

CONTROL AND STABILIZATION OF THE PERIODIC FIFTH ORDER KORTEWEG-DE VRIES EQUATION

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Abstract. We establish local exact control and local exponential stability of periodic solutions of fifth order Korteweg-de Vries type equations in $H^s(\mathbb{T})$, $s > 2$. A dissipative term is incorporated into the control which, along with a propagation of regularity property, yields a smoothing effect permitting the application of the contraction principle.

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1. INTRODUCTION

We study control of the fifth order Korteweg de-Vries (KdV) equation

$$\partial_t u - \partial_x^5 u - 30u^2 \partial_x u + 20\partial_x u \partial_x^2 u, + 10u \partial_x^3 u = 0, \quad (1.1)$$

where $u = u(x, t)$ denotes a real-valued function. This equation appears in the sequence of nonlinear dispersive equations

$$\partial_t u + \partial_x^{2j+1} u + Q_j(u, \partial_x u, \dots, \partial_x^{j-1} u) = 0, \quad j \in \mathbb{Z}^+, \quad (1.2)$$

known as the KdV hierarchy. The specification of the polynomials Q_j arises from the observation in [13] that the eigenvalues of the Schrödinger operator $L(u) = \frac{d^2}{dx^2} - u(x, \cdot)$ are independent of time when u evolves as a solution to the usual KdV equation

$$\partial_t u + \partial_x^3 u + u \partial_x u = 0. \quad (1.3)$$

By imposing a Lax pair structure

$$\partial_t u = [B_j; L(u)],$$

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the same statement holds for any equation in the sequence (1.2) when B_j is a skew-symmetric operator chosen so that $[B_j; L(u)]$ has degree zero [38]. The resulting hierarchy consists of a family of completely integrable equations which can be solved by the inverse scattering method.

This paper focuses on a fifth order equation generalizing (1.1)

$$\partial_t u - \partial_x^5 u + \beta_0 \partial_x^3 u + \beta_1 \partial_x u + c_0 u \partial_x u + c_1 u^2 \partial_x u + c_2 \partial_x u \partial_x^2 u + c_3 u \partial_x^3 u = 0, \quad (1.4)$$

where $\beta_0, \beta_1, c_0, c_1, c_2, c_3$ are real constants. Solutions to this equation formally conserve volume and arise in a number of physical situations. With $c_1 = 0$, this equation was shown to model the water wave problem for long, small amplitude waves over a shallow bottom [42]; see also [9]. The family (1.4) also contains Benney's model of short and long wave interaction [2]. Letting $c_0 = 1$ and $c_1 = c_2 = c_3 = 0$ yields the Kawahara equation

$$\partial_t u - \partial_x^5 u + \beta_0 \partial_x^3 u + \beta_1 \partial_x u + u \partial_x u = 0, \quad (1.5)$$

which describes the propagation of magneto-acoustic waves in a plasma [24].

The initial value problem (IVP) associated to these equations is naturally studied in the Sobolev scale

$$H^s(\mathbb{K}) = (1 - \partial_x^2)^{-s/2} L^2(\mathbb{K}), \quad s \in \mathbb{R},$$

$\mathbb{K} = \mathbb{T}$ or \mathbb{R} . For equations (1.1), (1.3) and (1.5), the problem of determining the minimal Sobolev regularity required to ensure well-posedness of the associated IVP has been studied extensively. The KdV equation is globally well-posed in $H^s(\mathbb{R})$ for $s \geq -3/4$ and $H^s(\mathbb{T})$ for $s \geq -1/2$. We mention the works [5–7, 16, 27, 30] in this regard. The initial-boundary value problem (IBVP) for the KdV equation on a finite interval with nonhomogeneous boundary conditions is locally well-posed in $H^s(0, L)$ with $-3/4 < s < 3/2$, $s \neq 1/2$, [19] and globally well-posed for $s \geq 0$ [4]. For the Kawahara equation we have local well-posedness in $H^s(\mathbb{R})$ for $s \geq -2$ [22] and global well-posedness in $H^s(\mathbb{R})$ for $s > -38/21$ [23]. In the periodic setting, the equation is locally well-posedness in $H^s(\mathbb{T})$ for $s \geq -3/2$ and globally well-posedness in $H^s(\mathbb{T})$, $s \geq -1$ [23]. On a finite interval with homogeneous boundary conditions, the authors in [33] establish the existence and uniqueness of global weak $L^2(0, L)$ solutions of the IBVP.

Ponce [44] established local well-posedness for the fifth order equation (1.4) in $H^s(\mathbb{R})$, $s \geq 4$, using sharp linear estimates and the Bona-Smith argument [3]. Kenig, Ponce and Vega investigated a class of equations containing the KdV hierarchy in polynomially weighted Sobolev spaces by combining a commuting vector field identity with the contraction principle [28, 29]. Pilod [43] showed that for each $j \geq 2$, the solution map $H^s(\mathbb{R}) \ni u_0 \mapsto u \in C([0, T]; H^s(\mathbb{R}))$ corresponding to the IVP for the equation

$$\partial_t u + \partial_x^{2j+1} u \pm u \partial_x^k u = 0, \quad k > j,$$

is not C^2 at the origin for any $s \in \mathbb{R}$. In fact, it is not even uniformly continuous [32]. Thus, in contrast to the KdV equation, higher order members of the KdV hierarchy cannot be solved using the contraction principle in $H^s(\mathbb{R})$ alone. Kwon [32] introduced a corrected energy and refined Strichartz estimate to establish local well-posedness for the fifth order KdV equation (1.1) in $H^s(\mathbb{R})$, $s > 5/2$. Kenig and Pilod [26] applied this technique to a class of equations containing the KdV hierarchy, obtaining local well-posedness in $H^s(\mathbb{R})$ for $s > 4j - 9/2$, $j \geq 2$. Global well-posedness of the fifth order KdV equation in $H^2(\mathbb{R})$ was established simultaneously by Guo, Kwak and Kwon [17] and Kenig and Pilod [25] using a Bourgain space approach. In the periodic setting, Schwarz [49] obtained the existence of weak solutions to equations in the KdV hierarchy (1.2) corresponding to data in $H^m(\mathbb{T})$, $m \geq 2j + 1$ and $m \in \mathbb{Z}^+$, with uniqueness holding under the condition $m \geq 3j + 1$. Recently, Kwak [31] obtained global well-posedness of equation (1.1) in $H^2(\mathbb{T})$. The proof relies somewhat on the completely integrable structure of the equation. On a finite domain, the IBVP for equation (1.4) with nonhomogeneous boundary conditions is locally well-posed in $L^2(0, L)$ [14]. In the Hamiltonian case with homogeneous boundary conditions, the IBVP is found to be globally well-posed in $H^s(0, L)$ for $0 \leq s \leq 5$ [52].

For the forced fifth order equation

$$\partial_t u - \partial_x^5 u + \beta_0 \partial_x^3 u + \beta_1 \partial_x u + c_0 u \partial_x u + c_1 u^2 \partial_x u + c_2 \partial_x u \partial_x^2 u + c_3 u \partial_x^3 u = F(x, t), \quad (1.6)$$

on a periodic domain, we investigate two questions central to control theory.

Exact Control Problem: Given an initial state u_0 and terminal state u_1 , can one find an appropriate forcing function F so that equation (1.6) admits a solution $u = u(x, t)$ which satisfies $u(\cdot, 0) = u_0$ and $u(\cdot, T) = u_1$?

Stabilization Problem: Can one find a feedback law $F = Ku$ so that the resulting closed-loop system is asymptotically stable as $t \rightarrow +\infty$?

As solutions to (1.6) satisfy the identity

$$\frac{d}{dt} \int_{\mathbb{T}} u(x, t) dx = \int_{\mathbb{T}} F(x, t) dx,$$

we achieve conservation of volume by choosing F of the form

$$(Gh)(x, t) := g(x) \left(h(x, t) - \int_{\mathbb{T}} g(y) h(y, t) dy \right), \quad (1.7)$$

where $g \in C^\infty(\mathbb{T})$ is nonnegative, has the mean value property

$$2\pi[g] = \int_{\mathbb{T}} g(x) dx = 1,$$

and is allowed to be supported in a proper subinterval of the torus.

Russell and Zhang obtained the first control results for an equation in the KdV hierarchy. Using the smoothing effect discovered by Bourgain [5, 6] they established *local exact controllability* for the KdV equation.

Theorem 1.1 ([47]). *Let $T > 0$ and $s \geq 0$ be given. Then there exists $\delta > 0$ such that for any $u_0, u_1 \in H^s(\mathbb{T})$ with $[u_0] = [u_1]$ and*

$$\|u_0\|_s + \|u_1\|_s \leq \delta,$$

one can find a control $h \in L^2(0, T, H^s(\mathbb{T}))$ such that the equation

$$\partial_t u + \partial_x^3 u + u \partial_x u = Gh$$

has a solution $u \in C([0, T]; H^s(\mathbb{T}))$ satisfying

$$u(\cdot, 0) = u_0 \quad \text{and} \quad u(\cdot, T) = u_1.$$

Additionally, they proved *local exponential stability*.

Theorem 1.2 ([47]). *Let $\kappa > 1$ and $s = 0$ or $s \geq 1$ be given. Then there exists positive constants, C , δ and λ such that if $u_0 \in H^s(\mathbb{T})$ with $\|u_0 - [u_0]\|_s \leq \delta$, then the corresponding solution u of the system*

$$\partial_t u + \partial_x^3 u + u \partial_x u = -\kappa Gu$$

satisfies

$$\|u(\cdot, t) - [u_0]\|_s \leq C e^{-\lambda t} \|u_0 - [u_0]\|_s$$

for all $t \geq 0$.

Laurent, Rosier and Zhang [37] later proved *global exact controllability* and *global exponential stability* for the KdV equation (1.3) in $H^s(\mathbb{T})$, $s \geq 0$. Their technique relied on the structure of the commutator $[\phi; \partial_x^3]$ for $\phi \in C^\infty(\mathbb{T})$. On the real line, Kato [21] utilized this structure to conclude that the solution to (1.3) corresponding to data $u_0 \in H^s(\mathbb{R})$ lies in $L^2(0, T; H_{\text{loc}}^{s+1}(\mathbb{R}))$. Though such a smoothing effect is false on the torus, the same formal computation reveals that if the solution lies in $L^2(0, T; H^{s+1}(\Omega))$ for some open set $\Omega \subset \mathbb{T}$, then one may conclude that it also lies in $L^2(0, T; H^{s+1}(\mathbb{T}))$. This propagation of regularity, along with a similar propagation of compactness lemma, was previously applied to control of wave [10] and Schrödinger equations [34, 35]. Though not discussed further here, we mention the extensive work on the control theory of the Korteweg-de Vries equation on a bounded domain, a review of which may be found in the survey [45].

