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ON THE TOTAL VARIATION WASSERSTEIN GRADIENT FLOW AND THE TV-JKO SCHEME

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Abstract. We study the JKO scheme for the total variation, characterize the optimizers, prove some of their qualitative properties (in particular a form of maximum principle and in some cases, a minimum principle as well). Finally, we establish a convergence result as the time step goes to zero to a solution of a fourth-order nonlinear evolution equation, under the additional assumption that the density remains bounded away from zero, this lower bound is shown in dimension one and in the radially symmetric case.

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1. Introduction

Variational schemes based on total variation are extremely popular in image processing for denoising purposes, in particular the seminal work of Rudin *et al.* [25] has been extremely influential and is still the object of an intense stream of research, see [10] and the references therein. Continuous-time counterparts are well-known to be related to the L^2 gradient flow of the total variation, see Bellettini *et al.* [3] and the mean-curvature flow, see Evans and Spruck [14]. The gradient flow of the total variation for other Hilbertian structures may be natural as well and in particular the H^{-1} case, leads to a singular fourth-order evolution equation studied by Giga and Giga [15], Giga *et al.* [16]. In the present work, we consider another metric, namely the Wasserstein one.

Given an open subset Ω of $\mathbb{R}^d$ and $\rho \in L^1(\Omega)$, recall that the total variation of ρ is given by

$$J(\rho) := \sup \left\{ \int_\Omega \operatorname{div}(z)\rho \ : \ z \in C_c^1(\Omega), \ \|z\|_{L^\infty} \leq 1 \right\} \tag{1.1}$$

and $\mathrm{BV}(\Omega)$ is by definition the subspace of $L^1(\Omega)$ consisting of those ρ's in $L^1(\Omega)$ such that $J(\rho)$ is finite. The following fourth-order nonlinear evolution equation

$$\partial_t \rho + \operatorname{div}\left(\rho \, \nabla \operatorname{div}\left(\frac{\nabla \rho}{|\nabla \rho|} \right) \right) = 0, \ \text{in } (0,T) \times \Omega, \ \rho_{|t=0} = \rho_0, \tag{1.2}$$

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supplemented by the zero-flux boundary condition

$$\rho\nabla\text{div}\left(\frac{\nabla\rho}{|\nabla\rho|}\right)\cdot\nu=0 \text{ on } \partial\Omega \tag{1.3}$$

has been proposed in [7] for the purpose of denoising image densities. Numerical schemes for approximating the solutions of this equation have been investigated in [4, 7, 13]. One should consider weak solutions and in particular interpret the nonlinear term div $\left(\frac{\nabla\rho}{|\nabla\rho|}\right)$ as the negative of an element of the subdifferential of J at ρ.

At least formally, when ρ_0 is a probability density on Ω, (1.2)–(1.3) can be viewed as the Wasserstein gradient flow of J (we refer to the textbooks of Ambrosio *et al.* [1] and Santambrogio [26], for a detailed exposition). Following the seminal work of Jordan *et al.* [17] for the Fokker–Planck equation, it is reasonable to expect that solutions of (1.2) can be obtained, at the limit $\tau\to 0^+$, of the JKO Euler implicit scheme:

$$\rho_0^\tau=\rho_0,\ \rho_{k+1}^\tau\in\operatorname{argmin}\left\{\frac{1}{2\tau}W_2^2(\rho_k^\tau,\rho)+J(\rho),\ \rho\in\text{BV}(\Omega)\cap\mathcal{P}_2(\overline{\Omega})\right\}, \tag{1.4}$$

where $\mathcal{P}_2(\overline{\Omega})$ is the space of Borel probability measures $\overline{\Omega}$ with finite second moment and W_2 is the quadratic Wasserstein distance:

$$W_2^2(\rho_0,\rho_1):=\inf_{\gamma\in\Pi(\rho_0,\rho_1)}\left\{\int_{\mathbb{R}^d\times\mathbb{R}^d}|x-y|^2\text{d}\gamma(x,y)\right\}, \tag{1.5}$$

$\Pi(\rho_0,\rho_1)$ denoting the set of transport plans between ρ_0 and ρ_1 *i.e.* the set of probability measures on $\mathbb{R}^d\times\mathbb{R}^d$ having ρ_0 and ρ_1 as marginals. Our aim is to study in detail the discrete TV-JKO scheme (1.4) as well as its connection with (suitable weak solutions) of the PDE (1.2). Although the assertion that (1.2) is the TV Wasserstein gradient flow is central to the numerical schemes described in [4, 7, 13], there has been so far, to the best of our knowledge, no theoretical justification of this fact.

Fourth-order equations which are Wasserstein gradient flows of functionals involving the gradient of ρ, such as the Dirichlet energy or the Fisher information, have been studied by McCann *et al.* [22] who found a new method, *the flow interchange technique*, to prove higher-order estimates, we refer to [18] for a recent reference on this topic. The total variation is however too singular for such arguments to be directly applicable, as far as we know. We shall prove the convergence of JKO steps as $\tau\to 0^+$ under the extra assumption that densities remain bounded aways from zero. Whether this extra assumption is reasonable or not is related to a minimum principle issue, interesting in its own right, namely the monotonicty of the infimum along JKO steps. We shall see that, in a convex domain, JKO steps obey a maximum principle (the maximum of the density is nonincreasing along JKO steps). The corresponding minimum principle seems more difficult to prove and we have been able to establish it only in some particular cases, namely in dimension one and in the radially symmetric case, eventhough we conjecture it is satisfied in more general situations.

The paper is organized as follows. In Section 2, we start with the discussion of a few examples. Section 3 establishes optimality conditions for JKO steps thanks to an entropic regularization scheme. Section 4 is devoted to some properties of solutions of JKO steps and in particular a maximum principle based on a result of [11], we also establish a minimum principle in dimension one and in the radially symmetric case. Finally, in Section 5, we prove a conditional convergence result, we establish convergence of the JKO scheme, as $\tau\to 0^+$, under the extra assumption that the density remains away from zero, this covers the unidimensional case as well as the radially symmetric case when the initial condition is strictly positive.

2. Some examples

We first recall the Kantorovich dual formulation of W_2^2:

$$\frac{1}{2}W_2^2(\mu_0,\mu_1) = \sup\left\{\int_{\mathbb{R}^d}\psi \mathrm{d}\mu_0 + \int_{\mathbb{R}^d}\varphi \mathrm{d}\mu_1 \ : \ \psi(x)+\varphi(y) \leq \frac{|x-y|^2}{2}\right\} \tag{2.1}$$

an optimal pair (ψ,φ) for this problem is called a pair of Kantorovich potentials. The existence of Kantorovich potentials is well-known and such potentials can be taken to be conjugates of each other, *i.e.* such that

$$\varphi(x) = \inf_{y\in\mathbb{R}^d}\left\{\frac{1}{2}|x-y|^2 - \psi(y)\right\},\ \psi(y) = \inf_{x\in\mathbb{R}^d}\left\{\frac{1}{2}|x-y|^2 - \varphi(x)\right\},$$

which implies that φ and ψ are semi-concave (more precisely $\frac{1}{2}|.|^2-\varphi$ is convex). If μ_1 is absolutely continuous with respect to the d-dimensional Lebesgue measure, φ is differentiable μ_1 a.e. and the map $T=\mathrm{id}-\nabla\varphi$ is the gradient of a convex function pushing forward μ_1 to μ_0 which is in fact the optimal transport between μ_0 and μ_1 thanks to Brenier's theorem [5]. In such a case, we will simply refer to φ as a Kantorovich potential between μ_1 and μ_0. We refer the reader to [28] and [26] for details.

In this section, we will consider some explicit examples which rely on the following sufficient optimality condition (details for a rigorous derivation of the Euler–Lagrange equation for JKO steps will be given in Sect. 3) in the case of the whole space *i.e.* $\Omega=\mathbb{R}^d$. Let us also recall that by Sobolev inequality $\mathrm{BV}(\mathbb{R}^d)$ is continuously embedded in $L^{\frac{d}{d-1}}(\mathbb{R}^d)$.

Lemma 2.1. *Let $\rho_0\in\mathcal{P}_2(\mathbb{R}^d)$, $\tau>0$ and $\Omega=\mathbb{R}^d$ (so J is the total variation on the whole space), if $\rho_1\in\mathrm{BV}(\mathbb{R}^d)\cap\mathcal{P}_2(\mathbb{R}^d)$ is such that*

$$\frac{\varphi}{\tau}+\mathrm{div}(z)\geq 0, \text{with equality} \rho_1-\text{a.e.}, \tag{2.2}$$

where φ is a Kantorovich potential between ρ_1 and ρ_0 and $z\in C^1(\mathbb{R}^d)$, with $\|z\|_{L^\infty}\leq 1$, $\mathrm{div}(z)\in L^d(\mathbb{R}^d)$ (so that $\mathrm{div}(z)\rho_1\in L^1(\mathbb{R}^d)$), and

$$J(\rho_1) = \int_{\mathbb{R}^d}\mathrm{div}(z)\rho_1. \tag{2.3}$$

Then, setting

$$\Phi_{\tau,\rho_0}(\rho) := \frac{1}{2\tau}W_2^2(\rho_0,\rho)+J(\rho),\ \forall\rho\in\mathrm{BV}(\mathbb{R}^d)\cap\mathcal{P}_2(\mathbb{R}^d) \tag{2.4}$$

one has

$$\Phi_{\tau,\rho_0}(\rho_1)\leq\Phi_{\tau,\rho_0}(\rho),\ \forall\rho\in\mathrm{BV}(\mathbb{R}^d)\cap\mathcal{P}_2(\mathbb{R}^d).$$

Proof. For all $\rho\in\mathrm{BV}(\mathbb{R}^d)\cap\mathcal{P}_2(\mathbb{R}^d)$, $J(\rho)\geq\int_{\mathbb{R}^d}\mathrm{div}(z)\rho = J(\rho_1)+\int_{\mathbb{R}^d}\mathrm{div}(z)(\rho-\rho_1)$, and it follows from the Kantorovich duality formula that

$$\frac{1}{2\tau}W_2^2(\rho_0,\rho)\geq\frac{1}{2\tau}W_2^2(\rho_0,\rho_1)+\int_{\mathbb{R}^d}\frac{\varphi}{\tau}(\rho-\rho_1).$$

