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STAMPACCHIA–CALDÉRON–ZYGMUND THEORY FOR LINEAR ELLIPTIC EQUATIONS WITH DISCONTINUOUS COEFFICIENTS AND SINGULAR DRIFT

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Abstract. In this paper, the existence and properties of solutions of the boundary value problem (1.4) are studied. No regularity assumptions on the coefficients of the matrix $M(x)$ are used (in particular we do not require that the principal part is $-\Delta$), no assumptions on the size of $\|E\|_{L^N}$ are needed.

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1. Introduction

Let Ω be a bounded, open subset of $\mathbb{R}^N$, $N > 2$ and $M : \Omega \to \mathbb{R}^{N^2}$, be a bounded and measurable matrix such that

$$\alpha|\xi|^2 \leq M(x)\xi \cdot \xi, \quad |M(x)| \leq \beta, \quad \text{a.e. } x \in \Omega, \quad \forall\ \xi \in \mathbb{R}^N. \tag{1.1}$$

We assume that $E(x)$ is a vector field and $f(x)$ is a function such that

$$E \in (L^N(\Omega))^N, \tag{1.2}$$

$$f \in L^m(\Omega), \ \ m > 1, \tag{1.3}$$

and we consider the following boundary value problem with drift term

$$\begin{cases} -\text{div}(M(x)\nabla u) = E(x) \cdot \nabla u + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

The above boundary problem is linear, but the differential operator may be not coercive, unless $\|E\|_{(L^N(\Omega))^N}$ is sufficiently small or $\text{div}(E) = 0$ (see also [4, 8]).

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First results on this problem can be found in [14, 23, 33]. In the second paper, it is handled for the first time the case $E \in (L^N(\Omega))^N$, with no restrictions on the norm and no assumptions of the type $\mathrm{div}(E) = 0$, and it proves the existence of solutions belonging to $W_0^{1,2}(\Omega)$. This existence result was generalized to the nonlinear case in [19].

In the present paper, we prove that the boundary value problem (1.4)

1. (*Stampacchia theory*) has a unique *weak* solution $u \in W_0^{1,2}(\Omega) \cap L^{m^{**}}(\Omega)$, $m^{**} = \frac{Nm}{N-2m}$, if $\frac{2N}{N+2} \le m < \frac{N}{2}$; if $m > \frac{N}{2}$, then $u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$;
2. (*Caldéron–Zygmund theory*) has a *distributional* solution $u \in W_0^{1,m^*}(\Omega)$, $m^* = \frac{Nm}{N-m}$, if $1 < m < \frac{2N}{N+2}$;

that is the results proved in [33, 34] (case 1) and [10] (case 2) under the assumption $E = 0$.

Moreover, in this paper, we do not use the weighted-Sobolev functional space framework of [34], but the standard-Sobolev functional space framework, so that our first estimate (1.7) is not easy at all (even in the case 1 of $W_0^{1,2}(\Omega)$ solutions).

If $E = 0$, also in the vectorial case, with no coefficient but considering the nonlinear case, the same problem has been considered in the paper [21]. There, existence results and Lipschitz estimates are obtained under the assumption that the vectorial field E belongs to the Lorentz space $L^{N,1}$, with a smallness condition on the $L^{N,1}$-norm.

Furthermore, we discuss the uniqueness, the stability of the solution with respect to the G-convergence and the positivity. In particular, with respect to the positivity of the solutions, we can follow the method of [15] and we prove a "strong maximum principle".

The main difficulty of the study is, even for data $f(x)$ with "good" summability (*i.e.* $f(x) \in L^m(\Omega)$, $m \ge \frac{2N}{N+2}$), the proof of *a priori* bounds of the type

$$\|u\|_{\mathcal{B}} \le C\Big(\|f\|_{L^m(\Omega)}, \|E\|_{(L^N(\Omega))^N}\Big), \tag{1.5}$$

where $\mathcal{B}$ is the Banach space, possibly as large as $L^1(\Omega)$.