We now discuss control results pertaining to equation (1.4). The situation is most developed for the Kawahara equation (1.5); in particular, Zhao and Zhang [51] applied the method of [37] to obtain global exact control and global exponential stability for periodic solutions in $H^s(\mathbb{T})$, $s \geq 0$. Moreover, exponential stability has been demonstrated for the initial-boundary value problem associated to the Kawahara equation on an interval in a number of situations. In the presence of a feedback term $F = -g(x)u$, we mention the works [50] and [1]. Without a feedback term, see [11] for the case of zero boundary conditions and [12] for a boundary dissipation mechanism.

In the case of $c_2^2 + c_3^2 > 0$, Glass and Guerrero [14] established local controllability to trajectories for the boundary value problem associated to equation (1.4) by using Carleman estimates and a smoothing effect of Kato type derived from the boundary conditions. To the best of our knowledge, there are no results concerning the exact control or exponential stability of equation (1.4) in the periodic setting when $c_2^2 + c_3^2 > 0$.

In this paper, we present affirmative answers to the exact control and stabilization problems for equation (1.6) on a periodic domain. To overcome the lack of an adequate smoothing effect, we adopt an approach previously applied to the Benjamin-Ono equation [39]. To stabilize (1.6) we consider a feedback law

$$F = -GD^3Gu.$$

In the linear homogeneous case, scaling the resulting equation

$$\partial_t v - \partial_x^5 v + \beta_0 \partial_x^3 v + \beta_1 \partial_x v + GD^3Gv = 0 \tag{1.8}$$

by v yields

$$\frac{1}{2} \|v(T)\|_{L^2(\mathbb{T})}^2 + \int_0^T \|D^{3/2}(Gv)\|_{L^2(\mathbb{T})}^2 dt = \frac{1}{2} \|v_0\|_{L^2(\mathbb{T})}^2,$$

which suggests a gain of 3/2 derivatives in the control region $\omega = \{x \in \mathbb{T} : g(x) > 0\}$. Using a propagation of regularity property we conclude

$$\|v\|_{L^2(0, T; H^{3/2}(\mathbb{T}))} \leq c(T; \|v_0\|_{L^2(\mathbb{T})}). \tag{1.9}$$

By considering the forcing term $F = -GD^3Gv + Gh$, a similar smoothing effect holds and we obtain exact controllability of the resulting linear equation by a classical observability argument. Thus a contraction principle argument yields the following nonlinear result.

Theorem 1.3. *Let $T > 0$ and $s > 2$ be given. Then there exists $\delta > 0$ such that for any $u_0, u_1 \in H^s(\mathbb{T})$ with $[u_0] = [u_1]$ and*

$$\|u_0\|_s + \|u_1\|_s \leq \delta,$$

one can find a control $h \in L^2(0, T; H^{s-3/2}(\mathbb{T}))$ such that the equation (1.6) with $F = -GD^3Gu + Gh$ has a solution $u \in C([0, T]; H^s(\mathbb{T}))$ satisfying

$$u(\cdot, 0) = u_0 \quad \text{and} \quad u(\cdot, T) = u_1.$$

Similarly, the linear exponential stability of solutions to (1.8) combined with the contraction principle in an appropriate space yields exponential stability for solutions to the nonlinear equation.

Theorem 1.4. *Let $s > 2$ be given. Then there exists constants $\rho, \lambda, C > 0$, such that for any $u_0 \in H^s(\mathbb{T})$ with $\|u_0 - [u_0]\|_s \leq \rho$, equation (1.6) with $F = -GD^3Gu$ admits a unique solution satisfying*

$$u \in C([0, T]; H^s(\mathbb{T})) \cap L^2(0, T; H^{s+3/2}(\mathbb{T}))$$

for any $T > 0$ and such that

$$\|u(t) - [u_0]\|_s \leq Ce^{-\lambda t} \|u_0 - [u_0]\|_s.$$

Here we draw a comparison to the completely integrable Benjamin-Ono equation, which is globally well-posed in $L^2(\mathbb{R})$ [20] and $L^2(\mathbb{T})$ [40] via the gauge transform, yet exhibits quasilinear behavior similar to the higher order KdV equations [41]. In their study of the Benjamin-Ono equation, Linares and Rosier [39] incorporated dissipation into the feedback law, obtaining semi-global exponential stability of vanishing viscosity solutions of the Benjamin-Ono equation in $L^2(\mathbb{T})$. Despite the non-local nature of the Hilbert transform \mathcal{H} , a Calderon estimate provided bounds on the commutator $[\partial_t + \mathcal{H}\partial_x^2; \phi]$ necessary to prove a propagation of regularity result. Laurent, Linares and Rosier [36] subsequently established global stabilization in $L^2(\mathbb{T})$ by use of the gauge transform and a more natural feedback law $-GG^*u$. In light of the works [36, 37], one would expect global exponential controllability and stabilization for the (Hamiltonian) fifth order KdV equation (1.1) in the energy space $H^2(\mathbb{T})$ without the need for dissipation in the control term, however, it is not clear how to incorporate a control term into the existing well-posedness result [31]. Also, our Theorem 1.3 inherits the limitation of [39] in that the control h is realized in the space $L^2(0, T; H^{s-3/2}(\mathbb{T}))$ instead of $L^2(0, T; H^s(\mathbb{T}))$.

The remainder of the paper is organized as follows. Section 2 contains estimates for solutions to the linear equation (1.8). The proof of Theorem 1.4 is found in Section 3 and the proof of Theorem 1.3 is found in Section 4. Additionally, Section 3 describes how to extend these results to a family of equations containing the KdV hierarchy.

2. PRELIMINARIES AND LINEAR ESTIMATES

Throughout the sequel, it suffices to consider only the case $[u_0] = 0$; the change of dependent variable $\tilde{u} = u - [u_0]$ in equation (1.4) leads to an equation in \tilde{u} of the same type. We denote $H_0^s(\mathbb{T}) = \{u \in H^s(\mathbb{T}) : [u] = 0\}$. The usual $L^2(\mathbb{T})$ scalar product is written $(u, v) = \int_{\mathbb{T}} u(x)v(x) dx$ and in $H^s(\mathbb{T})$, $s \in \mathbb{R}$, $(u, v)_s = ((1 - \partial_x^2)^{s/2}u, (1 - \partial_x^2)^{s/2}v)$. The norm in $H^s(\mathbb{T})$ is given by $\|u\|_s = (u, u)_s^{1/2}$ where we abbreviate $\|u\|_0 = \|u\|$. It is convenient to define the operator D^r , $r \in \mathbb{R}$, as

$$\widehat{D^r u}(k) = \begin{cases} |k|^r \hat{u}(k) & \text{if } k \neq 0, \\ \hat{u}(0) & \text{if } k = 0, \end{cases} \quad (2.1)$$

so that $\|u\|_s \cong \|D^s u\|$ for $u \in H_0^s(\mathbb{T})$. This operator satisfies the following commutator estimate.

Lemma 2.1 ([34], Lem. A.1). *If $\psi \in C^\infty(\mathbb{T})$, then for any $r, s \in \mathbb{R}$*

$$\|D^r [D^s; \psi] f\| \leq c(r; s; \psi) \|f\|_{r+s-1}. \quad (2.2)$$

We will make use of the Hilbert transform \mathcal{H} , defined as a Fourier multiplier via the formula

$$\widehat{\mathcal{H}f}(k) = -i \operatorname{sgn}(k) \hat{f}(k), \quad k \in \mathbb{Z}.$$

In addition to being volume-preserving and self-adjoint on $L^2(\mathbb{T})$, one sees using (2.2) that the operator G is a bounded operator on $H_0^s(\mathbb{T})$ for any $s \in \mathbb{R}$. The definition (1.7) yields for $\psi \in C^\infty(\mathbb{T})$

$$\begin{aligned} G(\psi h) &= \psi Gh + g \left(\psi \int gh \, dy - \int \psi gh \, dy \right) \\ &=: \psi Gh + \tilde{h}, \end{aligned} \quad (2.3)$$

where $\|D^s \tilde{h}\| \leq c(s; g; \psi) \|h\|$ for any $s \in \mathbb{R}$. Similarly,

$$[D^s; G] D^r f = [D^s; g] D^r f - D^s g \int f D^r g + g \int f D^{r+s} g \quad (2.4)$$

for any $s, r \in \mathbb{R}$. Thus, writing $r = r_1 + r_2$,

$$\|D^{r_1} [D^s; G] D^{r_2} f\| \leq c(r; s; g) \|f\|_{r+s-1}. \quad (2.5)$$

Next, we shall deduce estimates of solutions to the linear problem for $\epsilon > 0$.

$$\begin{cases} \partial_t v + (\epsilon D^5 - \partial_x^5) v + \beta_0 \partial_x^3 v + \beta_1 \partial_x v + GD^3 Gv = F, & x \in \mathbb{T}, t \geq 0, \\ v(x, 0) = v_0(x). \end{cases} \quad (2.6)$$

As it does not affect the analysis we assume $\beta_0 = \beta_1 = 0$. We first uncover *a priori* $H_0^s(\mathbb{T})$ bounds on smooth solutions to the above IVP by incorporating a propagation of regularity argument in the same vein as [10, 34, 35]. Writing $w = D^s v$ with v smooth, we see that w solves

$$\partial_t w + (\epsilon D^5 - \partial_x^5) w + GD^3 Gw + Ew = D^s F,$$

where the “remainder” operator

$$E = GD^3 [D^s; G] D^{-s} + [D^s; G] D^3 G D^{-s} \quad (2.7)$$

has order 2. The following weighted energy identity will be utilized.

Lemma 2.2. *A smooth solution $v = v(x, t)$ to IVP (2.6) satisfies*

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int w^2 \psi \, dx + \frac{5}{2} \int (\partial_x^2 w)^2 \psi' \, dx + \int D^{3/2}(Gw) D^{3/2} G(\psi w) \, dx \\ & \quad + \int w E w \psi \, dx + \epsilon \left\{ \int (D^{5/2} w)^2 \psi \, dx + \int D^{5/2} w [D^{5/2}; \psi] w \, dx \right\} \\ & = \frac{5}{2} \int (\partial_x w)^2 \psi^{(3)} \, dx + \frac{1}{2} \int w^2 \psi^{(5)} \, dx + \int w D^s F \psi \, dx, \end{aligned} \quad (2.8)$$

where $w = D^s v$ and $\psi \in C^\infty(\mathbb{T})$.

Motivated by the gain of $3/2$ -derivatives suggested by the form of the control term, we study solutions to the IVP (2.6) in the spaces

$$Z_{s,T} = C(0, T; H_0^s(\mathbb{T})) \cap L^2(0, T; H_0^{s+3/2}(\mathbb{T})), \quad (2.9)$$

with $s \in \mathbb{R}, T > 0$, endowed with the norm

$$\|v\|_{s,T} = \|v\|_{L^\infty(0,T;H^s(\mathbb{T}))} + \|v\|_{L^2(0,T;H^{s+3/2}(\mathbb{T}))}. \quad (2.10)$$

The next proposition establishes ϵ -uniform bounds in $Z_{s,T}$.