The claim then directly follows from (2.2). □

2.1. The case of a characteristic function

A simple illustration of Lemma 2.1 in dimension 1 concerns the case of a uniform ρ_0, (here and in the sequel we shall denote by χ_A the characteristic function of the set A):

$$\rho_0 = \rho_{\alpha_0},\ \alpha_0 > 0,\ \rho_\alpha := \frac{1}{2\alpha}\chi_{[-\alpha,\alpha]}.$$

It is natural to make the ansatz that the minimizer of Φ_{τ,ρ_0} defined by (2.4) remains of the form $\rho_1 = \rho_{\alpha_1}$ for some $\alpha_1 > \alpha_0$. The optimal transport between ρ_{α_1} and ρ_0 being the linear map $T = \frac{\alpha_0}{\alpha_1}\mathrm{id}$, a direct computation gives

$$\Phi_{\tau,\rho_0}(\rho_{\alpha_1}) = \frac{1}{\alpha_1} + \frac{1}{6\tau}(\alpha_1 - \alpha_0)^2,$$

which is minimal when α_1 is the only root in $(\alpha_0, +\infty)$ of

$$\alpha_1^2(\alpha_1 - \alpha_0) = 3\tau. \tag{2.5}$$

To check that this is the correct guess, we shall check that the conditions of Lemma 2.1 are met. It is easy to check that the potential defined by

$$\varphi(x) = \frac{1}{2\alpha_1}(\alpha_1 - \alpha_0)x^2 - \frac{3\tau}{2\alpha_1}$$

is a Kantorovich potential between $\rho_1 = \rho_{\alpha_1}$ and ρ_0. Define[1] then z_1 by

$$\tau z_1(x) := -\frac{(\alpha_1 - \alpha_0)}{6\alpha_1}x^3 + \frac{3\tau x}{2\alpha_1},\ x \in [-\alpha_1, \alpha_1]$$

extended by 1 on $[\alpha_1, +\infty)$ and -1 on $(-\infty, -\alpha_1]$. By construction $-1 \leq z_1 \leq 1$ (use the fact that it is odd and nondecreasing on $[0, \alpha_1]$ thanks to (2.5)), also $z_1'(\pm\alpha_1) = 0$ so that $z_1 \in C^1(\mathbb{R})$ and $z_1(\alpha_1) = 1$, $z_1(-\alpha_1) = -1$ and one easily checks that $J(\rho_1) = -\int_{\mathbb{R}} z_1 D\rho_1 = \int_{\mathbb{R}} z_1'\rho_1$ (here and in the sequel $D\rho_1$ denotes the Radon measure which is the distributional derivative of the BV function ρ_1). Moreover $\tau z_1' + \varphi \geq 0$ with an equality on $[-\alpha_1, \alpha_1]$. The optimality of $\rho_1 = \rho_{\alpha_1}$ then directly follows from Lemma 2.1.

Of course, the argument can be iterated so as to obtain the full TV-JKO sequence:

$$\rho_{k+1}^\tau = \mathrm{argmin}\ \Phi_{\tau,\rho_k^\tau} = \left(\frac{\alpha_{k+1}^\tau}{\alpha_k^\tau}\mathrm{id}\right)_\# \rho_k^\tau = \left(\frac{\alpha_{k+1}^\tau}{\alpha_0}\mathrm{id}\right)_\# \rho_0,$$

where α_k^τ is defined inductively by

$$(\alpha_{k+1}^\tau - \alpha_k^\tau)(\alpha_{k+1}^\tau)^2 = 3\tau,\ \alpha_0^\tau = \alpha_0,$$

which is nothing but the implicit Euler discretization of the ODE

$$\alpha'\alpha^2 = 3,\ \alpha(0) = \alpha_0,$$

whose solution is $\alpha(t) = (\alpha_0^3 + 9t)^{\frac{1}{3}}$. Extending ρ_k^τ in a piecewise constant way: $\rho^\tau(t) = \rho_{k+1}^\tau$ for $t \in (k\tau, (k+1)\tau]$, it is not difficult to check that ρ^τ converges (in $L^\infty((0,T), (\mathcal{P}_2(\mathbb{R}), W_2))$ and in $L^p((0,T) \times \mathbb{R})$ for any $p \in (1, \infty)$

[1] The guess for this construction is by integrating the Euler–Lagrange equation on the support of ρ_{α_1}.

and any $T>0$) to ρ given by $\rho(t,.) = (\frac{\alpha(t)}{\alpha_0}\mathrm{id})_{\#}\rho_0$. Since $v(t,x) = \frac{\alpha'(t)}{\alpha(t)}x$ is the velocity field associated to $X(t,x) = \frac{\alpha(t)}{\alpha_0}x$, ρ solves the continuity equation

$$\partial_t \rho + (\rho v)_x = 0.$$

In addition, $\rho v = -\rho z_{xx}$ where

$$z(t,x) = \frac{-\alpha'(t)}{6\alpha(t)}x^3 + \frac{3x}{2\alpha(t)},\ x \in [-\alpha(t), \alpha(t)],$$

extended by 1 (respectively -1) on $[\alpha(t), +\infty)$ (respectively $(-\infty, -\alpha(t)]$). The function z is C^1, $\|z\|_{L^\infty} \leq 1$ and $z \cdot D\rho = -|D\rho|$ (in the sense of measures). In other words the limit ρ of ρ^τ satisfies

$$\partial_t \rho - (\rho z_{xx})_x = 0$$

with $|z| \leq 1$ and $z \cdot D\rho = -|D\rho|$ which is the natural weak form of (1.2) since $z_{xx} = \nabla \mathrm{div}(z)$ in dimension one.

2.2. Instantaneous creation of discontinuities

We now consider the case, where $\rho_0(x) = (1-|x|)_+$ and will show that the JKO scheme instantaneously creates a discontinuity at the level of ρ_1, the minimizer of Φ_{τ,ρ_0} when τ is small enough. We indeed look for ρ_1 in the form (Fig. 1):

$$\rho_1(x) = \begin{cases} 1-\beta/2 & \text{if } |x| < \beta, \\ (1-|x|)_+ & \text{if } |x| \geq \beta, \end{cases}$$

for some well-chosen $\beta \in (0,1)$. The optimal transport map T between such a ρ_1 and ρ_0 is odd and given explicitly by

$$T(x) = \begin{cases} 1 - \sqrt{1-x(2-\beta)} & \text{if } x \in [0,\beta), \\ x & \text{if } x \geq \beta. \end{cases}$$

The Kantorovich potential which vanishes at β (extended in an even way to $\mathbb{R}_-$) is then given by

$$\varphi(x) = \begin{cases} \frac{x^2}{2} - x - \frac{(1-x(2-\beta))^{3/2}}{3(1-\beta/2)} + C & \text{if } x \in [0,\beta), \\ 0 & \text{if } x > \beta, \end{cases}$$

where

$$C = -\frac{\beta^2}{2} + \beta + \frac{2(1-\beta)^3}{3(2-\beta)}.$$

Let us now integrate $\tau z' = -\varphi$ on $[0,\beta]$ with initial condition $z(0)=0$, *i.e.* for $x \in [0,\beta]$

$$\tau z(x) = -\frac{x^3}{6} + \frac{x^2}{2} - \frac{4}{15(2-\beta)^2}[1-(1-2\beta)x]^{\frac{5}{2}} + \left(\frac{\beta^2}{2} - \beta - \frac{2(1-\beta)^3}{3(2-\beta)}\right)x + \frac{4}{15(2-\beta)^2}$$

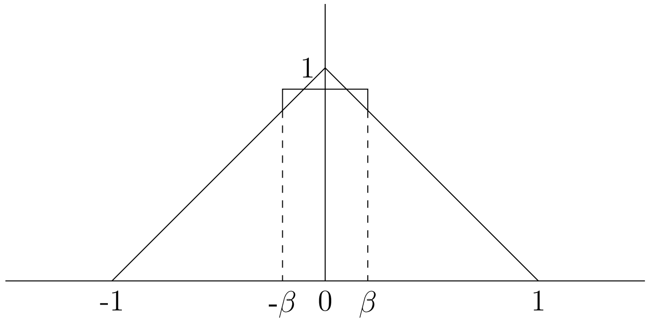

FIGURE 1. The probability density functions ρ_0 and ρ_1 from Section 2.2.

Note that z is nondecreasing on $[0,\beta]$ (because $\varphi(0)<0$, $\varphi(\beta)=0$ and φ is convex on $[0,\beta]$ so that $\varphi\leq 0$ on $[0,\beta]$), our aim now is to find $\beta\in(0,1)$ in such a way that $z(\beta)=1$ *i.e.* replacing in the previous formula

$$\tau=\frac{\beta^3}{3}-\frac{\beta^2}{2}+\frac{4(1-(1-\beta)^5)}{15(2-\beta)^2}-\frac{2(1-\beta)^3\beta}{3(2-\beta)}$$

the right hand-side is a continuous function of $\beta\in[0,1]$ taking value 0 for $\beta=0$ and $\frac{1}{10}$ for $\beta=1$, hence as soon as $10\tau<1$ one may find a $\beta\in(0,1)$ such that indeed $z(\beta)=1$. Extend then z by 1 on $[\beta,+\infty)$ and to $\mathbb{R}_-$ in an odd way. We then have built a function z which is C^1 ($\varphi(\beta)=0$), such that $|z|\leq 1$, $z\cdot D\rho_1=-|D\rho_1|$ and such that $z'+\frac{\varphi}{\tau}=0$. Thanks to Lemma 2.1, we conclude that ρ_1 is optimal. This example shows that discontinuities may appear at the very first iteration of the TV-JKO scheme.

3. Euler–Lagrange equation for JKO steps

The aim of this section is to establish optimality conditions for (3.1). Despite the fact that it is a convex minimization problem, it involves two nonsmooth terms J and $W_2^2(\rho_0,.)$, so some care should be taken of to justify rigorously the arguments. In the next subsection, we introduce an entropic regularization, the advantage of this strategy is that the minimizer will be positive everywhere, giving some differentiability of the transport term.