In [5], existence and properties of the solutions of the boundary value problem are studied

$$\begin{cases} -\mathrm{div}(M(x)\nabla\psi) = -\mathrm{div}(\psi\, E(x)) + g(x) & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

Although the above problem (1.6) is the dual problem of (1.4), it is not possible to study completely our problem (1.4) by a duality method, since the existence results about the linear problem (1.6) are obtained with a nonlinear approach, so that estimates of the type

$$\|\psi\|_X \le C\,\|g\|_Y$$

are not proved.

Nevertheless, in the first step of our proof, we are able to use a duality approach in order to prove our basic *a priori* bound

$$\|u\|_{L^1(\Omega)} \le C\Big(\|f\|_{L^m(\Omega)}, \|E\|_{(L^N(\Omega))^N}\Big), \tag{1.7}$$

thanks to some estimates proved in [5]. This starting point is different from [3, 7, 19].

Even if our first inequality (1.7) is "poor" (it is not a Sobolev space estimate), it is important in order to bypass the difficulty that we do not assume bounds on the size of $\|E\|_{(L^N(\Omega))^N}$ and it plays the same role of the log-estimate for the problem (1.6). We recall that log-estimate for the problem (1.6) was presented by the author in some talk at the beginning of the century and then written in [5] and used by many authors (see [6, 20, 24, 26, 28, 35]).

We point out that this paper is related with [2], where the Stampacchia–Caldéron–Zygmund theory is studied, also for nonlinear elliptic equations, under assumptions on the size of $\|E\|_{(L^N(\Omega))^N}$.

Nonlinear problems with drift term of form $\frac{x}{|x|^2}$ (which does not belong to L^N) are studied in [27].

In [17], it is proved the existence if $N = 3$ and E and f belong to a Morrey space (E with sufficiently small norm).

In [25], two-sided pointwise estimate by the truncated Riesz potential for nonnegative weak solutions to (1.4) are proved.

2. A nonlinear approach to a linear and noncoercive problem

Consider the following Dirichlet problems

$$u_n \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega) : \ -\mathrm{div}(M(x)\nabla u_n) = \frac{E(x) \cdot \nabla u_n}{(1 + \frac{1}{n}|E|)(1 + \frac{1}{n}|u_n|)^2} + \frac{f(x)}{1 + \frac{1}{n}|f|}. \tag{2.1}$$

For our study, we need to start with bounded weak solutions, so that we do not use the existence result of [14], since there is proved existence of weak solutions, but not the boundedness, if the data are bounded. In our approach (see Appendix A), the existence of u_n is guaranteed by the Schaefer fixed point theorem, which is a consequence of the Schauder theorem ([30]; see also [22], Thm. 4, Sect. 9.2.2), and the boundedness of u_n is guaranteed by the Stampacchia summability theorem (see [33]).

This is our nonlinear approach to a linear and noncoercive problem. Furthermore, note that we can rewrite (2.1) as

$$u_n \in W_0^{1,2}(\Omega) : \ -\mathrm{div}(M(x)\nabla u_n) = \frac{E(x) \cdot \nabla \frac{u_n}{1 + \frac{1}{n}|u_n|}}{1 + \frac{1}{n}|E|} + \frac{f(x)}{1 + \frac{1}{n}|f|}. \tag{2.2}$$

Lemma 2.1. *Assume (1.1), (1.2), (1.3). Then for any $k > 0$ there exists a constant $C_0\big(m, \|E\|_{L^N(\Omega)}\big) > 0$ such that*

$$\text{measure}\,\{x : k \le |u_n(x)|\} \le \frac{\|f\|_m C_0\big(m, \|E\|_N\big)}{k}. \tag{2.3}$$

