

EIGENCURVES FOR LINEAR ELLIPTIC EQUATIONS

MAURICIO A. RIVAS^{1,*} AND STEPHEN B. ROBINSON²

Abstract. This paper provides results for *variational eigencurves* associated with self-adjoint linear elliptic boundary value problems. The elliptic problems are treated as a general two-parameter eigenproblem for a triple (a, b, m) of continuous symmetric bilinear forms on a real separable Hilbert space V . *Geometric characterizations* of eigencurves associated with (a, b, m) are obtained and are based on their variational characterizations described here. Continuity, differentiability, as well as asymptotic, results for these eigencurves are proved. Finally, two-parameter Robin–Steklov eigenproblems are treated to illustrate the theory.

Mathematics Subject Classification. 35J20, 35P15, 58J20.

Received May 27, 2017. Accepted June 18, 2018.

1. INTRODUCTION

This paper is motivated by the study of *variational eigencurves* associated with two-parameter eigenproblems such as the *Robin–Steklov* problem given by

$$\begin{aligned} -\operatorname{div}(A\nabla u) + c(x)u &= \mu m_0(x)u && \text{in } \Omega, \\ (A\nabla u) \cdot \nu + b_c(x)u &= \lambda b_0(x)u && \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $(\lambda, \mu) \in \mathbb{R}^2$ is said to be an *eigenpair* provided (1.1) has a non-trivial solution u , and eigencurves are comprised entirely of eigenpairs. The given functions c, b_c, b_0, m_0 are assumed to lie in appropriate L^p -spaces, Ω is a bounded region in \mathbb{R}^N satisfying mild boundary regularity requirements, A is a matrix-valued field, and ν denotes the outward unit normal to $\partial\Omega$. We assume m_0 is strictly positive and allow b_0 to be sign-changing. For such problems, the boundary and interior equations may be combined in weak form using bilinear forms; the weak form of (1.1) is (10.1). Our focus in this paper is, therefore, on the analysis of abstract eigencurve problems associated with triples (a, b, m) of continuous symmetric bilinear forms on a real Hilbert space V .

Our main result generalizes the *geometric characterization of eigencurves* for Sturm–Liouville problems given in Binding and Volkmer [7]. To obtain our main result, we considerably expand on the issues treated in [7] regarding continuity, differentiability, and asymptotics of eigencurves, and also provide results for various issues not appearing in the ODE case.

Keywords and phrases: Two-parameter eigenproblems, variational eigencurves, Robin–Steklov eigenproblems.

¹ Department of Mathematics, North Carolina Agricultural and Technical State University, 232 Marteen Hall, 1601 East Market Street, Greensboro, NC 27411, USA.

² Department of Mathematics and Statistics, Wake Forest University, PO Box 7388, 127 Manchester Hall, Winston-Salem, NC 27109, USA.

* Corresponding author: marivas@ncat.edu

The analysis in this paper is based on the use of spectral results for bilinear forms obtained in Auchmuty [3]. This bilinear and variational approach provides a simpler alternative to the usual operator-theoretic approach to eigencurves as it avoids the use of dual, or other, Sobolev spaces and closed densely defined operators. For such a classical treatment of eigencurves see Kato [13] and, in particular, Theorems 1.8 and 4.2 of Chapter 7 for abstract results on *analytic eigencurves*. As weak forms to problems such as (1.1) incorporate both domain and boundary integrals, our variational analysis is easily applicable to problems satisfying general boundary conditions other than the common Dirichlet condition. For a treatment of one-parameter eigenproblems using bilinear forms invoking properties of associated linear operators see Attouch *et al.* [2], or Blanchard and Brüning [8].

It is worth noting that the assumptions on the bilinear forms used here also are sufficient to treat problems on exterior regions $U \subset \mathbb{R}^N$. In the recent paper [4], Auchmuty and Qi Han use variational methods and a suitable real Hilbert space $E^1(U)$ of *finite energy functions* to treat the well-posedness and representation of solutions of Poisson's equation on exterior regions subject to various boundary conditions. Moreover, the direct use of bilinear forms, instead of boundary integral linear operators or single and double layer potentials, greatly simplifies the analysis of related harmonic boundary value problems on exterior regions. We treat problem (1.1) on bounded regions Ω merely to exemplify our abstract results, but it is of great interest to interpret the results for problems on exterior regions. It is well known that eigenvalue problems on unbounded domains may also give rise to an essential spectrum. In this case, we emphasize that our results generalize results about the discrete spectrum and do not include the essential spectrum.

In this paper, we use variational analysis to study abstract eigencurves for bilinear forms on real Hilbert space. This leads to a more self-contained study of eigencurves, our proofs are different and lead to important additional details such as precise formulae for the slopes of the (variational) curves. More importantly, our main geometrical characterizations given in Sections 8 and 9 are based substantially on the variational characterizations.

Eigencurves have been studied and used in a variety of ways for more than a century (see Binding and Volkmer [7] for a nice summary), and renewed interest has emerged due to their effective use in recent advanced applications. For instance, in [15] Lacey *et al.* carry out an asymptotic analysis for the first eigencurve of a Robin–Steklov problem for the Laplacian to describe the stability of steady states of reaction diffusion models with *nonlinear boundary sources*. The asymptotic results for this first Robin–Steklov–Laplacian eigencurve were then generalized to higher eigencurves by Daners and Kennedy in [9]; a paper that takes a variational approach to these Robin–Steklov–Laplacian eigencurves is that of Filinovskiy [11]. In [12], Giorgi and Smits derive estimates for the first Robin–Steklov–Laplacian eigencurve to provide results for the loss of stability of the normal state in zero fields for systems with enhanced surface superconductivity, and for the long time behaviour of a Brownian motion with the creation of particles. Bandle and Wagner in [6] consider the Robin–Steklov–Laplacian problem to derive an isoperimetric inequality for arbitrary domains in \mathbb{R}^N using the harmonic transplantation method – they illustrate this method by means of some elliptic equations with *inhomogeneous* Neumann boundary conditions.

A paper that uses eigencurves to establish existence results for some indefinite weight semilinear elliptic problems is Ko and Brown [14]; those problems arise, for instance, in population genetics. Their results are based on results for the principle eigencurve(s) for linear boundary value problems with indefinite weight and Robin boundary conditions given in Afrouzi and Brown [1]. The paper of Binding and Volkmer [7] makes a brief discussion on connections to indefinite inner product spaces such as Pontryagin and Krein spaces.

The papers of Mavinga and Nkashama [17] and Mavinga [16] establish existence results for some nonlinear elliptic equations with nonlinear boundary equations where the nonlinearities interact with the associated generalized Robin–Steklov spectrum. The spectra considered in [16, 17] may be regarded as the set of points along a diagonal line intersecting the eigencurves associated to their equations. We note that the principle as well as *higher* eigenvalues of the related linear problems are considered in those papers.

Recently, two-parameter problems for the Laplacian have been used in Section 10 of Auchmuty and Rivas [5] to provide representations of Sobolev spaces on product regions as *tensor products* of Sobolev spaces on the individual factor regions. Those problems arise, for instance, in fluid mechanics, electromagnetic theory and elsewhere, when the method of separation of variables is used to reduce the dimensionality of the studied equation.

By exemplifying our abstract results through the treatment of (1.1), we hope to motivate the effective use of variational eigencurves to further advance the study of the above and other physical and mathematical models.

The current paper is organized as follows. The specific two-parameter eigenvalue problem considered in this paper is described in Section 2, as well as associated terminology and definitions. Section 3 details the variational characterizations for the eigencurves to be used in this paper. A first result that follows is the concavity of the first eigencurve.

Special inner products are used in Section 4 to establish various orthogonality results for certain eigenspaces associated with the eigencurves. In Section 5, the continuity of each eigencurve is established. In particular, they are shown to be Lipschitz continuous.

In Section 6, eigencurves are proved to be differentiable except possibly where they intersect, and that an intersection point (λ_*, μ_*) is the intersection of finitely many differentiable curves. That is, variationally characterized curves may not be differentiable at (λ_*, μ_*) because they may follow one differentiable curve into (λ_*, μ_*) from the left and a different differentiable curve out of (λ_*, μ_*) on the right. We give explicit formulae for one-sided derivatives at each point of the eigencurve in terms of spectral data for the pair (b, m) of bilinear forms. As noted above Kato's results in [13] show that the spectrum is a union of analytic curves. However, we believe it is of value to have a self-contained treatment emphasizing variational methods, and we note that our results provide precise formulae for derivatives.

Section 7 treats the issue of describing the asymptotic behaviour of eigencurves. A difficulty arises as the b, m forms are not necessarily variants of each other as in (10.1), the weak form of (1.1), where the associated bilinear form $b(u, v) := \int_{\partial\Omega} b_0 u v \, d\sigma$ comprises a boundary integral whereas the form $m(u, v) := \int_{\Omega} m_0 u v \, dx$ comprises an interior-region integral. We show the asymptotic behaviour of eigencurves is governed by the spectrum of (b, m) and that the sign of the quadratic form $u \mapsto b(u, u)$ plays a role in these matters.

An interesting side issue encountered in our analysis is the possible appearance of straight lines within the *spectrum* (collection of graphs of eigencurves) of (a, b, m) due to the degeneracy of b . Degeneracy may occur in elliptic problems when weight functions are zero on a set of positive measure or when the equations involve boundary integrals; the function $b_0(x)$ in (1.1) may be zero on a set of positive $(N - 1)$ -dimensional Hausdorff measure, but the associated form $b(u, v) := \int_{\partial\Omega} b_0 u v \, d\sigma$ is already degenerate as it is defined (in Sect. 10) on the Sobolev space $H^1(\Omega)$. Results for straight lines within the spectrum are given in Section 8 and are shown to be related to a question of linear independence of certain eigenvectors. Simple examples of triples (a, b, m) are also provided in this section to illustrate these results.

In Section 9, we prove a generalization of a main result in [7] stating that any straight line intersects the first n eigencurves in at most $2n$ points. The straight line is assumed not a subset of the spectrum of (a, b, m) . It is worth noting that even in the most degenerate case, there are at most countably many lines in the spectrum. As eigenvalues in our general setting may have multiplicity greater than one, for our main result we focus on the number of connected components the graph of an eigencurve may have above the intersecting line instead of considering points of intersections. This approach combined with the results of previous sections yields our *generalized geometrical characterization of eigencurves*, Theorem 9.6.

The final section treats in detail the two-parameter Robin–Steklov eigenproblem (1.1). The analysis demonstrates how the theory established here is applicable to quite general elliptic two-parameter eigenproblems in divergence form.

2. TWO-PARAMETER EIGENVALUE PROBLEMS FOR BILINEAR FORMS

The two-parameter eigenproblem will be studied in the framework of bilinear forms on Hilbert space with the following definitions and notation. V will denote a real, separable, Hilbert space with inner product and norm on V denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively.

Our interest is in describing the pairs $(\lambda, \mu) \in \mathbb{R}^2$ for which there is a nonzero u in V satisfying

$$a(u, v) = \lambda b(u, v) + \mu m(u, v) \quad \text{for all } v \in V, \quad (2.1)$$

where a, b, m are bilinear forms on V subject to the conditions described below. This will be called the (a, b, m) -*eigenproblem*. The pair (λ, μ) is said to be an *eigenpair* of (a, b, m) if there is a nonzero vector $u \in V$ satisfying (2.1), and such u will be called an *eigenvector* of (a, b, m) corresponding to (λ, μ) . The subset $\sigma(a, b, m)$ of \mathbb{R}^2 consisting of all eigenpairs (λ, μ) will be called the *spectrum* of (a, b, m) , and (2.1) will be called the *eigenpair equation*.

Define $\mathcal{A}, \mathcal{B}, \mathcal{M}$ to be the quadratic forms on V associated with a, b, m so that

$$\mathcal{A}(u) := a(u, u), \quad \mathcal{B}(u) := b(u, u), \quad \text{and} \quad \mathcal{M}(u) := m(u, u). \quad (2.2)$$

The assumptions on the bilinear forms to be used in this paper will include

(A1): $a(\cdot, \cdot)$ is a continuous, symmetric, bilinear form on V that is also V -coercive. That is, there are constants $0 < \kappa_1 \leq \kappa_2 < \infty$ such that

$$\kappa_1 \|u\|^2 \leq \mathcal{A}(u) \leq \kappa_2 \|u\|^2 \quad \text{for all } u \in V. \quad (2.3)$$

(A2): $b(\cdot, \cdot)$ is a weakly continuous, symmetric bilinear form on V ; the form b is weakly continuous if $u_n \rightharpoonup u$ in V implies $b(u_n, u_n) \rightarrow b(u, u)$, where \rightharpoonup indicates weak convergence.

(A3): $m(\cdot, \cdot)$ is a weakly continuous, symmetric, bilinear form that satisfies

$$\mathcal{M}(u) > 0 \quad \text{for all nonzero } u \in V.$$

When (A1) holds, then $a(\cdot, \cdot)$ defines an inner product on V equivalent to the V -inner product and is called the a -*inner product*; the associated norm will be denoted by $\|\cdot\|_a$. When (A3) holds, the quadratic form \mathcal{M} is strictly positive on V so that m is an inner product on V and the associated norm will be denoted by $\|u\|_m$. A vector u in V will be said to be m -*normalized* provided $m(u, u) = 1$. A bilinear form b satisfying (A2) may be negative or zero for some $v \in V$, and is said to be an *indefinite form* whenever \mathcal{B} attains both positive and negative values.

Note that the assumptions imply the existence of a compact embedding: *If $(V, a(\cdot, \cdot))$ is the Hilbert space described by (A1), $m(\cdot, \cdot)$ is as in (A3), and $(H, m(\cdot, \cdot))$ is the completion of V with respect to m , then it follows that V is a dense subspace of H and embeds compactly.* It is this compactness that is implicitly used in our constructive variational approach to eigencurves, and it is imperative to verify carefully that assumptions (A1)–(A3) hold when applying our results to particular situations; the assumptions do not hold, for example, for the choices of

$$V = H^1(\mathbb{R}), \quad a(u, v) := \int_{\mathbb{R}} [u' v' + uv] dx, \quad b(u, v) := \int_{\mathbb{R}} b_0 uv dx, \quad m(u, v) := \int_{\mathbb{R}} m_0 uv dx,$$

with $b_0, m_0 \in L^\infty(\mathbb{R})$ such that $m_0(x) > 0$ for all $x \in \mathbb{R}$ as the bilinear form m is not weakly continuous in this standard setting.