3.1. Entropic approximation

In this whole section, we assume that Ω is an open bounded connected (not necessarily convex) subset of $\mathbb{R}^d$ with Lipschitz boundary and denote by $\mathcal{P}_{\rm ac}(\Omega)$ the set of Borel probability measures on Ω that are absolutely continuous with respect to the Lebesgue measure (and will use the same notation for $\mu\in\mathcal{P}_{\rm ac}(\Omega)$ both for the measure μ and its density). Given $\rho_0\in\mathcal{P}_{\rm ac}(\Omega)$ and $\tau>0$, we consider one step of the TV-JKO scheme:

$$\inf_{\rho\in\mathcal{P}_{\rm ac}(\Omega)}\left\{\frac{1}{2\tau}W_2^2(\rho_0,\rho)+J(\rho)\right\}. \tag{3.1}$$

It is easy by the direct method of the calculus of variations to see that (3.1) has at least one solution, moreover J being convex and $\rho\mapsto W_2^2(\rho,\rho_0)$ being strictly convex whenever $\rho_0\in\mathcal{P}_{\rm ac}(\Omega)$ (see [26]), the minimizer is in fact unique, and in the sequel we denote it by ρ_1. Given $h>0$ we consider the following approximation of (3.1):

$$\inf_{\rho\in\mathcal{P}_{\rm ac}(\Omega)}\left\{\mathcal{F}_h(\rho):=\frac{1}{2\tau}W_2^2(\rho_0,\rho)+J(\rho)+h\mathcal{E}(\rho)\right\}, \tag{3.2}$$

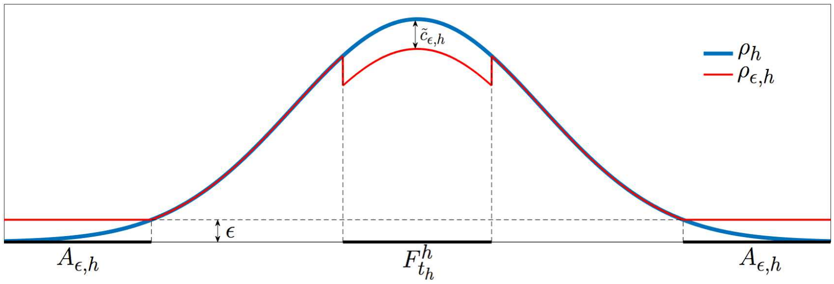

FIGURE 2. The perturbation $\rho_{\varepsilon,h}$ from (3.5).

where

$$\mathcal{E}(\rho) := \int_\Omega \rho(x)\log(\rho(x))\mathrm{d}x.$$

It is easy to see that (3.2) admits a unique solution ρ_h. Moreover, since Ω is bounded, $\mathcal{E}$ is lower bounded, hence $J(\rho_h)$ is bounded. Recalling that the embedding $BV(\Omega) \subset L^p(\Omega)$ is compact for every $p \in [1, \frac{d}{d-1})$, one may therefore (up to extraction) assume that ρ_h converges as $h \to 0$ a.e. and strongly in $L^p(\Omega)$ for every $p \in [1, \frac{d}{d-1})$ to some ρ_1, which, by a standard Γ-convergence argument, is easily seen to be the solution of (3.1). The advantage of this regularization is that not only each ρ_h is bounded from below but also that $h\log(\rho_h)$ is bounded from below uniformly in h (but not in τ which is fixed throughout this section):

Proposition 3.1. *Up to passing to a subsequence, the family $\beta_h := h\log(\rho_h)$ is uniformly bounded from below. Moreover, β_h is bounded in $L^p(\Omega)$ for any $p > 1$ and $\max(0, \beta_h)$ converges strongly to 0 in $L^p(\Omega)$ for any $p > 1$.*

Proof. Let $t_h > 0$ be such that the set $F^h_{t_h} := \{\rho_h > t_h\}$ has positive measure and finite perimeter (recall that $\rho_h \in \mathrm{BV}$). Let us assume that there is an $\varepsilon \in (0, 1)$ such that

$$\varepsilon \leq \frac{t_h|F^h_{t_h}|}{2|\Omega|}, \tag{3.3}$$

and

$$|A_{\varepsilon,h}| > 0 \text{ with } A_{\varepsilon,h} := \{\rho_h \leq \varepsilon\}. \tag{3.4}$$

We aim to show that ε cannot be arbitrarily small. Define then $\mu_{\varepsilon,h} := \max(\rho_h, \varepsilon)$ that is ε on $A_{\varepsilon,h}$ and ρ_h elsewhere. Defining $c_{\varepsilon,h} := \int_\Omega(\mu_{\varepsilon,h} - \rho_h)$ and observing that $c_{\varepsilon,h} \leq \varepsilon|\Omega|$, we see that (3.3) implies that $c_{\varepsilon,h} \leq \frac{1}{2}t_h|F^h_{t_h}|$ and $t_h \geq 2\varepsilon$ so that $A_{\varepsilon,h}$ and $F^h_{t_h}$ are disjoint. Finally, set

$$\rho_{\varepsilon,h} := \mu_{\varepsilon,h} - c_{\varepsilon,h}\frac{\chi_{F^h_{t_h}}}{|F^h_{t_h}|}. \tag{3.5}$$

See Figure 2, where we set $\tilde{c}_{\varepsilon,h} := c_{\varepsilon,h}/|F^h_{t_h}|$.

By construction $\rho_{\varepsilon,h} \in \mathcal{P}(\Omega)$ hence $0 \leq \mathcal{F}_h(\rho_{\varepsilon,h}) - \mathcal{F}_h(\rho_h)$, in this difference we have four terms, namely

- the Wasserstein term, which, using the Kantorovich duality formula (2.1) and the fact that Ω is bounded can be estimated in terms of $\|\rho_{\varepsilon,h} - \rho_h\|_{L^1} = 2c_{\varepsilon,h}$:

$$\frac{1}{2\tau}W_2^2(\rho_{\varepsilon,h}, \rho_0) - \frac{1}{2\tau}W_2^2(\rho_h, \rho_0) \leq \frac{C}{\tau}c_{\varepsilon,h}. \tag{3.6}$$

for a constant C that depends on Ω but neither on ε nor h,

- the TV term: $J(\rho_{\varepsilon,h}) - J(\rho_h)$: outside $F^h_{t_h}$ we have replaced ρ_h by a 1-Lipschitz function of ρ_h which decreases the TV semi-norm, on $F^h_{t_h}$ on the contrary we have created a jump of magnitude $c_{\varepsilon,h}/|F^h_{t_h}|$ so

$$J(\rho_{\varepsilon,h}) - J(\rho_h) \leq c_{\varepsilon,h} \frac{\mathrm{Per}(F^h_{t_h})}{|F^h_{t_h}|}, \tag{3.7}$$

where $\mathrm{Per}(F^h_{t_h}) = J(\chi_{F^h_{t_h}})$ denotes the perimeter of $F^h_{t_h}$ (in Ω),

- the entropy variation on $A_{\varepsilon,h}$, on this set both $\rho_{\varepsilon,h}$ and ρ_h are less than ε so that $(1+\log(t)) \leq (1+\log(\varepsilon))$ whenever $t \in [\rho_h, \rho_{\varepsilon,h}]$ which by the mean value theorem yields

$$\int_{A_{\varepsilon,h}} (\rho_{\varepsilon,h}\log(\rho_{\varepsilon,h}) - \rho_h \log(\rho_h)) \leq (1+\log(\varepsilon))c_{\varepsilon,h} \tag{3.8}$$

- the last term is the entropy variation on $F^h_{t_h}$. It is convenient to split $F^h_{t_h}$ into $F^h_{t_h} \cap \{\rho_{\varepsilon,h} \geq \frac{1}{e}\}$ and $F^h_{t_h} \cap \{\rho_{\varepsilon,h} < \frac{1}{e}\}$. The entropy variation on the first part is easy to control. Indeed, $t \mapsto t\log(t)$ is nondecreasing on $[\frac{1}{e}, +\infty)$. Since, on $F^h_{t_h} \cap \{\rho_{\varepsilon,h} \geq \frac{1}{e}\}$, $\rho_h \geq \rho_{\varepsilon,h} \geq \frac{1}{e}$, we have $(\rho_{\varepsilon,h}\log(\rho_{\varepsilon,h}) - \rho_h\log(\rho_h)) \leq 0$. As for the second part, we observe that $F^h_{t_h} \cap \{\rho_{\varepsilon,h} < \frac{1}{e}\} \subset \{\rho_h \leq \frac{1}{e} + \frac{t_h}{2}\}$, so on this set, both $\rho_{\varepsilon,h}$ and ρ_h remain in the interval $[\frac{t_h}{2}, \frac{1}{e} + \frac{t_h}{2}]$. We thus have

$$\int_{F^h_{t_h}} (\rho_{\varepsilon,h}\log(\rho_{\varepsilon,h}) - \rho_h \log(\rho_h)) \leq C_h(t_h) c_{\varepsilon,h}, \tag{3.9}$$

where

$$C_h(t_h) := \max\left\{ |1+\log(t)| \ : \ \frac{t_h}{2} \leq t \leq \frac{1}{e} + \frac{t_h}{2} \right\}. \tag{3.10}$$

Putting together (3.6)–(3.9), we arrive at

$$0 \leq \left(\frac{C}{\tau} + \frac{\mathrm{Per}(F^h_{t_h})}{|F^h_{t_h}|} + hC_h(t_h) + h\log(\varepsilon) + h \right) c_{\varepsilon,h},$$

which for small enough ε is possible only when $c_{\varepsilon,h} = 0$ *i.e.*, $|A_{\varepsilon,h}| = 0$. More precisely, either we have the lower bound:

$$h\log(\rho_h) \geq -\frac{C}{\tau} - hC_h(t_h) - \frac{\mathrm{Per}(F^h_{t_h})}{|F^h_{t_h}|} - h \tag{3.11}$$

or (3.3) is impossible *i.e.*, $\rho_h \geq \frac{t_h |F^h_{t_h}|}{2|\Omega|}$. To prove that $\beta_h = h\log(\rho_h)$ is bounded from below uniformly in h, it is therefore enough to show that we can find a family t_h, bounded and bounded away from 0, such that $|F^h_{t_h}|$ remains bounded away from 0, and $\mathrm{Per}(F^h_{t_h})$ is uniformly bounded from above as $h \to 0$. First note that, since $J(\rho_h)$ is bounded, there exists ρ such that $\rho_h \to \rho$ in L^1 and a.e. up to a subsequence, note also that $\rho \in \mathrm{BV}$ and ρ is a probability density. Setting $F_t := \{\rho > t\}$, $F^h_t := \{\rho_h > t\}$, if $s > t$, since ρ_h converges a.e. to ρ, we have a.e. $\liminf_h \chi_{F^h_t} \geq \chi_{F_s}$. It then follows from Fatou's Lemma that when $s > t$, $\liminf_h |F^h_t| \geq |F_s|$, hence choosing $0 < \beta_1 < \beta_2 < \beta$ so that $|F_\beta| > 0$, we deduce that there exists $h_0 > 0$ and $c_1 > 0$ such that for all