Proof. Consider the dual Dirichlet problem

$$\psi_n \in W_0^{1,2}(\Omega) : -\mathrm{div}(M^*(x)\nabla\psi_n) = -\mathrm{div}\left(\frac{\psi_n}{(1 + \frac{1}{n}|u_n|)^2}\frac{E(x)}{1 + \frac{1}{n}|E|}\right) + \frac{u_n(x)}{|u_n(x)|}, \tag{2.4}$$

where M^* denotes the adjoint matrix of M. We recall that in [5] it is proved the existence of a weak solution of the Dirichlet problem

$$\psi \in W_0^{1,2}(\Omega) : \ -\mathrm{div}(M^*(x)\nabla\psi) = -\mathrm{div}(\psi\ E(x)) + g$$

with $E \in L^N(\Omega)^N$ and $g \in L^q(\Omega)$; if $q \ge \frac{2N}{N+2}$ it is proved (Cor. 5.2) the estimate

$$\int_\Omega |\nabla\psi|^2 \le C_1(E, g, \alpha, \beta) \int_\Omega |g| + C_2(E, g, \alpha, \beta) \int_\Omega |E|^2 + C_3(E, g, \alpha, \beta)\Big[\int_\Omega |g|^{\frac{2N}{N+2}}\Big]^{\frac{N+2}{N}}, \tag{2.5}$$

where $C_i > 0$; if $1 < q < \frac{2N}{N+2}$, it is proved the estimate

$$\|\psi\|_{W_0^{1,m^*}(\Omega)} \leq \tilde{C}\Big(\|E\|_{L^N(\Omega)}, \|g\|_{L^m(\Omega)}, \alpha, \beta\Big).^{1} \tag{2.6}$$

Thus, the existence of ψ_n is a consequence of the results of [5]. Moreover, taking ψ_n as test function in (2.1) (use the equivalent form (2.2)) and u_n in (2.4), we get (use (2.6))

$$\int_\Omega |u_n| = \int_\Omega \frac{f(x)}{1+\frac{1}{n}|f|}\psi_n \leq \|f\|_{L^m(\Omega)}\|\psi_n\|_{L^{q^{**}}(\Omega)} \leq \|f\|_{L^m(\Omega)}\,\tilde{C}\Big(\|E\|_{L^N(\Omega)}\Big\|\frac{u_n}{|u_n|}\Big\|_{L^q(\Omega)}, \alpha, \beta\Big)$$

where $1 = \frac{1}{m} + \frac{1}{q^{**}}$, with $m > 1$ and $q < \frac{N}{2}$. Then we deduce

$$\|u_n\|_{L^1(\Omega)} \leq \|f\|_{L^m(\Omega)}\,\tilde{C}(\|E\|_{L^N(\Omega)}, \Omega, \alpha, \beta) \tag{2.7}$$

and thus (2.3) follows. □

Remark 2.2. With the above method, it is possible to handle the case $m = 1$ (see [9]), but then the summability $p > N$ of E is needed.

Lemma 2.3. *Assume (1.1), (1.2), (1.3). Then there exists a constant $C_m(E, f) > 0$ such that*

$$\|u_n\|_{L^{m^{**}}(\Omega)} \leq C_m\Big(\|E\|_{L^N(\Omega)}, \|f\|_{L^m(\Omega)}\Big).^{2}$$

Proof. First step – Recall Stampacchia's definitions: $\forall\, k \geq 0$,

$$T_k(s) = \begin{cases} s, & \text{if } |s| \leq k, \\ k\,\frac{s}{|s|}, & \text{if } |s| > k; \end{cases} \qquad \text{and} \qquad G_k(s) = s - T_k(s).$$

Here, we adapt the approach of [10]. We choose as test function in (2.1), for any $\delta > 0$, $\frac{1}{2\lambda-1}\big[(|G_k(u_n)| + \delta)^{2\lambda-1} - \delta^{2\lambda-1}\big]\,\mathrm{sign}(u_n)$, where $\lambda = \frac{m^{**}}{2^*}$. Then we use (1.1) and the Young inequality, so that we have