When working merely with a pair of bilinear forms, the standard (a, m) -*eigenproblem* for a pair (a, m) of symmetric, continuous bilinear forms on V is that of finding $\mu \in \mathbb{R}$ and nontrivial $u \in V$ satisfying

$$a(u, v) = \mu m(u, v) \quad \text{for all } v \in V. \quad (2.4)$$

A nonzero u satisfying (2.4) is called an *eigenvector* associated to the *eigenvalue* μ of (a, m) . The number of linearly independent eigenvectors of (a, m) corresponding to μ is called the *multiplicity* of μ . When the multiplicity of μ is one, then μ is said to be a *simple eigenvalue*. The set $\sigma(a, m)$ of all distinct eigenvalues of (a, m) will be called the *spectrum* of (a, m) .

When (A1), (A3) hold, there is an increasing sequence $\lambda_0 := \{\mu_{0n} : n \in \mathbb{N}\}$ of strictly positive eigenvalues of (a, m) , repeated according to (finite) multiplicity with $\mu_{0n} \rightarrow \infty$ as $n \rightarrow \infty$, and there is an associated sequence $\mathcal{E}_0 := \{e_{0n} : n \in \mathbb{N}\}$ of eigenvectors constituting a basis for V that is orthogonal with respect to both

a and m . These results follow from the analysis of Section 4 of [3] and the notation here is chosen to simplify the presentation.

3. VARIATIONAL CHARACTERIZATION OF EIGENCURVES

In this section, the variational eigencurves associated to (a, b, m) to be studied in this paper are obtained using the constructive algorithm given in Auchmuty [3]. The construction yields a one-parameter family of sequences of eigenpairs for (a, b, m) as well as a corresponding family of sequences of eigenvectors.

For $\lambda, \tau \in \mathbb{R}$ define $a_{\lambda, \tau}$ to be the bilinear form on V given by

$$a_{\lambda, \tau}(u, v) := a(u, v) - \lambda b(u, v) + \tau m(u, v) \quad \text{for } u, v \in V. \quad (3.1)$$

The basic coercivity result for these forms is the following.

Theorem 3.1. *Assume a, b, m satisfy (A1)–(A3) and \mathcal{A} is defined by (2.2). Then for fixed $r_0 > 0$ there exists a constant $\tau > 0$ such that for each $\lambda \in [-r_0, r_0]$ the bilinear form $a_{\lambda, \tau}$ given by (3.1) satisfies*

$$a_{\lambda, \tau}(u, u) \geq \frac{1}{2} \mathcal{A}(u) \quad \text{for all } u \in V. \quad (3.2)$$

Proof. Arguing by contradiction, suppose there are sequences $\tau_n > 0$, $\lambda_n \in [-r_0, r_0]$ and $u_n \in V$, with τ_n increasing to $+\infty$, satisfying

$$a_{\lambda_n, \tau_n}(u_n, u_n) < \frac{1}{2} \mathcal{A}(u_n) \quad \text{for all } n \in \mathbb{N}. \quad (3.3)$$

The form $a_{\lambda_n, \tau_n}(u_n, u_n)$ is equal to $\mathcal{A}(u_n) - \lambda_n \mathcal{B}(u_n) + \tau_n \mathcal{M}(u_n)$ where $\mathcal{A}, \mathcal{B}, \mathcal{M}$ are given by (2.2). Without loss of generality, suppose $\lambda_n \rightarrow \lambda$ in $[-r_0, r_0]$, and the u_n are a -normalized so that $\mathcal{A}(u_n) = 1$. By assumption (A1) the set $\{u \in V : \mathcal{A}(u) = 1\}$ is closed and bounded so we may also assume without loss of generality that u_n converges weakly to u in V . Then (3.3) becomes

$$\frac{1}{2} - \lambda_n \mathcal{B}(u_n) + \tau_n \mathcal{M}(u_n) < 0 \quad \text{for all } n \in \mathbb{N}. \quad (3.4)$$

By weak continuity of \mathcal{M}, \mathcal{B} we see that $\mathcal{M}(u_n) \rightarrow \mathcal{M}(u)$ and $\mathcal{B}(u_n) \rightarrow \mathcal{B}(u)$. If $u \neq 0$, then $\tau_n \mathcal{M}(u_n) \rightarrow +\infty$ which contradicts (3.4). If, on the other hand, $u = 0$, then $\lambda_n \mathcal{B}(u_n) \rightarrow 0$ which implies

$$\frac{1}{2} - \lambda_n \mathcal{B}(u_n) + \tau_n \mathcal{M}(u_n) > 0,$$

for large enough n . This again contradicts (3.4), so the assertion of the theorem holds. \square

This result says that for $\lambda \in \mathbb{R}$ fixed, $a_{\lambda, \tau}$ will be a coercive bilinear form whenever τ is large enough and provided (A1) and (A2) also hold. For the remainder of this section suppose $\lambda \in \mathbb{R}$ and $\tau > 0$ have been chosen to satisfy this criteria. Then $a_{\lambda, \tau}$ defines an inner product on V equivalent to the V -inner product.

Denote by $\mu_1(\lambda) + \tau$ the first eigenvalue of $(a_{\lambda, \tau}, m)$. It may be found by maximizing the weakly continuous functional \mathcal{M} defined by (2.2) on the closed convex subset $C_{1, \lambda, \tau}$ of V defined by $C_{1, \lambda, \tau} := \{u \in V : a_{\lambda, \tau}(u, u) \leq 1\}$. The maximizers of this problem are eigenvectors of $(a_{\lambda, \tau}, m)$ corresponding to the eigenvalue $\mu_1(\lambda) + \tau$ whereas the value of this problem is

$$\frac{1}{\mu_1(\lambda) + \tau} = \sup_{u \in C_{1, \lambda, \tau}} \mathcal{M}(u). \quad (3.5)$$

These results follow directly from Section 3 of [3] with the appropriate notational modifications. Then using the construction of Section 4 of that paper an infinite sequence of eigenvalues and eigenvectors of $(a_{\lambda,\tau}, m)$ may be found.

Let $\mu_n(\lambda) + \tau > 0$, with $n = 1, 2, \dots, k$, be the first k successive smallest eigenvalues of $(a_{\lambda,\tau}, m)$ and let $e_{\lambda n}$ be associated $a_{\lambda,\tau}$ -orthonormal eigenvectors. The $(k+1)$ st eigenvector of $(a_{\lambda,\tau}, m)$ will be a maximizer of \mathcal{M} over the subset

$$C_{k+1,\lambda,\tau} := \{u \in C_{1,\lambda,\tau} : a_{\lambda,\tau}(u, e_{\lambda n}) = 0 \text{ for } n = 1, 2, \dots, k\}, \quad (3.6)$$

and the value of this $(k+1)$ st problem is

$$\frac{1}{\mu_{k+1}(\lambda) + \tau} = \sup_{u \in C_{k+1,\lambda,\tau}} \mathcal{M}(u). \quad (3.7)$$

Let $A_\lambda := \{\mu_n(\lambda) : n \in \mathbb{N}\}$ be the sequence of such values repeated according to multiplicity and in increasing order and $\mathcal{E}_\lambda := \{e_{\lambda n} : n \in \mathbb{N}\}$ be a sequence of associated eigenvectors constructed by this iterative process. The main result for this eigendata is the following.

Theorem 3.2. *Assume a, b, m satisfy (A1)–(A3), $a_{\lambda,\tau}$ is defined by (3.1) for τ large enough, and $A_\lambda, \mathcal{E}_\lambda$ are the sequences defined as above. Then each $\mu_n(\lambda)$ in A_λ is of finite multiplicity with $-\infty < \mu_1(\lambda) \leq \mu_2(\lambda) \leq \dots \rightarrow +\infty$ and \mathcal{E}_λ is an $a_{\lambda,\tau}$ -orthonormal basis of V consisting of vectors satisfying*

$$a(e_{\lambda n}, v) = \lambda b(e_{\lambda n}, v) + \mu_n(\lambda) m(e_{\lambda n}, v) \quad \text{for all } v \in V. \quad (3.8)$$

Proof. Conditions (A1) and (A4) of [3] hold with the bilinear forms a, m there taken to be the pair $(a_{\lambda,\tau}, m)$ in this paper, so the analysis of Section 4 there is applicable. This yields the sequences A_λ and \mathcal{E}_λ for the pair $(a_{\lambda,\tau}, m)$ and their properties. Rearranging the corresponding eigenequation for $(a_{\lambda,\tau}, m)$ gives the eigenpair equation (3.8), so that the $e_{\lambda n}$ are indeed eigenvectors of (a, b, m) corresponding to eigenpairs $(\lambda, \mu_n(\lambda))$. \square

Define the n th-variational eigencurve associated to (a, b, m) to be the graph of the function $\lambda \mapsto \mu_n(\lambda)$. This will be a subset of $\sigma(a, b, m)$ in \mathbb{R}^2 , and we shall often say $\mu_n(\lambda)$ is the n th eigencurve associated to (a, b, m) . The point $(\lambda_*, \mu_*) \in \sigma(a, b, m)$ is said to be an *eigenpair of multiplicity k_0* for (a, b, m) accordingly as the value μ_* is an eigenvalue of multiplicity k_0 for the pair $(a_{\lambda_* b}, m)$ where $a_{\lambda_* b} := a - \lambda_* b$.

When a, b, m satisfy (A1)–(A3) and $\lambda \in \mathbb{R}$ is fixed, the following *minimax* characterization of eigenvalues will be useful for the analysis of these variational eigencurves:

$$\mu_k(\lambda) = \inf_{Z_k} \sup_{\substack{v \in Z_k \\ v \neq 0}} \left\{ \frac{\mathcal{A}(v) - \lambda \mathcal{B}(v)}{\mathcal{M}(v)} \right\}, \quad (3.9)$$

where the infimum is taken over all subspaces Z_k of V of dimension k . The supremum in (3.9) is equal to $\mu_k(\lambda)$ when Z_k is a subspace generated by the first k m -orthonormal eigenvectors corresponding to the smallest eigenvalues of $(a_{\lambda b}, m)$ where $a_{\lambda b} := a - \lambda b$. These results follow from Section 4 of [3] with $(a_{\lambda,\tau}, m)$ in place of (a, m) and rearranging the relations obtained there to eliminate the parameter τ .

A direct consequence of this characterization is the following.

Lemma 3.3. *Assume a, b, m satisfy (A1)–(A3). Then the first eigencurve $\mu_1(\lambda)$ is concave.*

Proof. It follows from (3.9) that $\mu_1(\lambda)$ is the infimum of affine functions on \mathbb{R} . \square

4. ORTHOGONALITY RELATIONS AND EIGENSPACES

Here, some *orthogonality relations* among eigenspaces of (a, b, m) will be described. In particular, distinct eigenspaces are shown to be orthogonal with respect to various bilinear forms whenever the corresponding eigenpairs lie on the same horizontal line or the same vertical line. These results simplify the analysis of eigencurves.

Two vectors $v, w \in V$ are said to be orthogonal with respect to a bilinear form \tilde{a} provided $\tilde{a}(v, w) = 0$. The forms \tilde{a} arising here are, in general, indefinite on the whole space V and are therefore not necessarily inner products on V . A first result is the following.

Theorem 4.1. *Assume a, b, m satisfy (A1)–(A3). If e_1, e_2 are eigenvectors corresponding to eigenpairs (λ_1, μ_1) and (λ_2, μ_2) in $\sigma(a, b, m)$, then*

$$(\lambda_2 - \lambda_1)b(e_1, e_2) + (\mu_2 - \mu_1)m(e_1, e_2) = 0. \quad (4.1)$$

Proof. The eigenpair equation (2.1) shows that for all $v \in V$ the following relations hold:

$$a(e_1, v) = \lambda_1 b(e_1, v) + \mu_1 m(e_1, v) \quad \text{and} \quad a(e_2, v) = \lambda_2 b(e_2, v) + \mu_2 m(e_2, v).$$

Take $v = e_2$ in the first equation and $v = e_1$ in the second and subtract to obtain (4.1). \square

When distinct eigenpairs lie on the same horizontal line, (4.1) reduces to the following.

Corollary 4.2. *Assume a, b, m satisfy (A1)–(A3), and let $\mu \in \mathbb{R}$ be fixed. If e_1, e_2 are eigenvectors corresponding to distinct eigenpairs (λ_1, μ) and (λ_2, μ) in $\sigma(a, b, m)$, then*

$$b(e_1, e_2) = 0 \quad \text{and} \quad a(e_1, e_2) - \mu m(e_1, e_2) = 0. \quad (4.2)$$

Proof. Take $\mu_1 = \mu_2 = \mu$ in (4.1) to get $(\lambda_2 - \lambda_1)b(e_1, e_2) = 0$. Then $\lambda_2 \neq \lambda_1$ implies $b(e_1, e_2) = 0$. Substituting this into the eigenpair equation (2.1) gives (4.2). \square

Let $E_{\lambda\mu}$ be the eigenspace in V associated with the eigenpair (λ, μ) of (a, b, m) . That is, let $E_{\lambda\mu}$ be the subspace generated by eigenvectors $e_{\lambda n}$ in Theorem 3.2 associated to the eigenvalues $\mu_n(\lambda)$ satisfying $\mu_n(\lambda) = \mu$ for fixed λ . The orthogonality relations of Corollary 4.2 may then be expressed as

$$E_{\lambda_1\mu} \perp_b E_{\lambda_2\mu} \quad \text{and} \quad E_{\lambda_1\mu} \perp_{a_{\mu m}} E_{\lambda_2\mu}, \quad (4.3)$$

where \perp_b and $\perp_{a_{\mu m}}$ indicate orthogonality with respect to b and $a_{\mu m} := a - \mu m$, respectively.

In the case that distinct eigenpairs lie on the same vertical line, (4.1) reduces instead to the following orthogonality relations.

