive number, $\langle y^*, h \rangle \leq 0$ for all $h \in C$. Hence $y^* \in -C^+$. This shows that (i) holds.

To prove (ii), suppose that F is continuous at x. Since (2.3) follows easily from (2.2) and the definitions concerned, we only need to prove (2.2). To do this, let $(u, v) \in T(\mathrm{epi}_C(F), (x, F(x)))$. Then, for any $(x_n, y_n) \stackrel{\mathrm{epi}_C(F)}{\longrightarrow} (x, F(x))$ and $t_n \to 0^+$ there exists a sequence $\{(u_n, v_n)\}$ such that $(u_n, v_n) \to (u, v)$ and

$$(x_n, y_n) + t_n(u_n, v_n) \in \mathrm{epi}_C(F) \qquad \forall n \in \mathbb{N},$$

and so

$$(x_n, y_n) + t_n(u_n, v_n + c) \in \mathrm{epi}_C(F) \qquad \forall (n, c) \in \mathbb{N} \times C.$$

Therefore, $(u, v + c) \in T(\mathrm{epi}_C(F), (x, F(x)))$ for all $c \in C$ and the equality in (2.2) is shown (noting that $\{0\} \times C$ contains the origin). To show the inclusion in (2.2), let $(u, v) \in T(\mathrm{epi}_C(F), (x, F(x)))$, and take any $(x_n, y_n) \stackrel{\mathrm{epi}_C(F)}{\longrightarrow} (x, F(x) + c)$ and $t_n \to 0^+$. Then, noting that $(x_n, F(x_n)) \to (x, F(x))$ by the continuity of F at x, there exists a sequence $(u_n, v_n) \to (u, v)$ such that $(x_n, F(x_n)) + t_n(u_n, v_n) \in \mathrm{epi}_C(F)$ for all $n \in \mathbb{N}$, and so $(x_n, y_n) + t_n(u_n, v_n) \in \mathrm{epi}_C(F)$ for all $n \in \mathbb{N}$ (because $y_n - F(x_n) \in C$). This implies that $(u, v) \in T(\mathrm{epi}_C(F), (x, F(x) + c))$, and the inclusion in (2.2) is shown. The proof is complete. □

The following lemma, essentially known (*cf.* [17, 30]), is also useful for us.

Lemma 2.4. *Let X, Y, Z be Banach spaces and let $F : X \to Z$ and $G : Z \to Y^\bullet$ be a continuously differentiable function and a C-convex function, respectively. Suppose that $\bar{x} \in F^{-1}(\mathrm{dom}(G))$ satisfies the following Robinson qualification condition:*

$$F'(\bar{x})(X) + \mathbb{R}_+(\mathrm{dom}(G) - F(\bar{x})) = Z, \tag{2.4}$$

where $F'(\bar{x})$ denotes the derivative of F at $\bar{x}$. Then

$$N(\mathrm{epi}_C(G \circ F), (\bar{x}, G(F(\bar{x})))) = \Big\{ (F'(\bar{x})^*(z^*), y^*) : \ (z^*, y^*) \in N(\mathrm{epi}_C(G), (F(\bar{x}), G(F(\bar{x})))) \Big\}.$$

In the vector-valued setting, we adopt the following coderivative (*cf.* [17])

$$D_e^*F(x)(y^*) := \Big\{x^* \in X^* : \ (x^*, -y^*) \in N(\mathrm{epi}_C(F), (x, F(x)))\Big\} \quad \forall y^* \in Y^*. \tag{2.5}$$

Since the Clarke normal cone $N(\mathrm{epi}_C(F), (x, F(x)))$ is a weak*-closed convex cone, $D_e^*F(\bar{x})$ is a weak*-closed sublinear multifunction (and so a closed convex multifunction). Let $\mathcal{F}_C(x) := F(x) + C$ for all $x \in X$. Then $\mathrm{gph}(\mathcal{F}_C) = \mathrm{epi}_C(F)$ and so $D_e^*F(x)$ coincides with the coderivative $D^*\mathcal{F}_C(x, F(x))$ of the multifunction $\mathcal{F}_C$ at $(x, F(x))$. For convenience, we will use the following subdifferential of F:

$$\begin{aligned}\partial_C F(x) :&= D_e^*F(x)(\mathcal{I}_{C^+})\\ &= \Big\{x^* \in X^* : \ (x^*, -y^*) \in N(\mathrm{epi}_C(F), (x, F(x))) \text{ for some } y^* \in \mathcal{I}_{C^+}\Big\}\end{aligned} \tag{2.6}$$

(*cf.* [27]). It is easy to verify that $\partial_C F(x) = \bigcup\limits_{y^* \in \mathcal{I}_{C^+}} \partial(y^* \circ F)(x)$ if F is C-convex. In the special case when $Y = \mathbb{R}$ and $C = \mathbb{R}_+$, it is clear from (2.1) that $\partial_C F(x)$ reduces to the Clarke subdifferential of F and hence is weak*-closed. It is worth mentioning that $\partial_C F(x)$ is not necessarily closed when Y is a general Banach space. In the C-convex case, the following subdifferential is known:

$$\partial F(\bar{x}) := \Big\{T \in \mathcal{L}(X, Y) : \ T(x - \bar{x}) \leq_C F(x) - F(\bar{x}) \quad \forall x \in X\Big\},$$

where $\mathcal{L}(X, Y)$ denote the space of all continuous linear operators from X to Y. In fact, several kinds of subdifferentials for vector-valued functions have been introduced and studied (for the detail see [5, 7, 11] and the references therein).

Recall that a function $F : X \to Y^\bullet$ is locally Lipschitz at $\bar{x} \in \mathrm{dom}(F)$ if $\|F(x_1) - F(x_2)\| \leq L\|x_1 - x_2\|$ for all $x_1, x_2 \in B(\bar{x}, \delta)$ and some $L, \delta \in (0, +\infty)$. Next we adopt the following weaker notion: F is said to be locally C-Lipschitz at $\bar{x}$ if there exist $L, \delta \in (0, +\infty)$ such that F is continuous on $B(\bar{x}, \delta)$ and

$$|\langle y^*, F(x_1) - F(x_2)\rangle| \leq L\|x_1 - x_2\| \quad \forall x_1, x_2 \in B(\bar{x}, \delta) \text{ and } \forall y^* \in \mathcal{I}_{C^+}. \tag{2.7}$$

With notations given in (2.5) and (2.6), we present the following:

Lemma 2.5. *Let X, Y be Banach spaces with Y ordered by a closed convex cone C and let $F : X \to Y^\bullet$ be locally C-Lipschitz at $\bar{x} \in \mathrm{dom}(F)$. Then*

$$\partial(y^* \circ F)(\bar{x}) \subset D_e^*F(\bar{x})(y^*) \quad \forall y^* \in \mathcal{I}_{C^+}. \tag{2.8}$$

Consequently,

$$\bigcup_{y^* \in \mathcal{I}_{C^+}} \partial(y^* \circ F)(\bar{x}) \subset \partial_C F(\bar{x}). \tag{2.9}$$

Proof. Let $y^* \in \mathcal{I}_{C^+}$ and $x^* \in \partial(y^* \circ F)(\bar{x})$. We need to show that $x^* \in D_e^*F(\bar{x})(y^*)$, namely $(x^*, -y^*) \in N(\mathrm{epi}_C(F), (\bar{x}, F(\bar{x})))$. To do this, let

$$(u, v) \in T(\mathrm{epi}_C(F), (\bar{x}, F(\bar{x}))). \tag{2.10}$$

We have to show that

$$\langle x^*, u\rangle - \langle y^*, v\rangle \leq 0. \tag{2.11}$$

Since F is locally C-Lipschitz at $\bar{x}$, there exist $L, \delta \in (0,\ +\infty)$ such that F is continuous on $B(\bar{x}, \delta)$ and (2.7) holds. Hence there exists a sequences $\{(x_n, t_n)\}$ in $X \times (0,\ +\infty)$ convergent to $(\bar{x}, 0)$ such that

$$(y^* \circ F)^\circ(\bar{x}, u) = \lim_{n\to\infty} \frac{\langle y^*, F(x_n + t_n u) - F(x_n)\rangle}{t_n}. \tag{2.12}$$

Since $(x_n, F(x_n)) \to (\bar{x}, F(\bar{x}))$, (2.10) implies that there exists a sequence $\{(u_n, v_n)\}$ in $X \times Y$ convergent to (u, v) such that $(x_n, F(x_n)) + t_n(u_n, v_n) \in \mathrm{epi}_C(F)$ for all $n \in \mathbb{N}$. Then there exists a sequence $\{c_n\}$ in C such that

$$F(x_n) + t_n v_n = F(x_n + t_n u_n) + c_n \quad \forall n \in \mathbb{N},$$

namely, $v_n = \frac{F(x_n+t_n u_n)-F(x_n)+c_n}{t_n}$ for all $n \in \mathbb{N}$, and so

$$\langle y^*, v_n\rangle \geq \frac{\langle y^*, F(x_n + t_n u_n) - F(x_n)\rangle}{t_n} \quad \forall n \in \mathbb{N}. \tag{2.13}$$

On the other hand, by (2.7), one has

$$\frac{\langle y^*, F(x_n + t_n u_n) - F(x_n)\rangle}{t_n} \geq \frac{\langle y^*, F(x_n + t_n u) - F(x_n)\rangle}{t_n} - L\|u - u_n\|$$

for all sufficiently large $n \in \mathbb{N}$. Since $(u_n, v_n) \to (u, v)$, this and (2.13) imply that

$$\langle y^*, v\rangle = \lim_{n\to\infty} \langle y^*, v_n\rangle \geq \lim_{n\to\infty} \frac{\langle y^*, F(x_n + t_n u) - F(x_n)\rangle}{t_n}.$$

It follows from (2.12) that $\langle y^*, v\rangle \geq (y^* \circ F)^\circ(\bar{x}, u)$. Since $x^* \in \partial(y^* \circ F)(\bar{x})$, this verifies (2.11). The proof is complete.

The main results in this paper require that ordering cone C of Y has a nonempty interior. For convenience, we provide some notations and properties on the ordering cone C under the assumption $\mathrm{int}(C) \neq \emptyset$. Given $y \in C$, let

$$C_y^+ := \{y^* \in C^+ :\ \langle y^*, y\rangle = 1\}, \tag{2.14}$$

$$r_y := \sup\{r > 0 :\ B(y, r) \subset C\} \ \text{ and } \ \gamma_C := \sup\{r_y :\ y \in C \text{ and } \|y\| = 1\}. \tag{2.15}$$

It is clear that $\gamma_C > 0$ if and only if C has a nonempty interior.

The following lemma plays an important role in the proofs of the main results and might be of independent interest.

Lemma 2.6. *Let $y \in Y$ with $\|y\| = 1$ and $r > 0$ be such that $B(y, r) \subset C$, and let $F : X \to Y^\bullet$ be a function such that its epigraph $\mathrm{epi}_C(F)$ is closed. Then the following statements hold.*
(i) $C_y^+ \subset [1,\ \frac{1}{r}]\mathcal{I}_{C^+}$ and $\mathcal{I}_{C^+} \subset [r,\ 1]C_y^+$; consequently

$$D_e^* F(x)(C_y^+) \subset \left[1,\ \frac{1}{r}\right]\partial_C F(x) \ \text{ and } \ \partial_C F(x) \subset [r,\ 1] D_e^* F(x)(C_y^+) \quad \forall x \in \mathrm{dom}(F).$$

(ii) C_y^+ *is a weak***-compact convex set.*
(iii) $D_e^*F(x)(C_y^+)$ *is a weak***-closed convex set for all* $x \in \mathrm{dom}(F)$*.*
(iv) $d(v, -C) \le \max\{\langle y^*, v\rangle : \ y^* \in C_y^+\} \le \frac{1}{r} d(v, -C)$ *for all* $v \in Y \setminus -C$*.*
(v) If $\{y_n^*\} \subset C^+$ *satisfies* $\|y_n^*\| \to 1$ *and* $y_n^* \xrightarrow{w^*} y^*$*, then* $\|y^*\| \ge \gamma_C$*.*

Proof. By the assumption that $B(y, r) \subset C$, one has

$$\langle y^*, y\rangle - r\|y^*\| = \inf\{\langle y^*, z\rangle : \ z \in B(y, r)\} \ge 0 \quad \forall y^* \in C^+.$$

Hence, $r\|y^*\| \le \langle y^*, y\rangle \le \|y^*\|$ for all $y^* \in C^+$ (thanks to $\|y\| = 1$). This implies that (i) holds. By (2.14), we have that C_y^+ is a weak*-closed convex set. Since every weak*-closed bounded set is weak*-compact, it follows from (i) that (ii) holds. Noting that

$$\mathrm{gph}(D_e^*F(x)) = \{(y^*, x^*) : \ (x^*, -y^*) \in N(\mathrm{epi}_C(F), (x, F(x)))\}$$

is a weak*-closed convex cone, it is easy from (ii) to verify that $D_e^*F(\bar{x})(C_y^+)$ is a weak*-closed convex set. This shows that (iii) holds.

To prove (iv), let $v \in Y \setminus -C$. Then, $d(v, -C) > 0$ and $B(v, d(v, -C)) \cap -C = \emptyset$. By the separation theorem, there exists $y_0^* \in Y^*$ with $\|y_0^*\| = 1$ such that $\inf\limits_{z \in B(v, d(v, -C))} \langle y_0^*, z\rangle = \sup\limits_{z \in -C} \langle y_0^*, z\rangle = 0$. This implies that $y_0^* \in \mathcal{I}_{C^+}$ and $\langle y_0^*, v\rangle - d(v, -C) = 0$. By (i), there exists $t \in [1, \ \frac{1}{r}]$ such that $t y_0^* \in C_y^+$. Hence

$$d(v, -C) \le \langle t y_0^*, v\rangle \le \max\{\langle y^*, v\rangle : \ y^* \in C_y^+\}. \tag{2.16}$$

Let $y^* \in C_y^+$. Then, by (i),

$$\langle y^*, v\rangle \le \langle y^*, v + c\rangle \le \|y^*\|\|v + c\| \le \frac{1}{r}\|v + c\| \quad \forall c \in C$$

and so $\langle y^*, v\rangle \le \frac{1}{r} d(v, -C)$. This and (2.16) imply that (iv) holds.