$$\left|\begin{aligned} &\alpha\int_\Omega (|G_k(u_n)|+\delta)^{2(\lambda-1)}|\nabla G_k(u_n)|^2 \\ &\leq \frac{1}{2\lambda-1}\int_\Omega \frac{|E(x)|}{1+\frac{1}{n}|E|}\frac{|\nabla u_n|}{(1+\frac{1}{n}|u_n|)^2}\big(|G_k(u_n)|+\delta\big)^{2\lambda-1} + \frac{1}{2\lambda-1}\int_\Omega |f||G_k(u_n)|^{2\lambda-1} \\ &\leq \frac{\alpha}{2}\int_\Omega \big(|G_k(u_n)|+\delta\big)^{2(\lambda-1)}|\nabla G_k(u_n)|^2 + C_\lambda \int_\Omega \big(|G_k(u_n)|+\delta\big)^{2\lambda}|E|^2 \\ &\quad + \frac{\|f\|_m}{2\lambda-1}\Big[\int_\Omega |G_k(u_n)|^{(2\lambda-1)m'}\Big]^{\frac{1}{m'}}. \end{aligned}\right. \tag{2.8}$$

[1] $\tilde{C}$ is nondecreasing w.r.t. $\|E\|_{L^N(\Omega)}$.

[2] From now on, we do not write α, β.

Let $A_n(k) = \{x : k \le |u_n(x)|\}$. Hence we have, for some positive constants C_i (independent of n, but depending of $\|E\|_{L^N(\Omega)}$) and thanks to Sobolev inequality, letting $\delta \to 0$,

$$\begin{aligned} &\left[\int_\Omega |G_k(u_n)|^{\lambda 2^*}\right]^{\frac{2}{2^*}} \\ &\le C_1\left[\int_{A_n(k)} |E|^N\right]^{\frac{2}{N}}\left[\int_\Omega |G_k(u_n)|^{\lambda 2^*}\right]^{\frac{2}{2^*}} + C_2\|f\|_{L^m(A_n(k))}\left[\int_\Omega |G_k(u_n)|^{(2\lambda-1)m'}\right]^{\frac{1}{m'}}. \end{aligned} \tag{2.9}$$

Then we have the inequality

$$\left(1 - C_1\left[\int_{A_n(k)} |E|^N\right]^{\frac{2}{N}}\right)\left[\int_\Omega |G_k(u_n)|^{\lambda 2^*}\right]^{\frac{2}{2^*}} \le C_2\|f\|_m\left[\int_\Omega |G_k(u_n)|^{(2\lambda-1)m'}\right]^{\frac{1}{m'}},$$

which implies, recalling that $\lambda 2^* = m^{**} = (2\lambda - 1)m'$,

$$\left(1 - C_1\left[\int_{A_n(k)} |E|^N\right]^{\frac{2}{N}}\right)\|G_k(u_n)\|_{L^{m^{**}}(\Omega)} \le C_1\|f\|_{L^m(\Omega)}.$$

Recall (2.7) and now fix k such that

$$\frac{1}{2} > C_1\left[\int_{A_n(k)} |E|^N\right]^{\frac{2}{N}}, \tag{2.10}$$

that is possible thanks to (2.3). Note that k depends on $\|E\|_{L^N}$ and $\|f\|_{L^m}$. Thus, we obtain that

$$\|G_k(u_n)\|_{L^{m^{**}}(\Omega)} \le C_3\|f\|_{L^m(\Omega)}.$$

Second step – We note that, since $u_n = T_k(u_n) + G_k(u_n)$,

$$\|u_n\|_{L^{m^{**}}(\Omega)} \le \|G_k(u_n)\|_{L^{m^{**}}(\Omega)} + \|T_k(u_n)\|_{L^{m^{**}}(\Omega)} \le C_3\|f\|_{L^m(\Omega)} + k\,\text{measure}\,(\Omega)^{\frac{1}{m^{**}}}.$$

□

Lemma 2.4. *Assume (1.1), (1.2) and $m > \frac{N}{2}$, then there exists a constant $C_m(E, f) > 0$ such that*

$$\|u_n\|_{L^\infty(\Omega)} \le C_m\Big(\|E\|_{L^N(\Omega)}, \|f\|_{L^m(\Omega)}\Big).$$