Corollary 4.3. *Assume a, b, m satisfy (A1)–(A3), and let $\lambda \in \mathbb{R}$ be fixed. If e_1, e_2 are eigenvectors corresponding to distinct eigenpairs (λ, μ_1) and (λ, μ_2) in $\sigma(a, b, m)$, then*

$$m(e_1, e_2) = 0 \quad \text{and} \quad a(e_1, e_2) - \lambda b(e_1, e_2) = 0. \quad (4.4)$$

Proof. Take $\lambda_1 = \lambda_2 = \lambda$ in (4.1) to get $(\mu_2 - \mu_1)m(e_1, e_2) = 0$. Then $\mu_2 \neq \mu_1$ implies $m(e_1, e_2) = 0$. Substituting this into the eigenpair equation (2.1) gives (4.4). \square

These results may be expressed as

$$E_{\lambda\mu_1} \perp_m E_{\lambda\mu_2} \quad \text{and} \quad E_{\lambda\mu_1} \perp_{a_{\lambda b}} E_{\lambda\mu_2}, \quad (4.5)$$

where \perp_m and $\perp_{a\lambda b}$ indicate orthogonality with respect to m and $a\lambda b$, respectively.

From Theorem 3.2, each eigenspace $E_{\lambda\mu}$ is finite dimensional for any eigenpair (λ, μ) of (a, b, m) , so that each linear subspace in (4.3) and (4.5) is finite dimensional. These finite dimensional subspaces will be particularly useful in the following sections.

5. CONTINUITY OF EIGENCURVES

This section describes the continuity of eigencurves associated to (a, b, m) and provides a first result on differentiability of eigencurves when simplicity of the eigenpair in question is assumed. The following is the main result on continuity.

Theorem 5.1. *Assume a, b, m satisfy (A1)–(A3). Then each variational eigencurve $\mu_n(\lambda)$ associated to (a, b, m) is Lipschitz continuous.*

The proof of this will be a straightforward consequence of the following result that is based on the general orthogonality result of Theorem 4.1 of the previous section. Here, the open ball in \mathbb{R}^2 of radius $\varepsilon > 0$ centered at (λ_*, μ_*) will be denoted by $B_\varepsilon(\lambda_*, \mu_*)$.

Theorem 5.2. *Assume a, b, m satisfy (A1)–(A3), and let (λ_*, μ_*) be a fixed eigenpair in $\sigma(a, b, m)$. Then there is an $\varepsilon > 0$ and a constant $C > 0$ such that*

$$|\mu - \mu_*| \leq C |\lambda - \lambda_*| \quad \text{for all } (\lambda, \mu) \in B_\varepsilon(\lambda_*, \mu_*) \cap \sigma(a, b, m). \quad (5.1)$$

Proof. Let $\{(\lambda_i, \mu_i) : i \in \mathbb{N}\}$ be a sequence of eigenpairs converging to (λ_*, μ_*) in \mathbb{R}^2 , and let e_i be associated a -normalized eigenvectors. Going to a subsequence if necessary, suppose e_i converges weakly to some e_* in V . Take $u = e_i$ in the eigenpair equation (2.1) and let $i \rightarrow \infty$ to get

$$a(e_*, v) = \lambda_* b(e_*, v) + \mu_* m(e_*, v) \quad \text{for each fixed } v \in V, \quad (5.2)$$

using the continuity of the linear functionals associated with a, b , and m when v is fixed. Since $1 = \lambda_i \mathcal{B}(e_i) + \mu_i \mathcal{M}(e_i)$ holds as each e_i is a -normalized, we get $1 = \lambda_* \mathcal{B}(e_*) + \mu_* \mathcal{M}(e_*)$ by weak continuity. This and taking $v = e_*$ in (5.2) then gives

$$1 = \lambda_* \mathcal{B}(e_*) + \mu_* \mathcal{M}(e_*) = \mathcal{A}(e_*).$$

Thus, $\mathcal{A}(e_i) \rightarrow \mathcal{A}(e_*)$, which implies $e_i \rightarrow e_*$. This shows e_* is nonzero and thus an eigenvector of (a, b, m) corresponding to (λ_*, μ_*) as (A1) and (5.2) hold.

Using the orthogonality relation (4.1) with (λ_*, μ_*) and (λ_i, μ_i) in place of (λ_1, μ_1) and (λ_2, μ_2) there, respectively, gives $(\lambda_i - \lambda_*) b(e_i, e_*) + (\mu_i - \mu_*) m(e_i, e_*) = 0$. Since $\mathcal{M}(e_*) > 0$, a rearrangement of this relation and taking limits then yields

$$\lim_{i \rightarrow \infty} \frac{\mu_i - \mu_*}{\lambda_i - \lambda_*} = -\frac{\mathcal{B}(e_*)}{\mathcal{M}(e_*)}. \quad (5.3)$$

This implies the original sequence $\{(\lambda_i, \mu_i) : i \in \mathbb{N}\}$ satisfies

$$\min_{\substack{v \in E_{\lambda_* \mu_*} \\ v \neq 0}} \frac{-\mathcal{B}(v)}{\mathcal{M}(v)} \leq \liminf_{i \rightarrow \infty} \frac{\mu_i - \mu_*}{\lambda_i - \lambda_*} \leq \limsup_{i \rightarrow \infty} \frac{\mu_i - \mu_*}{\lambda_i - \lambda_*} \leq \max_{\substack{v \in E_{\lambda_* \mu_*} \\ v \neq 0}} \frac{-\mathcal{B}(v)}{\mathcal{M}(v)}, \quad (5.4)$$

with the extreme sides of this relation being finite as the eigenspace $E_{\lambda_* \mu_*}$ is finite dimensional. The desired result then follows from these estimates. \square

When simplicity of $\mu_n(\lambda_*)$ is assumed the following is the basic differentiability result.

Corollary 5.3. *Assume a, b, m satisfy (A1)–(A3), and let (λ_*, μ_*) be a simple eigenpair of (a, b, m) with corresponding eigenvector e_* . If $\mu_n(\lambda_*) = \mu_*$, then the eigencurve $\mu_n(\lambda)$ is differentiable at λ_* with*

$$\left. \frac{d\mu_n}{d\lambda} \right|_{\lambda_*} = - \frac{\mathcal{B}(e_*)}{\mathcal{M}(e_*)}. \quad (5.5)$$

Proof. Take $\mu_i = \mu_n(\lambda_i)$ in the previous proof. Since $E_{\lambda_*\mu_*} = \{c e_* : c \in \mathbb{R}\}$, the inequalities in (5.4) are equalities so the assertions follow as this result holds for any sequence $\lambda_i \rightarrow \lambda_*$. \square

Differentiability at points (λ_*, μ_*) where $\mu_n(\lambda_*) = \mu_*$ has multiplicity two or more, is more complicated, and so we devote the next section to their investigation.

6. DIFFERENTIABILITY RESULTS FOR EIGENCURVES

This section provides a local characterization of eigencurves in the case of an eigenpair with finite multiplicity greater than one. Specifically, eigencurves are shown to possess well-defined one-sided derivatives. This result is then interpreted as saying that at any eigenpair of (a, b, m) , the spectrum $\sigma(a, b, m)$ locally is entirely composed of smooth curves crossing at that particular point; see Figure 1 corresponding to the matrix problem (6.5) and (6.6) for reference.

Let (λ_*, μ_*) be an eigenpair of multiplicity $k_0 \geq 2$ for the triple (a, b, m) and let $\mu_{n_0+1}(\lambda) \leq \mu_{n_0+2}(\lambda) \leq \dots \leq \mu_{n_0+k_0}(\lambda)$ be the eigencurves intersecting at (λ_*, μ_*) so that

$$\mu_{n_0}(\lambda_*) < \mu_* = \mu_{n_0+1}(\lambda_*) = \dots = \mu_{n_0+k_0}(\lambda_*) < \mu_{n_0+k_0+1}(\lambda_*).$$

Diagonalize $-b$ with respect to m on $E_{\lambda_*\mu_*}$ to obtain m -orthonormal vectors u_{*1}, \dots, u_{*k_0} and values

$$b_k := \min_{Z_k} \max_{\substack{v \in Z_k \\ v \neq 0}} \frac{-\mathcal{B}(v)}{\mathcal{M}(v)}, \quad (6.1)$$

with the minimization taken over all subspaces Z_k of $E_{\lambda_*\mu_*}$ of dimension k , satisfying

$$b_1 \leq b_2 \leq \dots \leq b_{k_0} \quad \text{and} \quad b(u_{*k}, v) = -b_k m(u_{*k}, v) \quad \text{for all } v \in E_{\lambda_*\mu_*}, \quad (6.2)$$

for each $1 \leq k \leq k_0$. The main result on differentiability of eigencurves is the following.

Theorem 6.1. *Assume a, b, m satisfy (A1)–(A3). Let (λ_*, μ_*) be a fixed eigenpair in $\sigma(a, b, m)$ and define b_k by (6.1). Then each eigencurve $\mu_{n_0+k}(\lambda)$ intersecting (λ_*, μ_*) satisfies*

$$\lim_{\lambda \rightarrow \lambda_*^-} \frac{\mu_{n_0+k}(\lambda) - \mu_*}{\lambda - \lambda_*} = b_{k_0-k+1} \quad \text{and} \quad \lim_{\lambda \rightarrow \lambda_*^+} \frac{\mu_{n_0+k}(\lambda) - \mu_*}{\lambda - \lambda_*} = b_k. \quad (6.3)$$

The proof of this theorem will follow as a consequence of several lemmas. Only the second limit in (6.3) will be considered as the proofs to establish each are similar.

Consider the map $F : V \times \mathbb{R}^2 \rightarrow V^*$, with V^* the dual space of V , defined by $F(u, \lambda, \mu) := a(u, \cdot) - \lambda b(u, \cdot) - \mu m(u, \cdot)$. We will use the implicit function theorem to study the level set $L := \{(u, \lambda, \mu) \in V \times \mathbb{R}^2 : F(u, \lambda, \mu) = 0\}$ in a neighborhood of (e_*, λ_*, μ_*) for some nontrivial $e_* \in E_{\lambda_*\mu_*}$ with $m(e_*, e_*) = 1$. Clearly $\{(e, \lambda_*, \mu_*) : e \in E_{\lambda_*\mu_*}\} \subseteq L$. Observe that $DF(e_*, \lambda_*, \mu_*) : V \times \mathbb{R}^2 \rightarrow V^*$ takes the form

$$DF(e_*, \lambda_*, \mu_*)(v, s, t) = a(v, \cdot) - \lambda_* b(v, \cdot) - \mu_* m(v, \cdot) - sb(e_*, \cdot) - tm(e_*, \cdot).$$

Using the decomposition $V = E_{\lambda_*\mu_*} \oplus E_{\lambda_*\mu_*}^\perp$, with $E_{\lambda_*\mu_*}^\perp$ the orthogonal complement of $E_{\lambda_*\mu_*}$, write $v = x + y \in E_{\lambda_*\mu_*} \oplus E_{\lambda_*\mu_*}^\perp$ to get

$$DF(e_*, \lambda_*, \mu_*)(v, s, t) = a(y, \cdot) - \lambda_* b(y, \cdot) - \mu_* m(y, \cdot) - sb(e_*, \cdot) - tm(e_*, \cdot),$$

because $a(x, \cdot) - \lambda_* b(x, \cdot) - \mu_* m(x, \cdot) \equiv 0$. Thus, it is natural to study the restriction of $DF(e_*, \lambda_*, \mu_*)$ to $E_{\lambda_*\mu_*}^\perp \times \mathbb{R}^2$.

Lemma 6.2. *Assume a, b , and m satisfy (A1)–(A3). Let (λ_*, μ_*) be a fixed eigenpair in $\sigma(a, b, m)$. If $e_* \in E_{\lambda_*\mu_*} \setminus \{0\}$ is not a scalar multiple of u_{*i} for any $i \in \{1, \dots, k_0\}$, i.e. e_* is not an eigenvector associated with (b, m) on $E_{\lambda_*\mu_*}$, then $DF(e_*, \lambda_*, \mu_*) : E_{\lambda_*\mu_*}^\perp \times \mathbb{R}^2 \rightarrow V^*$ is an injection.*

Proof. Suppose that $DF(e_*, \lambda_*, \mu_*)(y, s, t) = 0$. Then, in particular, $a(y, x) - \lambda_* b(y, x) - \mu_* m(y, x) - sb(e_*, x) - tm(e_*, x) = 0$ for all $x \in E_{\lambda_*\mu_*}$. But $a(y, x) - \lambda_* b(y, x) - \mu_* m(y, x) = 0$ for any $x \in E_{\lambda_*\mu_*}$, so $-sb(e_*, x) - tm(e_*, x) = 0$ for all $x \in E_{\lambda_*\mu_*}$. By the linear independence of $b(e_*, \cdot)$ and $m(e_*, \cdot)$ we know that $s = t = 0$. We now have that $a(y, \cdot) - \lambda_* b(y, \cdot) - \mu_* m(y, \cdot) = 0$, and thus $y \in E_{\lambda_*\mu_*}$. Hence $y \in E_{\lambda_*\mu_*} \cap E_{\lambda_*\mu_*}^\perp = \{0\}$. Therefore, $(y, s, t) = (0, 0, 0)$ and the lemma is proved. \square

Thus, if e_* is not an eigenvector as in the previous lemma and $W \subset V^*$ denotes the range of $DF(e_*, \lambda_*, \mu_*)$ restricted to $E_{\lambda_*\mu_*}^\perp \times \mathbb{R}^2$, then $DF(e_*, \lambda_*, \mu_*) : E_{\lambda_*\mu_*}^\perp \times \mathbb{R}^2 \rightarrow W$ is an invertible linear map.

Lemma 6.3. *Assume a, b , and m satisfy (A1)–(A3). Let (λ_*, μ_*) be a fixed eigenpair in $\sigma(a, b, m)$. If $e_* \in E_{\lambda_*\mu_*} \setminus \{0\}$ is not a scalar multiple of u_{*i} for any $i \in \{1, \dots, k_0\}$, then the range of $DF(e_*, \lambda_*, \mu_*) : E_{\lambda_*\mu_*}^\perp \times \mathbb{R}^2 \rightarrow V^*$ is closed in V^* .*

Proof. Let W represent the range, let $w_n = DF(e_*, \lambda_*, \mu_*)(y_n, s_n, t_n) \in W$, and assume $w_n \rightarrow w$ in V^* . We must show that $w \in W$. For any $x \in E_{\lambda_*\mu_*}$ we have $\langle w_n, x \rangle = -s_n b(e_*, x) - t_n m(e_*, x)$. Let $\alpha_n = \sqrt{s_n^2 + t_n^2}$. If $\alpha_n \rightarrow \infty$, then, by passing to a subsequence, we may assume $\alpha_n \nearrow \infty$, $\frac{s_n}{\alpha_n} \rightarrow \bar{s}$, and $\frac{t_n}{\alpha_n} \rightarrow \bar{t}$, with \bar{s} and \bar{t} not both zero. Dividing $\langle w_n, x \rangle = -s_n b(e_*, x) - t_n m(e_*, x)$ by α_n and taking a limit gives $0 = -\bar{s}b(e_*, x) - \bar{t}m(e_*, x)$ for all $x \in E_{\lambda_*\mu_*}$. This contradicts the linear independence hypothesis for e_* . Hence $\{\alpha_n : n \in \mathbb{N}\}$ is bounded, and, by passing to a subsequence, we may assume $s_n \rightarrow \bar{s}$ and $t_n \rightarrow \bar{t}$.