Proposition 2.3. *Let $s \in \mathbb{R}$ and $0 < \epsilon < 1$. A smooth solution to IVP (2.6) corresponding to data $v_0 \in H_0^s(\mathbb{T})$ satisfies $v \in Z_{s,T}$ with*

$$\|v\|_{s,T} \leq c(s, T) (\|v_0\|_s + \|F\|_{L^2(0,T;H^{s-3/2}(\mathbb{T}))}) \quad (2.11)$$

for any $T > 0$ and $c(s, T)$ nondecreasing in T .

Proof. We show the details for the case $s = 0$ and demonstrate the necessary modifications when $s \neq 0$.

(Case $s = 0$). In order to justify the following computations, assume $v_0 \in H_0^5(\mathbb{T})$ and $F \in C([0, T]; H_0^5(\mathbb{T}))$ so that

$$v \in C([0, T]; H_0^5(\mathbb{T})) \cap C^1([0, T]; H_0^0(\mathbb{T})).$$

Scaling the equation (2.6) by v , and for all $\tau < T$,

$$\frac{1}{2} \|v(\tau)\|^2 + \int_0^\tau \|D^{3/2}(Gv)\|^2 \, dt + \epsilon \int_0^\tau \|D^{5/2}v\|^2 \, dt \leq \frac{1}{2} \|v_0\|^2 + \int_0^\tau \|v\|_{3/2} \|F\|_{-3/2} \, dt.$$

and so

$$\|v\|_{L^\infty(0,T;H^0(\mathbb{T}))}^2 + \int_0^T \|D^{3/2}(Gv)\|^2 \, dt + \epsilon \int_0^T \|D^{5/2}v\|^2 \, dt \leq \|v_0\|^2 + 2 \int_0^T \|v\|_{3/2} \|F\|_{-3/2} \, dt. \quad (2.12)$$

We next apply a propagation of regularity argument to account for the extra $3/2$ -derivatives above. We begin by introducing a function $b \in C_0^\infty(\omega)$, $\omega = \{x \in \mathbb{T} : g(x) > 0\}$, which forms a partition of unity. Picking $t_0 \in (0, T)$,

$$\int_{t_0}^T \|D^{3/2}v\|^2 \, dt = \sum_k \int_{t_0}^T \left(b^2(x - x_k) D^{3/2}v, D^{3/2}v \right) \, dt. \quad (2.13)$$

Notice for each k , there exists a primitive $\phi \in C^\infty(\mathbb{T})$ which satisfies

$$b^2(x) - b^2(x - x_k) = \partial_x \phi(x). \quad (2.14)$$

As each of the k terms are estimated similarly inserting (2.14) yields

$$\int_{t_0}^T \|D^{3/2}v\|^2 dt \lesssim \left| \int_{t_0}^T \left(b^2 D^{3/2}v, D^{3/2}v \right) dt \right| + \left| \int_{t_0}^T \left(\partial_x \phi D^{3/2}v, D^{3/2}v \right) dt \right| =: M_1 + M_2. \quad (2.15)$$

Following [39], observe that by definition $b = g\tilde{b}$ for some $\tilde{b} \in C_0^\infty(\omega)$, so that applying the commutator estimate (2.2) and interpolating

$$\begin{aligned} M_1 &= \int_{t_0}^T \|bD^{3/2}v\|^2 dt \leq 2 \int_{t_0}^T \|\tilde{b}D^{3/2}(gv)\|^2 + \|\tilde{b}[D^{3/2}; g]v\|^2 dt \\ &\leq c \int_{t_0}^T \|D^{3/2}(gv)\|^2 dt + c(\delta) \int_{t_0}^T \|v\|^2 dt + \delta \int_{t_0}^T \|D^{3/2}v\|^2 dt. \end{aligned} \quad (2.16)$$

Using the definition (1.7) of G produces

$$\begin{aligned} \int_{t_0}^T \|D^{3/2}(gv)\|^2 dt &\leq \int_{t_0}^T \|D^{3/2}(Gv)\|^2 dt + \int_{t_0}^T \left| \int g(y)v(y, t) dy \right|^2 \|D^{3/2}g\|^2 dt \\ &\leq \int_{t_0}^T \|D^{3/2}(Gv)\|^2 dt + c \int_{t_0}^T \|v\|^2 dt. \end{aligned} \quad (2.17)$$

Combining (2.16) and (2.17), then applying (2.12) we have

$$\begin{aligned} M_1 &\leq c \int_{t_0}^T \|D^{3/2}(Gv)\|^2 dt + c(\delta) \int_{t_0}^T \|v\|^2 dt + \delta \int_{t_0}^T \|D^{3/2}v\|^2 dt \\ &\leq c\|v_0\|^2 + c \int_0^T \|v\|_{3/2} \|F\|_{-3/2} dt + c(\delta) \int_{t_0}^T \|v\|^2 dt + \delta \int_{t_0}^T \|D^{3/2}v\|^2 dt \end{aligned} \quad (2.18)$$

for any $\delta > 0$ and with $c, c(\delta)$ independent of ϵ and T .

Because v has mean value zero, $D = \mathcal{H}\partial_x$ and

$$M_2 = \left| \int_{t_0}^T \int \partial_x \phi (\partial_x^2 D^{-1/2}v)^2 dx dt \right|. \quad (2.19)$$

Taking $w = D^{-1/2}v$ in (2.8), integrating in time and applying the Sobolev embedding

$$\begin{aligned} M_2 &\leq c(\|v(t_0)\|^2 + \|v(T)\|^2) + \left| \int_{t_0}^T \int D^{3/2}(Gw)D^{3/2}G(\phi w) dx dt \right| \\ &\quad + \left| \int_{t_0}^T \int wEw\phi dx dt \right| + c\epsilon \int_{t_0}^T \|v\|^2 + \|D^{5/2}v\|^2 dt + c(\delta) \int_{t_0}^T \|v\|^2 dt \\ &\quad + \delta \int_{t_0}^T \|D^{3/2}v\|^2 dt + c \int_{t_0}^T \|v\|_{3/2} \|F\|_{-3/2} dt. \end{aligned} \quad (2.20)$$

Assuming $0 < \epsilon < 1$, then applying (2.12) produces

$$M_2 \leq c\|v_0\|^2 + c \int_0^T \|v\|_{3/2} \|F\|_{-3/2} dt + c(\delta) \int_{t_0}^T \|v\|^2 dt + \delta \int_{t_0}^T \|D^{3/2}v\|^2 dt + M_{21} + M_{22}, \quad (2.21)$$

where M_{21} and M_{22} are subsequently defined and estimated. First note,

$$\begin{aligned} M_{21} &:= \left| \int_{t_0}^T \int D^{3/2}(Gw) D^{3/2}G(\phi w) dx dt \right| \leq \frac{1}{2} \int_{t_0}^T \|D^{3/2}(Gw)\|^2 + \|D^{3/2}G(\phi w)\|^2 dt \\ &\leq c \int_{t_0}^T \|D^{3/2}(Gw)\|^2 dt + c \int_{t_0}^T \|v\|^2 dt, \end{aligned} \quad (2.22)$$

using identity (2.3), the commutator estimate (2.2) and the Sobolev embedding. As G is bounded on $H_0^s(\mathbb{T})$, applying the commutator estimate (2.5) yields

$$\begin{aligned} \int_{t_0}^T \|D^{3/2}(Gw)\|^2 dt &\leq 2 \int_{t_0}^T \|GD^{3/2}w\|^2 + \|[D^{3/2}; G]w\|^2 dt \\ &\leq c(\delta) \int_{t_0}^T \|v\|^2 dt + \delta \int_{t_0}^T \|D^{3/2}v\|^2 dt, \end{aligned} \quad (2.23)$$

and so

$$M_{21} \leq c(\delta) \int_{t_0}^T \|v\|^2 dt + \delta \int_{t_0}^T \|D^{3/2}v\|^2 dt. \quad (2.24)$$

Recalling the definition (2.7) of E

$$\begin{aligned} M_{22} &:= \left| \int_{t_0}^T \int w E w \phi dx dt \right| \leq \left| \int_{t_0}^T (\phi w, GD^3[D^{-1/2}; G]D^{1/2}w) dt \right| + \left| \int_{t_0}^T (\phi w, [D^{-1/2}; G]D^3GD^{1/2}w) dt \right| \\ &=: M_{221} + M_{222}. \end{aligned} \quad (2.25)$$

Using the commutator estimate (2.5), (2.22) and (2.23) yields

$$\begin{aligned} M_{221} &= \left| \int_{t_0}^T (D^{3/2}G(\phi w), D^{3/2}[D^{-1/2}; G]v) dt \right| \\ &\leq \int_{t_0}^T \|D^{3/2}G(\phi w)\| \|D^{3/2}[D^{-1/2}; G]v\| dt \\ &\leq c(\delta) \int_{t_0}^T \|v\|^2 dt + \delta \int_{t_0}^T \|D^{3/2}v\|^2 dt. \end{aligned} \quad (2.26)$$

Using the identity (2.4) with $r = 3$ and $f = Gv$ produces

$$\begin{aligned}
M_{222} &\leq \left| \int_{t_0}^T (\phi w, [D^{-1/2}; g] D^3 Gv) dt \right| + c \int_{t_0}^T \|v\|^2 dt \\
&\leq \int_{t_0}^T \|D^{3/2} [D^{-1/2}; g] \phi w\| \|D^{3/2}(Gv)\| dt + c \int_{t_0}^T \|v\|^2 dt \\
&\leq c \int_{t_0}^T \|v\|^2 dt + \frac{1}{2} \int_{t_0}^T \|D^{3/2}(Gv)\|^2 dt,
\end{aligned} \tag{2.27}$$

after utilizing the commutator estimate (2.2). Collecting (2.18)–(2.27), then applying (2.12) we have

$$\begin{aligned}
\int_{t_0}^T \|D^{3/2}v\|^2 dt &\leq c \int_{t_0}^T \|D^{3/2}(Gv)\|^2 dt + c \int_0^T \|v\|_{3/2} \|F\|_{-3/2} dt + c(\delta) \int_{t_0}^T \|v\|^2 dt + \delta \int_{t_0}^T \|D^{3/2}v\|^2 dt \\
&\leq c \|v_0\|^2 + c \int_0^T \|v\|_{3/2} \|F\|_{-3/2} dt + c(\delta) \int_{t_0}^T \|v\|^2 dt + \delta \int_{t_0}^T \|D^{3/2}v\|^2 dt
\end{aligned} \tag{2.28}$$

for any $\delta > 0$ and with $c, c(\delta)$ independent of ϵ and T . Thus fixing $\delta = 1/2$ and taking the limit $t_0 \rightarrow 0$ produces

$$\int_0^T \|D^{3/2}v\|^2 dt \leq c(T) \left(\|v_0\|^2 + \int_0^T \|v\|_{3/2} \|F\|_{-3/2} dt \right), \tag{2.29}$$

after again applying (2.12), for some $c(T) > 0$ nondecreasing in T and independent of $0 < \epsilon < 1$. Adding this to (2.12),

$$\begin{aligned}
\|v\|_{L^\infty(0,T;H^0(\mathbb{T}))}^2 + \int_0^T \|D^{3/2}v\|^2 dt &\leq c(T) \left(\|v_0\|^2 + \int_0^T \|v\|_{3/2} \|F\|_{-3/2} dt \right) \\
&\leq c(T) \left(\|v_0\|^2 + \|F\|_{L^2(0,T;H^{-3/2}(\mathbb{T}))}^2 \right) + \frac{1}{2} \int_0^T \|D^{3/2}v\|^2 dt.
\end{aligned} \tag{2.30}$$

The result holds for $v_0 \in H_0^0(\mathbb{T})$ and $F \in L^2(0, T; H^{-3/2}(\mathbb{T}))$ by density.