Proof. Note that (2.9) with the simple choice $\lambda = 1$ joint with (2.10) gives

$$\frac{1}{2}\left[\int_\Omega |G_k(u_n)|^{2^*}\right]^{\frac{2}{2^*}} \le C_2\|f\|_{L^{\frac{2N}{N+2}}(A_n(k))}\left[\int_\Omega |G_k(u_n)|^{2^*}\right]^{\frac{1}{2^*}},$$

that is

$$\left[\int_\Omega |G_k(u_n)|^{2^*}\right]^{\frac{1}{2^*}} \le C_3\|f\|_{L^{\frac{2N}{N+2}}(A_n(k))},$$

which implies

$$\left[\int_\Omega |G_k(u_n)|^{2^*}\right]^{\frac{1}{2^*}} \leq C_3\|f\|_{L^m(\Omega)} \text{ measure } (A_n(k))^{\frac{1}{2}+\frac{1}{N}-\frac{1}{m}}.$$

Then, since our assumption $m > \frac{N}{2}$ implies that $\frac{1}{2^*} < \frac{1}{2} + \frac{1}{N} - \frac{1}{m}$, from now it is enough to copy the Stampacchia proof (see [33]) and our estimate is proved. □

2.1. Positivity

Lemma 2.5. *Assume (1.1), (1.2), (1.3). Let $f \geq 0$. Then $u_n \geq 0$.*

Proof. Recall the definition of u_n, solution of (2.1) and consider the auxiliary problem

$$0 \leq z_n \in W_0^{1,2}(\Omega) : -\mathrm{div}(M^*(x)\nabla z_n) = -\mathrm{div}\left(\frac{z_n}{(1+\frac{1}{n}|u_n|)^2}\frac{E(x)}{1+\frac{1}{n}|E|}\right) + \frac{u_n(x)^-}{|u_n(x)|}.$$

The positivity of z_n is proved in [6]. The combined use of z_n as test function in (2.1) and of u_n in the above Dirichlet problem yields

$$\int_\Omega \frac{f(x)}{1+\frac{1}{n}|f|} z_n = \int_\Omega \frac{u_n(x)^-}{|u_n(x)|} u_n.$$

Note that this equality gives

$$0 \leq \int_\Omega \frac{f(x)}{1+\frac{1}{n}|f|} z_n = \int_\Omega \frac{u_n(x)^-}{|u_n(x)|} u_n = -\int_\Omega \frac{[u_n(x)^-]^2}{|u_n(x)|} \leq 0$$

which implies $u_n(x)^- = 0$. Thus, if $f \geq 0$, then $u_n \geq 0$.

3. Finite energy solutions (Stampacchia theory)

3.1. Existence of weak solutions

If $E \in (L^N(\Omega))^N$ and $f \in L^m(\Omega)$, $m \geq \frac{2N}{N+2}$, the existence of weak solutions is proved in [14]; our result on the existence is based on a different proof; our main contribution in this section is the Stampacchia type result on the summability of the solutions.

We point out that, even with $E = 0$, the summability of ∇u does not increase if the summability of f increases, when $m > \frac{N}{2}$ (see [7]), in contrast with the following section (where $1 < m < \frac{2N}{N+2}$).

Lemma 3.1. *Assume (1.1), (1.2), $m \geq \frac{2N}{N+2}$. Then the sequence $\{u_n\}$ is bounded in $W_0^{1,2}(\Omega)$.*

Proof. We use u_n as test function in (2.1) and then we use Lemma 2.3, so that we have

$$\begin{aligned}
\alpha\int_\Omega |\nabla u_n|^2 &\leq \int_\Omega |E(x)||\nabla u_n||u_n| + \|f\|_{L^m(\Omega)}\|u_n\|_{L^{2^*}(\Omega)} \\
&\leq \|E\|_{(L^N(\Omega))^N}\left[\int_\Omega |\nabla u_n|^2\right]^{\frac{1}{2}}\|u_n\|_{L^{2^*}(\Omega)} + \|f\|_{L^m(\Omega)}\|u_n\|_{L^{2^*}(\Omega)} \\
&\leq \tilde{C}_m\big(\|E\|_{L^N(\Omega)}, \|f\|_{L^m(\Omega)}\big)\left(\|E\|_{(L^N(\Omega))^N}\left[\int_\Omega |\nabla u_n|^2\right]^{\frac{1}{2}} + \|f\|_{L^m(\Omega)}\right).
\end{aligned}$$