Now suppose $\{\|y_n\|_a : n \in \mathbb{N}\}$ is unbounded. By passing to a subsequence, assume $\|y_n\|_a \nearrow \infty$ and $\bar{y}_n := \frac{y_n}{\|y_n\|_a} \rightarrow \bar{y}$ in V . Dividing $w_n = DF(e_*, \lambda_*, \mu_*)(y_n, s_n, t_n)$ by $\|y_n\|_a$ and taking a limit gives $0 = a(\bar{y}, \cdot) - \lambda_* b(\bar{y}, \cdot) - \mu_* m(\bar{y}, \cdot)$. Hence $\bar{y} \in E_{\lambda_*\mu_*}^\perp \cap E_{\lambda_*\mu_*} = \{0\}$. Since $-s_n b(e_*, x) - t_n m(e_*, x)$ is bounded we have $\left(\frac{1}{\|y_n\|_a}\right)(-s_n b(e_*, x) - t_n m(e_*, x)) \rightarrow 0$, and thus $a(\bar{y}_n, \cdot) - \lambda_* b(\bar{y}_n, \cdot) - \mu_* m(\bar{y}_n, \cdot) \rightarrow 0$ in V^* . By weak continuity, $b(\bar{y}_n, \bar{y}_n) \rightarrow 0$ and $m(\bar{y}_n, \bar{y}_n) \rightarrow 0$. However, $a(\bar{y}_n, \bar{y}_n) \equiv 1$, so

$$1 - \lambda_* b(\bar{y}_n, \bar{y}_n) - \mu_* m(\bar{y}_n, \bar{y}_n) \rightarrow 0,$$

and $1 = 0$, a contradiction. Hence the $\|y_n\|_a$ are bounded and we may assume without loss of generality that $y_n \rightarrow y \in E_{\lambda_*\mu_*}^\perp$ in V .

Therefore, for every $v \in V$, we have

$$w(v) = \lim_{n \rightarrow \infty} w_n(v) = a(y, v) - \lambda_* b(y, v) - \mu_* m(y, v) - \bar{s}b(e_*, v) - \bar{t}m(e_*, v).$$

Thus $w \in W$. \square

Lemma 6.4. *Assume a, b , and m satisfy (A1)–(A3). Let (λ_*, μ_*) be a fixed eigenpair in $\sigma(a, b, m)$. If $e_* \in E_{\lambda_*\mu_*} \setminus \{0\}$ is not a scalar multiple of u_{*i} for any $i \in \{1, \dots, k_0\}$, then $DF(e_*, \lambda_*, \mu_*) : E_{\lambda_*\mu_*}^\perp \times \mathbb{R}^2 \rightarrow W$ has a continuous inverse.*

Proof. If not then there exists $(y_n, s_n, t_n) \subset E_{\lambda_* \mu_*}^\perp \times \mathbb{R}^2$ with $\|y_n\|_a + |s_n| + |t_n| = 1$ and $w_n = DF(y_n, s_n, t_n) \rightarrow 0$ in V^* . We may assume $y_n \rightarrow y$ in V , $s_n \rightarrow s$, and $t_n \rightarrow t$. Thus $a(y_n, \cdot) - \lambda_* b(y_n, \cdot) - \mu_* m(y_n, \cdot) - s_n b(e_*, \cdot) - t_n m(e_*, \cdot) \rightarrow a(y, \cdot) - \lambda_* b(y, \cdot) - \mu_* m(y, \cdot) - s b(e_*, \cdot) - t m(e_*, \cdot) = 0$ in V^* .

For all $x \in E_{\lambda_* \mu_*}$ we have $-s b(e_*, x) - t m(e_*, x) = 0$, so $s = t = 0$. Thus $\|y_n\|_a \rightarrow 1$, and $a(y, \cdot) - \lambda_* b(y, \cdot) - \mu_* m(y, \cdot) = 0$, so $y \in E_{\lambda_* \mu_*} \cap E_{\lambda_* \mu_*}^\perp = \{0\}$. Therefore $y_n \rightarrow 0$, $b(y_n, y_n) \rightarrow 0$, and $m(y_n, y_n) \rightarrow 0$. Thus $0 = \lim_{n \rightarrow 0} \langle w_n, y_n \rangle = \lim_{n \rightarrow \infty} (a(y_n, y_n) - \lambda_* b(y_n, y_n) - \mu_* m(y_n, y_n)) = 1$, a contradiction. \square

Let $P : V^* \rightarrow W$ represent orthogonal projection, then $\bar{F} := PF : V \times \mathbb{R}^2 \rightarrow W$ is such that $D\bar{F}(e_*, \lambda_*, \mu_*) = P D F(e_*, \lambda_*, \mu_*) = DF(e_*, \lambda_*, \mu_*)$. Moreover, $\bar{F}(e_*, \lambda_*, \mu_*) = PF(e_*, \lambda_*, \mu_*) = 0$. By the implicit function theorem, there is a neighborhood, $N_1 \subset E_{\lambda_* \mu_*}$, of e_* , a neighborhood $N_2 \subset V \times \mathbb{R}^2$ of (e_*, λ_*, μ_*) , and a differentiable function $g : N_1 \rightarrow V \times \mathbb{R}^2$ of the form $g(x) = (x + y(x), \lambda(x), \mu(x))$ with $g(e_*) = (e_*, \lambda_*, \mu_*)$, such that $\{g(x) : x \in N_1\} = \{(x + y, \lambda, \mu) \in V \times \mathbb{R}^2 : \bar{F}(x + y, \lambda, \mu) = 0\} \cap N_2$. However, we already have $\bar{F}(x, \lambda_*, \mu_*) = PF(x, \lambda_*, \mu_*) = 0$ for all $x \in E_{\lambda_* \mu_*}$, so we must have $g(x) = (x, \lambda_*, \mu_*)$ for all $x \in N_1$. Since $F(x + y, \lambda, \mu) = 0$ implies $\bar{F}(x + y, \lambda, \mu) = 0$ we conclude that $L \cap N_2 = \{(x, \lambda_*, \mu_*) : x \in N_1\}$.

The discussion above leads to the following lemma.

Lemma 6.5. *Assume a, b, m satisfy (A1)–(A3), and let $\mu_n(\lambda)$ be a fixed eigencurve intersecting the eigenpair (λ_*, μ_*) . Suppose $\lambda_i \rightarrow \lambda_*$ and let $\{e_{in} : i \in \mathbb{N}\}$ be a sequence of m -normalized eigenvectors associated with $\mu_n(\lambda_i)$ with $e_{in} \rightarrow e$ in V . Then $e \in \{\pm u_{*1}, \dots, \pm u_{*k_0}\}$.*

Proof. Straight forward limits yield that e is an m -normalized element of $E_{\lambda_* \mu_*}$. Moreover, Lemma 6.2 shows that $(\lambda_i, \mu_n(\lambda_i), e_i)$ can only converge to (λ_*, μ_*, e) where e is an eigenvector of (b, m) on $E_{\lambda_* \mu_*}$. The m -normalized eigenvectors are precisely $\{\pm u_{*1}, \dots, \pm u_{*k_0}\}$. \square

To following lemma simply verifies that we can extract convergent subsequences when $(\lambda_i, \mu_n(\lambda_i)) \in \sigma(a, b, m)$ with corresponding eigenvectors e_i in V are such that $\lambda_i \rightarrow \lambda_*$.

Lemma 6.6. *Assume a, b, m satisfy (A1)–(A3), and let $\mu_n(\lambda)$ be a fixed eigencurve intersecting the eigenpair (λ_*, μ_*) . Suppose $\lambda_i \rightarrow \lambda_*$ and let $\{e_{in} : i \in \mathbb{N}\}$ be a sequence of m -normalized eigenvectors associated to $\mu_n(\lambda_i)$. Then there is a subsequence of $\{e_{in} : i \in \mathbb{N}\}$ that converges strongly to an eigenvector v_{*n} associated to $\mu_n(\lambda_*)$ satisfying $\mathcal{M}(v_{*n}) = 1$.*

Proof. Let $r_0 > \sup\{|\lambda_i| : i \in \mathbb{N}\}$ to obtain a $\tau > 0$ as in Theorem 3.1. The bilinear forms $a_{\lambda_i, \tau} := a - \lambda_i b + \tau m$ satisfy the identity $a_{\lambda_i, \tau}(e_{in}, e_{in}) = \mu_n(\lambda_i) + \tau$ upon using the eigenpair equation (2.1) with $u = e_{in}$. Equation (3.1) then gives

$$\mu_n(\lambda_i) + \tau \geq \frac{1}{2} \mathcal{A}(e_{in}),$$

for each $i \in \mathbb{N}$. This and continuity of $\mu_n(\lambda)$ on $[-r_0, r_0]$ imply $\{e_{in} : i \in \mathbb{N}\}$ is a bounded set and thus a weakly precompact set in V as (A1) holds. Extract a subsequence, denoted again by $\{e_{in} : i \in \mathbb{N}\}$, weakly converging to a vector v_{*n} in V . Since $\mathcal{M}(e_{in}) = 1$ for each $i \in \mathbb{N}$, it follows that $\mathcal{M}(v_{*n}) = 1$. Take $i \rightarrow \infty$ in the eigenpair equation (2.1) with $u = e_{in}$ to see that v_{*n} is an eigenvector associated to $\mu_n(\lambda_*)$.

Since $\|v_{*n} - e_{in}\|_a^2 = \mathcal{A}(v_{*n}) - 2a(v_{*n}, e_{in}) + \mathcal{A}(e_{in})$, it suffices to show $\mathcal{A}(e_{in}) \rightarrow \mathcal{A}(v_{*n})$ to establish strong convergence of e_{in} to v_{*n} . Therefore, from

$$\mathcal{A}(v_{*n}) = \lambda_* \mathcal{B}(v_{*n}) + \mu_n(\lambda_*) \mathcal{M}(v_{*n}) = \lim_{i \rightarrow \infty} (\lambda_i \mathcal{B}(e_{in}) + \mu_n(\lambda_i) \mathcal{M}(e_{in})) = \lim_{i \rightarrow \infty} \mathcal{A}(e_{in}),$$

the assertions of the lemma hold. \square

Lemma 6.7. *Assume a, b, m satisfy (A1)–(A3). Then for each $k \in \{1, \dots, k_0\}$,*

$$\lim_{\lambda \rightarrow \lambda_*^+} \frac{\mu_{n_0+k}(\lambda) - \mu_*}{\lambda - \lambda_*} = b_k. \quad (6.4)$$

Proof. Suppose that

$$l^- := \liminf_{\lambda \rightarrow \lambda_*^+} \frac{\mu_{n_0+k}(\lambda) - \mu_*}{\lambda - \lambda_*} < \limsup_{\lambda \rightarrow \lambda_*^+} \frac{\mu_{n_0+k}(\lambda) - \mu_*}{\lambda - \lambda_*} =: l^+.$$

Select $\beta \in (l^-, l^+)$ with $\beta \notin \{b_1, \dots, b_{k_0}\}$. There are $\lambda_i \subset \mathbb{R}$ such that $\lambda_i \searrow \lambda_*$ and

$$\lim_{i \rightarrow \infty} \frac{\mu_{n_0+k}(\lambda_i) - \mu_*}{\lambda_i - \lambda_*} = \beta.$$

By Lemma 6.6, we can also select e_i in $E_{\lambda_i, \mu_{n_0+k}(\lambda_i)}$ with $m(e_i, e_i) = 1$ and $e_i \rightarrow e$ in V , where e belongs to E_{λ_*, μ_*} and $m(e, e) = 1$ and $-b(e, e) = \beta$. However, this contradicts Lemma 6.5. Hence, $l^- = l^+$ and the limit on the left side of (6.4) exists. A similar argument by contradiction shows that this limit must be in the set $\{b_1, \dots, b_{k_0}\}$.

Let

$$\lim_{\lambda \rightarrow \lambda_*^+} \frac{\mu_{n_0+k}(\lambda) - \mu_*}{\lambda - \lambda_*} =: b_{m_k} \in \{b_1, \dots, b_{k_0}\}.$$

Since the variational curves are ordered it is clear that $b_{m_1} \leq b_{m_2} \leq \dots \leq b_{m_{k_0}}$. Let $\lambda_i \in \mathbb{R}$, $e_i^k \in E_{\lambda_i, \mu_{n_0+k}(\lambda_i)}$ be such that $\lambda_i \searrow \lambda_*$, $e_i^k \rightarrow e^k \in E_{\lambda_*, \mu_*}$ for each k . By previous lemmas we may assume that e^k is an m -normalized eigenvector of (b, m) associated with the eigenvalue b_{m_k} . It clearly follows that $b_{m_k} = b_k$ and $e^k = \pm u_{*k}$ for each k . Hence the proof is done. \square

The result of Theorem 6.1 says that an eigencurve $\mu_{n_0+k}(\lambda)$ intersecting an eigenpair (λ_*, μ_*) of multiplicity k_0 greater than one possesses well-defined one-sided derivatives at that eigenpair given by (6.3). Since the left- and right-derivative values b_{k_0-k+1} and b_k are not equal to each other in general at such points, this implies such an eigencurve $\mu_{n_0+k}(\lambda)$ is in general not differentiable at such an eigenpair.

However, since the eigencurve $\mu_{k_0-k+1}(\lambda)$ has b_k as its left-derivative at (λ_*, μ_*) , which is equal to the right-derivative of $\mu_{n_0+k}(\lambda)$ at this eigenpair, the theorem implies the spectrum of (a, b, m) can be characterized locally around (λ_*, μ_*) as being composed of k_0 curves that are differentiable at (λ_*, μ_*) .