(General case $s \neq 0$). Again assume $v_0 \in H_0^{s+5}(\mathbb{T})$ and $F \in C([0, T]; H_0^{s+5}(\mathbb{T}))$ to justify what follows. Applying D^s to the equation (2.6) and scaling by $w = D^s v$ yields

$$\begin{aligned}
\|w\|_{L^\infty(0,T;H^0(\mathbb{T}))}^2 + \int_0^T \|D^{3/2}(Gw)\|^2 dt + \epsilon \int_0^T \|D^{5/2}w\|^2 dt \\
\leq \|w_0\|^2 + 2 \int_0^T \|w\|_{3/2} \|F\|_{s-3/2} dt + 2 \left| \int_0^T (w, Ew) dt \right|.
\end{aligned} \tag{2.31}$$

Recalling the definition (2.7) of E we write

$$\begin{aligned}
N_1 &= \left| \int_0^T \int w E w dx dt \right| \leq \left| \int_0^T (w, GD^3[D^s; G]D^{-s}w) dt \right| + \left| \int_0^T (w, [D^s; G]D^3GD^{-s}w) dt \right| \\
&=: N_{11} + N_{12}.
\end{aligned} \tag{2.32}$$

Proceeding as in (2.26),

$$\begin{aligned} N_{11} &= \left| \int_0^T (D^{3/2}(Gw), D^{3/2}[D^s; G]D^{-s}w) dt \right| \leq \int_0^T \|D^{3/2}(Gw)\| \|D^{3/2}[D^s; G]D^{-s}w\| dt \\ &\leq \frac{1}{2} \int_0^T \|D^{3/2}(Gw)\|^2 dt + c \int_0^T \|D^{1/2}w\|^2 dt. \end{aligned} \quad (2.33)$$

and as in (2.27)

$$\begin{aligned} N_{12} &\leq \left| \int_0^T (w, [D^s; g]D^3GD^{-s}w) dt \right| + c \int_0^T \|w\|^2 dt \\ &\leq \int_0^T \|D^{2-s}[D^s; g]w\| \|D^{1+s}GD^{-s}w\|^2 dt + c \int_0^T \|w\|^2 dt \\ &\leq c \int_0^T \|Dw\|^2 dt. \end{aligned} \quad (2.34)$$

Collecting (2.32)–(2.34) we have

$$\begin{aligned} \|w\|_{L^\infty(0,T); H^0(\mathbb{T})}^2 &+ \frac{1}{2} \int_0^T \|D^{3/2}(Gw)\|^2 dt + \epsilon \int_0^T \|D^{5/2}w\|^2 dt \\ &\leq \|w_0\|^2 + c \int_0^T \|Dw\|^2 dt + 2 \int_0^T \|w\|_{3/2} \|F\|_{s-3/2} dt. \end{aligned} \quad (2.35)$$

The same propagation of regularity argument as in the $s = 0$ case reveals

$$\int_0^T \|D^{3/2}w\|^2 dt \leq c(T) \left(\|w_0\|^2 + \int_0^T \|w\|_{3/2} \|F\|_{s-3/2} dt \right), \quad (2.36)$$

for some $c(T) > 0$ nondecreasing in T and independent of $0 < \epsilon < 1$. Combining (2.35) and (2.36), a density argument shows the estimate (2.11) holds for any $s > 0$. \square

Solutions to the IVP (2.6) are obtained via semigroup theory by writing $L_\epsilon = A + B$ where

$$A := \epsilon D^5 - \partial_x^5 \quad \text{and} \quad B := GD^3G.$$

A perturbation argument shows that L_ϵ is sectorial.

Proposition 2.4. *Let $\epsilon > 0$. The operator L_ϵ is sectorial in $H_0^0(\mathbb{T})$ and thus $-L_\epsilon$ generates an analytic semigroup denoted $\{\mathcal{S}_\epsilon(t)\}_{t \geq 0}$. Moreover, this semigroup acts on $H_0^s(\mathbb{T})$ for any $s \geq 0$.*

Proof. The operator A has domain $\mathcal{D}(A) = H_0^5(\mathbb{T}) \subset H_0^0(\mathbb{T})$. Fixing $\theta \in (\arctan \epsilon^{-1}, \pi)$, it is clear that the sector

$$S_\theta = \{\lambda : \theta < |\arg \lambda| \leq \pi, \lambda \neq 0\},$$

lies in its resolvent. Moreover, there exists $C > 0$ so that for any $\lambda \in S_\theta$,

$$\|(A - \lambda)^{-1}\| \leq \sup_{k \neq 0} |(\epsilon - i)k^5 - \lambda|^{-1} \leq \frac{C}{|\lambda|}.$$

Thus A is sectorial in $H_0^0(\mathbb{T})$ [18], Definition 1.3.1.

Observe that $\sigma(A) = \{(\epsilon - i)k^5 : k \in \mathbb{Z}^*\}$ so that $\operatorname{Re} \sigma(A) \geq \epsilon$. Therefore, $A^{-\omega}$ is defined for all $\omega > 0$ and, in particular,

$$\|BA^{-3/5}f\|^2 \leq c\|A^{-3/5}f\|_{H_0^3(\mathbb{T})}^2 \leq c \sum_{k \neq 0} (1+k^2)^3 |(\epsilon - i)k^5|^{-6/5} |\hat{f}_k|^2 \leq c\|f\|^2.$$

It follows that $L_\epsilon = A + B$ is a sectorial operator on $H_0^0(\mathbb{T})$ [18], Corollary 1.4.5. Therefore $-L_\epsilon$ generates an analytic semigroup $\{\mathcal{S}_\epsilon(t)\}_{t \geq 0}$ on $H_0^0(\mathbb{T})$ [18], Theorem 1.3.4. Using [18], Theorem 1.4.8, we can compute explicitly $\mathcal{D}((A + B + \lambda)^\beta) = \mathcal{D}(A^\beta) = H_0^{5\beta}(\mathbb{T})$ for all $\beta \geq 0$ and $\lambda > 0$ large enough, hence for all $t > 0$ and $s \geq 0$,

$$\mathcal{S}_\epsilon(t)H_0^s(\mathbb{T}) \subset H_0^s(\mathbb{T}),$$

as desired. \square

The following unique continuation principle leads to exponential stability and exact control results for solutions to IVP (2.6).

Proposition 2.5. *Let $c \in L^2(0, T)$ and $v \in L^2(0, T; H_0^0(\mathbb{T}))$ be such that*

$$\begin{cases} \partial_t v + (\epsilon D^5 - \partial_x^5)v = 0, & \text{in } \mathbb{T} \times (0, T) \\ v(x, t) = c(t) & \text{for a.e. } (x, t) \in (a, b) \times (0, T), \end{cases} \quad (2.37)$$

for some numbers $T > 0$ and $0 \leq a < b \leq 2\pi$. Then $v \equiv 0$ for a.e. $(x, t) \in \mathbb{T} \times (0, T)$.

Proof. (Case $\epsilon = 0$). By assumption, $\partial_x^2 v = 0$ a.e. in $(a, b) \times (0, T)$ and so a propagation of regularity argument as in Proposition 2.3 implies $v \in L^2(0, T; H_0^2(\mathbb{T}))$. Thus for every $\delta > 0$, there exists $0 < t < \delta$ such that $v(t) \in H_0^2(\mathbb{T})$.

In fact $\partial_x^k v = 0$ a.e. in $(a, b) \times (0, T)$ for every $k \in \mathbb{Z}^+$. Repeating the above argument and using the equation we conclude that $v \in C^\infty((0, T) \times \mathbb{T})$. The unique continuation property now follows from the result in [48].

(Case $\epsilon > 0$). From (2.37) it follows that for a.e. $(x, t) \in (a, b) \times (0, T)$,

$$\partial_t v = (1 + \epsilon \mathcal{H})\partial_x^5 v = c'(t).$$

Moreover, for a.e. $t \in (0, T)$

$$\begin{aligned} \partial_x^6 v(\cdot, t) &\in H^{-6}(\mathbb{T}), \\ \partial_x^6 v(\cdot, t) &= 0, \\ \mathcal{H}\partial_x^6 v(\cdot, t) &= 0, \end{aligned}$$

since $\epsilon > 0$. Picking such a $t \in (0, T)$ and setting $w(\cdot) = \partial_x^6 v(\cdot, t)$, write

$$w = \sum_{k \in \mathbb{Z}} \hat{w}_k e^{ikx},$$

where the convergence occurs in $H^{-6}(\mathbb{T})$. Now

$$0 = iw - \mathcal{H}w = 2i \sum_{k > 0} \hat{w}_k e^{ikx}.$$

As w is real, we use the following result.

Lemma 2.6 ([39], Lem. 2.9). *Let $s \in \mathbb{R}$ and $w(x) = \sum_{k \geq 0} \widehat{w}_k e^{ikx} \in H^s(\mathbb{T})$ and $w = 0$ a.e. (a, b) . Then $w \equiv 0$ in \mathbb{T} .*

Thus $\partial_x^6 v \equiv 0$ in \mathbb{T} which implies that $v(x, t) = c(t)$ a.e. $\mathbb{T} \times (0, T)$. Furthermore, $(1 + \epsilon \mathcal{H})\partial_x^6 v = c_t(t) = 0$ and since v has mean value zero we have shown the desired outcome that $v \equiv 0$ in $\mathbb{T} \times (0, T)$. \square

The above propositions imply an observability inequality leading to the following result.

Proposition 2.7. *Let $0 < \epsilon < 1$, $s \geq 0$. There exists constants $C, \lambda > 0$ independent of ϵ such that*

$$\|\mathcal{S}_\epsilon(t)v_0\|_s \leq C e^{-\lambda t} \|v_0\|_s,$$

for all $v_0 \in H_0^s(\mathbb{T})$.