To prove (v), take an arbitrary γ in $(0, \ \gamma_C)$. By the definition of γ_C, there exists $y_0 \in C$ such that $\|y_0\| = 1$ and $B(y_0, \gamma) \subset C$. Hence

$$\langle v^*, y_0\rangle - \gamma\|v^*\| = \inf\{\langle v^*, z\rangle : \ z \in B(y_0, \gamma)\} \ge 0 \quad \forall v^* \in C^+. \tag{2.17}$$

Let $\{y_n^*\} \subset C^+$ be such that $\|y_n^*\| \to 1$ and $y_n^* \xrightarrow{w^*} y^*$. Then, by (2.17),

$$\|y^*\| \ge \langle y^*, y_0\rangle = \lim_{n\to\infty} \langle y_n^*, y_0\rangle \ge \lim_{n\to\infty} \gamma\|y_n^*\| = \gamma.$$

Hence $\|y^*\| \ge \gamma_C$. This shows (v). The proof is complete. □

3. Quasi-subsmooth vector-valued function

The notion of the subsmoothness of a closed set, a useful extension of convexity and smoothness, was introduced by Aussel *et al.* [1] and then was further studied by some authors (see [29, 31] and references therein). Let A be a closed subset of a Banach space X, and recall that A is said to be subsmooth at $\bar{x} \in A$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\langle x^*, u - x\rangle \le \varepsilon\|u - x\| \quad \forall u, x \in A \cap B(\bar{x}, \delta) \text{ and } \forall x^* \in N(A, x) \cap B_{X^*}. \tag{3.1}$$

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function. Motivated by (3.1) and the primal-lower-nice property introduced by Poliquin (*cf.* [23]), Zheng and Wei [31] considered the following notion of quasi-subsmoothness: *f is said to be quasi-subsmooth at $\bar{x}$ if for any $\varepsilon, M \in (0,\ +\infty)$ there exists $\delta > 0$ such that*

$$\langle x^*, u - x\rangle \leq f(u) - f(x) + \varepsilon\|u - x\| \ \ \forall u, x \in B(\bar{x}, \delta) \text{ and } x^* \in \partial f(x) \cap MB_{X^*}. \tag{3.2}$$

In extending to the vector-valued case, we have the corresponding notions in Definitions 3.1 and 3.2, where one considers $F : X \to Y^\bullet$ with Banach spaces X and Y and $C \subset Y$ being a closed convex cone.

Definition 3.1. We say that F is quasi-subsmooth at $\bar{x} \in \mathrm{dom}(F)$ with respect to C if for any $\varepsilon, M \in (0,\ +\infty)$ there exists $\delta > 0$ such that

$$\langle x^*, u - x\rangle \leq \langle y^*, F(u) - F(x)\rangle + \varepsilon\|u - x\| \tag{3.3}$$

whenever $u, x \in B(\bar{x}, \delta),\ y^* \in \mathcal{I}_{C^+}$ and $x^* \in D_e^* F(x)(y^*) \cap MB_{X^*}$.

It is routine to check that F is quasi-subsmooth at $\bar{x}$ with respect to C if $\mathrm{epi}_C(F)$ is subsmooth at $(\bar{x}, F(\bar{x}))$ and F is locally Lipschitz at $\bar{x}$.

Setting $x = \bar{x}$ in (3.3), we introduce the following weaker notion.

Definition 3.2. We say that F is w-quasi-subsmooth at $\bar{x} \in \mathrm{dom}(F)$ with respect to C if for any $\varepsilon, M \in (0,\ +\infty)$ there exists $\delta > 0$ such that

$$\langle x^*, u - \bar{x}\rangle \leq \langle y^*, F(u) - F(\bar{x})\rangle + \varepsilon\|u - \bar{x}\| \tag{3.4}$$

whenever $u \in B(\bar{x}, \delta),\ y^* \in \mathcal{I}_{C^+}$ and $x^* \in D_e^* F(\bar{x})(y^*) \cap MB_{X^*}$.

For convenience, let $\Upsilon(X, Y; C)$ denote the family of all functions $F : X \to Y^\bullet$ such that $\mathrm{epi}_C(F)$ is closed, and let

$$\Gamma(X, Y; C) := \{F \in \Upsilon(X, Y; C) :\ F \text{ is C} - \text{convex}\}.$$

In the case when $Y = \mathbb{R}$ and $C = \mathbb{R}_+$, the closedness of $\mathrm{epi}_C(F)$ means that F is lower semicontinuous. For $\bar{x} \in \mathrm{dom}(F)$, let

$$\begin{aligned} \mathcal{QS}_{\bar{x}}(X, Y; C) &: = \{F \in \Upsilon(X, Y; C) :\ F \text{ is quasi-subsmooth at } \bar{x}\},\\ \mathcal{QS}^w_{\bar{x}}(X, Y; C) &: = \{F \in \Upsilon(X, Y; C) :\ F \text{ is } w\text{-quasi-subsmooth at } \bar{x}\}. \end{aligned}$$

It is clear that

$$\Gamma(X, Y; C) \subset \mathcal{QS}_{\bar{x}}(X, Y; C) \subset \mathcal{QS}^w_{\bar{x}}(X, Y; C). \tag{3.5}$$

Since $D_e^* F(\bar{x})(y^*) \supset \hat{\partial}(y^* \circ F)(\bar{x})$ trivially holds for all $y^* \in C^+$, it is easy to verify from the sublinearity of $D_e^* F(\bar{x})$ and Definition 3.2 that

$$\Big[F \in \mathcal{QS}^w_{\bar{x}}(X, Y; C)\Big] \Longrightarrow \Big[D_e^* F(\bar{x})(y^*) = \hat{\partial}(y^* \circ F)(\bar{x}) \text{ for all } y^* \in C^+ \setminus \{0\}\Big]. \tag{3.6}$$

The following proposition provides a useful subclass of vector-valued quasi-subsmooth functions.

Proposition 3.3. *Let F be a continuously differentiable function between Banach spaces X and Z and let $G \in \Upsilon(Z, Y; C)$ be C-convex. Let $\bar{x} \in F^{-1}(\mathrm{dom}(G))$ be such that the Robinson qualification (2.4) is satisfied. Then the following statements hold.*
(i) $G \circ F \in \mathcal{QS}^w_{\bar{x}}(X, Y; C)$.
(ii) If, in addition, $F(\bar{x}) \in \mathrm{int}(\mathrm{dom}(G))$, then $G \circ F \in \mathcal{QS}_{\bar{x}}(X, Y; C)$.

Proof. From (2.4) and Lemma 2.4, it is easy to verify that

$$\partial(y^* \circ G \circ F)(\bar{x}) = D^*_e(G \circ F)(\bar{x})(y^*) = F'(\bar{x})^*(\partial(y^* \circ G)(F(\bar{x}))) \quad \forall y^* \in C^+. \tag{3.7}$$

To prove (i), let $\Lambda : X \times Y \times \mathbb{R} \rightrightarrows Z \times Y$ be such that

$$\Lambda(x, y, t) := \begin{cases} (F'(\bar{x})(x), y) + t(\mathrm{epi}_C(G) - (F(\bar{x}), \bar{y})), & \text{if } (x, y, t) \in X \times Y \times \mathbb{R}_+ \\ \emptyset, & \text{otherwise} \end{cases}$$

where $\bar{y} := G(F(\bar{x}))$. Then, Λ is a closed convex multifunction and $\Lambda(X \times Y \times \mathbb{R}) = Z \times Y$ (thanks to (2.4)). It follows from the Robinson–Ursecu theorem that there exists $r > 0$ such that

$$\begin{aligned} r(B_Z \times B_Y) \subset\ & \Lambda(B_X \times B_Y \times B_{\mathbb{R}}) \\ &= F'(\bar{x})(B_X) \times B_Y + [0,\ 1](\mathrm{epi}_C(G) - (F(\bar{x}), \bar{y})) \\ &= F'(\bar{x})(B_X) \times B_Y + \mathrm{epi}_C(G) - (F(\bar{x}), \bar{y}) \end{aligned} \tag{3.8}$$

(the last equality holds because $\mathrm{epi}_C(G)$ is a convex set containing $(F(\bar{x}), \bar{y})$). Let $\varepsilon, M \in (0,\ +\infty)$. Since F is continuously differentiable, take $\delta > 0$ such that

$$\|F(x) - F(x') - F'(x')(x - x')\| \le \frac{r\varepsilon}{M + 1}\|x - x'\| \quad \forall x, x' \in B(\bar{x}, \delta). \tag{3.9}$$

Let $y^* \in \mathcal{I}_{C^+}$ and $x^* \in D^*_e(G \circ F)(\bar{x})(y^*) \cap MB_{X^*} = \partial(y^* \circ G \circ F)(\bar{x}) \cap MB_{X^*}$ (thanks to (3.7)). To show that $G \circ F$ is w-quasi-subsmooth at $\bar{x}$, we only need to show that

$$\langle x^*, x - \bar{x}\rangle \le \langle y^*, G(F(x)) - G(F(\bar{x}))\rangle + \varepsilon\|x - \bar{x}\| \quad \forall x \in B(\bar{x}, \delta). \tag{3.10}$$

To do this, we use (3.7) to express x^* as $x^* := F'(\bar{x})^*(z^*)$ with some $z^* \in \partial(y^* \circ G)(F(\bar{x}))$. We claim that $\|z^*\| \le \frac{M+1}{r}$. Granting this and setting $x' = \bar{x}$ in (3.9), one has

$$\langle x^*, x - \bar{x}\rangle = \langle z^*, F'(\bar{x})(x - \bar{x})\rangle \le \langle z^*, F(x) - F(\bar{x})\rangle + \varepsilon\|x - \bar{x}\| \ \ \forall x \in B(\bar{x}, \delta);$$

this implies that (3.10) holds (thanks to $z^* \in \partial(y^* \circ G)(F(\bar{x}))$ and the convexity of $y^* \circ G$). To prove the inequality claimed, we consider an arbitrary z in B_Z. Then, by (3.8), there exist $(u, y') \in B_X \times B_Y$, $z' \in Z$ and $c' \in C$ such that $(rz, 0) = (F'(\bar{x})(u) + z' - F(\bar{x}), y' + G(z') + c' - \bar{y})$. Hence,

$$\begin{aligned} \langle z^*, rz\rangle &= \langle z^*, F'(\bar{x})(u) + z' - F(\bar{x})\rangle \\ &\le \langle z^*, F'(\bar{x})(u)\rangle + (y^* \circ G)(z') - (y^* \circ G)(F(\bar{x})) \\ &\le \langle F'(\bar{x})(z^*), u\rangle + \langle y^*, G(z') + c' - \bar{y}\rangle \\ &= \langle x^*, u\rangle - \langle y^*, y'\rangle \\ &\le M + 1, \end{aligned}$$

and so $\|z^*\| \le \frac{M+1}{r}$ is proved.

To prove (ii), suppose that $F(\bar{x}) \in \mathrm{int}(\mathrm{dom}(G))$. Since F is continuously differentiable, there exist $\delta' > 0$ and $1 > r > 0$ such that $rB_Z \subset \mathrm{int}(\mathrm{dom}(G)) - F(x)$ for all $x \in B_X(\bar{x}, \delta')$. Hence, as in the proof of part (i), the Robinson–Ursescu theorem enables us to strengthen (3.8) by replacing $\bar{x}$ and $\bar{y} = G(F(\bar{x}))$ with any $x \in B_X(\bar{x}, \delta')$ and $y = G(F(x))$, respectively. This, together with (3.9) and (3.7), implies that (ii) holds. The proof is completed. □

It is well known that if $F : X \to \mathbb{R}$ is locally Lipschitz at $\bar{x}$ then ∂F is bounded on a neighborhood of $\bar{x}$. We do not know whether or not the corresponding conclusion holds for vector-valued functions. However, under the quasi-subsmoothness assumption, we have the following result.

Proposition 3.4. *Let $F \in \Upsilon(X, Y; C)$ be locally Lipschitz at $\bar{x}$. Then the following statements hold:*
(i) If $F \in \mathcal{QS}_{\bar{x}}(X, Y; C)$, there exist $\delta, M \in (0, +\infty)$ such that

$$\sup\{\|x^*\| : \ x^* \in D^*_e F(x)(y^*)\} \le M\|y^*\| \quad \forall x \in B(\bar{x}, \delta) \text{ and } y^* \in C^+. \tag{3.11}$$

(ii) If $F \in \mathcal{QS}^w_{\bar{x}}(X, Y; C)$, there exists $M \in (0, +\infty)$ such that

$$\sup\{\|x^*\| : \ x^* \in D^*_e F(\bar{x})(y^*)\} \le M\|y^*\| \quad \forall y^* \in C^+.$$

Proof. Since F is locally Lipschitz at $\bar{x}$, there exist $L, r \in (0, +\infty)$ such that

$$\|F(x_1) - F(x_2)\| \le L\|x_1 - x_2\| \quad \forall x_1, x_2 \in B(\bar{x}, r). \tag{3.12}$$

First suppose that $F \in \mathcal{QS}_{\bar{x}}(X, Y; C)$. Then there exists $\delta \in (0, r)$ such that

$$\langle x^*, u - x\rangle \le \langle y^*, F(u) - F(x)\rangle + \|u - x\| \tag{3.13}$$

for all $u, x \in B(\bar{x}, \delta)$, $y^* \in \mathcal{I}_{C^+}$ and $x^* \in D^*_e F(x)(y^*) \cap (L + 2)B_{X^*}$. Since $D^*_e F(x)$ is sublinear, to prove (i), it suffices to show that

$$\sup\{\|x^*\| : \ x^* \in D^*_e F(x)(y^*)\} \le L + 2 \quad \forall (x, y^*) \in B(\bar{x}, \delta) \times \mathcal{I}_{C^+}.$$