Now we use Lemma 2.3, $m \geq \frac{2N}{N+2}$ and the above inequalities give the boundedness in $W_0^{1,2}(\Omega)$ of the sequence $\{u_n\}$. □

Remark 3.2. If $m = \frac{2N}{N+2}$, this Lemma is proved in [14], with a different proof.

As a consequence of the previous lemma, there exist $u \in W_0^{1,2}(\Omega)$ and a subsequence (still denoted by u_n) such that

$$\{u_n\} \text{ converges weakly to } u \text{ in } W_0^{1,2}(\Omega). \tag{3.1}$$

Theorem 3.3. *Assume (1.1), (1.2), $m \geq \frac{2N}{N+2}$. Then there exist a weak solution u of (1.4); that is*

$$u \in W_0^{1,2}(\Omega) : \int_\Omega M(x)\nabla u \nabla v = \int_\Omega v\, E(x) \cdot \nabla u + \int_\Omega f\, v,\ \forall\, v \in W_0^{1,2}(\Omega), \tag{3.2}$$

and

$$\alpha \int_\Omega |\nabla u|^2 \leq C_m^*\Big(\|E\|_{L^N(\Omega)}, \|f\|_{L^m(\Omega)}\Big). \tag{3.3}$$

Moreover

1. *if $\frac{2N}{N+2} < m < \frac{N}{2}$, u belongs to $L^{m^{**}}(\Omega)$;*
2. *if $m > \frac{N}{2}$, u belongs to $L^\infty(\Omega)$;*
3. *if $f \geq 0$, then $u \geq 0$.*

Proof. Thanks to the linearity of the problem and (3.1) it is easy to pass to the limit in the weak form of (2.1) and we obtain (3.2) and (3.3). Moreover Lemmas 2.3, 2.4, and 3.1 give 1 and 2, (3.3) and Lemma 2.5 gives 3. □

Remark 3.4. We point out that, in 1 of the above theorem, the proof of the summability $u \in L^{m^{**}}(\Omega)$ does not follow the Stampacchia's proof of [33], since it is genuinely linear; our proof is a little bit similar to the approach used in [7, 11], where furthermore more general results, with respect to [11, 33], are contained, with proofs which remain true in the nonlinear framework.

Remark 3.5. In the next section, we will prove a positivity result (even for less regular data f) stronger than the positivity result given in Theorem 3.3.

3.2. Uniqueness of weak solutions

Theorem 3.6. *Assume (1.1), (1.2), $m \geq \frac{2N}{N+2}$. Then the weak solution $u \in W_0^{1,2}(\Omega)$ of (3.2) is unique.*

Proof. Assume that there exists u, w weak solutions of (3.2). Then we have

$$\int_\Omega M(x)\nabla(u-w)\nabla v = \int_\Omega v\, E(x) \cdot \nabla(u-w),\ \forall\, v \in W_0^{1,2}(\Omega).$$

Let $\psi \in W_0^{1,2}(\Omega)$ be the weak solution of the auxiliary problem

$$\int_\Omega M^*(x)\nabla\psi\nabla\phi = \int_\Omega \psi\, E(x)\nabla\phi + \int_\Omega \frac{(u-w)}{|u-w|}\,\frac{1}{1+|u-w|}\,\chi_{u\neq w}\phi,$$

$\forall\, \phi \in W_0^{1,2}(\Omega)$. The existence of ψ is proved in [5]. The simple choice $v = \psi$ and $\phi = u - w$ yields

$$0 = \int_{u \neq w} \frac{|u-w|}{1+|u-w|} = \int_\Omega \frac{|u-w|}{1+|u-w|}. \tag{3.4}$$