$t \in [\beta_1, \beta_2]$ and all $h \in (0, h_0]$, we have $c_1 \leq |F_t^h| \leq |\Omega|$. For an upper bound on perimeters, we observe that since $J(\rho_h) \leq C$, thanks to the co-area formula, we have

$$\int_{\beta_1}^{\beta_2} \mathrm{Per}(F_t^h)\mathrm{d}t \leq J(\rho_h) \leq C.$$

So, there exists $t_h \in [\beta_1, \beta_2]$ such that $\mathrm{Per}(F_{t_h}^h) \leq C/(\beta_2 - \beta_1)$. Finally, since ρ_h converges in L^1, we may assume that, up to a subsequence, $\rho_h \leq \phi$ for some $\phi \in L^1$ (see [6], Thm. IV.9). Then, by dominated convergence and since $\log(\max(\phi, 1)) \in L^p(\Omega)$ for every $p > 1$, we have that $\log(\max(\rho_h, 1))$ converges a.e. and in L^p, in particular this implies that $\max(0, \beta_h)$ converges to 0 strongly in $L^p(\Omega)$, and we have just seen that $\min(0, \beta_h)$ is bounded in $L^\infty(\Omega)$. □

Let us also recall some well-known facts (see [9]) about the total variation functional J viewed as a convex lower semi-continuous (l.s.c.) and one-homogeneous functional on $L^{\frac{d}{d-1}}(\Omega)$. Define

$$\Gamma_d := \left\{\xi \in L^d(\Omega) \ : \ \exists z \in L^\infty(\Omega, \mathbb{R}^d), \ \|z\|_{L^\infty} \leq 1, \ \mathrm{div}(z) = \xi, \ \ z \cdot \nu = 0 \text{ on } \partial\Omega, \right\} \tag{3.12}$$

where $\mathrm{div}(z) = \xi, \ z \cdot \nu = 0$ on $\partial\Omega$ are to be understood in the weak sense

$$\int_\Omega \xi u = -\int_\Omega z \cdot \nabla u, \ \forall u \in C^1(\overline{\Omega}).$$

Note that Γ_d is closed and convex in $L^d(\Omega)$ and J is its support function:

$$J(\mu) = \sup_{\xi \in \Gamma_d} \int_\Omega \xi\mu, \ \forall \mu \in L^{\frac{d}{d-1}}(\Omega). \tag{3.13}$$

As for the Wasserstein term, recalling Kantorovich dual formulation (2.1), the derivative of the Wasserstein term $\rho \mapsto W_2^2(\rho_0, \rho)$ term will be expressed in terms of a Kantorovich potential between ρ and ρ_0.

We then have the following characterization for ρ_h:

Proposition 3.2. *There exists $z_h \in L^\infty(\Omega, \mathbb{R}^d)$ such that $\mathrm{div}(z_h) \in L^p(\Omega)$ for every $p \in [1, +\infty)$, $\|z_h\|_{L^\infty} \leq 1$, $z_h \cdot \nu = 0$ on $\partial\Omega$, $J(\rho_h) = \int_\Omega \mathrm{div}(z_h)\rho_h$ and*

$$\frac{\varphi_h}{\tau} + \mathrm{div}(z_h) + h\log(\rho_h) = 0, \ \text{ a.e. in } \Omega, \tag{3.14}$$

where φ_h is the Kantorovich potential between ρ_h and ρ_0.

Proof. Let $\mu \in L^\infty(\Omega) \cap \mathrm{BV}(\Omega)$ such that $\int_\Omega \mu = 0$. Thanks to Proposition 3.1, we know that ρ_h is bounded away from 0 hence for small enough $t > 0$, $\rho_h + t\mu$ is positive hence a probability density. Also, as a consequence of Theorem 1.52 in [26], we have that

$$\lim_{t \to 0^+} \frac{1}{2t}\left[W_2^2(\rho_0, \rho_h + t\mu) - W_2^2(\rho_0, \rho_h)\right] = \int_\Omega \varphi_h\mu, \tag{3.15}$$

where φ_h is the (unique up to an additive constant) Kantorovich potential between ρ_h and ρ_0, in particular φ_h is Lipschitz and semi concave ($D^2\varphi_h \leq \mathrm{id}$ in the sense of measures and $\mathrm{id} - \nabla\varphi_h$ is the optimal transport between ρ_h and ρ_1). By the optimality of ρ_h and the fact that J is a semi-norm, we get

$$J(\mu) \geq J(\rho_h + \mu) - J(\rho_h) \geq \lim_{t \to 0^+} t^{-1}(J(\rho_h + t\mu) - J(\rho_h)) \geq \int_\Omega \xi_h\mu, \tag{3.16}$$

where

$$\xi_h := -\frac{\varphi_h}{\tau} - h\log(\rho_h).$$

Since φ_h is defined up to an additive constant, we may chose it in such a way that ξ_h has zero mean, doing so, (3.16) holds for any $\mu \in L^\infty(\Omega) \cap \mathrm{BV}(\Omega)$ (not necessarily with zero mean). Being Lipschitz, φ_h is bounded, also observe that $h(\log(\rho_h))_+ = h\log(\max(1, \rho_h))$ is in $L^p(\Omega)$ for every $p \in [1, +\infty)$ since $\rho_h \in L^{\frac{d}{d-1}}(\Omega)$ and $h\log(\rho_h)_- = -h\log(\min(1, \rho_h))$ is $L^\infty(\Omega)$ thanks to Proposition 3.1, hence we have $\xi_h \in L^p(\Omega)$ for every $p \in [1, +\infty)$.

By approximation and observing that $\xi_h \in L^d(\Omega)$, (3.16) extends to all $\mu \in L^{\frac{d}{d-1}}(\Omega)$. In particular, we have

$$\sup_{\xi \in \Gamma_d} \int_\Omega \xi\mu \geq \int_\Omega \xi_h \mu$$

but since Γ_d is convex and closed in $L^d(\Omega)$, it follows from Hahn–Banach's separation theorem that $\xi_h \in \Gamma_d$. Finally, getting back to (3.16) (without the zero mean restriction on μ) and taking $\mu = -\rho_h$ gives $J(\rho_h) \leq \int_\Omega \xi_h \rho_h$, and we then deduce that this should be an equality. □

3.2. Euler–Lagrange equation

We are now in position to rigorously establish the Euler–Lagrange equation for (3.1):

Theorem 3.3. *If ρ_1 solves (3.1), there exists φ a Kantorovich potential between ρ_1 and ρ_0 (in particular* $\mathrm{id} - \nabla\varphi$ *is the optimal transport between ρ_1 and ρ_0), $\beta \in L^\infty(\Omega)$, $\beta \geq 0$ and $z \in L^\infty(\Omega, \mathbb{R}^d)$ such that*

$$\frac{\varphi}{\tau} + \mathrm{div}(z) = \beta,\ z \cdot \nu = 0 \ \text{on } \partial\Omega, \tag{3.17}$$

and

$$\beta\rho_1 = 0,\ \ \|z\|_{L^\infty} \leq 1,\ J(\rho_1) = \int_\Omega \mathrm{div}(z)\rho_1. \tag{3.18}$$

Remark 3.4. It is not difficult (since (3.1) is a convex problem) to check that (3.17)–(3.18) are also sufficient optimality conditions. The main point here is that the right hand side β in (3.17) which is a multiplier associated with the nonnegativity constraint is better than a measure, it is actually an L^∞ function.

Proof. As in Section 3.1, we denote by ρ_h the solution of the entropic approximation (3.2). Up to passing to a subsequence (not explicitly written), we may assume that ρ_h converges a.e. and strongly in $L^p(\Omega)$ (for any $p \in [1, \frac{d}{d-1}))$ to ρ_1 (the solution of (3.1), again by a standard Γ-convergence argument). We then rewrite the Euler–Lagrange equation from Proposition 3.2 as

$$\frac{\varphi_h}{\tau} + \mathrm{div}(z_h) + \beta_h^+ = \beta_h^-, \tag{3.19}$$

where $\beta_h^+ := h\log(\max(\rho_h, 1))$, $\beta_h^- := -h\log(\min(\rho_h, 1))$, and

$$\|z_h\|_{L^\infty} \leq 1,\ z_h \cdot \nu = 0 \text{ on } \partial\Omega \text{ and } J(\rho_h) = \int_\Omega \mathrm{div}(z_h)\rho_h. \tag{3.20}$$

It follows from Proposition 3.1 that β_h^+ converges to 0 strongly in any L^p, $p \in [1, +\infty)$ and that β_h^- is bounded in L^∞. Up to subsequences, we may therefore assume that z_h and β_h^- weakly-$*$ converge in L^∞ respectively to some z and β with $\|z\|_{L^\infty} \leq 1$, $z \cdot \nu = 0$ on $\partial\Omega$ and $\beta \geq 0$. As for the Kantorovich potentials φ_h, since the transport map $(\mathrm{id} - \nabla\varphi_h)$ a.e. takes values in Ω, we have $\|\nabla\varphi_h\|_{L^\infty} \leq \mathrm{diam}(\Omega)$, hence φ_h is an equi-Lipschitz family because Ω is bounded. Moreover $\int_\Omega \varphi_h = \tau \int_\Omega (\beta_h^- - \beta_h^+)$ which remains bounded, hence we may assume that φ_h converges uniformly to some potential φ and it is well-known (see [26]) that φ is a Kantorovich potential between ρ_1 and ρ_0. Letting h tend to 0 gives (3.17).