To do this, suppose to the contrary that there exist $x_0 \in B(\bar{x}, \delta) \subset B(\bar{x}, r)$, $y^*_0 \in \mathcal{I}_{C^+}$ and $x^*_0 \in D^*_e F(x_0)(y^*_0)$ such that $L + 2 < \|x^*_0\|$. On the other hand, by (3.12) and Lemmas 2.1 and 2.5, there exists $x^*_1 \in \partial(y^*_0 \circ F)(x_0) \subset D^*_e F(x_0)(y^*_0)$ such that $\|x^*_1\| \le L$. Since $D^*_e F(x_0)(y^*_0)$ is convex, there exists $t \in (0, 1)$ such that $\|tx^*_1 + (1 - t)x^*_0\| = L + 2$ and $tx^*_1 + (1 - t)x^*_0 \in D^*_e F(y^*_0)$. It follows from (3.12) and (3.13) that

$$\langle tx^*_1 + (1 - t)x^*_0, u - x_0\rangle \le \langle y^*_0, F(u) - F(x_0)\rangle + \|u - x_0\| \le (L + 1)\|u - x_0\| \quad \forall u \in B(\bar{x}, \delta),$$

and so $\|tx^*_1 + (1 - t)x^*_0\| \le L + 1$, contradicting the choice of t. This shows that (i) holds. Similarly, (ii) can be proved. □

Corollary 3.5. *Let $F \in \mathcal{QS}^w_{\bar{x}}(X, Y; C)$ be locally Lipschitz at $\bar{x}$. Then,*

$$N(\mathrm{epi}_C(F), (\bar{x}, F(\bar{x}))) = \hat{N}(\mathrm{epi}_C(F), (\bar{x}, F(\bar{x}))).$$

Proof. Let $(x^*, -y^*) \in N(\mathrm{epi}_C(F), (\bar{x}, F(\bar{x})))$. We only need to show that

$$(x^*, -y^*) \subset \hat{N}(\mathrm{epi}_C(F), (\bar{x}, F(\bar{x}))). \tag{3.14}$$

Note that $y^* \in C^+$ (by Lem. 2.3) and $x^* \in D_e^*F(\bar{x})(y^*)$. It follows from (3.6) that if $y^* \neq 0$ then $x^* \in \hat{\partial}(y^* \circ F)(\bar{x}) \subset \hat{D}_e^*F(\bar{x})(y^*)$, which implies that (3.14) holds. If $y^* = 0$, then $x^* = 0$ (see Prop. 3.4(ii)); so (3.14) also holds in this case. The proof is complete. □

Next, in the case when the ordering cone C has a nonempty interior, we prove that the quasi-subsmoothness and weak qausi-subsmoothness admit better properties. To do this, we need the following lemma.

Lemma 3.6. *Let $F \in \Upsilon(X, Y; C)$ be locally Lipschitz at $\bar{x} \in \mathrm{dom}(F)$ and let $y \in \mathrm{int}(C)$. Then the following statements hold.*
(i) $F \in \mathcal{QS}_{\bar{x}}(X, Y; C)$ if and only if for any $\varepsilon > 0$ there exists $\delta > 0$ such that (3.3) holds whenever $u, x \in B(\bar{x}, \delta)$, $y^* \in C_y^+$ and $x^* \in D_e^*F(x)(y^*)$, where C_y^+ is defined by (2.14).
(ii) $F \in \mathcal{QS}_{\bar{x}}^w(X, Y; C)$ if and only if for any $\varepsilon > 0$ there exists $\delta > 0$ such that (3.4) holds whenever $u \in B(\bar{x}, \delta)$, $y^* \in C_y^+$ and $x^* \in D_e^*F(\bar{x})(y^*)$.

Proof. We only need to show (i) (because the proof of (ii) is similar). Since $y \in \mathrm{int}(C)$, there exists $r > 0$ such that $B(y, r) \subset C$. Hence, by Lemma 2.6(i), $\mathcal{I}_{C^+} \subset [r,\ 1]C_y^+$. Since $D_e^*F(x)$ is positively homogeneous, the sufficient part holds trivially. To prove the necessity part, suppose that $F \in \mathcal{QS}_{\bar{x}}(X, Y; C)$. Since F is locally Lipschitz at $\bar{x}$, by Proposition 3.4(i), there exist $M, \delta \in (0,\ +\infty)$ such that $D_e^*F(x)(\mathcal{I}_{C^+}) \subset MB_{X^*}$ for all $x \in B(\bar{x}, \delta)$, and hence for any $\varepsilon > 0$ there exists $\delta_0 \in (0,\ \delta)$ such that (3.3) holds whenever $u, x \in B(\bar{x}, \delta_0)$, $y^* \in \mathcal{I}_{C^+}$ and $x^* \in D_e^*F(x)(y^*)$. Noting (by Lem. 2.6(i)) that $C_y^+ \subset [1,\ \frac{1}{r}]\mathcal{I}_{C^+}$, one sees that the necessity part holds. The proof is complete. □

Let T be a compact topological space and $C(X \times T, \mathbb{R})$ denote the space of all continuous functions on $X \times T$. For $\psi \in C(X \times T, \mathbb{R})$, recall that the family $\{\psi(\cdot, t)\}_{t \in T}$ is locally Lipschitz at $\bar{x}$ if there exist $L, \gamma \in (0,\ +\infty)$ such that

$$|\psi(x_1, t) - \psi(x_2, t)| \leq L\|x_1 - x_2\| \ \ \forall x_1, x_2 \in B(\bar{x}, \gamma) \text{ and } \forall t \in T. \tag{3.15}$$

Further recall (*cf.* [29]) that the family $\{\psi(\cdot, t)\}_{t \in T}$ is uniformly quasi-subsmooth at $\bar{x} \in X$ if for any $\varepsilon, M \in (0,\ +\infty)$ there exists $\delta > 0$ such that

$$\langle x^*, u - x\rangle \leq \psi(u, t) - \psi(x, t) + \varepsilon\|u - x\| \tag{3.16}$$

whenever $(u, t), (x, t) \in B(\bar{x}, \delta) \times T$ and $x^* \in \partial\psi(\cdot, t)(x) \cap MB_{X^*}$.

Note that (3.15) implies $\partial\psi(\cdot, t)(x) \subset LB_{X^*}$ for all $(x, t) \in B(\bar{x}, \delta) \times T$. Thus, in the case when $\{\psi(\cdot, t)\}_{t \in T}$ is locally Lipschitz at $\bar{x}$, $\{\psi(\cdot, t)\}_{t \in T}$ is uniformly quasi-subsmooth at $\bar{x} \in X$ if and only if for any $\varepsilon \in (0,\ +\infty)$ there exists $\delta > 0$ such that (3.16) holds whenever $(u, t), (x, t) \in B(\bar{x}, \delta) \times T$ and $x^* \in \partial\psi(\cdot, t)(x)$. It is known (*cf.*, Thm. 3.1 from [29]) that if $\{\psi(\cdot, t)\}_{t \in T}$ is uniformly quasi-subsmooth and locally Lipschitz at $\bar{x}$ then

$$\partial\Phi(\bar{x}) = \overline{\mathrm{co}}^{w^*}\left(\bigcup_{t \in T(\bar{x})} \partial\psi(\cdot, t)(\bar{x})\right), \tag{3.17}$$

where $\Phi(x) := \max\limits_{t \in T} \psi(x, t)$ and $T(\bar{x}) = \{t \in T :\ \Phi(\bar{x}) = \psi(\bar{x}, t)\}$.

The following proposition will play important roles in our later analysis.

Proposition 3.7. *Let X, Y be Banach spaces with the ordering cone C of Y having a nonempty interior. Let $\bar{x} \in X$ and $F \in \mathcal{QS}_{\bar{x}}(X, Y; C)$ be such that F is locally Lipschitz at $\bar{x}$. Let $y \in \mathrm{int}(C)$ with $\|y\| = 1$, and define $\psi : X \times C_y^+ \to \mathbb{R} \cup \{+\infty\}$ such that*

$$\psi(x, y^*) := \langle y^*, F(x)\rangle \quad \forall (x, y^*) \in X \times C_y^+,$$

where C_y^+ is defined by (2.14). Then the family $\{\psi(\cdot, y^*)\}_{y^*\in C_y^+}$ is uniformly quasi-smooth at $\bar{x}$ and

$$\partial\phi(\bar{x}) = \hat{\partial}\phi(\bar{x}) = D_e^*F(\bar{x})(\Lambda(\bar{x})), \tag{3.18}$$

where $\phi(x) := \max_{y^*\in C_y^+} \psi(x, y^*)$ for all $x \in X$ and $\Lambda(\bar{x}) := \Big\{v^* \in C_y^+ : \langle v^*, F(\bar{x})\rangle = \max_{y^*\in C_y^+} \langle y^*, F(\bar{x})\rangle\Big\}$.

Proof. Since F is locally Lipschitz at $\bar{x}$, it is easy to verify from Lemma 2.6(i) and Lemma 2.5 that there exist $L, \gamma \in (0, +\infty)$ such that (3.15) holds with $T = C_y^+$ and $\partial(y^* \circ F)(x) \subset D_e^*F(x)(y^*)$ for all $(x, y^*) \in B(\bar{x}, \gamma) \times C^+$. From Lemma 3.6(i) and the assumption that $F \in \mathcal{QS}_{\bar{x}}(X, Y; C)$, it follows that $\{\psi(\cdot, y^*)\}_{y^*\in C_y^+}$ is uniformly quasi-smooth at $\bar{x}$. Thus, by the Lipschitz property of F at $\bar{x}$ and ([29], Thm. 3.1), one has $\partial\phi(\bar{x}) = \overline{\text{co}}^{w^*}\big(\bigcup_{y^*\in\Lambda(\bar{x})} \partial(y^* \circ F)(\bar{x})\big) = \overline{\text{co}}^{w^*}\big(\bigcup_{y^*\in\Lambda(\bar{x})} \hat{\partial}(y^* \circ F)(\bar{x})\big)$. This, together with (3.5) and (3.6), implies that

$$\partial\phi(\bar{x}) = \overline{\text{co}}^{w^*}(D_e^*F(\bar{x})(\Lambda(\bar{x}))). \tag{3.19}$$

By Lemma 2.6(i) and the definition of $\Lambda(\bar{x})$, $\Lambda(\bar{x})$ is a bounded weak*-closed convex set and so is a weak*-compact convex set. Since $\text{gph}(D_e^*F(\bar{x}))$ is a weak*-closed convex cone, $D_e^*F(\bar{x})(\Lambda(\bar{x}))$ is a weak*-closed convex set. This and (3.19) imply that $\partial\phi(\bar{x}) = D_e^*F(\bar{x})(\Lambda(\bar{x}))$. Since $\hat{\partial}\phi(\bar{x})$ is trivially contained in $\partial\phi(\bar{x})$, it suffices to show (for establishing (3.18)) that

$$D_e^*F(\bar{x})(\Lambda(\bar{x})) \subset \hat{\partial}\phi(\bar{x}). \tag{3.20}$$

To prove this, let $x^* \in D_e^*F(\bar{x})(\Lambda(\bar{x}))$. Then there exists $y^* \in \Lambda(\bar{x}) \subset C_y^+$ such that $x^* \in D_e^*F(\bar{x})(y^*)$. Thus, by Lemma 3.6(i), for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\langle x^*, x - \bar{x}\rangle \le \langle y^*, F(x) - F(\bar{x})\rangle + \varepsilon\|x - \bar{x}\| = \langle y^*, F(x)\rangle - \phi(\bar{x}) + \varepsilon\|x - \bar{x}\| \le \phi(x) - \phi(\bar{x}) + \varepsilon\|x - \bar{x}\|$$

for all $x \in B(\bar{x}, \delta)$ (the above equality holds because $y^* \in \Lambda(\bar{x})$). Hence, $x^* \in \hat{\phi}(\bar{x})$. This shows that (3.20) holds. □

The following proposition provides the sum rule for subsmoothness and plays a key role in the proof of the main result in Section 4.

Proposition 3.8. *Let X, Y be Asplund spaces and the ordering cone C of Y have a nonempty interior. Let $F_1, F_2 \in \Upsilon(X, Y; C)$ be locally Lipschitz at $\bar{x}$. Then the following statements hold:*
(i) If $F_1, F_2 \in \mathcal{QS}^w_{\bar{x}}(X, Y; C)$, then

$$F_1 + F_2 \in \mathcal{QS}^w_{\bar{x}}(X, Y; C). \tag{3.21}$$

(ii) If there exists a neighborhood V of $\bar{x}$ such that $F_1, F_2 \in \mathcal{QS}_{\bar{x}}(X, Y; C) \cap \mathcal{QS}^w_x(X, Y; C)$ for all $x \in V$, then $F_1 + F_2 \in \mathcal{QS}_{\bar{x}}(X, Y; C)$.