Proof. (Case $s = 0$). Setting $F \equiv 0$ in (2.6) and scaling by v yields for any $T > 0$

$$\frac{1}{2} \|v(T)\|^2 + \epsilon \int_0^T \|D^{5/2}v\|^2 d\tau + \int_0^T \|D^{3/2}(Gv)\|^2 d\tau = \frac{1}{2} \|v_0\|^2, \quad (2.38)$$

and so stability follows from the observability inequality

$$\|v_0\|^2 \leq c \left(\epsilon \int_0^T \|D^{5/2}v\|^2 dt + \int_0^T \|D^{3/2}(Gv)(t)\|^2 dt \right). \quad (2.39)$$

For the sake of a contradiction, suppose that (2.39) fails. Then there is a sequence $\{v_0^n\}_{n=1}^\infty \subset H_0^0(\mathbb{T})$, (up to scaling) such that

$$1 = \|v_0^n\|^2 > n \left(\epsilon \int_0^T \|D^{5/2}v^n\|^2 dt + \int_0^T \|D^{3/2}(Gv^n)(t)\|^2 dt \right), \quad (2.40)$$

with v^n denoting the solution to (2.6) corresponding to data v_0^n . For any $\delta > 0$, denote $\gamma = -\frac{7}{2} - \delta$. Applying the Sobolev embedding,

$$\|(\epsilon D^5 - \partial_x^5)v^n\|_{L^2(0,T;H^\gamma(\mathbb{T}))} \leq c \|v^n\|_{L^2(0,T;H^{3/2}(\mathbb{T}))},$$

which is uniformly bounded by the estimates (2.11). Using commutator estimates, the Sobolev embedding and the fact that G is bounded on $H_0^0(\mathbb{T})$,

$$\begin{aligned} \|D^\gamma G(D^3 Gv^n)\| &\leq \|G(D^{3+\gamma} Gv^n)\| + \|[D^\gamma, G]D^3 Gv^n\| \\ &\leq c(\|Gv^n\|_{3+\gamma} + \|Gv^n\|_{2+\gamma}) \\ &\leq c\|Gv^n\|_{3/2}. \end{aligned}$$

Consequently,

$$\|GD^3 Gv^n\|_{L^2(0,T;H^\gamma(\mathbb{T}))}^2 \leq c \int_0^T \|D^{3/2}(Gv^n)\|^2 dt \leq C,$$

using (2.38). Combining these estimates and using the equation produces

$$\|v_t^n\|_{L^2(0,T;H^\gamma(\mathbb{T}))} \leq C,$$

for some $C > 0$ independent of n . Note that $\{v_t^n\}$ is bounded in $L^2(0, T; H^\gamma(\mathbb{T}))$ and, from (2.11), the sequence $\{v^n\}$ is bounded in $L^2(0, T; H^{3/2}(\mathbb{T}))$. Applying the Banach-Alaoglu theorem and the Aubin-Lions lemma, we obtain a subsequence with the following properties:

$$\begin{aligned} v^n &\rightharpoonup v && \text{in } L^2(0, T; H^\beta(\mathbb{T})), && \forall \beta < 3/2, \\ v^n &\rightharpoonup v && \text{in } L^2(0, T; H^{3/2}(\mathbb{T})) \text{ weak}, \\ v^n &\rightharpoonup v && \text{in } L^\infty(0, T; L^2(\mathbb{T})) \text{ weak*}, \end{aligned}$$

where $v \in L^2(0, T; H_0^\beta(\mathbb{T})) \cap L^\infty(0, T; L^2(\mathbb{T}))$. In particular, taking $\beta = 0$

$$(v^n)^2 \rightarrow v^2 \quad \text{in } L^1(\mathbb{T} \times (0, T)).$$

Letting $n \rightarrow \infty$ in (2.40) we have that

$$\int_0^T \|D^{3/2}(Gv)\|^2 dt = 0.$$

Hence $Gv = 0$ a.e. $\mathbb{T} \times (0, T)$ and using (1.7) we may write

$$v(x, t) = \int_{\mathbb{T}} g(y)v(y, t) dy := c(t) \quad \text{for all } (x, t) \in \omega \times (0, T),$$

where $\omega = \{x \in \mathbb{T} : g(x) > 0\}$ and $c \in L^\infty(0, T)$. Thus v satisfies the hypothesis of Proposition 2.5, implying that $v \equiv 0$ and contradicting the fact that $\|v^n(0)\| = \|v_0^n\| = 1$.

(Case $s = 5$). Assume $v_0 \in H_0^5(\mathbb{T})$ and denote $v(t) = \mathcal{S}(t)v_0$. Let $w = \partial_t v$, which solves

$$\begin{cases} \partial_t w + (\epsilon D^5 - \partial_x^5)w + GD^3Gw = 0, & x \in \mathbb{T}, t \geq 0, \\ w(x, 0) = w_0(x) := (\epsilon D^5 - \partial_x^5 + GD^3G)v_0, \end{cases}$$

and so by the $s = 0$ case

$$\|w(t)\| = \|\mathcal{S}(t)w_0\| \leq Ce^{-\lambda t}\|w_0\|.$$

Using the equation (2.6) and the previous estimate

$$\begin{aligned} \|D^5v\| &\leq C\|(\epsilon D^5 - \partial_x^5)v\| \\ &\leq C\|\partial_t v + GD^3Gv\| \\ &\leq (Ce^{-\lambda t}\|w_0\| + c(\delta)\|v\|) + \delta\|D^5v\|, \end{aligned}$$

for any $\delta > 0$. Choosing $\delta > 0$ small enough,

$$\|D^5v\| \leq (1 - \delta)^{-1} (Ce^{-\lambda t}\|v_0\|_5 + c(\delta)e^{-\lambda t}\|v_0\|) \leq Ce^{-\lambda t}\|v_0\|_5,$$

where $\lambda > 0$ is as in the case $s = 0$. Interpolating produces the desired result for $0 \leq s \leq 5$, with the case of $s > 5$ following by induction. Thus the constant appearing above will be nondecreasing in s . \square

We now establish solutions as $\epsilon \searrow 0$ using a Bona-Smith argument [3]. The resulting homogeneous solutions to (2.41) will be denoted $\mathcal{S}(t)v_0$.

Proposition 2.8. *Fix $s \in \mathbb{R}$ and $T > 0$. Let $v_0 \in H_0^s(\mathbb{T})$ and $F \in L^2(0, T; H_0^{s-3/2}(\mathbb{T}))$. Then there exists a unique solution $v \in Z_{s, T}$ to the IVP*

$$\begin{cases} \partial_t v - \partial_x^5 v + GD^3 Gv = F, & x \in \mathbb{T}, t \geq 0, \\ v(x, 0) = v_0(x) \end{cases} \quad (2.41)$$

satisfying

$$\|v\|_{s, T} \leq c(s, T) (\|v_0\|_s + \|F\|_{L^2(0, T; H^{s-3/2}(\mathbb{T}))}) \quad (2.42)$$

with $c(s, T)$ nondecreasing in T . Moreover, there exists constants $C, \lambda > 0$, $C = C(s)$, such that

$$\|\mathcal{S}(t)v_0\|_s \leq Ce^{-\lambda t} \|v_0\|_s \quad (2.43)$$

for all $v_0 \in H_0^s(\mathbb{T})$, $s \geq 0$.

Proof. We follow the argument of Bona and Smith to establish existence of solutions to the IVP (2.41). Define the regularization

$$\widehat{v}_0^\epsilon(k) = \exp(-\epsilon^{1/10} k^2) \widehat{v}_0(k) \quad (2.44)$$

and observe that $v_0^\epsilon \in H^\infty(\mathbb{T})$ and for ϵ sufficiently small

$$\epsilon^{\gamma/10} \|v_0^\epsilon\|_{s+\gamma} \leq c(\gamma) \|v_0\|_s \quad (2.45)$$

for any $\gamma > 0$. Let $\{\epsilon_n\} \subset (0, 1)$ be a monotonic sequence satisfying $\epsilon^n \searrow 0$ and denote $v_0^n = v_0^{\epsilon^n}$. Observe that $v_0^n \rightarrow v_0$ strongly in $H^s(\mathbb{T})$. Let $F^n \in C([0, T]; H^\infty(\mathbb{T}))$ be a sequence converging strongly to F in $L^2(0, T; H^{s-3/2}(\mathbb{T}))$. Let v^n be the associated solution to the IVP

$$\partial_t v^n + L_{\epsilon^n} v^n = F^n, \quad v^n(0) = v_0^n \quad (2.46)$$

provided by Proposition 2.3.

We now demonstrate that the sequence $\{v^n\}$ is Cauchy in $Z_{s, T}$ by considering

$$\|v^n - v^m\|_{s, T}$$

assuming $n \leq m$ (so that $0 < \epsilon^m < \epsilon^n$). The difference $w = v^n - v^m$ is a smooth solution to

$$\partial_t w + L_{\epsilon^m} w + (\epsilon^m - \epsilon^n) v^m = F^n - F^m, \quad w(0) = v_0^n - v_0^m. \quad (2.47)$$

Thus taking

$$F := F^n - F^m - (\epsilon^n - \epsilon^m) D^5 v^n$$

in (2.11) produces

$$\|w\|_{s, T} \leq c(s, T) (\|w_0\|_s^2 + \|F^n - F^m\|_{L^2(0, T; H^{s-3/2}(\mathbb{T}))}) + (\epsilon^n - \epsilon^m) c(s, T) \|D^{s+5} v^n\|_{L^2(0, T; H^{s-3/2}(\mathbb{T}))}. \quad (2.48)$$

Applying (2.11) to v^n

$$\begin{aligned}
(\epsilon^n - \epsilon^m) \|D^{s+5} v^n\|_{L^2(0,T;H^{s-3/2}(\mathbb{T}))} &\leq \epsilon^n \|v^n\|_{s+2,T} \\
&\leq (\epsilon^n)^{3/10} c(s,T) \left((\epsilon^n)^{7/20} \|v_0^n\|_{s+2} + (\epsilon^n)^{7/20} \|F^n\|_{L^2(0,T;H^{s+2}(\mathbb{T}))} \right) \\
&\leq (\epsilon^n)^{3/10} c(s,T) (\|v_0\|_s + \|F\|_{L^2(0,T;H^{s-3/2}(\mathbb{T}))}), \tag{2.49}
\end{aligned}$$

where we utilized (2.45) with $\gamma = 7/2$, $\epsilon^n - \epsilon^m \leq \epsilon^n < 1$ and that $F^n \rightarrow F$ strongly. Inserting (2.49) into (2.48) yields

$$\begin{aligned}
\|w\|_{s,T} &\leq c(s,T) (\|w_0\|_s + \|F^n - F^m\|_{L^2(0,T;H^{s-3/2}(\mathbb{T}))}) \\
&\quad + (\epsilon^n)^{3/10} c(s,T) (\|v_0\|_s + \|F\|_{L^2(0,T;H^{s-3/2}(\mathbb{T}))}).
\end{aligned}$$

This proves that $\{v^n\}$ is Cauchy in $Z_{s,T}$, thus $v^n \rightarrow v$ for some $v \in Z_{s,T}$. Choosing n large enough,

$$\begin{aligned}
\|v\|_{s,T} &\leq \|v^n\|_{s,T} + \delta \\
&\leq c(s,T) (\|v_0^n\|_s + \|F^n\|_{L^2(0,T;H^{s-3/2}(\mathbb{T}))}) + \delta \\
&\leq c(s,T) (\|v_0\|_s + \|F\|_{L^2(0,T;H^{s-3/2}(\mathbb{T}))}) + 3\delta,
\end{aligned}$$

and so v satisfies (2.42). Moreover, v is a distributional solution of IVP (2.41) with $v(\cdot, t) \rightarrow v_0$ strongly in $H^s(\mathbb{T})$ as $t \rightarrow 0$. Uniqueness and continuous dependence on the initial data follow easily from (2.42). Finally, (2.43) holds as the results of Proposition 2.7 are independent of $0 < \epsilon < 1$. \square

3. EXPONENTIAL STABILIZATION

This section is concerned with local well-posedness and stabilization of solutions to the following nonlinear equation

$$\begin{cases} \partial_t u - \partial_x^5 u + u \partial_x^3 u = -GD^3 G u, & x \in \mathbb{T}, t \geq 0, \\ u(x, 0) = u_0(x). \end{cases} \tag{3.1}$$

The linear estimates given in Proposition 2.8, when combined with the contraction principle yield local well-posedness for small data in $H_0^s(\mathbb{T})$ for $s > 2$.