These one-sided differentiability results can be exemplified by (finite-dimensional) *two-parameter generalized matrix eigenproblems*: find $(\lambda, \mu) \in \mathbb{R}^2$ and nonzero $x \in \mathbb{R}^N$ satisfying

$$Ax = \lambda Bx + \mu x, \quad (6.5)$$

where A is a symmetric positive definite matrix, such as the Laplacian matrix, and B a symmetric matrix on \mathbb{R}^N . When A, B are, for example, the 3×3 matrices given by

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 0 \end{bmatrix}, \quad (6.6)$$

the eigencurves may be found (implicitly) and are plotted in Figure 1. For this choice of A, B , the eigencurves $\mu_1(\lambda), \mu_2(\lambda)$ are not differentiable at $(\lambda_*, \mu_*) = (1, 0)$, but locally around this point their graphs comprise two curves that are differentiable at (λ_*, μ_*) with one-sided derivatives given by (6.3) in Theorem 6.1.

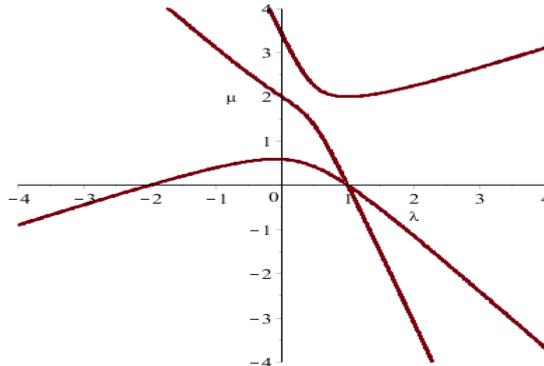


FIGURE 1. Eigencurves for the matrix eigenproblem (6.5) and (6.6).

7. ASYMPTOTIC RESULTS

The previous two sections dealt with regularity issues pertaining to eigencurves. In the next few sections, the interest is on providing a *geometrical description* of these variational eigencurves which is the main contribution of this paper.

First, we shall treat the asymptotic behaviour of eigencurves. In particular, it is shown that the dimension of subspaces on which the quadratic form \mathcal{B} is strictly positive, or strictly negative, controls the number of eigencurves that go asymptotically down to negative infinity as $\lambda \rightarrow \infty$. Using a more careful analysis, explicit formulae are obtained to describe the asymptotic behaviour, the results depending on spectral data for the pair (b, m) of bilinear forms.

For the first eigencurve, the result is the following.

Lemma 7.1. *Assume a, b, m satisfy (A1)–(A3), and let $\mu_1(\lambda)$ be the first variational eigencurve associated with (a, b, m) .*

(i) *If there is a vector $\hat{v} \in V$ such that $\mathcal{B}(\hat{v}) > 0$, then*

$$\lim_{\lambda \rightarrow \infty} \mu_1(\lambda) = -\infty.$$

(ii) *If there is a vector $\hat{w} \in V$ such that $\mathcal{B}(\hat{w}) < 0$, then*

$$\lim_{\lambda \rightarrow -\infty} \mu_1(\lambda) = -\infty.$$

Proof. The variational characterization (3.9) for $\mu_1(\lambda)$ gives

$$\mu_1(\lambda) \leq \frac{\mathcal{A}(v) - \lambda \mathcal{B}(v)}{\mathcal{M}(v)} \quad \text{for all nonzero } v \in V.$$

For such v fixed, the right side of this inequality is an affine function in λ with the sign of its slope determined by the sign of $\mathcal{B}(v)$. Considering \hat{v}, \hat{w} as in the theorem and taking the appropriate limit in λ gives the desired results. \square

This result may be interpreted as saying that the sign of the quadratic form \mathcal{B} on one-dimensional subspaces (spanned by such \hat{v} or \hat{w}) may be used to determine the asymptotic behaviour of the first variational eigencurve $\mu_1(\lambda)$. The next result says that the asymptotic behaviour of the k th eigencurve is determined by the sign of \mathcal{B} on k -dimensional subspaces.

Lemma 7.2. *Assume a, b, m satisfy (A1)–(A3), and let $\mu_k(\lambda)$ be the k th-variational eigencurve associated with (a, b, m) .*

(i) *If $\hat{v}_1, \dots, \hat{v}_k$ are k linearly independent vectors at which \mathcal{B} is strictly positive, then*

$$\lim_{\lambda \rightarrow \infty} \mu_k(\lambda) = -\infty.$$

(ii) *If $\hat{w}_1, \dots, \hat{w}_k$ are k linearly independent vectors at which \mathcal{B} is strictly negative, then*

$$\lim_{\lambda \rightarrow -\infty} \mu_k(\lambda) = -\infty.$$

Proof. Let $\lambda > 0$ and let \hat{Z}_k denote the subspace spanned by the vectors $\hat{v}_1, \dots, \hat{v}_k$ at which \mathcal{B} is strictly positive. The variational characterization (3.9) for $\mu_k(\lambda)$ gives

$$\mu_k(\lambda) \leq \sup_{\substack{v \in \hat{Z}_k \\ v \neq 0}} \frac{\mathcal{A}(v) - \lambda \mathcal{B}(v)}{\mathcal{M}(v)} \leq \sup_{\substack{v \in \hat{Z}_k \\ v \neq 0}} \left\{ \frac{\mathcal{A}(v)}{\mathcal{M}(v)} \right\} + \lambda \sup_{\substack{v \in \hat{Z}_k \\ v \neq 0}} \left\{ -\frac{\mathcal{B}(v)}{\mathcal{M}(v)} \right\}.$$

This holds as \mathcal{B} is strictly positive on the m -unit sphere in the finite dimensional space \hat{Z}_k . The right side of this relation is an affine function in λ . Part (i) then follows upon taking the limit, and part (ii) is similarly proved. \square

When \mathcal{B} is strictly positive on some k -dimensional subspace of V , it follows from this result that the first k eigencurves asymptotically decrease to $-\infty$ as λ increases to $+\infty$. The positivity of \mathcal{B} is easily verifiable, so this asymptotic information for eigencurves is straightforward to establish in applications. A similar statement can be made when \mathcal{B} is strictly negative on a finite dimensional subspace.

To determine a more precise asymptotic description, define the values

$$\eta_k := \inf_{Z_k} \sup_{\substack{v \in Z_k \\ v \neq 0}} \left\{ -\frac{\mathcal{B}(v)}{\mathcal{M}(v)} \right\}, \quad (7.1)$$

where the infimum is taken over all k -dimensional subspaces Z_k of V . It is worth noting that these values may equal $-\infty$, but never $+\infty$. The following theorem shows that the asymptotic behaviour of eigencurves are prescribed by these spectral values η_k , and that the sign of \mathcal{B} plays a role here as in the previous lemmas.

Theorem 7.3. *Assume a, b, m satisfy (A1)–(A3). Then the k th-eigencurve $\mu_k(\lambda)$ satisfies*

$$\lim_{\lambda \rightarrow \infty} \frac{\mu_k(\lambda)}{\lambda} = \eta_k. \quad (7.2)$$

Proof. Assume $\lambda > 0$ and let Z_k be a k -dimensional subspace of V . Then

$$\sup_{\substack{v \in Z_k \\ v \neq 0}} \frac{\mathcal{A}(v) - \lambda \mathcal{B}(v)}{\mathcal{M}(v)} \geq \lambda \sup_{\substack{v \in Z_k \\ v \neq 0}} \left\{ -\frac{\mathcal{B}(v)}{\mathcal{M}(v)} \right\}.$$

Taking the infimum over all such Z_k , and taking the limit inferior after rearranging λ gives

$$\liminf_{\lambda \rightarrow \infty} \frac{\mu_k(\lambda)}{\lambda} \geq \eta_k,$$

using the variational characterization of $\mu_k(\lambda)$.

To establish the desired reverse inequality, consider first the case in which $\eta_k > -\infty$. Let $\varepsilon > 0$ and take \hat{Z}_k to be a k -dimensional subspace of V satisfying

$$\sup_{\substack{v \in \hat{Z}_k \\ v \neq 0}} \left\{ -\frac{\mathcal{B}(v)}{\mathcal{M}(v)} \right\} \leq \eta_k + \varepsilon.$$

The variational characterization (3.9) for $\mu_k(\lambda)$ shows that

$$\frac{\mu_k(\lambda)}{\lambda} \leq \frac{1}{\lambda} \sup_{\substack{v \in \hat{Z}_k \\ v \neq 0}} \left\{ \frac{\mathcal{A}(v)}{\mathcal{M}(v)} \right\} + \eta_k + \varepsilon,$$

which gives $\limsup_{\lambda \rightarrow \infty} \mu_k(\lambda)/\lambda \leq \eta_k + \varepsilon$. Taking ε down to zero then gives the desired converse inequality so that (7.2) holds in the case η_k is finite.

When $\eta_k = -\infty$, for $n \in \mathbb{N}$ take $Z_{k,n}$ to be a k -dimensional subspace of V such that

$$\sup_{\substack{v \in Z_{k,n} \\ v \neq 0}} \left\{ -\frac{\mathcal{B}(v)}{\mathcal{M}(v)} \right\} \leq -n.$$

The characterization (3.9) for the eigencurve $\mu_k(\lambda)$ gives in this case

$$\frac{\mu_k(\lambda)}{\lambda} \leq \frac{1}{\lambda} \sup_{\substack{v \in Z_{k,n} \\ v \neq 0}} \left\{ \frac{\mathcal{A}(v)}{\mathcal{M}(v)} \right\} - n.$$

Taking the limit superior and then taking n to $+\infty$ gives the desired result. \square

For the specific matrix problem (6.5) and (6.6) of Section 6, the quadratic form \mathcal{B} corresponding to the matrix B is strictly positive on a two-dimensional subspace of \mathbb{R}^3 . Hence, as seen in Figure 1, the first and second eigencurves go to $-\infty$ as $\lambda \rightarrow +\infty$; the eigencurves are asymptotic to lines whose slopes are the negative eigenvalues of B .

Richardson's equation is discussed in Section 3 of Binding and Vokmer [7] to exemplify their results on Sturm–Liouville problems. The quadratic form \mathcal{B} associated with that example is $\mathcal{B}(y) := \int_{-1}^1 \operatorname{sgn}(x) y(x)^2 dx$ defined on $L^2[-1, 1]$. It is obvious that \mathcal{B} is strictly positive on an infinite dimensional subspace of $L^2[-1, 1]$, so every eigencurve $\mu_n(\lambda)$ goes to $-\infty$ as $\lambda \rightarrow +\infty$; likewise, $\mu_n(\lambda) \rightarrow -\infty$ as $\lambda \rightarrow -\infty$. This behaviour is seen in Figure 3.1 of [7]. Their Theorem 2.2 provides explicit formulae for the asymptotic behaviour of the eigencurves.

8. STRAIGHT LINES WITHIN THE SPECTRUM AND LINEAR INDEPENDENCE

In this section, the degeneracy of the form b is used to determine when straight lines appear within the spectrum of (a, b, m) . When this holds, a further orthogonal relation is found for some of the associated eigenspaces. Related to these results is a description of linear independence of vectors associated with eigenspaces corresponding to eigenpairs lying on the same horizontal line.

We point out that the results imply that if the spectrum contains a line segment, then the spectrum actually contains the entire line containing that line segment. This leads to a technical, but interesting, point about eigencurves that can hopefully be clarified in future research.

To proceed with the analysis, recall that the *null space* $N(b)$ of a bilinear form b obeying (A2) is the set of all vectors $u \in V$ satisfying

$$b(u, v) = 0 \quad \text{for all } v \in V. \quad (8.1)$$

When $N(b) = \{0\}$ then b is said to be *non-degenerate*, otherwise b is said to be *degenerate*. It is worth noting that the degeneracy of b has played no role in the preceding analysis.

Theorem 8.1. *Assume a, b, m satisfy (A1)–(A3). Let $\mu_* \notin \sigma(a, m)$ and (λ_i, μ_*) be distinct eigenpairs in $\sigma(a, b, m)$ for $1 \leq i \leq k$, with $k \geq 2$. If e_i are eigenvectors associated with (λ_i, μ_*) , then the set $\mathcal{E}_k := \{e_1, \dots, e_k\}$ is linearly independent.*

Proof. Suppose \mathcal{E}_k is linearly dependent. Then without loss of generality we may assume $e_k \in \text{span}\{e_1, \dots, e_{k-1}\}$. If $e_k = \alpha e_i$ for some $1 \leq i \leq k-1$ and some nonzero constant α , then by subtracting the eigenpair equations that e_k, e_i satisfy shows that $b(e_k, v) = 0$ for all $v \in V$. This says $e_k \in N(b)$ which implies $(\lambda, \mu_*) \in \sigma(a, b, m)$ for any $\lambda \in \mathbb{R}$. In particular, $(0, \mu_*) \in \sigma(a, b, m)$ or that $\mu_* \in \sigma(a, m)$, in contradiction with $\mu_* \notin \sigma(a, m)$.

Without loss of generality assume now that $e_k = \sum_{i=1}^{k-1} \alpha_i e_i$ with $\alpha_i \neq 0$ for all i and that the dimension d of the span of \mathcal{E}_k satisfies $2 \leq d = k-1$. Substituting this expression for e_k into the eigenpair equation that e_k satisfies gives

$$\sum_{i=1}^{k-1} \alpha_i a(e_i, v) = \lambda_k b(e_k, v) + \mu_* \sum_{i=1}^{k-1} \alpha_i m(e_i, v),$$

holding for all $v \in V$. Considering the terms that involve the a, m bilinear forms, and using the eigenpair equations for e_1, \dots, e_{k-1} simplifies this expression to

$$\sum_{i=1}^{k-1} \alpha_i \lambda_i b(e_i, v) = \lambda_k b(e_k, v).$$

Using the expansion for e_k and rearranging terms reduces this last relation to

$$b(\tilde{e}, v) = 0 \quad \text{with} \quad \tilde{e} := \sum_{i=1}^{k-1} \alpha_i (\lambda_i - \lambda_k) e_i. \quad (8.2)$$

The vector \tilde{e} is nontrivial since the λ_i are distinct and $\{e_1, \dots, e_{k-1}\}$ is linearly independent. As (8.2) holds for all $v \in V$, it follows that \tilde{e} belongs to $N(b)$.