Since ρ_h converges strongly in L^1 to ρ_1 and β_h^- converges weakly-$*$ to β in L^∞ we have

$$\int_\Omega \rho_1 \beta = \lim_h \int_\Omega \rho_h \beta_h^- = \lim_h h \int_\Omega \rho_h |\log(\min(1, \rho_h))| = 0,$$

hence $\beta\rho_1 = 0$. Thanks to (3.13), we obviously have $J(\rho_1) \geq \int_\Omega \mathrm{div}(z)\rho_1$ (since $\mathrm{div}(z) \in L^\infty$, $\mathrm{div}(z) \in \mathit{\Gamma}_d$), for the converse inequality, it is enough to observe that

$$J(\rho_1) \leq \liminf_h J(\rho_h) = \liminf_h \int_\Omega \mathrm{div}(z_h)\rho_h$$

and that $\mathrm{div}(z_h) = -\frac{\varphi_h}{\tau} - \beta_h^+ + \beta_h^-$ converges to $\mathrm{div}(z)$ weakly in L^q for every $q \in [1, +\infty)$. Since ρ_h converges strongly to ρ_1 in L^q when $q \in [1, \frac{d}{d-1})$ we deduce that $J(\rho_1) = \int_\Omega \mathrm{div}(z)\rho_1$ which completes the proof of (3.18). □

A first consequence of the high integrability of $\mathrm{div}(z)$ is that one can give a meaning to $z \cdot \nabla u$ for any $u \in \mathrm{BV}(\Omega)$. Indeed, if $q \in [\frac{d}{d-1}, +\infty]$ and q' denotes its conjugate exponent, following Anzellotti [2], if $u \in \mathrm{BV}(\Omega) \cap L^q(\Omega)$ and $\sigma \in L^\infty(\Omega, \mathbb{R}^d)$ is such that $\mathrm{div}(\sigma) \in L^{q'}(\Omega)$, one can define the distribution $\sigma \cdot Du$ by

$$\langle \sigma \cdot Du, v\rangle = -\int_\Omega \mathrm{div}(\sigma)\, uv - \int_\Omega u\, \sigma \cdot \nabla v,\ \forall v \in C_c^1(\Omega).$$

Then $\sigma \cdot Du$ is a Radon measure which satisfies $|\sigma \cdot Du| \leq \|\sigma\|_{L^\infty}|Du|$ (in the sense of measures) hence is absolutely continuous with respect to $|Du|$. Moreover one can also define a weak notion of normal trace of σ, $\sigma \cdot \nu \in L^\infty(\partial\Omega)$ such that the following integration by parts formula holds

$$\int_\Omega \sigma \cdot Du = -\int_\Omega \mathrm{div}(\sigma)u + \int_{\partial\Omega} u(\sigma \cdot \nu).$$

We refer to [2] for proofs. These considerations of course apply to $\sigma = z$ and $u = \rho_1 \in \mathrm{BV}(\Omega)$ and in particular enable one to see $z \cdot D\rho_1$ as a measure and to interpret the optimality condition $J(\rho_1) = \int_\Omega \mathrm{div}(z)\rho_1$ as $|D\rho_1| = -z \cdot D\rho_1$ in the sense of measures. Finally, the fact that $\mathrm{div}(z) \in L^\infty$ in Theorem 3.3 and the theory of variational mean curvature (see Tamanini [27], Massari [19, 20], Gonzalez and Massari [21], Thm. 3.6) allows for conclusions about the regularity of the level sets, $F_t = \{\rho_1 > t\}$ of ρ_1, the solution of (3.1), we do not elaborate this regularity (which, anyway, only holds for fixed time step $\tau > 0$) further here.

4. Maximum and minimum principles for JKO steps

Throughout this section, we further assume that Ω is a convex open bounded subset of $\mathbb{R}^d$, our aim is to establish bounds on the TV-JKO iterates given by (3.1). Since, the TV-JKO scheme aims at minimizing total variation at the fastest rate in the Wasserstein metric, it is natural to wonder whether when the initial condition is bounded from above and from below then the JKO-iterates remain so (with the same bounds). We shall answer affirmatively for the upper bound (maximum principle), as for the propagation of the lower bound (minimum principle), we have been able to prove it only in special cases (dimension one and radially symmetric setting).

4.1. Convexity along generalized geodesics

Our aim is to deduce some bounds on ρ_1 from bounds on ρ_0. To do so, we shall combine some convexity arguments and a remarkable BV estimate due to De Philippis *et al.* [11]. First we recall the notion of generalized geodesic from Ambrosio *et al.* [1]. Given $\overline{\mu}$, μ_0 and μ_1 in $\mathcal{P}_{\rm ac}(\Omega)$, and denoting by T_0 (respectively T_1) the optimal transport (Brenier) map between $\overline{\mu}$ and μ_0 (respectively μ_1), the *generalized geodesic with base* $\overline{\mu}$ joining μ_0 to μ_1 is by definition the curve of measures:

$$\mu_t := ((1-t)T_0 + tT_1)_\# \overline{\mu},\ t \in [0,1]. \tag{4.1}$$

A key property of these curves introduced in [1] is the strong convexity of the squared distance estimate:

$$W_2^2(\overline{\mu}, \mu_t) \leq (1-t)W_2^2(\overline{\mu}, \mu_0) + tW_2^2(\overline{\mu}, \mu_1) - t(1-t)W_2^2(\mu_0, \mu_1). \tag{4.2}$$

It is well-known that if $G : \mathbb{R}_+ \to \mathbb{R} \cup \{+\infty\}$ is a proper convex l.s.c. internal energy density, bounded from below such that $G(0) = 0$ and which satisfies McCann's condition (see [23])

$$\lambda \in \mathbb{R}_+ \to \lambda^d G(\lambda^{-d}) \text{ is convex nonincreasing} \tag{4.3}$$

then defining the generalized geodesic curve $(\mu_t)_{t\in[0,1]}$ by (4.1), one has

$$\int_\Omega G(\mu_t(x))\mathrm{d}x \leq (1-t)\int_\Omega G(\mu_0(x))\mathrm{d}x + t\int_\Omega G(\mu_1(x))\mathrm{d}x. \tag{4.4}$$

In particular L^p and uniform bounds are stable along generalized geodesics:

$$\|\mu_t\|_{L^p}^p \leq (1-t)\|\mu_0\|_{L^p}^p + t\|\mu_0\|_{L^p}^p,\ \|\mu_t\|_{L^\infty} \leq \max(\|\mu_0\|_{L^\infty}, \|\mu_1\|_{L^\infty}), \tag{4.5}$$

and

$$\int_\Omega \mu_t(x)\log(\mu_t(x))\mathrm{d}x \leq (1-t)\int_\Omega \mu_0(x)\log(\mu_0(x))\mathrm{d}x + t\int_\Omega \mu_1(x)\log(\mu_1(x))\mathrm{d}x. \tag{4.6}$$

An immediate consequence of (4.2) (see Chap. 4 of [1] for general contraction estimates) is the following

Lemma 4.1. *Let K be a nonempty subset of $\mathcal{P}_{\rm ac}(\Omega)$, let $\mu_0 \in K$, $\mu_1 \in \mathcal{P}_{\rm ac}(\Omega)$, if $\hat{\mu}_1 \in \mathrm{argmin}_{\mu\in K} W_2^2(\mu_1, \mu)$ is a Wasserstein projection of μ_1 onto K, and if the generalized geodesic with base μ_1 joining μ_0 to $\hat{\mu}_1$ remains in K then*

$$W_2^2(\mu_0, \hat{\mu}_1) \leq W_2^2(\mu_0, \mu_1) - W_2^2(\mu_1, \hat{\mu}_1). \tag{4.7}$$

Proof. Since $\mu_t \in K$ we have $W_2^2(\mu_1, \hat{\mu}_1) \leq W_2^2(\mu_1, \mu_t)$, applying (4.2) to the generalized geodesics with base μ_1 joining μ_0 to $\hat{\mu}_1$ we thus get

$$(1-t)W_2^2(\mu_1, \hat{\mu}_1) \leq (1-t)W_2^2(\mu_1, \mu_0) - t(1-t)W_2^2(\mu_0, \hat{\mu}_1),$$

dividing by $(1-t)$ and then taking $t = 1$ therefore gives the desired result. □

The other result we shall use to derive bounds is a BV estimate of De Philippis *et al.* [11], which states that, given, $\mu \in \mathcal{P}_{\mathrm{ac}}(\Omega) \cap \mathrm{BV}(\Omega)$, and $G : \mathbb{R}_+ \to \mathbb{R} \cup \{+\infty\}$, proper convex l.s.c., the solution of

$$\inf_{\rho \in \mathcal{P}_{\mathrm{ac}}(\Omega)} \left\{ \frac{1}{2} W_2^2(\mu, \rho) + \int_\Omega G(\rho(x)) \mathrm{d}x \right\} \tag{4.8}$$

is BV with the bound

$$J(\rho) \leq J(\mu). \tag{4.9}$$

Taking in particular,

$$G(\rho) := \begin{cases} 0 \text{ if } \rho \leq M, \\ +\infty \text{ otherwise,} \end{cases}$$

this implies that the Wasserstein projection of μ onto the set defined by the constraint $\rho \leq M$ has a smaller total variation than μ.