Theorem 3.1. *Suppose $s > 2$ and $T > 0$. Then there exists $\rho = \rho(s, T) > 0$ such that for any $u_0 \in H_0^s(\mathbb{T})$ with $\|u_0\|_s \leq \rho$, the IVP (3.1) admits a unique solution in the space $Z_{s,T}$.*

Proof. We write (3.1) in the integral form

$$u(t) = \mathcal{S}(t)u_0 - \int_0^t \mathcal{S}(t-t')(u \partial_x^3 u)(t') dt' =: \Gamma(u)$$

and show that Γ defines a contraction on $B = \{v \in Z_{s,T} : \|v\|_{s,T} \leq R\}$ for appropriate choices of $R > 0$ and $\rho > 0$. Note that the restriction $s > 2$ ensures that $H^{s-3/2}(\mathbb{T})$ forms a Banach algebra. The estimate (2.11) yields

$$\|\Gamma(u)\|_{s,T} \leq c(s,T) (\|u_0\|_s + \|u \partial_x^3 u\|_{L^2(0,T;H^{s-3/2}(\mathbb{T}))}).$$

Assuming $u \in Z_{s,T}$, then

$$\begin{aligned} \int_0^T \|u \partial_x^3 u\|_{s-3/2}^2 dt &\leq c \int_0^T (\|u\|_{s-3/2} \|\partial_x^3 u\|_{s-3/2})^2 dt \\ &\leq c \int_0^T \|u\|_s^2 \|u\|_{s+3/2}^2 dt \\ &\leq c \|u\|_{L^\infty(0,T;H^s(\mathbb{T}))}^2 \|u\|_{L^2(0,T;H^{s+3/2}(\mathbb{T}))}^2 \\ &\leq c \|u\|_{s,T}^4. \end{aligned}$$

Therefore

$$\|\Gamma(u)\|_{s,T} \leq C_0 \|u_0\|_s + C_1 \|u\|_{s,T}^2$$

for some $C_0, C_1 > 0$ (which depend on T through estimate (2.11)). Next, assuming $u, v \in Z_{s,T}$ and writing

$$u \partial_x^3 u - v \partial_x^3 v = (u \partial_x^3 u - v \partial_x^3 u) + (v \partial_x^3 u - v \partial_x^3 v),$$

the same estimates as above reveal

$$\|\Gamma(u) - \Gamma(v)\|_{s,T} \leq C_1 (\|u\|_{s,T} + \|v\|_{s,T}) \|u - v\|_{s,T}.$$

Thus Γ forms a contraction on B provided

$$C_0 \|u_0\|_s + C_1 R^2 < R \quad \text{and} \quad 2C_1 R < 1.$$

It is sufficient to take

$$R = (4C_1)^{-1} \quad \text{and} \quad \|u_0\|_s \leq \rho := R/2C_0.$$

□

Following [39], the contraction principle is also used to establish local exponential stability of the solutions to the IVP (3.1). However, the estimates in Proposition 2.8 incorporate only the regularizing effects of the control term and not any stabilization. As a result, the H^s -estimates (2.42) possibly grow in time. This artifact is avoided by restricting (2.42) to (at most) unit length time intervals through use of the spaces

$$Z_{s,T}([n, n+1]) := C([n, n+1]; H_0^s(\mathbb{T})) \cap L^2(n, n+1; H_0^{s+3/2}(\mathbb{T}))$$

endowed with the norm

$$\| \|u\|_n := \|u\|_{L^\infty(n, n+1; H^s(\mathbb{T}))} + \|u\|_{L^2(n, n+1; H^{s+3/2}(\mathbb{T}))}.$$

Proposition 2.8 leads to the following linear estimates.

Proposition 3.2. *Let $0 \leq s \leq 5$ and $v_0 \in H_0^s(\mathbb{T})$. Then for some $\lambda, \tilde{c}_0, \tilde{c}_1 > 0$ independent of s and t ,*

$$\| \mathcal{S}(t)u_0 \|_n \leq \tilde{c}_0 e^{-\lambda t} \|u_0\|_s \tag{3.2}$$

and

$$\left\| \int_0^t \mathcal{S}(t-t')F(t') dt' \right\|_n \leq \tilde{c}_1 \left(\|F\|_{L^2(n,n+1;H^{s-3/2}(\mathbb{T}))} + \sum_{k=1}^n e^{-\lambda(n-k)} \|F\|_{L^2(k-1,k;H^{s-3/2}(\mathbb{T}))} \right). \quad (3.3)$$

Proof. From (2.43)

$$\|\mathcal{S}(n)u_0\|_s \leq ce^{-\lambda n}\|u_0\|_s$$

so that, combined with (2.42),

$$\|\mathcal{S}(t)u_0\|_n \leq \tilde{c}_0 e^{-\lambda n} \|u_0\|_s.$$

For the inhomogeneous estimates, write,

$$\begin{aligned} \left\| \int_0^t \mathcal{S}(t-t')F(t') dt' \right\|_n &\leq \left\| \mathcal{S}(t-n) \int_0^n \mathcal{S}(n-t')F(t') dt' \right\|_n + \left\| \int_n^t \mathcal{S}(t-t')F(t') dt' \right\|_n \\ &=: I_1 + I_2. \end{aligned}$$

Applying (2.42) and (2.43) repeatedly over the time intervals $[0, 1], [1, 2], \dots, [n-1, n], [n, t]$ yields, since $n \leq t < n+1$,

$$\begin{aligned} I_1 &\leq c \left\| \int_0^n \mathcal{S}(n-t')F(t') dt' \right\|_s \leq c \sum_{k=1}^n \left\| \mathcal{S}(n-k) \int_{k-1}^k \mathcal{S}(k-t')F(t') dt' \right\|_s \\ &\leq c \cdot \tilde{c}_0 \sum_{k=1}^n e^{-\lambda(n-k)} \left\| \int_{k-1}^k \mathcal{S}(k-t')F(t') dt' \right\|_s \leq \tilde{c}_1 \sum_{k=1}^n e^{-\lambda(n-k)} \|F\|_{L^2(k-1,k;H^{s-3/2}(\mathbb{T}))}, \end{aligned}$$

where we used (3.2). Similarly,

$$I_2 \leq \tilde{c}_1 \|F\|_{L^2(n,n+1;H^{s-3/2}(\mathbb{T}))},$$

completing the proof. \square

We now prove Theorem 1.4 under the assumption $[u_0] = 0$.

Proof. We proceed via the contraction principle in the Banach space

$$X := \{u \in C(\mathbb{R}^+; H_0^s(\mathbb{T})) \cap L_{\text{loc}}^2(\mathbb{R}^+; H_0^{s+3/2}(\mathbb{T})) : \|u\|_X < \infty\}$$

where

$$\|u\|_X := \sup_{n \geq 0} \{e^{n\lambda} \|u\|_n\}.$$

Writing (4.1) in the integral form

$$u(t) = \mathcal{S}(t)u_0 - \int_0^t \mathcal{S}(t-t')(u\partial_x^3 u)(t') dt' =: \Gamma(u),$$

then (3.2) and (3.3) produce

$$\| \Gamma(u) \|_X \leq \tilde{c}_0 \|u_0\|_s + \sup_{n \geq 0} \tilde{c}_1 e^{n\lambda} \left\{ \begin{aligned} & \|u \partial_x^3 u\|_{L^2(n, n+1; \mathbb{H}^{s-3/2}(\mathbb{T}))} \\ & + \sum_{k=1}^n e^{-\lambda(n-k)} \|u \partial_x^3 u\|_{L^2(k-1, k; \mathbb{H}^{s-3/2}(\mathbb{T}))} \end{aligned} \right\}. \quad (3.4)$$

Using the algebra property of $\mathbb{H}^{s-3/2}(\mathbb{T})$,

$$\begin{aligned} \int_{k-1}^k \|u \partial_x^3 u\|_{s-3/2}^2 dt &\leq c \int_{k-1}^k (\|u\|_{s-3/2} \|\partial_x^3 u\|_{s-3/2})^2 dt \\ &\leq c \int_{k-1}^k \|u\|_s^2 \|u\|_{s+3/2}^2 dt \\ &\leq c \|u\|_{L^\infty(k-1, k; \mathbb{H}^s(\mathbb{T}))}^2 \|u\|_{L^2(k-1, k; \mathbb{H}^{s+3/2}(\mathbb{T}))}^2 \\ &\leq c \|u\|_{k-1}^4. \end{aligned} \quad (3.5)$$

Assuming $u \in E$, inserting (3.5) into (3.4), we have

$$\begin{aligned} \| \Gamma(u) \|_X &\leq \tilde{c}_0 \|u_0\|_s + \sup_{n \geq 0} \tilde{c}_1 e^{n\lambda} \left\{ \|u\|_n^2 + \sum_{k=1}^n e^{-\lambda(n-k)} \|u\|_{k-1}^2 \right\} \\ &\leq \tilde{c}_0 \|u_0\|_s + \tilde{c}_1 \|u\|_X^2 + \tilde{c}_1 \sup_{n \geq 0} \left\{ \sum_{k=1}^n e^{k\lambda} \|u\|_{k-1}^2 \right\} \\ &\leq \tilde{c}_0 \|u_0\|_s + \tilde{c}_1 \|u\|_X^2 + \tilde{c}_1 e^{2\lambda} \|u\|_X^2 \left\{ \sup_{n \geq 0} \sum_{k=1}^n e^{-k\lambda} \right\} \\ &\leq \tilde{c}_0 \|u_0\|_s + \tilde{c}_1(\lambda) \|u\|_X^2. \end{aligned}$$

Similarly, assuming $u, v \in X$,

$$\| \Gamma(u) - \Gamma(v) \|_X \leq \tilde{c}_1 (\|u\|_X + \|v\|_X) \|u - v\|_X.$$

Thus Γ forms a contraction on $B = \{v \in X : \|u\|_X \leq R\}$ provided

$$\tilde{c}_0 \|u_0\|_s + \tilde{c}_1 R^2 < R \quad \text{and} \quad 2\tilde{c}_1 R < 1.$$

It is sufficient to take

$$R = (4\tilde{c}_1)^{-1} \quad \text{and} \quad \|u_0\|_s \leq \rho := R/2\tilde{c}_0.$$

□

Remark 3.3. We now consider the previous two theorems applied to

$$\partial_t u - \partial_x^5 u + c_1 u^2 \partial_x u + c_2 \partial_x u \partial_x^2 u + c_3 u \partial_x^3 u + GD^3 Gu = 0. \quad (3.6)$$

Observe that the nonlinearity $P(u)$ satisfies

$$\int_{\mathbb{T}} P(u) \, dx = 0,$$

and so solutions to (3.6) preserve volume. The restriction $s > 2$ arose from utilizing the algebra property of $H^{s-3/2}(\mathbb{T})$ in estimates of the form

$$\int_0^T \|P(u)\|_{s-3/2}^2 \, dt. \quad (3.7)$$

Hence Theorems 1.4 and 3.1 apply to equation (3.6) with the same technique. In fact, imposing $s > 7/2$ and replacing the algebra property with

$$\|fg\|_s \leq c (\|f\|_{L_x^\infty} \|g\|_s + \|f\|_s \|g\|_{L_x^\infty}),$$

the theorems extend to an even wider family of fifth order models.