Now, for a given value $\lambda \in \mathbb{R} \setminus \{\lambda_1, \dots, \lambda_{k-1}\}$, define the vector

$$e_\lambda := \sum_{i=1}^{k-1} \alpha_i \left(\frac{\lambda_i - \lambda_k}{\lambda_i - \lambda} \right) e_i.$$

For fixed $v \in V$, the expression $\lambda b(e_\lambda, v) + \mu_* m(e_\lambda, v)$ is then equal to

$$\sum_{i=1}^{k-1} \alpha_i \left(\frac{\lambda_i - \lambda_k}{\lambda_i - \lambda} \right) [\lambda b(e_i, v) + \mu_* m(e_i, v)], \quad (8.3)$$

upon collecting like terms. Since the quantity $\lambda b(e_i, v) + \mu_* m(e_i, v)$ is equal to the quantity $\lambda_i b(e_i, v) + \mu_* m(e_i, v) + (\lambda - \lambda_i)b(e_i, v)$, it follows from the eigenpair equation that e_i satisfies that the expression (8.3) is equal to

$$\sum_{i=1}^{k-1} \alpha_i \left(\frac{\lambda_i - \lambda_k}{\lambda_i - \lambda} \right) [a(e_i, v) + (\lambda - \lambda_i)b(e_i, v)].$$

Distributing and using the definitions of \tilde{e}, e_λ shows that this last expression is equal to the quantity $a(e_\lambda, v) + b(\tilde{e}, v)$, which reduces to $a(e_\lambda, v)$ since $\tilde{e} \in N(b)$. This shows e_λ satisfies $a(e_\lambda, v) = \lambda b(e_\lambda, v) + \mu_* m(e_\lambda, v)$ for all $v \in V$, so that e_λ is an eigenvector of (a, b, m) corresponding to the eigenvalue (λ, μ_*) . Taking $\lambda = 0$ implies that $(0, \mu_*) \in \sigma(a, b, m)$ which contradicts $\mu_* \notin \sigma(a, m)$. Therefore, \mathcal{E}_k is a linearly independent set. \square

The next results considers the case $\mu_* \in \sigma(a, m)$, and it says that when there is a nontrivial intersection of the corresponding eigenspace $E_{0\mu_*}$ and the null space $N(b)$ of b , then there is an entire straight line within the spectrum of (a, b, m) .

Theorem 8.2. *Assume a, b, m satisfy (A1)–(A3), respectively, and let $\mu_* \in \mathbb{R}$ be fixed. Then any nonzero vector $e \in E_{0\mu_*} \cap N(b)$ also belongs to the eigenspace $E_{\lambda\mu_*}$ for each $\lambda \in \mathbb{R}$, i.e., the straight line $\{(\lambda, \mu_*) : \lambda \in \mathbb{R}\}$ is contained in $\sigma(a, b, m)$ in this case.*

Proof. Such an e satisfies $a(e, v) = \mu_* m(e, v)$ and also $b(e, v) = 0$ for all $v \in V$. Multiply the second relation by λ and add this to the first relation to get that e is an eigenvector of (a, b, m) corresponding to (λ, μ_*) for any λ . The assertions then follow. \square

Let $N_{0\mu_*}(b)$ denote the subspace $E_{0\mu_*} \cap N(b)$ for fixed $\mu_* \in \mathbb{R}$. We point out that in many cases, $N_{0\mu_*}(b)$ is simply the trivial subspace. When it is nontrivial, denote by $\hat{E}_{\lambda\mu_*}$ the m -orthogonal complement of $N_{0\mu_*}(b)$ in $E_{\lambda\mu_*}$ to obtain the decomposition

$$E_{\lambda\mu_*} = \hat{E}_{\lambda\mu_*} \oplus_m N_{0\mu_*}(b), \quad (8.4)$$

where \oplus_m indicates orthogonality with respect to the m inner product on V . This yields a further orthogonality result to those given in Section 4.

It is instructive at this point to provide some easy concrete examples of spectra to illustrate the preceding results. In particular, we describe spectra consisting entirely of (not necessarily horizontal) straight lines.

Suppose a, m satisfy (A1) and (A3), and let $\Lambda_0 := \{\mu_{0n} : n \in \mathbb{N}\}$ and $\mathcal{E}_0 := \{e_{0n} : n \in \mathbb{N}\}$ be the eigendata described in Section 2 satisfying

$$a(e_{0n}, v) = \mu_{0n} m(e_{0n}, v) \quad \text{for all } v \in V.$$

Consider the form $b_\varepsilon := \varepsilon m$ defined on V for fixed $\varepsilon \in \mathbb{R}$. When $\varepsilon = 0$, the pair (a, b_ε) has no eigenvalues and then the horizontal straight lines $\mu_n(\lambda) = \mu_{0n}$ completely make up the spectrum $\sigma(a, b_\varepsilon, m)$.

When $\varepsilon \neq 0$, then $\{\varepsilon^{-1}\mu_{0n} : n \in \mathbb{N}\}$ is the sequence of eigenvalues of (a, b_ε) so that the spectrum $\sigma(a, b_\varepsilon, m)$ comprises precisely the nonhorizontal straight lines $\mu_n(\lambda) = \mu_{0n} - \varepsilon\lambda$. In these two examples, the functional forms of the eigencurves are linear.

As a third example, take $K \subset \mathbb{N}$ to be an indexing set of finite cardinality $|K|$, and let ε be a vector in $\mathbb{R}^{|K|}$ with nonzero entries ε_k . Consider the bilinear form $b_{\varepsilon K}$ defined by

$$b_{\varepsilon K}(u, v) := \sum_{k \in K} \varepsilon_k m(e_{0k}, u) m(e_{0k}, v) \quad \text{for } u, v \in V. \quad (8.5)$$

If $n \notin K$, then $b_{\varepsilon K}(e_{0n}, v) = 0$ for all $v \in V$. Using the eigenpair equation (2.1) for $(a, b_{\varepsilon K}, m)$ it is easy to see that the graph of the constant function $\lambda \mapsto \mu_{0n}$ belongs to $\sigma(a, b_{\varepsilon K}, m)$. When $n \in K$, then the eigenpair equation shows that the graph of the function $\lambda \mapsto \mu_{0n} - \varepsilon_n \lambda$ belongs to $\sigma(a, b_{\varepsilon K}, m)$. Here, the nonhorizontal lines are not necessarily parallel nor have slopes of the same sign. Therefore, in this example the variational eigencurves $\mu_n(\lambda)$ for this system are instead piecewise-linearly defined.

From these examples, it is easy to see that the degeneracy of b drastically affects the geometry of the spectrum of (a, b, m) as well as the asymptotic behaviour of the eigencurves. We also remark that since only straight lines through points $(0, \mu_*)$, with $\mu_* \in \sigma(a, m)$, can be in $\sigma(a, b, m)$, there are at most countably many straight lines in the spectrum of (a, b, m) .

9. INTERSECTING THE SPECTRUM WITH STRAIGHT LINES

This section deals with the *geometrical problem* of establishing an upper bound for the number of *components* the n th-eigencurve may have above a given a horizontal straight line $\mu = \mu_*$. The case where the intersecting line is not horizontal is discussed at the end of the section, and the results are seen to also hold for this case as a consequence of a simple change-of-variables.

For a fixed eigencurve $\mu_n(\lambda)$ and fixed $\mu_* \in \mathbb{R}$, denote by $\{\mu_n > \mu_*\}$ the superlevel set of $\mu_n(\lambda)$ given by $\{\lambda \in \mathbb{R} : \mu_n(\lambda) > \mu_*\}$. A finite interval $(s, t) \subset \mathbb{R}$ is said to be a *component* of $\{\mu_n > \mu_*\}$ provided $\mu_n(s) = \mu_n(t) = \mu_*$ and $\mu_n(\lambda) > \mu_*$ for all $s < \lambda < t$. A semi-infinite interval $(s, \infty) \subset \mathbb{R}$ is also said to be a *component* of $\{\mu_n > \mu_*\}$ provided $\mu_n(s) = \mu_*$ and $\mu_n(\lambda) > \mu_*$ for all $\lambda \in (s, \infty)$; a component of the form $(-\infty, t)$ is similarly defined. We shall often say a component (s, t) of $\{\mu_n > \mu_*\}$ is a component of $\mu_n(\lambda)$ above μ_* .

For the first eigencurve, the geometrical result is the following.

Lemma 9.1. *Assume a, b, m satisfy (A1)–(A3) and let $\mu_* \notin \sigma(a, m)$ be fixed. Then $\{\mu_1 > \mu_*\}$ has at most one component.*

Proof. This follows from the concavity of $\mu_1(\lambda)$ given by Lemma 3.3. □

To begin the geometrical description of higher eigencurves, we first show in the next lemma that the eigencurves satisfy a *nested property* and then in Lemma 9.3 we provide a *sign-condition* that eigenvectors associated with endpoints of a component must satisfy.

Lemma 9.2. *Assume a, b, m satisfy (A1)–(A3), and let $\mu_* \notin \sigma(a, m)$ and $n, k \in \mathbb{N}$ be fixed with $k > n$. Let I_n be a component of $\{\mu_n > \mu_*\}$ and suppose $\{\mu_k > \mu_*\}$ has a component. Then for each integer $n \leq i \leq k$, there is a component I_i of $\{\mu_i > \mu_*\}$ such that these components satisfy $I_n \subset I_{n+1} \subset \dots \subset I_i \subset \dots \subset I_k$.*

Proof. By definition of eigencurves, $\mu_k(\lambda) \geq \mu_n(\lambda)$ for each $\lambda \in \mathbb{R}$, so $\mu_k(\lambda) > \mu_*$ on I_n . Since $\{\mu_k > \mu_*\}$ has at least one component, there is a component I_k of $\{\mu_k > \mu_*\}$ containing I_n . Since $\mu_n(\lambda) \leq \mu_i(\lambda) \leq \mu_k(\lambda)$ holds for $n \leq i \leq k$, again by definition, it follows that $\{\mu_i > \mu_*\}$ contains a component I_i satisfying $I_k \supset I_i \supset I_n$ and the assertions then follow. □

It is worth noting that the components in Lemma 9.2 may be semi-infinite intervals.

Lemma 9.3. *Assume a, b, m satisfy (A1)–(A3) and let $\mu_* \notin \sigma(a, m)$ be fixed. If the bounded interval (s, t) is a component of $\{\mu_n > \mu_*\}$, then eigenvectors e_{s_n}, e_{t_n} associated with $\mu_n(s), \mu_n(t)$, respectively, satisfy*

$$\mathcal{B}(e_{s_n}) \leq 0 \leq \mathcal{B}(e_{t_n}). \tag{9.1}$$

Proof. Let $\lambda_i \in (s, t)$ be a sequence monotonically increasing to t . Using an argument similar to the proof of Lemma 6.6, let $v_i \in V$ be a sequence of eigenvectors of (a, b, m) associated with $\mu_n(\lambda_i)$ satisfying $v_i \rightarrow e_{t_n}$ in V as $i \rightarrow \infty$. The orthogonality relation of Theorem 4.1 then yields

$$\lim_{i \rightarrow \infty} \frac{\mu_n(\lambda_i) - \mu_n(t)}{\lambda_i - t} = \lim_{i \rightarrow \infty} \frac{-b(v_i, e_{t_n})}{m(v_i, e_{t_n})} = -\frac{\mathcal{B}(e_{t_n})}{\mathcal{M}(e_{t_n})}.$$

Since $\mu_n(\lambda_i) > \mu_* = \mu_n(t)$ and $\lambda_i < t$, it follows that $\mathcal{B}(e_{t_n}) \geq 0$. The argument that $\mathcal{B}(e_{s_n}) \leq 0$ is similar. \square

If a component of $\mu_n(\lambda)$ is of the form $(-\infty, t)$ instead, then the proof above is exactly the same to show that an eigenvector e_{t_n} associated with $\mu_n(t)$ satisfies $\mathcal{B}(e_{t_n}) \geq 0$ provided $\mu_* \notin \sigma(a, m)$. An analogous statement also holds for a component of $\mu_n(\lambda)$ of the form (s, ∞) .

Our main result for higher eigencurves is the following.

Theorem 9.4. *Assume a, b, m satisfy (A1)–(A3), and let $\mu_* \notin \sigma(a, m)$ be fixed. Suppose $(s_1, t_1), \dots, (s_n, t_n)$ are finite intervals with (s_i, t_i) a component of some (not necessarily the same) $\{\mu_j > \mu_*\}$. Two such intervals are either nested, with either endpoint possibly in common, or disjoint. If $(s_i, t_i) = (s_j, t_j)$, then these intervals are assumed to be components of two distinct eigencurves. Then*

$$\mu_n(\lambda) \leq \mu_* \quad \text{for all } \lambda \in (-\infty, \hat{s}] \cup [\hat{t}, \infty), \quad (9.2)$$

where $\hat{s} := \min\{s_1, \dots, s_n\}$ and $\hat{t} := \max\{t_1, \dots, t_n\}$.

Proof. For each t_i , with $1 \leq i \leq n$, use Lemma 9.3 to get eigenvectors e_{i_n} associated with $\mu_n(t_i)$ satisfying $\mathcal{B}(e_{i_n}) \geq 0$, and define $Z := \text{span}\{e_{1_n}, \dots, e_{n_n}\}$. By Theorem 8.1, the vectors e_{i_n} are linearly independent so that Z is an n -dimensional subspace of V . By Corollary 4.2, the vectors e_{i_n} are b -orthogonal as well as $(a - \mu_* m)$ -orthogonal. Let $v = \sum_{i=1}^n \alpha_i e_{i_n}$, with $\alpha_i \in \mathbb{R}$, be a vector in Z and let $\lambda \geq \hat{t}$. Using the eigenpair equation that each e_{i_n} satisfies and the orthogonality just mentioned yields the following:

$$\mathcal{A}(v) - \mu_* \mathcal{M}(v) - \lambda \mathcal{B}(v) = \sum_{i=1}^n \alpha_i^2 t_i \mathcal{B}(e_{i_n}) - \lambda \sum_{i=1}^n \alpha_i^2 \mathcal{B}(e_{i_n}) = \sum_{i=1}^n \alpha_i^2 (t_i - \lambda) \mathcal{B}(e_{i_n}) \leq 0.$$

Rearranging the extreme sides of this relation gives

$$\mu_n(\lambda) \leq \sup_{v \in Z \setminus \{0\}} \left\{ \frac{\mathcal{A}(v) - \lambda \mathcal{B}(v)}{\mathcal{M}(v)} \right\} \leq \mu_* \quad \text{for all } \lambda \geq \hat{t},$$

using the variational characterization (3.9) for $\mu_n(\lambda)$. A similar argument using left endpoints s_i shows that $\mu_n(\lambda) \leq \mu_*$ for all $\lambda \leq \hat{s}$ so that the desired result holds. \square

For fixed eigencurve $\mu_n(\lambda)$ and $\mu_* \in \mathbb{R}$ denote by $K_n(\mu_*)$ the number of components comprising the superlevel set $\{\mu_n > \mu_*\}$. Lemma 9.1 may then be expressed as saying that $K_1(\mu_*) \leq 1$ for all $\mu_* \in \mathbb{R}$.