4.2. Maximum principle

Theorem 4.2. *Let $\rho_0 \in \mathcal{P}_{\mathrm{ac}}(\Omega) \cap L^\infty(\Omega)$ and let ρ_1 be the solution of (3.1), then $\rho_1 \in L^\infty(\Omega)$ with*

$$\|\rho_1\|_{L^\infty(\Omega)} \leq \|\rho_0\|_{L^\infty(\Omega)}. \tag{4.10}$$

Proof. Thanks to (4.5) the set $K := \{\rho \in \mathcal{P}_{\mathrm{ac}}(\Omega) :\ \rho \leq \|\rho_0\|_{L^\infty(\Omega)} \text{ a.e.}\}$ has the property that the generalized geodesics (with any base) joining two of its points remains in K. Let then $\hat{\rho}_1$ be the W_2 projection of ρ_1 onto K *i.e.* the solution of $\inf_{\rho \in K} W_2^2(\rho_1, \rho)$. Thanks to Lemma 4.1 we have $W_2^2(\rho_0, \hat{\rho}_1) \leq W_2^2(\rho_0, \rho_1) - W_2^2(\rho_1, \hat{\rho}_1)$ and thanks to Theorem 1.1 of [11], $J(\hat{\rho}_1) \leq J(\rho_1)$. The optimality of ρ_1 for (3.1) therefore implies $W_2(\rho_1, \hat{\rho}_1) = 0$ *i.e.* $\rho_1 \leq \|\rho_0\|_{L^\infty(\Omega)}$. □

Remark 4.3. In Section 3, we have used an approximation of (3.1) with an additional small entropy term, the same bound as in Theorem 4.2 will remain valid in this case. Indeed, consider a proper convex l.s.c. and bounded from below internal energy density G and consider given $h \geq 0$, the variant of (3.1)

$$\inf_{\rho \in \mathcal{P}_{\mathrm{ac}}(\Omega)} \left\{ \frac{1}{2\tau} W_2^2(\rho_0, \rho) + J(\rho) + h \int_\Omega G(\rho(x)) \mathrm{d}x \right\}. \tag{4.11}$$

Then, we claim that the solution ρ_h still satisfies $\rho_h \leq \|\rho_0\|_{L^\infty(\Omega)}$. Indeed, we have seen in the previous proof that the Wasserstein projection $\hat{\rho}_h$ of ρ_h onto the constraint $\rho \leq \|\rho_0\|_{L^\infty(\Omega)}$ both diminishes J and the Wasserstein distance to ρ_0. It turns out that it also diminishes the internal energy. Indeed, thanks to Proposition 5.2 of [11], there is a measurable set A such that $\hat{\rho}_h = \chi_A \rho_h + \chi_{\Omega \setminus A} \|\rho_0\|_{L^\infty}$, it thus follows that $|\Omega \setminus A| \|\rho_0\|_{L^\infty} = \int_{\Omega \setminus A} \rho_h$. So, from the convexity of G and Jensen's inequality,

$$\int G(\hat{\rho}_h) = \int_A G(\rho_h) + |\Omega \setminus A| G\left(|\Omega \setminus A|^{-1} \int_{\Omega \setminus A} \rho_h \right) \leq \int G(\rho_h),$$

thus yielding the same conclusion as above.

4.3. Minimum principle in special cases

In dimension one, it turns out that we can obtain bounds from below by the same convexity arguments as for the maximum principle of Theorem 4.2:

Proposition 4.4. *Assume that $d = 1$, that Ω is a bounded interval and that $\rho_0 \geq \alpha > 0$ a.e. on Ω then the solution ρ_1 of (3.1) also satisfies $\rho_1 \geq \alpha > 0$ a.e. on Ω.*

Proof. The proof is similar to that of Theorem 4.2 but using the Wasserstein projection on the set $K := \{\rho \in \mathcal{P}_{\mathrm{ac}}(\Omega) \ : \ \rho \geq \alpha\}$, the only thing to check to be able to use Lemma 4.1 is that for any basepoint $\overline{\mu}$ and any μ_0 and μ_1 in K, the generalized geodesic with base point $\overline{\mu}$ joining μ_0 to μ_1 remains in K. The optimal transport maps T_0 and T_1 from $\overline{\mu}$ to μ_0 and μ_1 respectively are nondecreasing and continuous and setting $T_t := (1-t)T_0 + tT_1$, one has

$$\overline{\mu} = \mu_t(T_t)T_t' = \mu_0(T_0)T_0' = \mu_1 T_1' = (1-t)\mu_0(T_0)T_0' + t\mu_1(T_1)T_1' \geq \alpha T_t'$$

which is easily seen to imply that $\mu_t \geq \alpha$ a.e. □

As a consequence of the previous minimum principle, integrating the Euler–Lagrange equation one can deduce higher regularity for the dual variable z:

Corollary 4.5. *Assume that $d = 1$ and Ω is a bounded interval. If ρ_1 solves (3.1) and z is as in Theorem 3.3 then $z \in W_0^{1,\infty}(\Omega)$. If in addition $\rho_0 \geq \alpha > 0$ a.e. on Ω, then $z \in W^{3,\infty}(\Omega)$.*

Proof. The first claim is obvious because both φ and β (φ, β and z are as in Thm. 3.3) are bounded hence so is z'. As for the second one when $\rho_0 \geq \alpha > 0$, thanks to Proposition 4.4, we also have $\rho_1 \geq \alpha$ hence $\beta = 0$ in (3.17) and in this case $\mathrm{div}(z) = z' = -\frac{\varphi}{\tau}$ is Lipschitz *i.e.*, $z \in W^{2,\infty}$. One can actually go one step further because $x - \varphi'(x) = T(x)$ where T is the optimal (monotone) transport between ρ_1 and ρ_0. This map is explicit in terms of the cumulative distribution function of ρ_1, F_1, and F_0^{-1} the inverse of F_0, the cumulative distribution function of ρ_0, namely $T = F_0^{-1} \circ F_1$. But F_1 is Lipschitz since its derivative is ρ_1 which is BV hence bounded and F_0^{-1} is Lipschitz as well since $\rho_0 \geq \alpha > 0$. This gives that $\varphi \in W^{2,\infty}$ hence $z \in W^{3,\infty}$. □

The proof of Proposition 4.4 unfortunately does not generalize to higher dimensions, because densities which are bounded from below by α are not stable by generalized geodesics. In the radially symmetric case, we can use the Euler–Lagrange equation to derive a minimum principle. We believe that JKO steps preserve lower bounds in more general situations but have not been able to prove it.

Proposition 4.6. *Assume that $\Omega = B(0, R)$ is the ball centered at 0 or radius $R > 0$ in $\mathbb{R}^d$, and that ρ_0 is radially symmetric with $\rho_0 \geq \alpha > 0$ a.e. on Ω then the solution ρ_1 of (3.1) also satisfies $\rho_1 \geq \alpha > 0$ a.e. on Ω.*

Proof. Let us write $\rho_0(x) = \widetilde{\rho}_0(r)$ with $r = |x| \in [0, R]$, since (3.1) is invariant by rotation and strictly convex, it is easy to see that its unique solution ρ_1 is also radially symmetric, let us write it as $\rho_1(x) = \widetilde{\rho}_1(r)$. Denoting by c_d the $(d-1)$-Hausdorff measure of the unit sphere S^{d-1}, and setting $\widetilde{\mu}_0 := c_d r^{d-1}\widetilde{\rho}_0$, $\widetilde{\mu}_1 := c_d r^{d-1}\widetilde{\rho}_1$, observe that $\widetilde{\rho}_1$ is the minimizer of the one-dimensional convex functional

$$\mathcal{F}^{\mathrm{rad}}(\widetilde{\rho}) := \frac{1}{2\tau} W_2^2(\widetilde{\mu}_0, c_d r^{d-1}\widetilde{\rho}) + c_d \int_0^R r^{d-1}|D\widetilde{\rho}|$$

among nonnegative densities $\widetilde{\rho}$ on $(0, R)$ such that $c_d \int_0^R r^{d-1}\widetilde{\rho} = 1$ and $r^{d-1}D\widetilde{\rho}$ is a bounded Radon measure on $(0, R)$. Arguing as in the proof of Theorem 3.3, the minimizer $\widetilde{\rho}_1$ is characterized by the Euler–Lagrange equation

$$(\widetilde{z}r^{d-1})' + \frac{\widetilde{\varphi}}{\tau} r^{d-1} = \widetilde{\beta} \geq 0, \ \widetilde{\beta} \in L^\infty(0, R), \ \widetilde{\beta}\widetilde{\rho}_1 = 0, \tag{4.12}$$

where $\widetilde{\varphi}$ is a Kantorovich potential between $\widetilde{\mu}_1$ and $\widetilde{\mu}_0$ and $\widetilde{z} \in L^\infty(0,R)$ is such that

$$|\widetilde{z}| \leq 1 \text{ a.e. and } \int_0^R r^{d-1}|D\widetilde{\rho}_1| = \int_0^R (\widetilde{z}r^{d-1})'\widetilde{\rho}_1. \tag{4.13}$$

Note that (4.12) implies that $r^{d-1}\widetilde{z}$ is Lipschitz so that $\widetilde{z}$ is locally Lipschitz and

$$\int_0^R r^{d-1}|D\widetilde{\rho}_1| = -\int_0^R r^{d-1}\widetilde{z}D\widetilde{\rho}_1 \tag{4.14}$$

Since $\widetilde{\rho}_1 \in \mathrm{BV}_{\mathrm{loc}}(0,R)$, we can perform a Hahn–Jordan decomposition of $D\widetilde{\rho}_1$:

$$D\widetilde{\rho}_1 = \nu^+ - \nu^-,\ \nu^+ \geq 0,\ \nu^- \geq 0,\ \nu^+ \perp \nu^-, \tag{4.15}$$

and set

$$A := \mathrm{spt}(|D\widetilde{\rho}_1|) = A^+ \cup A^- \text{ with } A^+ := \mathrm{spt}(\nu^+),\ A^- := \mathrm{spt}(\nu^-). \tag{4.16}$$

Next, we observe that, using (4.14), we have $|D\widetilde{\rho}_1| = \nu^+ + \nu^- = -\widetilde{z}(\nu^+ - \nu^-)$, we thus deduce that $\widetilde{z} = -1 = \min \widetilde{z}$ ν^+-a.e. and since $\widetilde{z}$ is continuous we actually have $\widetilde{z} = -1$ on $A^+ = \mathrm{spt}(\nu^+)$. In a similar way, $\widetilde{z} = 1 = \max \widetilde{z}$ on $A^- := \mathrm{spt}(\nu^-)$.