Proof. Suppose that $F_1, F_2 \in \mathcal{QS}^w_{\bar{x}}(X, Y; C)$. Then, by Corollary 3.5, the three kinds of normal cones of $\text{epi}_C(F_i)$ to $(\bar{x}, F_i(\bar{x}))$ coincide:

$$N(\text{epi}_C(F_i), (\bar{x}, F_i(\bar{x}))) = \bar{N}(\text{epi}_C(F_i), (\bar{x}, F_i(\bar{x}))) = \hat{N}(\text{epi}_C(F_i), (\bar{x}, F_i(\bar{x}))),\ i = 1, 2 \tag{3.22}$$

(and all are weak*-closed convex cones). We claim that

$$D_e^*(F_1 + F_2)(\bar{x})(y^*) \subset D_e^*F_1(\bar{x})(y^*) + D_e^*F_2(\bar{x})(y^*) \quad \forall y^* \in C^+. \tag{3.23}$$

To do this, let K be the convex cone defined by

$$K := \big\{(x_1^*+x_2^*, v^*):\ (x_i^*, v^*)\in \bar{N}(\mathrm{epi}_C(F_i), (\bar{x}, F_i(\bar{x}))),\ i=1,2\big\}.$$

Then, by ([17], Cor. 3.11) and the Lipschitz assumption on F_i at $\bar{x}$, one has

$$\bar{N}(\mathrm{epi}_C(F_1+F_2), (\bar{x}, F_1(\bar{x})+F_2(\bar{x})))\subset K. \tag{3.24}$$

This and Lemma 2.2(i) imply that $N(\mathrm{epi}_C(F_1+F_2), (\bar{x}, F_1(\bar{x})+F_2(\bar{x})))\subset \overline{\mathrm{co}}^{w^*}(K)=\overline{K}^{w^*}$. Thus, in order to show (3.23), by (3.22) and the definition of D_e^*F (see (2.5)), we only need to show that K is weak*-closed. In turn, by the Krein–Smulian theorem, it suffices to show that $K\cap(B_{X^*}\times B_{Y^*})$ is a weak*-closed set. To do this, let $(x^*, -v^*)$ be a weak* cluster point of $K\cap(B_{X^*}\times B_{Y^*})$. Since $B_{X^*}\times B_{Y^*}$ is weak*-compact, it is sufficient to prove that $(x^*, -v^*)\in K$. Take a net $\{(x_\alpha^*, -v_\alpha^*)\}$ in $K\cap(B_{X^*}\times B_{Y^*})$ converging to $(x^*, -v^*)$ with respect to the weak* topology. Hence, for each α there exist $u_\alpha^*\in D_e^*F_1(\bar{x})(v_\alpha^*)$ and $w_\alpha^*\in D_e^*F_2(\bar{x})(v_\alpha^*)$ such that $x_\alpha^*=u_\alpha^*+w_\alpha^*$. It follows that $v_\alpha^*\in C^+$ and $\max\{\|u_\alpha^*\|, \|w_\alpha^*\|\}\le M\|v_\alpha^*\|\le M$ for all α and some constant M (thanks to Prop. 3.4(ii)). Now consider any $\varepsilon>0$. Since $F_1, F_2\in\mathcal{QS}_{\bar{x}}^w(X, Y; C)$, there exists $\delta>0$ such that

$$\langle u_\alpha^*, x-\bar{x}\rangle\le\langle v_\alpha^*, F_1(x)-F_1(\bar{x})\rangle+\varepsilon\|x-\bar{x}\|\ \text{ and }\ \langle w_\alpha^*, x-\bar{x}\rangle\le\langle v_\alpha^*, F_2(x)-F_2(\bar{x})\rangle+\varepsilon\|x-\bar{x}\|$$

for all $x\in B(\bar{x}, \delta)$. Without loss of generality, we can assume that $u_\alpha^*\xrightarrow{w^*}u^*$ and $w_\alpha^*\xrightarrow{w^*}z^*$ (taking a subnet if necessary). Then, $x_\alpha^*=u_\alpha^*+w_\alpha^*\xrightarrow{w^*}u^*+z^*=x^*$,

$$\langle u^*, x-\bar{x}\rangle\le\langle v^*, F_1(x)-F_1(\bar{x})\rangle+\varepsilon\|x-\bar{x}\|\ \text{ and }\ \langle z^*, x-\bar{x}\rangle\le\langle v^*, F_2(x)-F_2(\bar{x})\rangle+\varepsilon\|x-\bar{x}\|$$

for all $x\in B(\bar{x}, \delta)$. Hence

$$(u^*, -v^*)\in\hat{N}(\mathrm{epi}_C(F_1), (\bar{x}, F_1(\bar{x})))\ \text{ and }\ (z^*, -v^*)\in\hat{N}(\mathrm{epi}_C(F_2), (\bar{x}, F_2(\bar{x})))$$

and so $(x^*, -v^*)=(u^*, -v^*)+(z^*, -v^*)\in K$. This shows that (3.23) holds. To prove (3.21), let $y^*\in\mathcal{I}_{C^+}$ and $x^*\in D_e^*(F_1+F_2)(\bar{x})(y^*)$; then (3.23) implies that there exist $x_i^*\in D_e^*F_i(\bar{x})(y^*)$ $(i=1,2)$ such that $x^*=x_1^*+x_2^*$. From Proposition 3.4(ii) and the assumption that $F_1, F_2\in\mathcal{QS}_{\bar{x}}^w(X, Y; C)$ are locally Lipschitz at $\bar{x}$, it follows that for any $\varepsilon>0$ there exists $\delta'>0$ (independent on y^* and x^*) such that $\langle x_i^*, x-\bar{x}\rangle\le\langle y^*, F_i(x)-F_i(\bar{x})\rangle+\varepsilon\|x-\bar{x}\|$ for all $x\in B(\bar{x}, \delta')$, and so

$$\langle x^*, x-\bar{x}\rangle=\langle x_1^*, x-\bar{x}\rangle+\langle x_2^*, x-\bar{x}\rangle\le\langle y^*, F_1(x)+F_2(x)-F_1(\bar{x})-F_2(\bar{x})\rangle+2\varepsilon\|x-\bar{x}\|$$

for all $x\in B(\bar{x}, \delta')$. This implies that (3.21) holds and so (i) is shown. Similar to the proof of (3.21), (ii) is immediate from (3.23) (with $\bar{x}$ being replaced by each x in some neighborhood V of $\bar{x}$) and the definition of the quasi-subsmoothness. The proof is complete. □

Remark. Inclusion (3.24), as a consequence of ([17], Cor. 3.11), plays a key role in the above proof, while the proof of ([17], Cor. 3.11) relies on the assumption that X and Y are Asplund spaces. We do not know whether (3.24) holds with the Clarke normal cone replacing the limiting normal cone when X and Y are Banach spaces.

Let $F, G\in\Upsilon(X, Y; C)$ and $\bar{x}\in\mathrm{dom}(F)\cap\mathrm{dom}(G)$. We introduce the following notation

$$\|G-F\|_{\bar{x}} := \limsup_{x\to\bar{x}}\frac{\|G(x)-F(x)-(G(\bar{x})-F(\bar{x}))\|}{\|x-\bar{x}\|}.$$

Clearly, $\|G-F\|_{\bar{x}} < +\infty$ if and only if there exist $L, \delta \in (0,\ +\infty)$ such that

$$\|(G-F)(x) - (G-F)(\bar{x})\| \le L\|x-\bar{x}\| \quad \forall x \in B(\bar{x}, \delta),$$

which is weaker than the local Lipschitz property of $G-F$ at $\bar{x}$. In the case, when $Y = \mathbb{R}$, $\|G-F\|_{\bar{x}}$ was used in [12, 13, 20, 31].

Proposition 3.9. *Let $F \in \Upsilon(X, Y; C)$ and $G \in \mathcal{QS}^w_{\bar{x}}(X, Y; C)$ be such that $\|G-F\|_{\bar{x}} < +\infty$. Then*

$$D^*_e G(\bar{x})(y^*) \subset D^*_e F(\bar{x})(y^*) + \|y^*\|\|G-F\|_{\bar{x}} B_{X^*} \quad \forall y^* \in \mathcal{I}_{C^+} \tag{3.25}$$

Consequently, $\partial_C G(\bar{x}) \subset \partial_C F(\bar{x}) + \|G-F\|_{\bar{x}} B_{X^*}$ and

$$d(0, \partial_C F(\bar{x})) \le \|u^*\| + \|G-F\|_{\bar{x}} \quad \forall u^* \in \partial_C G(\bar{x}).$$

Proof. Let $y^* \in \mathcal{I}_{C^+}$ and $x^* \in D^*_e G(\bar{x})(y^*)$ and take an arbitrary $\varepsilon > 0$. Then, by $G \in \mathcal{QS}^w_{\bar{x}}(X, Y; C)$, there exists $\delta > 0$ such that

$$\langle x^*, x-\bar{x}\rangle \le \langle y^*, G(x) - G(\bar{x})\rangle + \varepsilon\|x-\bar{x}\| \quad \forall x \in B(\bar{x}, \delta). \tag{3.26}$$

Since $\|G-F\|_{\bar{x}} < +\infty$, there exists $\delta' \in (0,\ \delta)$ such that

$$\|G(x) - G(\bar{x}) - (F(x) - F(\bar{x}))\| \le (\|G-F\|_{\bar{x}} + \varepsilon)\|x-\bar{x}\| \quad \forall x \in B(\bar{x}, \delta').$$

Hence, for any $x \in B(\bar{x}, \delta')$,

$$\begin{aligned}\langle y^*, G(x) - G(\bar{x})\rangle &\le \langle y^*, F(x) - F(\bar{x})\rangle + \|y^*\|\|G(x) - G(\bar{x}) - (F(x) - F(\bar{x}))\| \\ &\le \langle y^*, F(x) - F(\bar{x})\rangle + \|y^*\|(\|G-F\|_{\bar{x}} + \varepsilon)\|x-\bar{x}\|.\end{aligned}$$

This and (3.26) imply that

$$\langle x^*, x-\bar{x}\rangle \le \langle y^*, F(x) - F(\bar{x})\rangle + \|y^*\|(\|G-F\|_{\bar{x}} + 2\varepsilon)\|x-\bar{x}\| \quad \forall x \in B(\bar{x}, \delta'). \tag{3.27}$$

Noting that $y^* \in C^+$, it follows that

$$\langle x^*, x-\bar{x}\rangle \le \langle y^*, F(x) + c - F(\bar{x})\rangle + \|y^*\|(\|G-F\|_{\bar{x}} + 2\varepsilon)\|x-\bar{x}\| \quad \forall (x, c) \in B(\bar{x}, \delta') \times C,$$

that is,

$$\langle x^*, x-\bar{x}\rangle - \langle y^*, y - F(\bar{x})\rangle \le \delta_{\mathrm{epi}_C(F)}(x, y) + \|y^*\|(\|G-F\|_{\bar{x}} + 2\varepsilon)\|x-\bar{x}\|$$

for all $(x, y) \in B(\bar{x}, \delta') \times Y$. Hence, by Lemma 2.1,

$$\begin{aligned}(x^*, -y^*) &\in \partial\delta_{\mathrm{epi}_C(F)}(\bar{x}, F(\bar{x})) + \|y^*\|(\|G-F\|_{\bar{x}} + \varepsilon)B_{X^*} \times \{0\} \\ &= N(\mathrm{epi}_C(F), (\bar{x}, F(\bar{x}))) + \|y^*\|(\|G-F\|_{\bar{x}} + \varepsilon)B_{X^*} \times \{0\}.\end{aligned}$$

This means that $x^* \in D^*_e F(\bar{x})(y^*) + \|y^*\|(\|G-F\|_{\bar{x}} + \varepsilon)B_{X^*}$. Letting $\varepsilon \to 0$ and noting that $D^*_e F(\bar{x})(y^*)$ and B_{X^*} are respectively weak*-closed and weak*-compact, we have $x^* \in D^*_e F(\bar{x})(y^*) + \|y^*\|\|G-F\|_{\bar{x}} B_{X^*}$. This shows that (3.25) holds. The proof is completed. □

4. Main results

Let $F \in \Upsilon(X, Y; C)$, that is, $F : X \to Y^\bullet$ be a function such that its epigraph $\mathrm{epi}_C(F)$ is closed. For $\bar{x} \in \mathrm{dom}(F)$, recall that F has an error bound at $\bar{x}$ if and only if $\tau(F, \bar{x}, C) < +\infty$, where the error bound modulus $\tau(F, \bar{x}, C)$ is defined by (1.4). We say that F has a stable error bound at $\bar{x}$ if there exist $\eta, \delta \in (0, +\infty)$ such that

$$\tau(G, \bar{x}, C) \leq \eta \quad \forall G \in \Upsilon(X, Y; C) \text{ with } \|G - F\|_{\bar{x}} < \delta. \tag{4.1}$$

We say that F has a stable strong error bound at $\bar{x}$ if there exist $\eta, \delta \in (0, +\infty)$ such that (4.1) holds and $\bar{x}$ is an isolated point of $S(G, \bar{x}, C)$ for all $G \in \Upsilon(X, Y; C)$ with $\|G - F\|_{\bar{x}} < \delta$, that is, for any $\varepsilon > 0$ there exists $r \in (0, +\infty)$ such that

$$\|x - \bar{x}\| \leq (\eta + \varepsilon) d(G(x) - G(\bar{x}), -C) \text{ for all } x \in B(\bar{x}, r).$$

In this section, in terms of subdifferential $\partial_C F$, we will consider the stability of error bounds for a quasi-subsmooth function F. Note that $0 \notin \mathrm{bd}(\partial_C F(\bar{x}))$ if and only if one of the following two properties holds:

$$\text{(i)}\ \ 0 \in \mathrm{int}(\mathrm{cl}(\partial_C F(\bar{x}))) \quad \text{and} \quad \text{(ii)}\ \ 0 \notin \mathrm{cl}(\partial_C F(\bar{x})).$$

It is easy to verify that

$$\text{(i)} \Longrightarrow d(0, \mathrm{bd}(\partial_C F(\bar{x}))) = \sup\{r > 0 : \ rB_{X^*} \subset \mathrm{cl}(\partial_C F(\bar{x}))\} > 0$$

and

$$\text{(ii)} \Longrightarrow d(0, \mathrm{bd}(\partial_C F(\bar{x}))) = d(0, \partial_C F(\bar{x})) > 0.$$

For case (i), we have the following result.

Theorem 4.1. *Let $F \in \mathcal{QS}^w_{\bar{x}}(X, Y; C)$ and $0 \in \mathrm{int}(\mathrm{cl}(\partial_C F(\bar{x})))$. Suppose that $G \in \Upsilon(X, Y; C)$ is such that $\|G - F\|_{\bar{x}} < d(0, \mathrm{bd}(\partial_C F(\bar{x})))$. Then, for any $\varepsilon \in (0, +\infty)$ there exists $\delta > 0$ such that*

$$\|x - \bar{x}\| \leq \frac{1 + \varepsilon}{d(0, \mathrm{bd}(\partial_C F(\bar{x}))) - \|G - F\|_{\bar{x}}} d(G(x) - G(\bar{x}), -C) \quad \forall x \in B(\bar{x}, \delta). \tag{4.2}$$

Consequently, F has a stable strong error bound at $\bar{x}$.