Remark 3.4. Next it is shown that Theorems 1.4 and 3.1 apply to a family of equations containing the KdV hierarchy. Following Section 3, we see that for each $l \in \mathbb{Z}^+$, $l \geq 2$, the linear equation

$$\partial_t v + (-1)^{l+1} \partial_x^{2l+1} v = -GD^{2l-1} Gv,$$

possesses a unique solution in the space

$$Z_{s,T}^l = C(0, T; H_0^s(\mathbb{T})) \cap L^2(0, T; H_0^{s+l-1/2}(\mathbb{T})),$$

which decays exponentially in $H_0^s(\mathbb{T})$ for $s \geq 0$. The algebra property holds for $H_0^{s-l+1/2}(\mathbb{T})$ assuming $s > l$, and in this case

$$\int_0^T \|u \partial_x^{2l-1} u\|_{s-l+1/2}^2 \, dt \leq c \|u\|_{L^\infty(0,T; H_0^s(\mathbb{T}))}^2 \|u\|_{L^2(0,T; H_0^{s+l-1/2}(\mathbb{T}))}^2.$$

Therefore the equation

$$\partial_t u + (-1)^{l+1} \partial_x^{2l+1} u + u \partial_x^{2l-1} u = -GD^{2l-1} Gu,$$

is locally well-posed and exponentially stabilizable for small data in $H_0^s(\mathbb{T})$, $s > l$. The nonlinearity $u \partial_x^{2l-1} u$ is the most difficult to control in the following family studied by Kenig and Pilod [26] and Grünrock [15]:

$$\partial_t u + (-1)^{l+1} \partial_x^{2l+1} u + GD^{2l-1} Gu = \sum_{k=2}^{l+1} N_{lk}(u),$$

where

$$N_{lk}(u) = \sum_{|n|=2(l-k)+3} c_{l,k,n} \partial_x^{n_0} \prod_{i=1}^k \partial_x^{n_i} u,$$

with $|n| = \sum_{i=0}^k n_i$, $n_i \in \mathbb{N}$, for $i = 0, \dots, k$ and $c_{l,k,n} \in \mathbb{R}$. Further imposing $n_0 > 0$, this describes a family of volume-preserving equations containing the KdV hierarchy to which we have extended Theorems 1.4 and 3.1.

4. EXACT CONTROLLABILITY

This section is devoted to establishing exact controllability of the equation

$$\partial_t u - \partial_x^5 u + u \partial_x^3 u = Gh,$$

where h is the control input. Following [39], we incorporate dissipation into the control input in order to obtain a suitable smoothing effect. We set

$$h = -D^3 Gu + D^{3/2} k,$$

viewing k as the new control, and focus on the system

$$\partial_t u - \partial_x^5 u + GD^3 Gu + u \partial_x^3 u = GD^{3/2} k, \quad u(0) = u_0. \quad (4.1)$$

We first establish control of the associated linear system in $H^s(\mathbb{T})$ using the Hilbert Uniqueness Method (as in [39, 46]) and then apply the contraction principle to obtain controllability of (4.1).

Proposition 4.1. *Let $s \geq 0$ and $T > 0$. Then for any $v_0, v_T \in H_0^s(\mathbb{T})$, there exists $k \in L^2(0, T; H_0^s(\mathbb{T}))$ such that*

$$\partial_t v - \partial_x^5 v + GD^3 Gv = GD^{3/2} k \quad (4.2)$$

admits a unique solution $v \in Z_{s,T}$ satisfying $v(0) = v_0$ and $v(T) = v_T$.

Proof. (Case $s = 0$.)

Note that for $v_0 \in H_0^s(\mathbb{T})$ and $k \in L^2(0, T; H_0^s(\mathbb{T}))$ the solution to (4.2) lies in $Z_{s,T}$. We associate to this equation the adjoint system

$$-\partial_t u + \partial_x^5 u + GD^3 Gu = 0, \quad u(T) = u_T. \quad (4.3)$$

Assuming $v_0, u_T \in H_0^5(\mathbb{T})$ and $k \in L^2(0, T; H_0^{13/2}(\mathbb{T}))$ to justify the computations, scaling (4.2) by u yields

$$\int_{\mathbb{T}} uv \, dx \Big|_0^T = \int_0^T \int_{\mathbb{T}} k D^{3/2}(Gu) \, dx dt, \quad (4.4)$$

assuming $u_0 = 0$. Hence duality implies that exact controllability of (4.2) follows from an observability inequality

$$\|u_T\|^2 \leq C \int_0^T \|D^{3/2}(Gu)\|^2 dt \quad (4.5)$$

for solutions to (4.3).

Demonstrating (4.5) requires a few properties of these solutions. Note that scaling the adjoint equation (4.3) by tu yields

$$\frac{T}{2} \|u_T\|^2 = \frac{1}{2} \int_0^T \|u(t)\|^2 dt + \int_0^T t \|D^{3/2}(Gu)\|^2 dt. \quad (4.6)$$

Moreover, a propagation of regularity argument similar to Proposition 2.3 (changing t to $T - t$) shows that solutions to (4.3) satisfy

$$\int_0^T \|D^{3/2}u\|^2 dt \leq C(T)\|u_T\|^2. \quad (4.7)$$

We now demonstrate (4.5). Proceeding by contradiction, suppose there is a sequence $\{u_T^n\}$ in $H_0^0(\mathbb{T})$ such that

$$1 = \|u_T^n\|^2 > n \int_0^T \|D^{3/2}(Gu^n)\|^2 dt. \quad (4.8)$$

with u^n denoting the solution to (4.3) corresponding to data u_T^n . Using the equation (4.3) and (4.7), the sequence $\{u^n\}$ is seen to be bounded in

$$L^2(0, T; H_0^{3/2}(\mathbb{T})) \cap H^1(0, T; H_0^0(\mathbb{T})).$$

The Aubin-Lions lemma implies the existence of a subsequence (still denoted u^n) converging strongly to a limit u in $L^2(0, T; H_0^0(\mathbb{T}))$.

Next, we verify that $\{u_T^n\}$ is Cauchy in $H_0^0(\mathbb{T})$. Estimate (4.6) applied to the difference of two solutions yields

$$\begin{aligned} \|u_T^n - u_T^m\| &\leq \frac{1}{T} \int_0^T \|u^n - u^m\|^2 dt + 2 \int_0^T \|D^{3/2}G(u^n - u^m)\|^2 dt \\ &\leq \frac{1}{T} \int_0^T \|u^n - u^m\|^2 dt + 4 \left(\frac{1}{n} + \frac{1}{m} \right) \end{aligned}$$

after applying (4.8). Thus $u_T^n \rightarrow u_T$ strongly in $H_0^0(\mathbb{T})$ and it follows that the solution of (4.3) associated to u_T agrees with the limit u of the sequence $\{u^n\}$. Letting $n \rightarrow \infty$ in (4.8) we have that

$$\int_0^T \|D^{3/2}(Gu)\|^2 dt = 0.$$

Hence $Gu = 0$ a.e. $\mathbb{T} \times (0, T)$ and using (1.7) we may write

$$u(x, t) = \int_{\mathbb{T}} g(y)u(y, t) dy := c(t) \quad \text{for all } (x, t) \in \omega \times (0, T),$$

where $\omega = \{x \in \mathbb{T} : g(x) > 0\}$ and $c \in L^\infty(0, T)$. Thus u satisfies the hypothesis of Proposition 2.5, implying that $u \equiv 0$ and contradicting the fact that $\|u_T\| = \|u_T^n\| = 1$.

(Case $s > 0$). As the IVP (4.3) is well-posed backwards in time, we have

$$\|u\|_{-s, T} \leq c(s, T)\|u_T\|_{-s}. \quad (4.9)$$

As in (4.4), scaling (4.2) by a solution u to (4.3) and supposing $v_0 = 0$, then

$$\langle u_T, v(T) \rangle_{-s, s} = \int_0^T \langle D^{3/2}(Gu), k \rangle_{-s, s} dt, \quad (4.10)$$

where $\langle \cdot, \cdot \rangle_{-s,s}$ denotes the pairing $\langle \cdot, \cdot \rangle_{H_0^{-s}(\mathbb{T}), H_0^s(\mathbb{T})}$. Thus it suffices to prove the following observability inequality

$$\|u_T\|_{-s}^2 \leq C \int_0^T \|D^{3/2}(Gu)\|_{-s}^2 dt \quad (4.11)$$

for solutions u to (4.3). We first show that $w = D^{-s}u$ satisfies

$$\|w_T\|^2 \leq C \left(\int_0^T \|D^{3/2}(Gw)\|^2 dt + \int_0^T \|Ew\|_{-3/2}^2 dt \right) \quad (4.12)$$

where

$$-\partial_t w + \partial_x^5 w + GD^3Gw = Ew, \quad w(T) = w_T := D^{-s}u_T \quad (4.13)$$

and $E := D^{-s}[D^s; GD^3G]$. To obtain a contradiction, suppose there is a sequence $\{w_T^n\}$ in $H_0^0(\mathbb{T})$ such that

$$1 = \|w_T^n\|^2 > n \left(\int_0^T \|D^{3/2}(Gw^n)\|^2 dt + \int_0^T \|Ew^n\|_{-3/2}^2 dt \right), \quad (4.14)$$