Now let us show that $\widetilde{\rho}_1 \geq \alpha$. Assume, by contradiction, that the set where $\widetilde{\rho}_1 < \alpha$ has positive measure in $(0,R)$, and let $r_0 \in (0,R)$ be a continuity point of $\widetilde{\rho}_1$ such that $\widetilde{\rho}_1(r_0) < \alpha$, define then

$$a_- := \inf\{r \in (0,r_0)\ :\ \widetilde{\rho}_1 \leq \alpha \text{ on } [r,r_0]\},$$
$$a_+ := \sup\{r \in (r_0,R)\ :\ \widetilde{\rho}_1 \leq \alpha \text{ on } [r_0,r]\}.$$

We then have $0 \leq a_- < a_+ \leq R$. Let us assume that $a_- > 0$, we claim then that $a_- \in A^-$ since otherwise, $\widetilde{\rho}_1$ would be nondecreasing in a neighbourhood of a_- which would imply $\widetilde{\rho}_1(a_- - \varepsilon) \leq \alpha$ for small $\varepsilon > 0$, contradicting the definition of a_-, we thus have $\widetilde{z}(a_-) = 1$. Since $\widetilde{\rho}_1$ is BV in a neighbourhood of a_-, it has a right and a left limit at a_-, again by minimality of a_-, the left limit of $\widetilde{\rho}_1$ at a_- cannot be strictly smaller than α, so there is an $\varepsilon > 0$ such that $\widetilde{\rho}_1 > 0$ on $I_- := [a_- - \varepsilon, a_-)$. Hence on I_-, (4.12) becomes

$$\widetilde{z}' + \frac{d-1}{r}\widetilde{z} + \frac{\widetilde{\varphi}}{\tau} = 0, \tag{4.17}$$

moreover, on I_-, $\widetilde{\varphi}$ is actually of class C^1 with $\widetilde{\varphi}'(r) = r - \widetilde{T}(r)$ where $\widetilde{T}$ is the (continuous) optimal transport between $\widetilde{\mu}_1$ and $\widetilde{\mu}_0$ obtained by the relation $F_{\widetilde{\mu}_0} \circ \widetilde{T} = F_{\widetilde{\mu}_1}$ (where $F_{\widetilde{\mu}_i}$ is the cumulative distribution function of $\widetilde{\mu}_i$ for $i = 0,1$). One can therefore differentiate (4.17) on I_- so as to obtain

$$\widetilde{z}'' + \frac{d-1}{r}\widetilde{z}' - \frac{(d-1)}{r^2}\widetilde{z}(r) + \frac{r - \widetilde{T}(r)}{\tau} = 0, \forall r \in I_-. \tag{4.18}$$

Since $\widetilde{z}$ is maximal at a_-, we first have

$$\lim_{\delta \to 0^+} \delta^{-1}[\widetilde{z}(a_-) - \widetilde{z}(a_- - \delta)] = -\frac{(d-1)\widetilde{z}(a_-)}{a_-} - \frac{\widetilde{\varphi}(a_-)}{\tau} \geq 0$$

but recalling (4.12) we also have

$$\begin{aligned}0 &\geq \limsup_{\delta\to 0^+} \delta^{-1}[\widetilde{z}(a_- + \delta) - \widetilde{z}(a_-)]\\ &\geq \lim_{\delta\to 0^+} \delta^{-1} \int_{a_-}^{a_-+\delta} \left[-(d-1)s^{-1}\widetilde{z}(s) - \tau^{-1}\widetilde{\varphi}(s)\right] \mathrm{d}s\\ &= -\frac{(d-1)\widetilde{z}(a_-)}{a_-} - \frac{\widetilde{\varphi}(a_-)}{\tau}\end{aligned}$$

which shows that $\widetilde{z}$ is differentiable at a_- with $\widetilde{z}'(a_-) = 0$, this enables us to deduce that $\widetilde{z}''(a_-^-) := \lim_{\delta\to 0^+} \widetilde{z}''(a_- - \delta) \leq 0$, with (4.18) this gives

$$\widetilde{T}(a_-) - a_- = \tau \left(\widetilde{z}''(a_-^-) - \frac{(d-1)}{a_-^2} \right) \leq 0.$$

If $a_- = 0$, since $\widetilde{T}(0) = 0$, the same conclusion is reached with an equality. In a similar way, we obtain $\widetilde{T}(a_+) \geq a_+$ (again with an equality in case $a_+ = R$). Using the fact that $\widetilde{\rho}_1 \leq \alpha$ on (a_-, a_+) (with strict inequality in a neighbourhood of r_0) together with $F_{\widetilde{\mu}_0} \circ \widetilde{T} = F_{\widetilde{\mu}_1}$ and $\widetilde{\rho}_0 \geq \alpha$, we get

$$\begin{aligned}\alpha c_d \frac{(a_+^d - a_-^d)}{d} &> F_{\widetilde{\mu}_1}(a_+) - F_{\widetilde{\mu}_1}(a_-) = F_{\widetilde{\mu}_0}(\widetilde{T}(a_+)) - F_{\widetilde{\mu}_0}(\widetilde{T}(a_-))\\ &\geq F_{\widetilde{\mu}_0}(a_+) - F_{\widetilde{\mu}_0}(a_-) \geq \alpha c_d \frac{(a_+^d - a_-^d)}{d}\end{aligned}$$

which yields the desired contradiction. □

Let us remark that the proof of Proposition 4.6 gives an alternative proof of the minimum principle in dimension one.

5. Convergence of the TV-JKO scheme under a lower bound estimate

We are now interested in the convergence of the TV-JKO scheme to a solution of the fourth-order nonlinear equation (1.2) as the time step τ goes to 0. Throughout this section, we assume that Ω is a bounded open convex subset of $\mathbb{R}^d$ and that the initial condition ρ_0 satisfies

$$\rho_0 \in \mathcal{P}_{\mathrm{ac}}(\Omega) \cap BV(\Omega) \cap L^\infty(\Omega),\ \rho_0 \geq \alpha > 0 \text{ a.e. on } \Omega. \tag{5.1}$$

We fix a time horizon T, and for small $\tau > 0$, define the sequence ρ_k^τ by

$$\rho_0^\tau = \rho_0,\ \rho_{k+1}^\tau \in \operatorname{argmin} \left\{ \frac{1}{2\tau} W_2^2(\rho_k^\tau, \rho) + J(\rho),\ \rho \in \mathrm{BV} \cap \mathcal{P}_{\mathrm{ac}}(\Omega) \right\} \tag{5.2}$$

for $k = 0, \ldots, N_\tau$ with $N_\tau := [\frac{T}{\tau}]$. Thanks to Theorem 4.2, (5.1) ensures that the JKO-iterates ρ_k^τ defined by (5.2) also remain bounded $\rho_k^\tau \leq \|\rho_0\|_{L^\infty(\Omega)}$. We shall also assume that ρ_k^τ remains bounded from below by α:

$$\rho_k^\tau \geq \alpha > 0 \text{ a.e. in } \Omega, \text{ for every } k \text{ and } \tau \tag{5.3}$$

which holds, as we have seen in Section 4.3 when $d = 1$ or when Ω is a ball and ρ_0 is radially symmetric.

We extend this discrete sequence by piecewise constant interpolation *i.e.*,

$$\rho^\tau(t,x) = \rho^\tau_{k+1}(x),\ t\in(k\tau,(k+1)\tau],\ k=0,\dots N_\tau,\ x\in\Omega. \tag{5.4}$$

We shall see that ρ^τ converges to a solution ρ of

$$\partial_t\rho + \operatorname{div}\left(\rho\,\nabla\operatorname{div}\left(\frac{\nabla\rho}{|\nabla\rho|}\right)\right) = 0,\ (t,x)\in(0,T)\times\Omega,\ \rho_{|t=0}=\rho_0, \tag{5.5}$$

with the no-flux boundary condition

$$\rho\,\nabla\operatorname{div}\left(\frac{\nabla\rho}{|\nabla\rho|}\right)\cdot\nu = 0,\quad \text{on } (0,T)\times\partial\Omega. \tag{5.6}$$

Let us introduce the spaces

$$H^1_{\operatorname{div}}(\Omega) := \{z\in L^2(\Omega,\mathbb{R}^d)\ :\ \operatorname{div}(z)\in L^2(\Omega)\},$$
$$H^2_{\operatorname{div}}(\Omega) := \{z\in H^1_{\operatorname{div}}(\Omega)\ :\ \operatorname{div}(z)\in H^1(\Omega).\}$$

Since ρ is no more than BV in x, one has to be slightly cautious in the meaning of div $\left(\frac{\nabla\rho}{|\nabla\rho|}\right)$ which be conveniently done by interpreting this term as the negative of an element in the subdifferential of J (in the L^2 sense). For every $\rho\in \mathrm{BV}(\Omega)\cap L^2(\Omega)$ let us define

$$\partial J(\rho) := \{\operatorname{div}(z)\ :\ z\in H^1_{\operatorname{div}}(\Omega),\ \|z\|_{L^\infty}\le 1,\ z\cdot\nu=0 \text{ on } \partial\Omega,\ J(\rho)=\int_\Omega \operatorname{div}(z)\rho\}.$$

This leads to the following definition:

Definition 5.1. A weak solution of (5.5)–(5.6) is a $\rho\in L^\infty((0,T),\mathrm{BV}(\Omega)\cap L^\infty(\Omega))\cap C^0([0,T],(\mathcal{P}(\overline{\Omega}),W_2))$ such that there exists $z\in L^\infty((0,T)\times\Omega)\cap L^2((0,T),H^2_{\operatorname{div}}(\Omega))$ with

$$\operatorname{div}(z(t,.))\in\partial J(\rho(t,.)) \text{ for a.e. } t\in(0,T), \tag{5.7}$$

and ρ is a weak solution of

$$\partial_t\rho - \operatorname{div}(\rho\nabla\operatorname{div}(z)) = 0,\ \rho_{|t=0}=\rho_0,\ \rho\nabla\operatorname{div}(z)\cdot\nu = 0 \text{ on } (0,T)\times\partial\Omega. \tag{5.8}$$

i.e., for every $u\in C^\infty_c([0,T)\times\overline{\Omega})$

$$\int_0^T\int_\Omega(\partial_t u\,\rho - \rho\nabla\operatorname{div}(z)\cdot\nabla u)\mathrm{d}x\mathrm{d}t = -\int_\Omega u(0,x)\rho_0(x)\mathrm{d}x.$$