Proof. Let $r := d(0, \mathrm{bd}(\partial_C F(\bar{x})))$. From the assumption that $0 \in \mathrm{int}(\mathrm{cl}(\partial_C F(\bar{x})))$, it follows that

$$r = \sup\Big\{t > 0 : \ tB_{X^*} \subset \mathrm{cl}(\partial_C F(\bar{x}))\Big\} > 0 \text{ and } rB_{X^*} \subset \mathrm{cl}(\partial_C F(\bar{x})). \tag{4.3}$$

Let $G \in \Upsilon(X, Y; C)$ with $\|G - F\|_{\bar{x}} < r$. Given an $\varepsilon \in (0, \infty)$, take $\eta \in \left(0, \ \frac{r - \|G - F\|_{\bar{x}}}{2}\right)$ and $\delta_1 > 0$ such that

$$\frac{r - \|G - F\|_{\bar{x}}}{1 + \varepsilon} < r - \|G - F\|_{\bar{x}} - 2\eta \tag{4.4}$$

and

$$\|G(x) - G(\bar{x}) - (F(x) - F(\bar{x}))\| \leq (\|G - F\|_{\bar{x}} + \eta)\|x - \bar{x}\| \quad \forall x \in B(\bar{x}, \delta_1).$$

Noting that $\langle y^*, F(x) - F(\bar{x})\rangle \leq d(F(x) - F(\bar{x}), -C)$ for all $y^* \in \mathcal{I}_{C^+}$, since $F \in \mathcal{QS}^w_{\bar{x}}(X, Y; C)$, there exists $\delta \in (0,\ \delta_1)$ such that

$$\begin{aligned}\langle x^*, x - \bar{x}\rangle &\leq d(F(x) - F(\bar{x}), -C) + \eta\|x - \bar{x}\| \\ &\leq d(G(x) - G(\bar{x}), -C) + (\|G - F\|_{\bar{x}} + 2\eta)\|x - \bar{x}\|\end{aligned}$$

for any $x^* \in rB_{X^*} \cap \partial_C F(\bar{x})$ and $x \in B(\bar{x}, \delta)$. It follows from the inclusion in (4.3) that

$$\langle x^*, x - \bar{x}\rangle \leq d(G(x) - G(\bar{x}), -C) + (\|G - F\|_{\bar{x}} + 2\eta)\|x - \bar{x}\| \quad \forall (x^*, x) \in rB_{X^*} \times B(\bar{x}, \delta).$$

This means that

$$r\|x - \bar{x}\| \leq d(G(x) - G(\bar{x}), -C) + (\|G - F\|_{\bar{x}} + 2\eta)\|x - \bar{x}\| \quad \forall x \in B(\bar{x}, \delta),$$

that is, $(r - \|G - F\|_{\bar{x}} - 2\eta)\|x - \bar{x}\| \leq d(G(x) - G(\bar{x}), -C)$ for all $x \in B(\bar{x}, \delta)$. This and (4.4) imply that (4.2) holds. The proof is complete. □

Under some mild assumptions, we will prove that $0 \in \text{int}(\text{cl}(\partial_C F(\bar{x})))$ is also a necessary condition for F to have a stable strong error bound at $\bar{x}$.

Theorem 4.2. *Suppose that the ordering cone C has a nonempty interior and that F is quasi-subsmooth at $\bar{x} \in \text{dom}(F)$. Further suppose that $F : X \to Y$ is locally Lipschitz at $\bar{x} \in X$. Then F has a stable strong error bound at $\bar{x}$ if and only if $0 \in \text{int}(\text{cl}(\partial_C F(\bar{x})))$.*

Proof. The sufficiency part is immediate from Theorem 4.1 (and (3.5)). Next suppose that F has a stable strong error bound at $\bar{x}$. Then there exist $\kappa, \delta \in (0,\ +\infty)$ such that

$$\kappa\|x - \bar{x}\| \leq d(F(x) - F(\bar{x}), -C) \quad \forall x \in B(\bar{x}, \delta),$$

and so $F(x) - F(\bar{x}) \notin -C$ for all $x \in B(\bar{x}, \delta) \setminus \{\bar{x}\}$. Fix $y \in \text{int}(C)$ with $\|y\| = 1$, and let $\phi(x) := \max_{y^* \in C_y^+} \langle y^*, F(x) - F(\bar{x})\rangle$ for all $x \in X$. Thus, by Lemma 2.6(iv), one has

$$\kappa\|x - \bar{x}\| \leq d(F(x) - F(\bar{x}), -C) \leq \phi(x) = \phi(x) - \phi(\bar{x}) \quad \forall x \in B(\bar{x}, \delta).$$

It follows that

$$\kappa B_{X^*} \subset \hat{\partial}\phi(\bar{x}). \tag{4.5}$$

Noting that $F - F(\bar{x})$ is locally Lipschitz and quasi-subsmooth at $\bar{x}$, one has, by Proposition 3.7 (applied to F being replaced by $F - F(\bar{x})$), that

$$\hat{\partial}\phi(\bar{x}) = D_e^*(F - F(\bar{x}))(\bar{x})(C_y^+) = D_e^* F(\bar{x})(C_y^+).$$

It follows from (4.5) and Lemma 2.6(i) that $\kappa r B_{X^*} \subset \partial_C F(\bar{x})$, and so $0 \in \operatorname{int}(\operatorname{cl}(\partial_C F(\bar{x})))$. The proof is complete. □

Next we consider the case when $0 \notin \operatorname{cl}(\partial_C F(\bar{x}))$. To do this, we need the following lemma.

Lemma 4.3. *Let $G \in \Upsilon(X, Y; C)$, $\bar{x} \in \operatorname{dom}(G)$ and let $\beta \in (0,\ \tau(G, \bar{x}, C))$. Then there exists a sequence $\{(\bar{x}_n, \bar{c}_n, \eta_n)\} \subset X \times C \times (0,\ +\infty)$ such that*

$$G(\bar{x}_n) + \bar{c}_n \neq G(\bar{x}) \quad \forall n \in \mathbb{N}, \tag{4.6}$$

$$(\bar{x}_n, G(\bar{x}_n) + \bar{c}_n, \eta_n) \to (\bar{x}, G(\bar{x}), 0) \tag{4.7}$$

and

$$0 \in \hat{\partial}\Big(f + \delta_{\operatorname{epi}_C(G)} + \frac{1}{\beta}\|\cdot - (\bar{x}_n, G(\bar{x}_n) + \bar{c}_n)\|_n\Big)(\bar{x}_n, G(\bar{x}_n) + \bar{c}_n) \quad \forall n \in \mathbb{N}, \tag{4.8}$$

where

$$f(x, y) := \|y - G(\bar{x})\| \ \text{ and } \ \|(x, y)\|_n := \|x\| + \eta_n\|y\| \quad \forall (x, y) \in X \times Y. \tag{4.9}$$

Proof. Since $\tau(G, \bar{x}, C) > \beta$, there exists a sequence $\{\tilde{x}_n\}$ in $X \setminus S(G, \bar{x}, C)$ converging to $\bar{x}$ such that

$$L_n := d(\tilde{x}_n, S(G, \bar{x}, C)) > \beta d(G(\tilde{x}_n) - G(\bar{x}), -C) \quad \forall n \in \mathbb{N}. \tag{4.10}$$

Hence, $0 < L_n \leq \|\tilde{x}_n - \bar{x}\| \to 0$, and for each $n \in \mathbb{N}$ there exists $\tilde{c}_n \in C$ such that

$$\|G(\tilde{x}_n) + \tilde{c}_n - G(\bar{x})\| < \frac{L_n}{\beta}. \tag{4.11}$$

This implies that $f(\tilde{x}_n, G(\tilde{x}_n) + \tilde{c}_n) < \inf\big\{f(x, y) + \delta_{\operatorname{epi}_C(G)}(x, y) :\ (x, y) \in X \times Y\big\} + \frac{L_n}{\beta}$, where f is defined by (4.9). Noting that $f + \delta_{\operatorname{epi}_C(G)}$ is lower semicontinuous, it follows from the Ekeland variational principle that there exists $(\bar{x}_n, \bar{c}_n) \in X \times C$ such that

$$\|(\bar{x}_n, G(\bar{x}_n) + \bar{c}_n) - (\tilde{x}_n, G(\tilde{x}_n) + \tilde{c}_n)\|_n < L_n$$

and

$$f(\bar{x}_n, G(\bar{x}_n) + \bar{c}_n) \leq f(x, y) + \delta_{\operatorname{epi}_C(G)}(x, y) + \frac{1}{\beta}\|(x, y) - (\bar{x}_n, G(\bar{x}_n) + \bar{c}_n)\|_n \quad \forall (x, y) \in X \times Y,$$

where $\|\cdot\|_n$ is defined by (4.9) with $\eta_n := \sqrt{L_n} \to 0$. Thus, (4.8) holds,

$$\|\bar{x}_n - \tilde{x}_n\| < L_n \to 0 \ \text{ and } \ \|G(\bar{x}_n) + \bar{c}_n - G(\tilde{x}_n) - \tilde{c}_n\| < \sqrt{L_n} \to 0.$$

By (4.10), (4.11) and $\bar{x} \in S(G, \bar{x}, C)$, we have

$$\bar{x}_n \to \bar{x}, \ \ G(\bar{x}_n) + \bar{c}_n \to G(\bar{x}), \ \bar{x}_n \notin S(G, \bar{x}, C)$$

(so $G(\bar{x}_n) + \bar{c}_n \neq G(\bar{x})$). The proof is complete. □

Theorem 4.4. *Let X, Y be Banach spaces and suppose that the ordering cone C of Y has a nonempty interior. Let $F, G \in \Upsilon(X, Y; C)$ and $\bar{x} \in \mathrm{dom}(F)$ be such that*

$$0 \notin \mathrm{cl}(\partial_C F(\bar{x})) \ \text{ and } \ \|G - F\|_{\bar{x}} < d(0, \partial_C F(\bar{x})).$$

Further suppose that one of the following three conditions is satisfied:
(i) G is continuous on some neighborhood of $\bar{x}$ and $G \in \mathcal{QS}_{\bar{x}}(X, Y; C)$.
(ii) $G \in \Gamma(X, Y; C)$.
(iii) X, Y are Asplund spaces and $G \in \mathcal{QS}^w_{\bar{x}}(X, Y; C)$.
Then

$$\tau(G, \bar{x}, C) \leq \frac{1}{\gamma_C\big(d(0, \partial_C F(\bar{x})) - \|G - F\|_{\bar{x}}\big)}.$$

Proof. Suppose to the contrary that $\tau(G, \bar{x}, C) > \frac{1}{\gamma_C\big(d(0,\partial_C F(\bar{x})) - \|G-F\|_{\bar{x}}\big)}$. Then there exists $r \in (0,\ \gamma_C)$ sufficiently close to γ_C such that

$$\tau(G, \bar{x}, C) > \tau_r := \frac{1}{r\big(d(0, \partial_C F(\bar{x})) - \|G - F\|_{\bar{x}}\big)}.$$

By Lemma 4.3, there exists $\{(\bar{x}_n, \bar{c}_n, \eta_n)\} \subset X \times C \times (0,\ +\infty)$ such that (4.6), (4.7) and (4.8) hold with $\beta = \tau_r$. By (4.8) and Lemma 2.1, one has

$$\begin{aligned}
(0,0) &\in \partial\Big(f + \delta_{\mathrm{epi}_C(G)} + \frac{1}{\tau_r}\|\cdot - (\bar{x}_n, G(\bar{x}_n) + \bar{c}_n)\|_n\Big)(\bar{x}_n, G(\bar{x}_n) + \bar{c}_n) \\
&\subset \partial f(\bar{x}_n, G(\bar{x}_n) + \bar{c}_n) + \partial \delta_{\mathrm{epi}_C(G)}(\bar{x}_n, G(\bar{x}_n) + \bar{c}_n) + \frac{1}{\tau_r}\partial\|\cdot - (\bar{x}_n, G(\bar{x}_n) + \bar{c}_n)\|_n(\bar{x}_n, G(\bar{x}_n) + \bar{c}_n) \\
&\subset \{0\} \times S_{Y^*} + N(\mathrm{epi}_C(G), (\bar{x}_n, G(\bar{x}_n) + \bar{c}_n)) + \frac{1}{\tau_r}(B_{X^*} \times \eta_n B_{Y^*}),
\end{aligned}$$

where f and $\|\cdot\|_n$ are defined by (4.9). Hence, there exist $\bar{y}_n^* \in S_{Y^*}$, $\bar{v}_n^* \in B_{Y^*}$ and $\bar{x}_n^* \in B_{X^*}$ such that

$$\left(\frac{\bar{x}_n^*}{\tau_r}, -\bar{y}_n^* - \frac{\eta_n \bar{v}_n^*}{\tau_r}\right) \in N(\mathrm{epi}_C(G), (\bar{x}, G(\bar{x}_n) + \bar{c}_n)). \tag{4.12}$$

First suppose that (i) holds. Then G is continuous on some neighborhood of $\bar{x}$. By (4.7), G is continuous at each $\bar{x}_n$ for all sufficiently large n. It follows from Lemma 2.3(ii) that

$$N(\mathrm{epi}_C(G), (\bar{x}_n, G(\bar{x}_n) + \bar{c}_n)) \subset N(\mathrm{epi}_C(G), (\bar{x}_n, G(\bar{x}_n))) \subset X^* \times -C^+$$

for all sufficiently large n. By (4.12), one has $\bar{y}_n^* + \frac{\eta_n \bar{v}_n^*}{\tau_r} \in C^+$ and $\frac{\bar{x}_n^*}{\tau_r} \in D_e^* G(\bar{x}_n)(\bar{y}_n^* + \frac{\eta_n \bar{v}_n^*}{\tau_r})$ for all sufficiently large n. Since $\|\bar{y}_n^* + \frac{\eta_n \bar{v}_n^*}{\tau_r}\| \to 1$ and G is quasi-subsmooth at $\bar{x}$, it follows that for any $\varepsilon > 0$ there exists $\delta' > 0$ such that

$$\left\langle \frac{\bar{x}_n^*}{\tau_r}, x - \bar{x}_n \right\rangle \leq \left\langle \bar{y}_n^* + \frac{\eta_n \bar{v}_n^*}{\tau_r}, G(x) - G(\bar{x}_n) \right\rangle + \varepsilon\|x - \bar{x}_n\|$$

for all $x \in B(\bar{x}, \delta')$ and all sufficiently large n. Noting that $\|\bar{x}_n - \bar{x}\| \to 0$ and $\|G(\bar{x}_n) - G(\bar{x})\| \to 0$, it follows that

$$\liminf_{n\to\infty} \left\langle \frac{\bar{x}_n^*}{\tau_r}, x - \bar{x} \right\rangle \leq \liminf_{n\to\infty} \left\langle \bar{y}_n^* + \frac{\eta_n \bar{v}_n^*}{\tau_r}, G(x) - G(\bar{x}) \right\rangle + \varepsilon \|x - \bar{x}\| \quad \forall x \in B(\bar{x}, \delta'). \tag{4.13}$$