The following result is an immediate consequence of Theorem 9.4.

Corollary 9.5. *Assume a, b, m satisfy (A1)–(A3) and let $\mu_* \notin \sigma(a, m)$ be fixed. Then for each $n \in \mathbb{N}$ we have $K_n(\mu_*) \leq n$.*

Proof. Suppose $(s_1, t_1), \dots, (s_n, t_n), (s_{n+1}, t_{n+1})$ are distinct components comprising the set $\{\mu_n > \mu_*\}$ with the endpoints satisfying $t_{i-1} < s_i < t_i < s_{i+1}$ for each i . We point out that we could have $s_1 = -\infty$ and also $t_{n+1} = +\infty$. Nevertheless, consider only the first n components. We conclude from the proof of Theorem 9.4 that $\mu_n(\lambda) \leq \mu_*$ for $\lambda \geq t_n$ in contradiction to (s_{n+1}, t_{n+1}) being a component of $\mu_n(\lambda)$. Therefore, $\{\mu_n > \mu_*\}$ can have at most n components so that $K_n(\mu_*) \leq n$ holds for each $n \in \mathbb{N}$. \square

Our main geometrical result for higher eigencurves is the following.

Theorem 9.6. *Assume a, b, m satisfy (A1)–(A3) and let $\mu_* \notin \sigma(a, m)$ be fixed. Then for each $n \in \mathbb{N}$ we have*

$$\sum_{i=1}^n K_i(\mu_*) \leq n. \quad (9.3)$$

Proof. Let $(s_1, t_1), \dots, (s_k, t_k)$ be distinct intervals, each a component of (not necessarily the same) $\{\mu_i > \mu_*\}$ for some $1 \leq i \leq n$. Define

$$\lambda_* := \inf\{s_i : 1 \leq i \leq k\} \quad \text{and} \quad \lambda^* := \sup\{t_i : 1 \leq i \leq k\}.$$

Assume without loss of generality that $(s_1, t_1), \dots, (s_{k_0}, t_{k_0})$ are all the components satisfying $t_i \leq t_{k_0} < \lambda^*$ for each $i = 1, \dots, k_0$. It follows from Theorem 9.4 that

$$\mu_{k_0}(\lambda) \leq \mu_* \quad \text{for all } \lambda \geq t_{k_0}.$$

Since $\mu_i(\lambda) \leq \mu_{k_0}(\lambda)$ for each $i = 1, \dots, k_0$, it follows that $\mu_i(\lambda) \leq \mu_*$ for all $\lambda \geq t_{k_0}$ for each $i \leq k_0$. For each $k_0 < i \leq k$, we have that $t_i = \lambda^*$ and that (s_i, t_i) is a component of one of the following

$$\{\mu_{k_0+1} > \mu_*\}, \quad \{\mu_{k_0+2} > \mu_*\}, \quad \dots, \quad \{\mu_k > \mu_*\}.$$

Two components $(s_i, t_i), (s_j, t_j)$ with $k_0 < i, j \leq k$ therefore belong to two different eigencurves above μ_* . Since the intervals considered here are components of $\{\mu_i > \mu_*\}$ for some $1 \leq i \leq n$, it follows that $k - k_0 \leq n - k_0$, so that $k \leq n$. This means $\sum_{i=1}^n K_i(\mu_*) \leq n$ holds for each $n \in \mathbb{N}$. \square

The above analysis has dealt with the case of a horizontal line $\mu = \mu_*$ intersecting the spectrum $\sigma(a, b, m)$. Consider now the case of a non-horizontal line $\mu = \alpha \lambda + \beta$ intersecting the spectrum of (a, b, m) . The previous work generalizes to this case using a simple change-of-variables as the relation

$$a(e, v) = \lambda b(e, v) + \mu m(e, v) \quad \text{for all } v \in V,$$

holds if and only if the following relation holds,

$$a(e, v) = \lambda b_\alpha(e, v) + \beta m(e, v) \quad \text{for all } v \in V,$$

where $b_\alpha := b + \alpha m$. In this case, the (a, b_α, m) -eigenproblem satisfies the same assumptions as the original (a, b, m) -eigenproblem. Denote the eigencurves associated with the triple (a, b_α, m) by $\beta_{\alpha, n}(\lambda)$. The variational characterization (3.9) shows that

$$\beta_{\alpha, n}(\lambda) = \inf_{Z_k} \sup_{\substack{v \in Z_k \\ v \neq 0}} \left\{ \frac{\mathcal{A}(v) - \lambda \mathcal{B}_\alpha(v)}{\mathcal{M}(v)} \right\} = \inf_{Z_k} \sup_{\substack{v \in Z_k \\ v \neq 0}} \left\{ \frac{\mathcal{A}(v) - \lambda \mathcal{B}(v)}{\mathcal{M}(v)} - \alpha \lambda \right\} = \mu_n(\lambda) - \alpha \lambda,$$

where the minimization is over all k -dimensional subspaces Z_k of V . Thus, for fixed $\lambda \in \mathbb{R}$, the eigenvalues $\mu_n(\lambda)$ and $\beta_{\alpha, n}(\lambda)$ only differ by the constant $\alpha \lambda$, and therefore the eigencurves $(\lambda, \mu_n(\lambda))$ and $(\lambda, \beta_{\alpha, n}(\lambda))$ identify the same curve. Hence, the results for horizontal-line-intersections generalizes.

10. EIGENCURVES FOR TWO-PARAMETER ROBIN–STEKLOV EIGENPROBLEMS

The abstract results of this paper apply to very general linear elliptic two-parameter eigenproblems. In this section, they are used to describe and analyze eigencurves for the (weak form of the) Robin–Steklov two-parameter eigenproblem (1.1) mentioned in the introduction.

In these problems, $|\cdot|$ and $\langle \cdot, \cdot \rangle$ denote the Euclidean norm and inner product on \mathbb{R}^N . The terminology and notation of Evans and Gariepy [10] will be used here except that $\sigma, d\sigma$ will denote Hausdorff $(N - 1)$ -dimensional measure and integration with respect to this measure, respectively. The real Lebesgue spaces $L^p(\Omega)$ and $L^p(\partial\Omega, d\sigma)$, $1 \leq p \leq \infty$, are defined in the standard manner with the usual p -norms denoted $\|u\|_{p, \Omega}$ and

$\|u\|_{p,\partial\Omega}$; the gradient of a function u will be denoted ∇u ; and $H^1(\Omega)$ is the usual real Sobolev space with inner product and norm given by

$$[u, v]_{1,2} := \int_{\Omega} [\nabla u \cdot \nabla v + u v] dx \quad \text{and} \quad \|u\|_{1,2} := \int_{\Omega} [|\nabla u|^2 + u^2] dx.$$

In the examples treated here, it is required that the embedding of $H^1(\Omega)$ into $L^p(\Omega)$ (for $1 \leq p < p_s$ where $p_s = 2N/(N-2)$ if $N \geq 3$ and $p_s = \infty$ if $N = 2$) and the trace operator $\Gamma : H^1(\Omega) \rightarrow L^2(\partial\Omega, d\sigma)$ are both compact. This requirement holds, for instance, when $\Omega \subset \mathbb{R}^N$ is a bounded domain such that its boundary consists of a finite number of closed Lipschitz surfaces of finite surface area (see Chapter 4 of [10] for details on these issues). This requirement on Ω is assumed for the remainder of the paper.

The *two-parameter Robin–Steklov eigenproblem* is that of finding $(\lambda, \mu) \in \mathbb{R}^2$ and nonzero $u \in H^1(\Omega)$ that satisfy, for each $v \in H^1(\Omega)$, the relation

$$\int_{\Omega} [(A\nabla u) \cdot \nabla v + c u v] dx + \int_{\partial\Omega} b_c u v d\sigma = \lambda \int_{\partial\Omega} b_0 u v d\sigma + \mu \int_{\Omega} m_0 u v dx. \quad (10.1)$$

This is the weak form of problem (1.1), and the interest is in describing and analyzing the associated eigencurves $\lambda \mapsto \mu_n(\lambda)$. The conditions on the given data include:

(C1): $A(x) := (a_{ij}(x))$ is a real symmetric matrix whose components are bounded Lebesgue-measurable function on Ω and there exist constants $0 < \kappa_3 \leq \kappa_4$ such that

$$\kappa_3 |\xi|^2 \leq \langle A(x)\xi, \xi \rangle \leq \kappa_4 |\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^N, x \in \Omega;$$

(C2): $c \geq 0$ on Ω with $c \in L^p(\Omega)$ for some $p \geq N/2$ when $N \geq 3$, or $p > 1$ when $N = 2$;

(C3): $b_c \geq 0$ on $\partial\Omega$ with $b \in L^\infty(\partial\Omega)$, and c, b_c satisfy

$$\int_{\Omega} c dx + \int_{\partial\Omega} b_c d\sigma > 0;$$

(C4): $m_0 > 0$ on Ω with $m_0 \in L^q(\Omega)$ for some $q > N/2$, and

$$\int_{\Omega} m_0 u^2 dx > 0 \quad \text{for all non-zero } u \in H^1(\Omega);$$

(C5): $b_0 \in L^\infty(\partial\Omega)$ and b_0 may be sign-changing on $\partial\Omega$.

The a -bilinear form associated with (10.1) is, with $u, v \in H^1(\Omega)$,

$$a(u, v) := \int_{\Omega} [(A\nabla u) \cdot \nabla v + c u v] dx + \int_{\partial\Omega} b_c u v d\sigma, \quad (10.2)$$

and the associated b, m -bilinear forms are, respectively,

$$b(u, v) := \int_{\partial\Omega} b_0 u v d\sigma \quad \text{and} \quad m(u, v) := \int_{\Omega} m_0 u v dx. \quad (10.3)$$

It is a nice exercise to show that the conditions on the given data imply these a, b, m bilinear forms satisfy (A1)–(A3) given in Section 2, so the eigencurve results of this paper apply to this two-parameter Robin–Steklov eigenproblem.

Let $\tau > 0$ and $\lambda_* \in \mathbb{R}$ be fixed and define $a_{\lambda_*, \tau} := a - \lambda_* b + \tau m$. When τ is large enough, it follows from Section 3, that the first eigenpair $(\lambda_*, \mu_1(\lambda_*))$ on the vertical line $\lambda = \lambda_*$ in \mathbb{R}^2 for problem (10.1) is obtained from the constrained variational problem

$$\frac{1}{\mu_1(\lambda_*) + \tau} = \sup \left\{ \int_{\Omega} m_0 u^2 dx : u \in H^1(\Omega), a_{\lambda_*, \tau}(u, u) \leq 1 \right\}, \quad (10.4)$$

and maximizers $e_{\lambda_*, 1}$ are Robin–Steklov eigenfunctions corresponding to $(\lambda_*, \mu_1(\lambda_*))$.

When $e_{\lambda_*, 1}, \dots, e_{\lambda_*, k}$ are eigenfunctions corresponding to the first k successive lowest Robin–Steklov eigenpairs on the vertical line $\lambda = \lambda_*$, then the $(k+1)^{st}$ -eigenpair $(\lambda_*, \mu_{k+1}(\lambda_*))$ for (10.1) is obtained from the constrained variational problem

$$\frac{1}{\mu_{k+1}(\lambda_*) + \tau} = \sup \left\{ \int_{\Omega} m_0 u^2 dx : u \in H^1(\Omega), a_{\lambda_*, \tau}(u, u) \leq 1, a_{\lambda_*, \tau}(u, e_{\lambda_*, n}) = 0, n = 1, \dots, k \right\}; \quad (10.5)$$

the maximizers $e_{\lambda_*, k+1}$ are Robin–Steklov eigenfunctions corresponding to $(\lambda_*, \mu_{k+1}(\lambda_*))$.

In this fashion, the eigencurves $\lambda \mapsto \mu_n(\lambda)$ for problem (10.1) are obtained variationally and their properties are summarized as follows.

Theorem 10.1. *Assume (C1)–(C5) hold and the bilinear forms a, b, m are given by (10.2) and (10.3). Let $a_{\lambda_*, \tau} := a - \lambda_* b + \tau m$ with τ large enough, and suppose $\mu_n(\lambda_*)$ and $e_{\lambda_*, n}$ are given as above. Then each $\mu_n(\lambda_*)$ is of finite multiplicity with $-\infty < \mu_1(\lambda_*) \leq \mu_2(\lambda_*) \leq \dots \rightarrow +\infty$ and the class $\{e_{\lambda_*, n} : n \in \mathbb{N}\}$ is an $a_{\lambda_*, \tau}$ -orthonormal basis for $H^1(\Omega)$ consisting of functions satisfying*

$$\int_{\Omega} [(A \nabla e_{\lambda_*, n}) \cdot \nabla v + c e_{\lambda_*, n} v] dx + \int_{\partial \Omega} b_c e_{\lambda_*, n} v d\sigma = \lambda_* \int_{\partial \Omega} b_0 e_{\lambda_*, n} v d\sigma + \mu_n(\lambda_*) \int_{\Omega} m_0 e_{\lambda_*, n} v dx. \quad (10.6)$$

for each $v \in H^1(\Omega)$.

Proof. As mentioned above, when the conditions (C1)–(C5) hold the bilinear forms a, b, m given by (10.2) and (10.3) satisfy the assumptions (A1)–(A3) of Section 2. So, the results are a special case of Theorem 3.2. \square

The orthogonality relations that Robin–Steklov eigenfunctions satisfy are as follows.