We then have

Theorem 5.2. *If ρ_0 satisfies (5.6) and the JKO iterates ρ^τ_k obey the lower bound (5.3), there exists a vanishing sequence of time steps $\tau_n\to 0$ such that the sequence ρ^{τ_n} constructed by (5.2)–(5.4) converges strongly in $L^p((0,T)\times(0,1))$ for any $p\in[1,+\infty)$ and in $L^\infty((0,T),(\mathcal{P}(\overline{\Omega}),W_2))$ to a weak solution of (5.5)–(5.6).*

Proof. First, ρ_0 being L^∞, we have a uniform L^∞ bound on ρ^τ thanks to Theorem 4.2, and from our extra lower bound assumption (5.3) we have

$$M := \|\rho_0\|_{L^\infty} \geq \rho^\tau(t,x) \geq \alpha, \ t \in [0,T], \ \text{ a.e. } x \in \Omega. \tag{5.9}$$

Moreover, by construction of the TV-JKO scheme (5.2), one has

$$\frac{1}{2\tau} \sum_{k=0}^{N_\tau} W_2^2(\rho_k^\tau, \rho_{k+1}^\tau) \leq J(\rho_0), \quad \sup_{t\in[0,T]} J(\rho^\tau(t,.)) \leq J(\rho_0) \tag{5.10}$$

By using an Aubin–Lions type compactness Theorem of Savaré and Rossi (Thm. 2 in [24]), the fact that the embedding of $\mathrm{BV}(\Omega)$ into $L^p(\Omega)$ is compact for every $p \in [1, \frac{d}{d-1})$ as well as a refinement of Arzèla–Ascoli Theorem (Prop. 3.3.1 in [1]), one obtains (see Sect. 4 of [12] or Sect. 5 of [8] for details) that, up to taking suitable sequence of vanishing times steps $\tau_n \to 0$, we may assume that

$$\rho^\tau \to \rho \text{ a.e. in } (0,T)\times\Omega \text{ and in } L^p((0,T)\times\Omega), \ \forall p \in \left[1, \frac{d}{d-1}\right) \tag{5.11}$$

and

$$\sup_{t\in[0,T]} W_2(\rho^\tau(t,.), \rho(t,.)) \to 0 \text{ as } \tau \to 0, \tag{5.12}$$

for some limit curve $\rho \in C^{0,\frac{1}{2}}([0,T], (\mathcal{P}(\overline{\Omega}), W_2)) \cap L^q((0,T)\times\Omega)$. From (5.9) and Lebesgue's dominated convergence Theorem, we deduce that the convergence in (5.11) actually holds for any $p \in [1,+\infty)$. It also follows from (5.9) and (5.10), that $\rho \in L^\infty((0,T), \mathrm{BV}(\Omega)\cap L^\infty(\Omega))$ and that $\rho \geq \alpha$.

We deduce from the fact that $\rho_k^\tau \geq \alpha > 0$ and Theorem 3.3 that for each $k = 0, \dots, N_\tau$, there exists $z_k^\tau \in L^\infty(\Omega, \mathbb{R}^d)$ such that $\mathrm{div}(z_k^\tau) \in W^{1,\infty}(\Omega)$ and

$$\|z_k^\tau\|_{L^\infty} \leq 1, \ z_k^\tau \cdot \nu = 0 \text{ on } \partial\Omega, \ J(\rho_k^\tau) = \int_\Omega \mathrm{div}(z_k^\tau)\rho_k^\tau, \tag{5.13}$$

and the optimal (backward) optimal transport T_{k+1}^τ from ρ_{k+1}^τ to ρ_k^τ is related to z_{k+1}^τ by

$$\mathrm{id} - T_{k+1}^\tau = -\tau \nabla \mathrm{div}(z_{k+1}^\tau). \tag{5.14}$$

We extend z_k^τ in a piecewise constant way *i.e.* set

$$z^\tau(t,x) = z_{k+1}^\tau(x), \ t \in (k\tau, (k+1)\tau], \ k = 0, \dots, N_\tau, \ x \in \Omega. \tag{5.15}$$

We then observe that

$$\begin{aligned} W_2^2(\rho_k^\tau, \rho_{k+1}^\tau) &= \int_\Omega |x - T_{k+1}^\tau(x)|^2 \rho_{k+1}^\tau(x)\mathrm{d}x \\ &= \tau^2 \int_\Omega |\nabla \mathrm{div}(z_{k+1}^\tau)|^2 \rho_{k+1}^\tau(x)\mathrm{d}x \\ &\geq \alpha\tau^2 \int_\Omega |\nabla \mathrm{div}(z_{k+1}^\tau)|^2 \mathrm{d}x. \end{aligned}$$

Thanks to (5.10) we thus deduce that $\nabla \mathrm{div} z^\tau$ is bounded in $L^2((0,T)\times\Omega)$, since $\mathrm{div}(z^\tau)$ has zero-mean, with Poincaré–Wirtinger inequality, we obtain

$$\int_0^T \|\mathrm{div}(z^\tau)\|^2_{H^1(\Omega)} \mathrm{d}t \leq C. \tag{5.16}$$

We may therefore assume (up to further suitable extractions) that there is some $z \in L^\infty((0,T)\times\Omega)\cap L^2((0,T), H^2_{\mathrm{div}}(\Omega))$ such that z^τ converges to z weakly $*$ in $L^\infty((0,T)\times\Omega)$ and $(\mathrm{div}(z^\tau), \nabla\mathrm{div}(z^\tau))$ converges weakly in $L^2((0,T)\times\Omega)$ to $(\mathrm{div}(z), \nabla\mathrm{div}(z))$. Of course $\|z\|_{L^\infty}\leq 1$ and $z(t,.)\cdot\nu = 0$ on $\partial\Omega$ for a.e. t. Note also that $\rho^\tau\nabla\mathrm{div}(z^\tau)$ converges weakly in $L^1((0,T)\times\Omega)$ to $\rho\nabla\mathrm{div}(z)$.

The limiting equation can now be derived using standard computations (see the proof of Theorem 5.1 of the seminal work [17], or Chap. 8 of [26]): Let $u\in C^2_c([0,T)\times\overline{\Omega})$ and observe that

$$\int_0^T \int_\Omega \partial_t u\ \rho^\tau \mathrm{d}x\mathrm{d}t = \sum_{k=1}^{N_\tau}\left(\int_\Omega u(k\tau,x)(\rho^\tau_k(x)-\rho^\tau_{k+1}(x))\mathrm{d}x\right) - \int_\Omega u(0,x)\rho^\tau_1(x)\mathrm{d}x.$$

Recalling that $\rho^\tau_k = T^\tau_{k+1\#}\rho^\tau_{k+1}$, and applying Taylor's theorem, we have

$$\begin{aligned}
&\sum_{k=1}^{N_\tau}\left(\int_\Omega u(k\tau,x)(\rho^\tau_k(x)-\rho^\tau_{k+1}(x))\mathrm{d}x\right)\\
&=\sum_{k=1}^{N_\tau}\left(\int_\Omega ((T^\tau_{k+1}(x)-x)\cdot\nabla u(k\tau,x)+\tilde{R}_\tau(x))\rho^\tau_{k+1}\mathrm{d}x\right)\\
&=\sum_{k=1}^{N_\tau}\left(\int_\Omega (\tau(\nabla\mathrm{div}(z^\tau_{k+1}))\cdot\nabla u(k\tau,x)+\tilde{R}_\tau(x))\rho^\tau_{k+1}\mathrm{d}x\right),
\end{aligned}$$

where $|\tilde{R}_\tau(x)|\leq C\|D^2u(k\tau,\cdot)\|_{L^\infty}|T^\tau_{k+1}(x)-x|^2$. Note also that for $t\in(k\tau,(k+1)\tau]$, $|\nabla u(k\tau,\cdot)-\nabla u(t,\cdot)|\leq \tau\|\partial_t\nabla u\|_{L^\infty}$. Therefore,

$$\int_0^T\int_\Omega(\partial_t u\ \rho^\tau - \rho^\tau\nabla\mathrm{div}(z^\tau)\cdot\nabla u)\mathrm{d}x\mathrm{d}t = -\int_\Omega u(0,x)\rho^\tau_1(x)\mathrm{d}x + R_\tau(u) \tag{5.17}$$

with

$$|R_\tau(u)|\leq C\max\{\|D^2u\|_{L^\infty}, \|\partial_t\nabla u\|_{L^\infty}\Big\}\sum_{k=0}^{N_\tau} W_2^2(\rho^\tau_k,\rho^\tau_{k+1})\leq C\tau. \tag{5.18}$$

Passing to the limit τ to 0 in (5.17) yields that ρ is a weak solution to

$$\partial_t\rho - \mathrm{div}\left(\rho\nabla\mathrm{div}(z)\right)=0,\ \rho_{|t=0}=\rho_0,\ \rho\nabla\mathrm{div}(z)\cdot\nu = 0 \text{ on } (0,T)\times\partial\Omega.$$

It remains to prove that $J(\rho(t,.)) = \int_\Omega \mathrm{div}(z(t,x))\rho(t,x)\mathrm{d}x$, for a.e. $t\in(0,T)$. The inequality $J(\rho(t,.))\geq \int_\Omega \mathrm{div}(z(t,x))\rho(t,x)\mathrm{d}x$ is obvious since $z(t,.)\in H^1_{\mathrm{div}}(\Omega)$, $z(t,.)\cdot\nu=0$ on $\partial\Omega$ and $\|z(t,.)\|_{L^\infty}\leq 1$. To prove the converse inequality, we use Fatou's Lemma, the lower semi-continuity of J, (5.13) and the weak-convergence in

$L^1((0,T)\times\Omega)$ of $\rho^\tau \mathrm{div}(z^\tau)$ to $\rho\mathrm{div}(z)$:

$$\begin{aligned}\int_0^T J(\rho(t,.))\mathrm{d}t &\leq \int_0^T \liminf_\tau J(\rho^\tau(t,.))\mathrm{d}t\\ &\leq \liminf_\tau \int_0^1 J(\rho^\tau(t,.))\mathrm{d}t\\ &= \liminf_\tau \int_0^T\int_\Omega \mathrm{div}(z^\tau(t,x))\rho^\tau(t,x)\mathrm{d}x\mathrm{d}t\\ &= \int_0^T\int_\Omega \mathrm{div}(z(t,x))\rho(t,x)\mathrm{d}x\mathrm{d}t\end{aligned}$$

which concludes the proof. □

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