Without loss of generality, we may assume that $\bar{x}_n^* \xrightarrow{w^*} x^* \in B_{X^*}$ and $\bar{y}_n^* + \frac{\eta_n \bar{v}_n^*}{\tau_r} \xrightarrow{w^*} y^* \in B_{Y^*} \cap C^+$ (passing to a subnet if necessary). Hence, by Lemma 2.6(v) and (4.13), one has $\gamma_C \leq \|y^*\|$ and

$$\left\langle \frac{x^*}{\tau_r}, x - \bar{x} \right\rangle \leq \left\langle y^*, G(x) - G(\bar{x}) \right\rangle + \varepsilon \|x - \bar{x}\| \quad \forall x \in B(\bar{x}, \delta').$$

It follows that $\frac{x^*}{\tau_r} \in \hat{\partial}(y^* \circ G)(\bar{x}) \subset D_e^* G(\bar{x})(y^*)$, and so

$$\frac{x^*}{\tau_r \|y^*\|} \in D_e^* G(\bar{x}) \Big(\frac{y^*}{\|y^*\|} \Big) \subset D_e^* G(\bar{x})(\mathcal{I}_{C^+}) = \partial_C G(\bar{x}). \tag{4.14}$$

Therefore, by Proposition 3.9,

$$d(0, \partial_C F(\bar{x})) \leq \frac{\|x^*\|}{\tau_r \|y^*\|} + \|G - F\|_{\bar{x}} \leq \frac{1}{\tau_r \gamma_C} + \|G - F\|_{\bar{x}} < \frac{1}{\tau_r r} + \|G - F\|_{\bar{x}} \tag{4.15}$$

(thanks to the choice of r). This implies that $\tau_r < \frac{1}{r(d(0, \partial_C F(\bar{x})) - \|G - F\|_{\bar{x}})}$, contradicting the definition of τ_r.

Next suppose that (ii) holds. The convexity of $\mathrm{epi}_C(G)$, together with (4.12) and Lemma 2.3(i), implies that $\bar{y}_n^* + \frac{\eta_n \bar{v}_n^*}{\tau_r} \in C^+$ and

$$\left\langle \frac{\bar{x}_n^*}{\tau_r}, x - \bar{x}_n \right\rangle - \left\langle \bar{y}_n^* + \frac{\eta_n \bar{v}_n^*}{\tau_r}, y - G(\bar{x}_n) - \bar{c}_n \right\rangle \leq 0 \quad \forall (x, y) \in \mathrm{epi}_C(G).$$

Let $(\frac{x^*}{\tau_r}, -y^*)$ be a weak*-cluster point of $\{(\frac{x_n^*}{\tau_r}, -\bar{y}_n^* - \frac{\eta_n \bar{v}_n^*}{\tau_r})\}$. Then, by Lemma 2.6(v) and (4.7), one has $\gamma_C \leq \|y^*\|$ and

$$\left\langle \frac{x^*}{\tau_r}, x - \bar{x} \right\rangle - \langle y^*, y - G(\bar{x}) \rangle \leq 0 \quad \forall (x, y) \in \mathrm{epi}_C(G).$$

It follows from the convexity of $\mathrm{epi}_C(G)$ and Lemma 2.3(i) that $(\frac{x^*}{\tau_r}, -y^*) \in N(\mathrm{epi}_C(G), (\bar{x}, G(\bar{x})))$ and $y^* \in C^+$. Hence (4.14) and (4.15) hold, which implies $\tau_r < \frac{1}{r(d(0, \partial_C F(\bar{x})) - \|G - F\|_{\bar{x}})}$, contradicting the definition of τ_r.

Finally suppose that (iii) holds. Since X and Y are Asplund space, Lemma 2.2(iv) together with (4.6)–(4.8) implies that there exist $\{(a_n, b_n)\}, \{(d_n, e_n\}), \{(u_n, v_n)\} \subset X \times Y$ converging to $(\bar{x}, G(\bar{x}))$ such that $b_n \neq G(\bar{x})$ and

$$(0,0) \in \hat{\partial} f(a_n, b_n) + \hat{\partial} \delta_{\mathrm{epi}_C(G)}(u_n, v_n) + \frac{1}{\tau_r} \hat{\partial} \| \cdot - (\bar{x}_n, G(\bar{x}_n) + \bar{c}_n) \|_n (d_n, e_n) + \frac{1}{n} (B_{X^*} \times B_{Y^*}).$$

It follows that

$$(0,0) \in \{0\} \times S_{Y^*} + \hat{N}\big(\mathrm{epi}_C(G), (u_n, v_n)\big) + \frac{1}{\tau_r} (B_{X^*} \times \eta_n B_{Y^*}) + \frac{1}{n} (B_{X^*} \times B_{Y^*}).$$

Hence, there exist $y_n^* \in S_{Y^*}$, $v_n^* \in B_{Y^*}$ and $x_n^* \in B_{X^*}$ such that

$$\left(\left(\frac{1}{\tau_r}+\frac{1}{n}\right)x_n^*, -y_n^* - \left(\frac{\eta_n}{\tau_r}+\frac{1}{n}\right)v_n^*\right) \in \hat{N}(\mathrm{epi}_C(G), (u_n, v_n)). \tag{4.16}$$

This and Lemma 2.3(i) imply that

$$\|y_n^* + \left(\frac{\eta_n}{\tau_r}+\frac{1}{n}\right)v_n^*\| \to 1 \quad \text{and} \quad y_n^* + \Big(\frac{\eta_n}{\tau_r}+\frac{1}{n}\Big)v_n^* \in C^+.$$

Without loss of generality, we may assume that $x_n^* \xrightarrow{w^*} x^* \in B_{X^*}$ and $y_n^* + (\frac{\eta_n}{\tau_r}+\frac{1}{n})v_n^* \xrightarrow{w^*} y^* \in C^+$ (passing to a subsequence if necessary). Thus, by Lemma 2.6(v) and (4.16), one has

$$r < \gamma_C \le \|y^*\|, \;\; \left(\frac{x^*}{\tau_r}, -y^*\right) \in \overline{N}(\mathrm{epi}_C(G), (\bar{x}, G(\bar{x}))) \subset N(\mathrm{epi}_C(G), (\bar{x}, G(\bar{x}))),$$

and so $\frac{x^*}{\tau_r\|y^*\|} \in D_e^*G(\bar{x})\left(\frac{y^*}{\|y^*\|}\right) \subset \partial_C G(\bar{x})$. This and Proposition 3.9 imply that

$$d(0, \partial_C F(\bar{x})) \le \frac{\|x^*\|}{\tau_r\|y^*\|} + \|G-F\|_{\bar{x}} < \frac{1}{\tau_r r} + \|G-F\|_{\bar{x}},$$

namely $\tau_r < \frac{1}{r(d(0,\partial_C F(\bar{x})) - \|G-F\|_{\bar{x}})}$, contradicting the definition of τ_r. □

By Theorems 4.1 and 4.4, under the stated assumptions, we know that $d(0, \partial_C F(\bar{x})) > 0$ is not only a sufficient condition for F to have a stable error bound at $\bar{x}$ but also yields the following quantitative estimate for error bound modulus

$$\tau(G, \bar{x}, C) \le \frac{1}{\gamma_C\big(d(0, \partial_C F(\bar{x})) - \|G-F\|_{\bar{x}}\big)}.$$

This leads to us to consider sufficient and necessary conditions for $d(0, \partial_C F(\bar{x})) > 0$. For convenience, we use the Clarke tangent derivative $D_C F(\bar{x}) : X \rightrightarrows Y$ defined by

$$D_C F(\bar{x})(u) := \{(v \in Y : \; (u, v) \in T(\mathrm{epi}_C(F), (\bar{x}, F(\bar{x}))\} \quad \forall u \in X.$$

Proposition 4.5. *Let X, Y be Banach spaces and suppose that the ordering cone C of Y has a nonempty interior. Given $F \in \Upsilon(X, Y; C)$ and $\bar{x} \in \mathrm{dom}(F)$, the following statements hold:*
(i) If the Clarke tangent derivative $D_C F(\bar{x})$ is surjective, that is, $D_C F(\bar{x})(X) = Y$, then $d(0, \partial_C F(\bar{x})) > 0$.
(ii) If F is continuous at $\bar{x}$, then $d(0, \partial_C F(\bar{x})) > 0$ if and only if $D_C F(\bar{x})(X) = Y$.

Proof. To prove (i), suppose that $D_C F(\bar{x})(X) = Y$. Since $\mathrm{gph}(D_C F(\bar{x})) = T(\mathrm{epi}_C(F), (\bar{x}, F(\bar{x})))$ is a closed convex cone, the Robinson–Ursescu theorem implies that there exists $r > 0$ such that $rB_Y \subset D_C F(\bar{x})(B_X)$. We claim that

$$d(0, \partial_C F(\bar{x})) \ge r. \tag{4.17}$$

Let x^* be an arbitrary element in $\partial_C F(\bar{x})$ with $y^* \in \mathcal{I}_{C^+}$ such that $x^* \in D_e^* F(\bar{x})(y^*)$. Then, for any $\varepsilon \in (0,\ 1)$ there exist $y_\varepsilon \in B_Y$ and $u_\varepsilon \in B_X$ such that

$$\langle y^*, -y_\varepsilon\rangle > \|y^*\| - \varepsilon = 1 - \varepsilon \;\; \text{and} \;\; \langle x^*, u_\varepsilon\rangle - \langle y^*, r y_\varepsilon\rangle \le 0$$

(because $ry_\varepsilon \in rB_Y \subset D_CF(\bar{x})(B_X)$); consequently $\|x^*\| \geq r(1-\varepsilon)$ and so $\|x^*\| \geq r$ as ε is arbitrary in $(0,\ 1)$. This shows that (4.17) holds.

To prove (ii), suppose that F is continuous at $\bar{x}$. Since the sufficiency part is just (i), it suffices to show the necessity part. To prove this, suppose that $d(0, \partial_C F(\bar{x})) > 0$. We claim that

$$\mathrm{cl}(D_CF(\bar{x})(X)) = Y. \tag{4.18}$$

Indeed, if this is not the case, $\mathrm{cl}(D_CF(\bar{x})(X)) \neq Y$. Since $\mathrm{cl}(D_CF(\bar{x})(X))$ is a closed convex cone in Y, it follows from the separation theorem that there exists $v^* \in Y^*$ with $\|v^*\| = 1$ such that $\sup\{\langle v^*, y\rangle :\ y \in D_CF(\bar{x})(X)\} = 0$. Noting that $D_CF(\bar{x})(X) = D_CF(\bar{x})(X) + C$ (thanks to Lemma 2.3(ii)), it follows that $y^* = -v^* \in \mathcal{I}_{C^+}$ and $(0, -y^*) \in N(\mathrm{epi}_C(F), (\bar{x}, F(\bar{x})))$ and so $0 \in D_e^*F(\bar{x})(y^*) \subset \partial_C F(\bar{x})$, contradicting $d(0, \partial_C F(\bar{x})) > 0$. This shows that (4.18) holds. Take an element c in $\mathrm{int}(C)$ and $\delta > 0$ such that $B(c, \delta) \subset C$. It follows from (4.18) that there exists $y \in D_CF(\bar{x})(X) \cap -B(c, \frac{\delta}{2})$. Hence

$$B\Big(0, \frac{\delta}{2}\Big) \subset y + B(c, \delta) \subset D_CF(\bar{x})(X) + C = D_CF(\bar{x})(X).$$

Since $D_CF(\bar{x})(X)$ is a cone, this implies that $D_CF(\bar{x})(X) = Y$. The proof is complete. □

From Theorem 4.4 and Proposition 4.5, we have the following corollary.