Theorem 10.2. *Assume (C1)–(C5) hold, and let $e_{\lambda_1 i}, e_{\lambda_2 j}$ be eigenfunctions corresponding to distinct eigenpairs $(\lambda_1, \mu_i(\lambda_1))$ and $(\lambda_2, \mu_j(\lambda_2))$ of (10.1).*

(i) *In general, the eigendata satisfies*

$$(\lambda_2 - \lambda_1) \int_{\partial \Omega} b_0 e_{\lambda_1 i} e_{\lambda_2 j} d\sigma + (\mu_j(\lambda_2) - \mu_i(\lambda_1)) \int_{\Omega} m_0 e_{\lambda_1 i} e_{\lambda_2 j} dx = 0. \quad (10.7)$$

(ii) *If $\lambda_1 = \lambda_2 = \lambda_*$ and $\mu_i(\lambda_*) \neq \mu_j(\lambda_*)$, then*

$$\int_{\Omega} m_0 e_{\lambda_* i} e_{\lambda_* j} dx = 0, \quad (10.8)$$

and

$$\int_{\Omega} [(A \nabla e_{\lambda_* i}) \cdot \nabla e_{\lambda_* j} + c e_{\lambda_* i} e_{\lambda_* j}] dx + \int_{\partial \Omega} (b_c - \lambda_* b_0) e_{\lambda_* i} e_{\lambda_* j} d\sigma = 0. \quad (10.9)$$

(iii) If $\lambda_1 \neq \lambda_2$ and $\mu_i(\lambda_1) = \mu_j(\lambda_2) = \mu_*$, then

$$\int_{\partial\Omega} b_0 e_{\lambda_1 i} e_{\lambda_2 j} d\sigma = 0 \quad (10.10)$$

and

$$\int_{\Omega} [(A\nabla e_{\lambda_1 i}) \cdot \nabla e_{\lambda_2 j} + (c - \mu_* m_0) e_{\lambda_1 i} e_{\lambda_2 j}] dx + \int_{\partial\Omega} b_c e_{\lambda_1 i} e_{\lambda_2 j} d\sigma = 0. \quad (10.11)$$

Proof. The bilinear forms a, b, m given by (10.2) and (10.3) satisfy (A1)–(A3), so the proof of (i) is similar to that of Theorem (4.1). From this relation, the orthogonal relations of (ii) and (iii) follow as in the proofs of Corollaries 4.2 and 4.3. \square

We emphasize that the weight function b_0 is allowed to be sign-changing, so the bilinear forms in these orthogonality relations are not necessarily inner products on $H^1(\Omega)$.

A first regularity result for Robin–Steklov eigencurves is summarized next.

Theorem 10.3. *Assume (C1)–(C5) hold and the values $\mu_n(\lambda)$ are given by (10.4) and (10.5). Then each eigencurve $\mu_n(\lambda)$ associated with the Robin–Steklov eigenproblem (10.1) is Lipschitz continuous. Moreover, if (λ_*, μ_*) is a simple eigenpair of (10.1) and $\mu_n(\lambda_*) = \mu_*$, then the eigencurve $\mu_n(\lambda)$ is differentiable at λ_* and has, with corresponding eigenfunction e_* ,*

$$\left. \frac{d\mu_n}{d\lambda} \right|_{\lambda_*} = - \frac{\int_{\partial\Omega} b_0 e_*^2 d\sigma}{\int_{\Omega} m_0 e_*^2 dx}. \quad (10.12)$$

Furthermore, the first Robin–Steklov eigencurve $\mu_1(\lambda)$ is concave on \mathbb{R} .

Proof. The first statement follows from Theorem 5.1, the second one from Corollary 5.3, while the concavity result follows from Lemma 3.3 when the bilinear forms a, b, m are taken to be those given by (10.2) and (10.3). \square

For higher multiplicities, the differentiability result is the following.

Theorem 10.4. *Assume (C1)–(C5) hold and the values $\mu_n(\lambda)$ are given by (10.4) and (10.5). Let (λ_*, μ_*) be an eigenpair of (10.1) of multiplicity $k_0 \geq 2$ and let*

$$\mu_{n_0+1}(\lambda) \leq \mu_{n_0+2}(\lambda) \leq \cdots \leq \mu_{n_0+k_0}(\lambda),$$

be the Robin–Steklov eigencurves intersecting (λ_*, μ_*) . The eigencurve $\mu_{n_0+k}(\lambda)$ intersecting (λ_*, μ_*) has right-sided derivative satisfying

$$\lim_{\lambda \rightarrow \lambda_*^+} \frac{\mu_{n_0+k}(\lambda) - \mu_*}{\lambda - \lambda_*} = \min_{Z_k} \max_{\substack{v \in Z_k \\ v \neq 0}} \left\{ \frac{-\int_{\partial\Omega} b_0 v^2 d\sigma}{\int_{\Omega} m_0 v^2 dx} \right\}, \quad (10.13)$$

where the minimization is taken over all k -dimensional subspaces Z_k in the subspace E_{λ_*, μ_*} associated with (λ_*, μ_*) ; the left-sided derivative of $\mu_{n_0+k}(\lambda)$ satisfies

$$\lim_{\lambda \rightarrow \lambda_*^-} \frac{\mu_{n_0+k}(\lambda) - \mu_*}{\lambda - \lambda_*} = \min_{Z_d} \max_{\substack{v \in Z_d \\ v \neq 0}} \left\{ \frac{-\int_{\partial\Omega} b_0 v^2 d\sigma}{\int_{\Omega} m_0 v^2 dx} \right\}, \quad (10.14)$$

where the minimization is taken over all $d = k_0 - k + 1$ dimensional subspaces Z_d of E_{λ_*, μ_*} .

This theorem follows from the results of Section 6, and the results characterize, locally, the spectrum (collection of graphs of eigencurves) of Robin–Steklov eigenproblems as comprising finitely many smooth curves. However, the eigencurves intersecting such points of higher multiplicities are in general not differentiable at such eigenpairs. This local characterization of eigencurves is concretely seen in the specific matrix eigenproblem (6.5) and (6.6) given in Section 6. There, the point $(\lambda, \mu) = (1, 0)$ in \mathbb{R}^2 in Figure 1 is the point of intersection for the first and second variational eigencurves $\mu_1(\lambda), \mu_2(\lambda)$. The graph of each curve has a corner at this point, so $\mu_1(\lambda), \mu_2(\lambda)$ are not differentiable at $\lambda = 1$. However, concatenating the piece of $\mu_1(\lambda)$ left of $(1, 0)$ in \mathbb{R}^2 with the piece of $\mu_2(\lambda)$ right of $(1, 0)$ yields a smooth curve.

Regardless of the multiplicity, it is interesting to note that differentiability of eigencurves is influenced by spectral data for the pair (b, m) given by (10.3). As seen in Theorem 10.6, the pair (b, m) also dictates the asymptotic behaviour of eigencurves.

The next result concerns the sign of the weight function b_0 in the bilinear form b .

Theorem 10.5. *Assume (C1)–(C5) hold and the values $\mu_n(\lambda)$ are given by (10.4) and (10.5).*

- (i) *If b_0 is strictly positive σ -a.e. on a subset of $\partial\Omega$ of strictly positive Hausdorff measure, then all the Robin–Steklov eigencurves $\mu_n(\lambda)$ decrease to $-\infty$ as $\lambda \rightarrow +\infty$.*
- (ii) *If b_0 is strictly negative σ -a.e. on a subset of $\partial\Omega$ of strictly positive Hausdorff measure, then all the Robin–Steklov eigencurves $\mu_n(\lambda)$ decrease to $-\infty$ as $\lambda \rightarrow -\infty$.*

Proof. The first result follows from part (i) of Lemma 7.2, since in this case the quadratic form $\mathcal{B}(u) := \int_{\partial\Omega} b_0 u^2 d\sigma$ is strictly positive on an infinite dimensional subspace of $H^1(\Omega)$. The second result follows similarly from part (ii) of that lemma. \square

A further asymptotic result is the following.

Theorem 10.6. *Assume (C1)–(C5) hold and the values $\mu_n(\lambda)$ are given by (10.4) and (10.5). Then the k th-eigencurve $\mu_k(\lambda)$ satisfies*

$$\lim_{\lambda \rightarrow +\infty} \frac{\mu_k(\lambda)}{\lambda} = \inf_{Z_k} \sup_{\substack{v \in Z_k \\ v \neq 0}} \left\{ \frac{-\int_{\partial\Omega} b_0 v^2 d\sigma}{\int_{\Omega} m_0 v^2 dx} \right\},$$

where the infimum is taken over all k -dimensional subspaces Z_k of $H^1(\Omega)$.

The asymptotic results and methodology behind Theorems 10.5 and 10.6 may be compared and contrasted with those of Daners and Kennedy [9]. The specific equation in [9] is the case where $c \equiv b_c \equiv 0$ and $m_0 \equiv b_0 \equiv 1$ in (10.1) of this paper, so that the compatibility condition (C3) above is not satisfied. The authors in [9] show that each eigencurve $\mu_k(\lambda)$, $k \in \mathbb{N}$, behaves asymptotically like $-\lambda^2$ as $\lambda \rightarrow \infty$. It is, therefore, of interest to find further natural assumptions on the data to obtain sharper asymptotic results for eigencurves of (10.1).

The above theorems provide regularity and asymptotic results for eigencurves of (10.1). The next result describes when straight lines may appear within the Robin–Steklov spectrum.

Theorem 10.7. *Assume (C1)–(C5) hold. If $\mu_* \in \mathbb{R}$ is such that there is a non-zero function $e_* \in H^1(\Omega)$ satisfying*

$$\int_{\partial\Omega} b_0 e_* v d\sigma = 0, \tag{10.15}$$

and

$$\int_{\Omega} [(A\nabla e_*) \cdot \nabla v + c e_* v] dx + \int_{\partial\Omega} b_c e_* v d\sigma = \mu_* \int_{\Omega} m_0 e_* v dx, \tag{10.16}$$

for all $v \in H^1(\Omega)$, then the straight line $\{(\lambda, \mu_*) : \lambda \in \mathbb{R}\}$ is contained within the spectrum of the Robin–Steklov eigenproblem (10.1).

Proof. When a, b, m are the bilinear forms given by (10.2) and (10.3), then μ_*, e_* in this theorem are such that $e_* \in E_{0\mu_*} \cap N(b)$, so the result is a special case of Theorem 8.2. \square

The data (μ_*, e_*) in this theorem is a weak solution to the Robin eigenvalue problem

$$\begin{aligned} -\operatorname{div}(A\nabla u) + c(x)u &= \mu m_0(x)u && \text{in } \Omega, \\ (A\nabla u) \cdot \nu + b_c(x)u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Denote by $E_{0\mu_*}$ the eigenspace in $H^1(\Omega)$ associated with such Robin eigenvalue μ_* . The theorem can be interpreted as saying that when Robin eigensubspaces $E_{0\mu_*}$ intersect non-trivially with the null space $N(b)$ of b , then there are straight lines within the two-parameter Robin–Steklov spectrum $\sigma(a, b, m)$.

Our *main geometrical result* for Robin–Steklov eigencurves is the following.

Theorem 10.8. *Assume (C1)–(C5) hold and the values $\mu_n(\lambda)$ are given by (10.4) and (10.5). If μ_* is not a Robin eigenvalue, then the total number of connected components of the graphs of the first $n \in \mathbb{N}$ Robin–Steklov eigencurves $\mu_1(\lambda) \leq \mu_2(\lambda) \leq \dots \leq \mu_n(\lambda)$ that lie strictly above the horizontal line $\mu = \mu_*$ is at most n .*

Proof. As the conditions (C1)–(C5) imply the bilinear forms a, b, m in (10.2) and (10.3) satisfy assumptions (A1)–(A3), the results follow from the construction given in Section 9, and in particular that of Theorem 9.6. \square

Acknowledgements. The authors are deeply grateful to the anonymous reviewers for their useful comments that helped improve the article, and for providing interesting references and questions that have led to further promising research directions.

REFERENCES

- [1] G.A. Afrouzi and K.J. Brown, On principle eigenvalues for boundary value problems with indefinite weight and Robin boundary conditions. *Proc. Amer. Math. Soc.* (1999) 125–130.
- [2] H. Attouch, G. Buttazzo and G. Michaille, Variational Analysis in Sobolev and BV Spaces: Applications to PDEs and Optimization. MPS-SIAM, Philadelphia (2005).
- [3] G. Auchmuty, Bases and comparison results for linear elliptic eigenproblems. *J. Math. Anal. Appl.* **390** (2012) 394–406.
- [4] G. Auchmuty and Q. Han, Representation of solutions of Laplacian boundary value problems on exterior regions. *Appl. Math. Optim.* **69** (2014) 21–45.
- [5] G. Auchmuty and M.A. Rivas, Laplacian eigenproblems on product regions and tensor products of Sobolev spaces. *J. Math. Anal. Appl.* **435** (2016) 842–859.
- [6] C. Bandle and A. Wagner, Isoperimetric inequalities for the principle eigenvalue of a membrane and the energy of problems with Robin boundary conditions. *J. Convex Anal.* **22** (2014) 627–640.
- [7] P. Binding and H. Volkmer, Eigencurves for two-parameter Sturm–Liouville equations. *SIAM Rev.* **38** (1996) 27–48.
- [8] P. Blanchard and E. Brüning, Variational Methods in Mathematical Physics. Springer-Verlag, Berlin (1992).
- [9] D. Daners and J.B. Kennedy, On the asymptotic behaviour of the eigenvalues of a Robin problem. *Differ. Integ. Equ.* **23** (2010) 659–669.
- [10] L.C. Evans and R.F. Gariepy, Measure Theory and Fine Properties of Functions. CRC Press, Boca Raton (1992).
- [11] M.A. Filinovskiy, On the eigenvalues of a Robin problem with a large parameter. *Math. Bohem.* **139** (2014) 341–352.
- [12] T. Giorgi and R. Smits, Eigenvalue estimates and critical temperatures in zero fields for enhanced surface superconductivity. *Z. Angew. Math. Phys.* **58** (2007) 224–245.
- [13] T. Kato, Perturbation Theory for Linear Operators, 2nd Edn. Springer-Verlag, New York (1976).
- [14] B. Ko and K.J. Brown, The existence of positive solutions for a class of indefinite weight semilinear elliptic boundary value problems. *Nonlin. Anal.: Theory Methods. Appl.* **39** (2000) 587–597.
- [15] A.A. Lacey, J.R. Ockendon and J. Sabina, Multidimensional reaction diffusion equations with nonlinear boundary conditions. *SIAM J. Appl. Math.* **58** (1998) 1622–1647.
- [16] N. Mavinga, Generalized eigenproblem and nonlinear elliptic equations with nonlinear boundary conditions. *Proc. R. Soc. Edinburgh* **142A** (2012) 137–153.
- [17] N. Mavinga and M.N. Nkashama, Steklov–Neumann eigenproblems and nonlinear elliptic equations with nonlinear boundary conditions. *J. Differ. Equ.* **248** (2010) 1212–1229.