with w^n denoting the solution to (4.13) corresponding to $w(T) = w_T^n$. Then (4.9), along with the equation satisfied by w^n , implies that the sequence $\{w^n\}$ is bounded in

$$L^2(0, T; H_0^{3/2}(\mathbb{T})) \cap H^1(0, T; H_0^{-7/2}(\mathbb{T})).$$

The Aubin-Lions lemma implies the existence of a subsequence (still denoted w^n) converging strongly to a limit w in $L^2(0, T; H_0^0(\mathbb{T}))$. Next we verify that $\{w_T^n\}$ is Cauchy in $H_0^0(\mathbb{T})$. Scaling equation (4.13) by tw yields

$$\frac{T}{2} \|w_T\|^2 = \frac{1}{2} \int_0^T \|w(t)\|^2 dt + \int_0^T t \|D^{3/2}(Gw)\|^2 dt + \int_0^T \int_{\mathbb{T}} twEw dx dt, \quad (4.15)$$

which also applies to the difference of two solutions so that

$$\begin{aligned} \|w_T^n - w_T^m\| &\leq \frac{1}{T} \int_0^T \|w^n - w^m\|^2 dt + 2 \int_0^T \|D^{3/2}G(w^n - w^m)\|^2 dt \\ &\quad + 2 \int_0^T \|w^n - w^m\|_{3/2} \|E(w^n - w^m)\|_{-3/2} dt \\ &\leq \frac{1}{T} \int_0^T \|w^n - w^m\|^2 dt + 4 \left(\int_0^T \|D^{3/2}(Gw^n)\|^2 dt + \int_0^T \|D^{3/2}(Gw^m)\|^2 dt \right) \\ &\quad + \delta \int_0^T \|w^n - w^m\|_{3/2}^2 dt + c(\delta) \left(\int_0^T \|Ew^n\|_{-3/2}^2 dt + \int_0^T \|Ew^m\|_{-3/2}^2 dt \right). \end{aligned}$$

Choosing $\delta > 0$ small enough, the claim follows from (4.9), (4.14) and the strong convergence of w^n in $L^2(0, T; H_0^0(\mathbb{T}))$. Thus $w_T^n \rightarrow w_T$ strongly in $H_0^0(\mathbb{T})$ and it follows that the solution of (4.13) associated to

w_T agrees with the limit w of the sequence $\{w^n\}$. Letting $n \rightarrow \infty$ in (4.14) we have that

$$\int_0^T \|D^{3/2}(Gw)\|^2 dt = 0.$$

Hence $Gw = 0$ a.e. $\mathbb{T} \times (0, T)$ and an application of Proposition 2.5 implies that $w \equiv 0$, thus contradicting the fact that $\|w_T\| = \|w_T^n\| = 1$. Thus (4.12) holds.

We now prove the following estimate of solutions u to equation (4.3)

$$\|u_T\|_{-s}^2 \leq C \left(\int_0^T \|D^{3/2}(Gu)\|_{-s}^2 dt, + \|u_T\|_{-s-1}^2 \right), \quad (4.16)$$

from which (4.11) will follow. To obtain a contradiction, suppose there is a sequence $\{u_T^n\}$ in $H_0^{-s}(\mathbb{T})$ such that

$$1 = \|u_T^n\|_{-s}^2 > n \left(\int_0^T \|D^{3/2}(Gu^n)\|_{-s}^2 dt + \|u_T^n\|_{-s-1}^2 \right), \quad (4.17)$$

with u^n denoting the solution to (4.3) corresponding to $u(T) = u_T^n$. This implies $u_T^n \rightarrow 0$ strongly in $H_0^{-s-1}(\mathbb{T})$ and so $u^n \rightarrow 0$ in $Z_{-s-1, T}$. Then

$$\int_0^T \|GD^{-s}u^n\|_{3/2}^2 dt \leq c \left(\int_0^T \|D^{3/2}(Gu^n)\|_{-s}^2 dt + \int_0^T \|D^{3/2}[D^{-s}; G]u^n\|^2 dt \right),$$

where the first term on the right-hand side tends towards zero by (4.17). Applying commutator estimate (2.5),

$$\int_0^T \|D^{3/2}[D^{-s}; G]u^n\|^2 dt \leq c \int_0^T \|D^{-s+1/2}u^n\|^2 dt \leq c(T)\|u_T^n\|_{-s-1}^2$$

by the propagation of regularity result for IVP (4.3). Therefore

$$\int_0^T \|GD^{-s}u^n\|_{3/2}^2 dt \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Inserting this into (4.12) and using that $u^n \rightarrow 0$ in $Z_{-s-1, T}$ we conclude that $u_T^n \rightarrow 0$ in $H_0^{-s}(\mathbb{T})$, thus contradicting the fact that $\|u_T\|_{-s} = \|u_T^n\|_{-s} = 1$. Thus (4.16) holds.

We now show that (4.16) implies (4.11). To obtain a contradiction, suppose there is a sequence $\{u_T^n\}$ in $H_0^{-s}(\mathbb{T})$ such that

$$1 = \|u_T^n\|_{-s}^2 > n \left(\int_0^T \|D^{3/2}(Gu^n)\|_{-s}^2 dt \right), \quad (4.18)$$

with u^n denoting the solution to (4.3) corresponding to $u(T) = u_T^n$. By compactness of the embedding $H_0^{-s}(\mathbb{T}) \hookrightarrow H_0^{-s-1}(\mathbb{T})$ then $u_T^n \rightarrow u_T$ in $H_0^{-s-1}(\mathbb{T})$. Applying (4.16) to the difference of two solutions,

$$\begin{aligned} \|u_T^n - u_T^m\|_{-s}^2 &\leq C \left(\int_0^T \|D^{3/2}G(u^n - u^m)\|_{-s}^2 dt + \|u_T^n - u_T^m\|_{-s-1}^2 \right) \\ &\leq C \left(\int_0^T \|D^{3/2}(Gu^n)\|_{-s}^2 dt + \int_0^T \|D^{3/2}(Gu^m)\|_{-s}^2 dt \right) + C\|u_T^n - u_T^m\|_{-s-1}^2. \end{aligned}$$

Combining this with (4.18) implies that $u_T^n \rightarrow u_T$ strongly in $H_0^{-s}(\mathbb{T})$. Letting $n \rightarrow \infty$ in (4.18) we have that

$$\int_0^T \|D^{3/2}(Gu)\|^2 dt = 0,$$

with u denoting the solution to (4.3) corresponding to $u(T) = u_T$. Hence $Gu = 0$ a.e. $\mathbb{T} \times (0, T)$ and an application of Proposition 2.5 implies that $u \equiv 0$, thus contradicting the fact that $\|u_T\|_{-s} = \|u_T^n\|_{-s} = 1$. Thus (4.11) holds. \square

We are now able to prove Theorem 1.3, local exact control of the nonlinear equation (1.6). As in the remarks following the proof of Theorem 1.4, the results in this section apply to equation (1.6) as well as a class of equations containing the KdV hierarchy.

Proof. For each $s \geq 0, T > 0$, Lemma 4.1 provides the existence of a continuous linear operator [8], Lemma 2.48, p. 58.

$$A : H_0^s(\mathbb{T}) \rightarrow L^2(0, T; H_0^s(\mathbb{T}))$$

such that given $v_T \in H_0^s(\mathbb{T})$, the solution v of (4.2) associated to $v_0 = 0$ and $k = A(v_T)$ satisfies $v(T) = v_T$. Denote this solution by

$$W(k)(t) = v(t) = \int_0^T \mathcal{S}(t-t')GD^{3/2}k(t') dt'.$$

From Proposition 2.8, it holds that $W : L^2(0, T; H_0^s(\mathbb{T})) \rightarrow Z_{s,T}$ is continuous.

Let $u_0, u_T \in H_0^s(\mathbb{T})$, $s > 2$, with

$$\|u_0\|_s \leq \rho \quad \text{and} \quad \|u_T\|_s \leq \rho,$$

for some $\rho > 0$ to be determined. For $v \in Z_{s,T}$, set

$$\omega(v) = \int_0^t \mathcal{S}(t-t')(v\partial_x^3 v)(t') dt'$$

and note that Proposition 2.8 yields

$$\|\omega(v)\|_s \leq c \left(\int_0^T \|v\partial_x^3 v\|_{s-3/2}^2 dt \right)^{1/2} \leq c\|v\|_{s,T}^2, \quad (4.19)$$

by the algebra property of $H^{s-3/2}(\mathbb{T})$ for $s > 2$. Thus $\omega(v)(T) \in H_0^s(\mathbb{T})$. Defining

$$\Gamma(v) := \mathcal{S}(t)u_0 - \int_0^t \mathcal{S}(t-t')(v\partial_x^3 v)(t') dt' + W(\Lambda(u_T - \mathcal{S}(T)u_0 + \omega(v)(T))),$$

it is clear that $\Gamma(v)(0) = u_0$ and $\Gamma(v)(T) = u_T$ for any $v \in Z_{s,T}$. Thus it suffices to establish a fixed point of the nonlinear map $v \mapsto \Gamma(v)$ in a closed ball in $Z_{s,T}$.

Repeating the argument of the proof of Theorem 3.1, we show that Γ defines a contraction on $B = \{v \in Z_{s,T} : \|v\|_{s < T} \leq R\}$ for appropriate choices of $R > 0$ and $\rho > 0$. The estimate (2.11) yields

$$\|\Gamma(u)\|_{s,T} \leq c(s,T) (\|u_0\|_s + \|u\partial_x^3 u\|_{L^2(0,T;H^{s-3/2}(\mathbb{T}))}) + \|W(\Lambda(u_T - \mathcal{S}(T)u_0 + \omega(v)(T)))\|_{s,T}.$$

Assuming $u \in Z_{s,T}$, then by the algebra property of $H_0^{s-3/2}(\mathbb{T})$,

$$\int_0^T \|u\partial_x^3 u\|_{s-3/2}^2 dt \leq c\|u\|_{s,T}^4.$$

Estimate (4.19), along with the continuity of Γ and W , yields

$$\|W(\Lambda(u_T - \mathcal{S}(T)u_0 + \omega(v)(T)))\|_{s,T} \leq C_0(\|u_0\|_s + \|u_T\|_s) + C_1\|u\|_{s,T}^2.$$

Therefore

$$\|\Gamma(u)\|_{s,T} \leq C_0(\|u_0\|_s + \|u_T\|_s) + C_1\|u\|_{s,T}^2$$

for some $C_0, C_1 > 0$. Similarly,

$$\|\Gamma(u) - \Gamma(v)\|_{s,T} \leq C_1(\|u\|_{s,T} + \|v\|_{s,T})\|u - v\|_{s,T}.$$

Thus Γ forms a contraction on B provided

$$C_0(\|u_0\|_s + \|u_T\|_s) + C_1R^2 < R \quad \text{and} \quad 2C_1R < 1.$$

It is sufficient to take

$$R = (4C_1)^{-1} \quad \text{and} \quad \|u_0\|_s \leq \rho := R/2C_0.$$

□

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