Corollary 4.6. *Let X, Y be Banach spaces and suppose that the ordering cone C of Y has a nonempty interior. Let $F \in \Upsilon(X, Y; C)$ and $\bar{x} \in \mathrm{dom}(F)$ be such that the sublinear conic inequality $D_CF(\bar{x})(u) <_C 0$ is solvable in the sense that there exists $u_0 \in X$ such that $D_CF(\bar{x})(u_0) \cap -\mathrm{int}(C) \neq \emptyset$. Further suppose that one of the following three conditions is satisfied:*
(i) G is continuous on some neighborhood of $\bar{x}$ and $G \in \mathcal{QS}_{\bar{x}}(X, Y; C)$.
(ii) $G \in \Gamma(X, Y; C)$.
(iii) X, Y are Asplund spaces and $G \in \mathcal{QS}^w_{\bar{x}}(X, Y; C)$.
Then

$$\tau(G, \bar{x}; C) \leq \frac{1}{\gamma_C\big[d(0, \partial_C F(\bar{x})) - \|G - F\|_{\bar{x}}\big]_+}.$$

Remark. Let $F : X \to Y^\bullet$ be smooth around $\bar{x} \in \mathrm{dom}(F)$. It is easy to verify that $D_CF(\bar{x})(u) = F'(\bar{x})(u) + C$ for all $u \in X$, where $F'(\bar{x})$ denotes the derivative of F at $\bar{x}$, and so $D_CF(\bar{x})(u) <_C 0$ is solvable if and only if $F'(\bar{x})(X) + C = Y$. Thus, as a byproduct of Corollary 4.6, we have the following result: *if F is smooth around $\bar{x}$ and $F'(\bar{x})(X) + C = Y$ then F has a stable error bound at $\bar{x}$ with respect to C*. Clearly, $F'(\bar{x})(X) + C = Y$ is weaker than $F'(\bar{x})(X) = Y$, the later condition being an important assumption for the famous Lyusternik–Graves theorem: *if F is smooth around $\bar{x}$ and $F'(\bar{x})(X) = Y$ then F is metrically regular at $(\bar{x}, F(\bar{x}))$, namely there exist $\tau, \delta \in (0,\ +\infty)$ such that*

$$d(x, F^{-1}(y)) \leq \tau\|y - F(x)\| \quad \forall (x, y) \in B(\bar{x}, \delta) \times B(F(\bar{x}), \delta)).$$

Let $L(X, Y)$ denote the space of all bounded linear operators from X to Y. Recall that $T \in L(X, Y)$ is said to satisfy the Slater condition with respect to C if

$$T(X) \cap -\mathrm{int}(C) \neq \emptyset \text{ (equivalently, } T(X) + C = Y). \tag{4.19}$$

By the Robinson–Ursescu theorem, the above Slater condition means

$$r(T, C) := \sup\{r \geq 0 :\ rB_Y \subset T(B_X) + C\} > 0.$$

In the case when $X = \mathbb{R}^m$, $Y = \mathbb{R}^n$ and $C = \mathbb{R}^n_+$, under the Slater condition, Luo and Tseng [16] considered the stability of error bounds for a conic linear inequality defined by a bounded linear operator T and $C = \mathbb{R}^n_+$. Under the Slater condition, with the help of $r(T, C)$, we provide the following quantitative estimate of local error bound moduli of conic linear inequalities:

$$\tau(\tilde{T}, \bar{x}, C) \leq \frac{1}{\gamma_C(r(T, C) - \|\tilde{T} - T\|)} \tag{4.20}$$

for all $(\tilde{T}, \bar{x}) \in L(X, Y) \times X$ with $\|\tilde{T} - T\| < r(T, C)$. Indeed, noting that $\mathrm{epi}_C(T) = \{(x, Tx + c) : (x, c) \in X \times C\}$ is a closed convex cone, it is easy to verify that

$$T(\mathrm{epi}_C(T), (x, Tx)) \supset \mathrm{epi}_C(T) \quad \forall x \in X.$$

Hence, by the definition of $r(T, C)$,

$$rB_Y \subset T(B_X) + C \subset D_C T(\bar{x})(B_X) \quad \forall (\bar{x}, r) \in X \times (0,\ r(T, C)).$$

This and (4.17) imply that $d(0, \partial_C T(\bar{x})) \geq r(T, C)$ for all $\bar{x} \in X$. It follows from Corollary 4.6 that (4.20) holds for all $(\tilde{T}, \bar{x}) \in L(X, Y) \times X$ with $\|\tilde{T} - T\| < r(T, C)$.

Theorems 4.1 and 4.4 show that, in a sense, $0 \notin \mathrm{bd}(\partial_C F(\bar{x}))$ is a sufficient condition for (CIE) to have a stable local error bound at $\bar{x}$. The following theorem shows that the converse also holds under some mild assumption.

Theorem 4.7. *Let X, Y be Asplund spaces with the ordering cone C of Y having a nonempty interior. Suppose that $F \in \Upsilon(X, Y; C)$ is quasi-subsmooth and locally Lipschitz on a neighborhood of $\bar{x} \in \mathrm{dom}(F)$. Then the following statements are equivalent:*

(i) $0 \notin \mathrm{bd}(\partial_C F(\bar{x}))$.

(ii) *There exist $L, \delta \in (0,\ +\infty)$ such that $\tau(G, \bar{x}, C) \leq L$ whenever $G \in \mathcal{QS}^w_{\bar{x}}(X, Y; C)$ and $\|G - F\|_{\bar{x}} < \delta$.*

(iii) *For any $y \in \mathrm{int}(C)$ with $\|y\| = 1$ there exist $L, \delta \in (0,\ +\infty)$ such that*

$$\tau\big(F + \varepsilon(\|\cdot - \bar{x}\| + \langle x^*, \cdot\rangle)y, \bar{x}, C\big) \leq L \ \ \forall (\varepsilon, x^*) \in [0,\ \delta) \times B_{X^*}. \tag{4.21}$$

(iv) *There exist $y \in \mathrm{int}(C)$ with $\|y\| = 1$ and $L, \delta \in (0,\ +\infty)$ such that (4.21) holds.*

Proof. (i)⇒(ii) is immediate from Theorems 4.1 and 4.4. By Proposition 3.8, one has

$$F + \varepsilon(\|\cdot - \bar{x}\| + x^*)y \in \mathcal{QS}^w_{\bar{x}}(X, Y; C) \quad \forall (\varepsilon, y, x^*) \in (0,\ +\infty) \times \mathrm{int}(C) \times B_{X^*}.$$

Therefore, (ii)⇒(iii) holds. Since (iii)⇒(iv) is trivial, it remains to show (iv)⇒(i). To do this, suppose to the contrary that $0 \in \mathrm{bd}(\partial_C F(\bar{x}))$, that is,

$$0 \in \mathrm{cl}(\partial_C F(\bar{x})) \ \text{ and } \ 0 \notin \mathrm{int}(\mathrm{cl}(\partial_C F(\bar{x}))). \tag{4.22}$$

Hence

$$\varepsilon B_{X^*} \not\subset \partial_C F(\bar{x}) \quad \forall \varepsilon \in (0,\ +\infty). \tag{4.23}$$

By (iv), there exist $y \in \mathrm{int}(C)$ with $\|y\| = 1$ and $L, \delta, r \in (0,\ +\infty)$ such that (4.21) holds and $y + rB_Y \subset C$. It follows from Lemma 2.6(i) that

$$1 \leq \|y^*\| \leq \frac{1}{r} \quad \forall y^* \in C^+_y. \tag{4.24}$$

We claim that

$$\varepsilon B_{X^*} \not\subset D_e^* F(\bar{x})(C_y^+) \quad \forall \varepsilon \in (0, +\infty). \tag{4.25}$$

Indeed, if not, namely there exists $\varepsilon_0 > 0$ such that

$$\varepsilon_0 B_{X^*} \subset D_e^* F(\bar{x})(C_y^+), \tag{4.26}$$

then

$$r\varepsilon_0 B_{X^*} \subset \partial_C F(\bar{x}) \tag{4.27}$$

(contradicting (4.23)). To see how (4.27) follows from (4.26), let $x^* \in B_{X^*}$. Then (4.26) implies that there exist $y_0^*, y^* \in C_y^+$ such that $0 \in D_e^* F(\bar{x})(y_0^*)$ and $\varepsilon_0 x^* \in D_e^* F(\bar{x})(y^*)$. Noting that, since $D_e^* F(\bar{x})$ is a convex multifunction, it follows that

$$t\varepsilon_0 x^* = (1-t)0 + t\varepsilon_0 x^* \subset D_e^* F(\bar{x})((1-t)y_0^* + ty^*) \;\; \forall t \in [0, 1],$$

and so

$$\frac{t\varepsilon_0 x^*}{\|(1-t)y_0^* + ty^*\|} \in D_e^* F(\bar{x}) \left(\frac{(1-t)y_0^* + ty^*}{\|(1-t)y_0^* + ty^*\|} \right) \subset \partial_C F(\bar{x}) \quad \forall t \in [0, 1]. \tag{4.28}$$

Since the continuous function $t \mapsto \frac{t}{\|(1-t)y_0^* + ty^*\|}$ takes value $\frac{1}{\|y^*\|}$ at $t = 1$, there exists $t_0 \in (0, 1]$ such that $\frac{t_0}{\|(1-t_0)y_0^* + t_0 y^*\|} = r$ (thanks to (4.24)). This and (4.28) imply that $r\varepsilon_0 x^* \in \partial_C F(\bar{x})$, showing (4.27). Therefore our claim (4.25) is valid. For each $x \in \mathrm{dom}(F)$, let $\Phi(x)$ and $\Lambda(x)$ be defined by

$$\Phi(x) := \max_{y^* \in C_y^+} \langle y^*, F(x) - F(\bar{x}) \rangle \;\text{ and }\; \Lambda(x) := \{y^* \in C_y^+ :\; \langle y^*, F(x) - F(\bar{x}) \rangle = \Phi(x)\}. \tag{4.29}$$

By assumption, we take $\delta' \in (0, \delta)$ such that F (and $F - F(\bar{x})$) are quasi-subsmooth and Lipschitz on $B(\bar{x}, \delta')$. From Proposition 3.7, it follows that

$$\hat{\partial}\Phi(x) = \partial\Phi(x) = D_e^* F(x)(\Lambda(x)) \quad \forall x \in B(\bar{x}, \delta') \tag{4.30}$$

and, in particular, $\mathrm{cl}(\hat{\partial}\Phi(\bar{x})) = \hat{\partial}\Phi(\bar{x}) = D_e^* F(\bar{x})(C_y^+)$ (because $\Lambda(\bar{x}) = C_y^+$). Hence, by (4.22) and (4.25), one has $0 \in \hat{\partial}\Phi(\bar{x})$ and $\varepsilon B_{X^*} \not\subset \hat{\partial}\Phi(\bar{x})$ for all $\varepsilon \in (0, +\infty)$. Let $\varepsilon \in (0, \delta)$ be such that $(2 + \frac{1}{r})\varepsilon < \frac{1}{4L}$, and take $u_\varepsilon^* \in \varepsilon B_{X^*} \setminus \hat{\partial}\Phi(\bar{x})$; then there exist $\eta \in (0, \delta')$ and a sequence $\{x_n\}$ in X converging to $\bar{x}$ such that

$$0 \le \Phi(x) - \Phi(\bar{x}) + \varepsilon\|x - \bar{x}\| = \Phi(x) + \varepsilon\|x - \bar{x}\| \quad \forall x \in B[\bar{x}, \eta], \tag{4.31}$$

$\langle u_\varepsilon^*, x_n - \bar{x} \rangle > \Phi(x_n) - \Phi(\bar{x})$, and so $\varepsilon\|x_n - \bar{x}\| > \Phi(x_n)$ for all $n \in \mathbb{N}$. Hence

$$\Phi(x_n) + \varepsilon\|x_n - \bar{x}\| < \inf\{\Phi(x) + \varepsilon\|x - \bar{x}\| :\; x \in B[\bar{x}, \eta]\} + 2\varepsilon\|x_n - \bar{x}\| \;\; \forall n \in \mathbb{N}.$$

By the Ekeland variational principle, for any $n \in \mathbb{N}$ there exists $a_n \in B[\bar{x}, \eta]$ such that $\|a_n - x_n\| < \|x_n - \bar{x}\|$ and

$$\Phi(a_n) + \varepsilon\|a_n - \bar{x}\| \le \Phi(x) + \varepsilon\|x - \bar{x}\| + 2\varepsilon\|x - a_n\| \;\; \forall x \in B[\bar{x}, \eta]. \tag{4.32}$$

Therefore, $a_n \neq \bar{x}$ for all $n \in \mathbb{N}$ and $a_n \to \bar{x}$ (as $x_n \to \bar{x}$). Without loss of generality we assume that $a_n \in B(\bar{x}, \eta)$ for all $n \in \mathbb{N}$. Thus, by (4.32), one has

$$0 \in \partial(\Phi + \varepsilon\| \cdot - \bar{x}\| + 2\varepsilon\| \cdot - a_n\|)(a_n) \subset \partial\Phi(a_n) + 3\varepsilon B_{X^*} \quad \forall n \in \mathbb{N}. \tag{4.33}$$

By the uniform boundedness principle in functional analysis (*cf.*, Thm. 1.6.9 from [18]), there exists $x^* \in B_{X^*}$ such that $\sup\left\{\left|\left\langle x^*, \frac{n(a_n-\bar{x})}{\|a_n-\bar{x}\|}\right\rangle\right| : n \in \mathbb{N}\right\} = +\infty$. Without loss of generality, we assume that $\langle x^*, a_n - \bar{x}\rangle > 0$ for all $n \in \mathbb{N}$ (taking a subsequence of $\{a_n\}$ and replacing x^* with $-x^*$ if necessary). It follows from (4.31) that

$$0 < \Phi(a_n) + \varepsilon\|a_n - \bar{x}\| + \varepsilon\langle x^*, a_n - \bar{x}\rangle \qquad \forall n \in \mathbb{N}. \tag{4.34}$$

Take $y_n^* \in \Lambda(a_n)$ (see (4.29)). Then $\langle y_n^*, y\rangle = 1$, $\Phi(a_n) = \langle y_n^*, F(a_n) - F(\bar{x})\rangle$, and hence we can rewrite (4.34) as

$$0 < \langle y_n^*, F(a_n) - F(\bar{x}) + \varepsilon(\|a_n - \bar{x}\| + \langle x^*, a_n - \bar{x}\rangle)y\rangle = \langle y_n^*, G(a_n) - G(\bar{x})\rangle,$$

where $G := F + \varepsilon(\| \cdot - \bar{x}\| + x^*)y$. It follows that

$$G(a_n) - G(\bar{x}) \notin -C. \tag{4.35}$$

By (4.21), there exists $\bar{\delta} > 0$ such that

$$d(x, S(G, \bar{x}, C)) \leq 2Ld(G(x) - G(\bar{x}), -C) \quad \forall x \in B(\bar{x}, \bar{\delta}). \tag{4.36}$$

Take a sequence $\{u_n\}$ in $S(G, \bar{x}, C)$ such that

$$\|a_n - u_n\| \leq \Big(1 + \frac{1}{n}\Big)d(a_n, S(G, \bar{x}, C)) \leq \Big(1 + \frac{1}{n}\Big)\|a_n - \bar{x}\| \to 0. \tag{4.37}$$

Noting that G is quasi-subsmooth and Lipschitz at $\bar{x}$ (thanks to Prop. 3.8(ii)), it follows from Lemma 3.6(i) that we can assume without loss of generality that

$$\langle a^*, u_n - a_n\rangle \leq \langle y^*, G(u_n) - G(a_n)\rangle + \varepsilon\|u_n - a_n\| \quad \forall y^* \in C_y^+ \text{ and } a^* \in D_e^*G(a_n)(y^*).$$

Since $G(u_n) \leq_C G(\bar{x})$, this implies that

$$\langle a^*, u_n - a_n\rangle \leq \langle y^*, G(\bar{x}) - G(a_n)\rangle + \varepsilon\|u_n - a_n\| \quad \forall y^* \in C_y^+ \text{ and } a^* \in D_e^*G(a_n)(y^*).$$

Hence

$$(\|a^*\| + \varepsilon)\|u_n - a_n\| \geq \langle y^*, G(a_n) - G(\bar{x})\rangle \quad \forall y^* \in C_y^+ \text{ and } a^* \in D_e^*G(a_n)(y^*). \tag{4.38}$$

From (4.29) and the definition of G, it is easy to verify that

$$\Lambda(a_n) = \Big\{v^* \in C_y^+ : \langle v^*, G(a_n) - G(\bar{x})\rangle = \max_{y^* \in C_y^+} \langle y^*, G(a_n) - G(\bar{x})\rangle\Big\},$$

and so $\langle y^*, G(a_n) - G(\bar{x})\rangle \geq d(G(a_n) - G(\bar{x}), -C)$ for all $y^* \in \Lambda(a_n)$ (thanks to Lemma 2.6(iv) and (4.35)). It follows from (4.38) that

$$(\|a^*\| + \varepsilon)\|u_n - a_n\| \geq d(G(a_n) - G(\bar{x}), -C) \quad \forall a^* \in D_e^*G(a_n)(\Lambda(a_n)).$$

Thus, by (4.36), one has

$$d(a_n, S(G,\bar{x},C)) \le 2L(\|a^*\|+\varepsilon)\|u_n-a_n\| \quad \forall a^* \in D_e^*G(a_n)(\Lambda(a_n)).$$

This, together with (4.37) and (4.35), implies that $\frac{n}{1+n} \le 2L(\|a^*\|+\varepsilon)$ for all $a^* \in D_e^*G(a_n)(\Lambda(a_n))$, namely $\frac{n}{2L(1+n)} - \varepsilon \le d(0, D_e^*G(\Lambda(a_n)))$. It follows from Proposition 3.9 and (4.24) that

$$\frac{n}{2L(1+n)} - \varepsilon \le d(0, D_e^*F(a_n)(\Lambda(a_n))) + \frac{1}{r}\|G-F\|_{a_n} \le d(0, D_e^*F(a_n)(\Lambda(a_n))) + \frac{2\varepsilon}{r}. \tag{4.39}$$

By (4.33) and (4.30), one has $0 \in \partial\Phi(a_n) + 3\varepsilon B_{X^*} = D_e^*F(a_n)(\Lambda(a_n)) + 3\varepsilon B_{X^*}$, and so $d(0, D_e^*F(a_n)(\Lambda(a_n))) \le 3\varepsilon$. This and (4.39) imply that $\frac{1}{2L} \le (4+\frac{2}{r})\varepsilon$, contradicting the choice of ε. The proof is complete. □

It is worth mentioning that the perturbations of F appearing in (iii) and (iv) of Theorem 4.7 are all in the same direction as y, which are different from the perturbations of F in (ii) of Theorem 4.7. In the special case when $Y=\mathbb{R}$ and $C=\mathbb{R}_+$, Theorems 4.1, 4.4 and 4.7 reduce to the corresponding results in [31].

In the convexity case, the Asplund space and Lipschitz assumptions of Theorem 4.7 can be relaxed; moreover, linear perturbation functions can be considered.

Theorem 4.8. *Let X, Y be Banach spaces with the ordering cone C of Y having a nonempty interior. Let $F \in \Gamma(X,Y;C) = \{H \in \Upsilon(X,Y;C) : \ \mathrm{epi}_C(H) \text{ is convex}\}$ and $\bar{x} \in \mathrm{dom}(F)$ be such that F is bounded above with respect to C at $\bar{x}$, namely there exist $\gamma, M \in (0, +\infty)$ such that*

$$F(B(\bar{x},\gamma)) \subset MB_Y - C. \tag{4.40}$$

Then the following statements are equivalent:

(i) $0 \notin \mathrm{bd}(\partial_C F(\bar{x}))$.

(ii) *There exist $L, \delta \in (0, +\infty)$ such that $\tau(G,\bar{x},C) \le L$ for all $G \in \Gamma(X,Y;C)$ with $\|G-F\|_{\bar{x}} < \delta$.*

(iii) *There exist $y \in \mathrm{int}(C)$ with $\|y\|=1$ and $L, \delta \in (0, +\infty)$ such that*

$$\tau\big(F + \varepsilon\langle x^*, \cdot\rangle y, \bar{x}, C\big) \le L \ \ \forall (\varepsilon, x^*) \in [0, \ \delta) \times B_{X^*}. \tag{4.41}$$

Proof. By the C-convexity of F, one has $F \in \mathcal{QS}_{\bar{x}}^w(X,Y,;C)$. Hence (i)⇒(ii) is immediate from Theorems 4.1 and 4.4. Let $y \in Y$ and $x^* \in X^*$. From the C-convexity of F, it is easy to verify that $F + \varepsilon\langle x^*, \cdot\rangle y$ is C-convex and $\big\|F + \varepsilon\langle x^*,\cdot\rangle y - F\big\|_{\bar{x}} \le \varepsilon\|x^*\|\|y\|$ for all $\varepsilon > 0$. Therefore, (ii)⇒(iii) is true. It remains to show (iii)⇒(i). To do this, suppose to the contrary that (iii) holds but (i) does not hold. Then, (4.22) holds, and similar to the corresponding part of the proof of Theorem 4.7, there exist $y \in \mathrm{int}(C)$ with $\|y\|=1$ and $L, \delta, r \in (0, +\infty)$ such that (4.41), (4.24) and (4.25) hold. Let $\Phi(x)$ and $\Lambda(x)$ be as in (4.29). Then, Φ is convex. By (4.40) and (4.24), one has $\langle y^*, F(x) - F(\bar{x})\rangle \le \frac{2M}{r}$ for all $(y^*, x) \in C_y^+ \times B(\bar{x},\gamma)$. Since each function $x \mapsto \langle y^*, F(x) - F(\bar{x})\rangle$ is convex, there exists $L' \in (0, +\infty)$ such that

$$|\langle y^*, F(x_1) - F(\bar{x})\rangle - \langle y^*, F(x_2) - F(\bar{x})\rangle| \le L'\|x_1 - x_2\| \quad \forall y^* \in C_y^+ \text{ and } \forall x_1, x_2 \in B\Big(\bar{x}, \frac{\gamma}{2}\Big).$$

Noting that C_y^+ is weak*-compact, Ioffe–Tikhomirov theorem (*cf.* Thm. 2 from [10]) implies that

$$\partial\Phi(\bar{x}) = \overline{\mathrm{co}}^{w^*}\left(\bigcup_{y^* \in C_y^+} \partial(y^* \circ F)(\bar{x})\right) = \overline{\mathrm{co}}^{w^*}\big(D_e^*F(\bar{x})(C_y^+)\big) = D_e^*F(\bar{x})(C_y^+),$$

where the last equality holds by Lemma 2.6(iii). Thus, by (4.22) and (4.25), one has

$$0 \in \partial\Phi(\bar{x}) \quad \text{and} \quad \varepsilon B_{X^*} \not\subset \partial\Phi(\bar{x}) \quad \forall \varepsilon \in (0,\ +\infty).$$

Hence

$$0 \leq \Phi(x) - \Phi(\bar{x}) = \Phi(x) \quad \forall x \in X. \tag{4.42}$$

Let $\varepsilon \in (0,\ \delta)$ be such that $\frac{1}{2L} > 2\varepsilon + \frac{\varepsilon}{r}$, and take $u_\varepsilon^* \in \varepsilon B_{X^*} \setminus \partial\Phi(\bar{x})$. Then there exists a sequence $\{x_n\}$ in X converging to $\bar{x}$ such that $\langle u_\varepsilon^*, x_n - \bar{x}\rangle > \Phi(x_n) - \Phi(\bar{x}) = \Phi(x_n)$ for all $n \in \mathbb{N}$. Hence

$$\Phi(x_n) < \inf_{x\in X} \Phi(x) + \langle u_\varepsilon^*, x_n - \bar{x}\rangle \leq \inf_{x\in X} \Phi(x) + \varepsilon\|x_n - \bar{x}\| \ \ \forall n \in \mathbb{N}.$$

By the Ekeland variational principle, for any $n \in \mathbb{N}$ there exists $a_n \in X$ such that $\|a_n - x_n\| < \|x_n - \bar{x}\|$ and $\Phi(a_n) \leq \Phi(x) + \varepsilon\|x - a_n\|$ for all $x \in X$. Therefore, $a_n \neq \bar{x}$, $a_n \to \bar{x}$ and

$$0 \in \partial(\Phi + \varepsilon\|\cdot - a_n\|)(a_n) \subset \partial\Phi(a_n) + \varepsilon B_{X^*} \quad \forall n \in \mathbb{N}. \tag{4.43}$$

Without loss of generality we assume that there exist $x^* \in B_{X^*}$ such that $\langle x^*, a_n - \bar{x}\rangle > 0$ for all $n \in \mathbb{N}$ (taking a subsequence of $\{a_n\}$ if necessary). It follows from (4.42) that

$$0 < \Phi(a_n) + \varepsilon\langle x^*, a_n - \bar{x}\rangle \qquad \forall n \in \mathbb{N}. \tag{4.44}$$

On the other hand, by (4.41), there exists $\delta' > 0$ such that

$$d(x, S(G, \bar{x}, C)) \leq 2Ld(G(x) - G(\bar{x}), -C) \quad \forall x \in B(\bar{x}, \delta'), \tag{4.45}$$

where $G := F + \varepsilon\langle x^*, \cdot\rangle y$. Similar to the corresponding part of the proof of Theorem 4.7, we have

$$\frac{n}{2L(1+n)} - \varepsilon \leq d(0, D_e^*F(a_n)(\Lambda(a_n))) + \frac{1}{r}\|G - F\|_{a_n} \leq d(0, D_e^*F(a_n)(\Lambda(a_n))) + \frac{\varepsilon}{r}. \tag{4.46}$$

Since the family $\{\langle y^*, F - F(\bar{x})\rangle :\ y^* \in C_y^+\}$ is convex and Lipschitz on $B(\bar{x}, \frac{\gamma}{2})$, one has $\partial\Phi(a_n) \subset D_e^*F(a_n)(\Lambda(a_n))$. It follows from (4.43) that $0 \in D_e^*F(a_n)(\Lambda(a_n)) + \varepsilon B_{X^*}$. This and (4.46) imply that $\frac{n}{2L(1+n)} - \varepsilon \leq \varepsilon + \frac{\varepsilon}{r}$, and so $\frac{1}{2L} \leq 2\varepsilon + \frac{\varepsilon}{r}$, contradicting the choice of ε. □

In the special case when $Y = \mathbb{R}$, $C = \mathbb{R}_+$ and F is convex, the stability of the local error bound for F at $\bar{x}$ was characterized in terms of $0 \notin \partial F(\bar{x})$ (as in Thm. 4.8) in [13, 20]. It is easy to verify that some results in [13, 20] can be recaptured by Theorems 4.1, 4.4 and 4.8. For example, in the case when $Y = \mathbb{R}$, $C = \mathbb{R}_+$ and F is convex, Theorem 4.4 recaptures ([13], Thm. 8(i)), and Theorem 4.8 extends ([13], Thm. 8(iii)) to the infinite-dimensional case from the finite-dimensional case.

Imitating the corresponding notion in a very recent paper [12] by Kruger *et al.*, one can adopt the following radii of error bound for a C-convex vector-valued functions F at $\bar{x}$ with respect to C:

$$\mathcal{R}_c(F, \bar{x}, C) := \inf\{\varepsilon > 0 :\ \tau_{\mathrm{Ptb}_c}(F, \bar{x}, C, \varepsilon) = +\infty\} \ \text{ and } \ \mathcal{R}_l(F, \bar{x}, C) := \inf\{\varepsilon > 0 :\ \tau_{\mathrm{Ptb}_l}(F, \bar{x}, C, \varepsilon) = +\infty\},$$

where

$$\tau_{\mathrm{Ptb}_c}(F, \bar{x}, C, \varepsilon) := \sup\{\tau(G, \bar{x}, C) :\ G \in F + \Gamma(X, Y; C) \text{ with } \|G - F\|_{\bar{x}} \leq \varepsilon\}$$

and

$$\tau_{\mathrm{Ptb}_l}(F,\bar{x},C,\varepsilon) := \sup\{\tau(F+T,\bar{x},C):\ T:X\to Y \text{ is continuous linear operator with } \|T\|\le\varepsilon\}.$$

In the case when $Y=\mathbb{R}$ and $C=\mathbb{R}_+$, it was proved in [12] that

$$d(0,\mathrm{bd}(\partial_C F(\bar{x}))) = \mathcal{R}_c(F,\bar{x},C) = \mathcal{R}_l(F,\bar{x},C),$$

which implies the following stability

$$\tau(G,\bar{x},C) < +\infty \quad \forall G\in F+\varGamma(X,Y;C) \text{ with } \|G-F\|_{\bar{x}} < \mathcal{R}_c(F,\bar{x},C).$$

Replacing $\tau(G,\bar{x},C)<+\infty$, Theorems 4.1 and 4.4 provide a better estimate for error bound modulus $\tau(G,\bar{x},C)$:

$$\tau(G,\bar{x},C) \le \frac{1}{\gamma_C\big(d(0,\mathrm{bd}(\partial_C F(\bar{x}))) - \|G-F\|_{\bar{x}}\big)} \quad \forall G\in \mathcal{QS}^w_{\bar{x}}(X,Y;C) \text{ with } \|G-F\|_{\bar{x}} < d(0,\mathrm{bd}(\partial_C F(\bar{x}))).$$

From Theorems 4.1 and 4.4, it is clear that

$$d(0,\mathrm{bd}(\partial_C F(\bar{x}))) \le \mathcal{R}_l(F,\bar{x},C) \le \mathcal{R}_c(F,\bar{x},C).$$

Moreover, in the convex case, it is easy to verify from Theorem 4.8 that

$$\gamma_C\mathcal{R}_l(F,\bar{x},C) \le d(0,\mathrm{bd}(\partial_C F(\bar{x}))).$$

Therefore, in the special case when $Y=\mathbb{R}^n$ and $C=\mathbb{R}^n_+$, $d(0,\mathrm{bd}(\partial_C F(\bar{x}))) = \mathcal{R}_l(F,\bar{x},C)$ (because $\gamma_C=1$).

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