The last integral is the distance equivalent to the convergence in measure; thus, we proved that $u = w$. □

Remark 3.7. With the same proof, it is possible to prove the following comparison result: in (3.2) we consider data f_i, with $f_2 \geq f_1$; then the solution u_2 is greater than the solution u_1. Indeed, following the proof of Lemma 2.5, define now $\tilde\psi \in W_0^{1,2}(\Omega)$, the weak solution of the auxiliary problem (the existence and the positivity of $\tilde\psi$ is proved in [6])

$$\int_\Omega M^*(x)\nabla\tilde\psi\nabla\phi = \int_\Omega \tilde\psi\, E(x)\nabla\phi + \int_\Omega \frac{(u_2-u_1)^-}{|u_2-u_1|}\,\frac{\phi}{1+|u_2-u_1|},\ \forall\, \phi \in W_0^{1,2}(\Omega).$$

On the other hand, we have

$$\int_\Omega M(x)\nabla(u_2-u_1)\nabla v = \int_\Omega v\, E(x)\cdot\nabla(u_2-u_1) + \int_\Omega (f_2-f_1)v,\ \forall\, v \in W_0^{1,2}(\Omega).$$

The choice $\phi = u_2 - u_1$, $v = \tilde\psi$ yields

$$\int_\Omega \frac{(u_2-u_1)^-}{|u_2-u_1|}\,\frac{u_2-u_1}{1+|u_2-u_1|} = \int_\Omega (f_2-f_1)\tilde\psi,$$

which implies

$$-\int_\Omega \frac{(u_2-u_1)^-}{|u_2-u_1|}\,\frac{(u_2-u_1)^-}{1+|u_2-u_1|} = \int_\Omega \frac{(u_2-u_1)^-}{|u_2-u_1|}\,\frac{u_2-u_1}{1+|u_2-u_1|} \geq 0,$$

that is $(u_2-u_1)^- = 0$.

4. Infinite energy solutions (Caldéron–Zygmund theory)

Theorem 4.1. *Assume (1.1), (1.2), $1 < m < \frac{2N}{N+2}$. Then there exist a distributional solution u of (1.4); that is*

$$u \in W_0^{1,m^*}(\Omega):\ \int_\Omega M(x)\nabla u\nabla\varphi = \int_\Omega \varphi\, E(x)\cdot\nabla u + \int_\Omega f\,\varphi, \quad \forall\, \varphi \in C_c^1(\Omega). \tag{4.1}$$

Proof. Recall that $\lambda = \frac{m^{**}}{2^*}$; in this theorem $1 < m < \frac{2N}{N+2}$, so that $\frac{1}{2} < \lambda < 1$. We use the inequality (2.8), joint with Lemma 2.3, and we have

$$\frac{\alpha}{2}\int_\Omega \frac{|\nabla G_k(u_n)|^2}{(|G_k(u_n)|+\delta)^{2(1-\lambda)}} \leq C_1, \quad \delta \in (0,1).$$

The use of the Fatou Lemma, as δ and k go to 0, gives

$$\frac{\alpha}{2}\int_\Omega \frac{|\nabla u_n|^2}{|u_n|^{2(1-\lambda)}} \leq C_1.$$

Now, we recall that in [10] it is proved that a consequence of the above inequality is the boundedness of the sequence $\{u_n\}$ in $W_0^{1,m^*}(\Omega)$. Then there exist $u \in W_0^{1,m^*}(\Omega)$ and a subsequence (still denoted by u_n) such that

$$u_n \text{ converges a.e. and weakly to } u \text{ in } W_0^{1,m^*}(\Omega). \tag{4.2}$$

Thanks to the linearity of the problem and (4.2) it is easy to pass to the limit in the weak form of (2.1) and we obtain (4.1).

Remark 4.2. In the present paper, we do not use the assumption $\operatorname{div}(E) = 0$, but we note that we can handle a drift term of the form $E + E_0$, with $\operatorname{div}(E_0) = 0$.

Moreover, we are able to obtain a general solvability result in $W_0^{1,p}(\Omega)$, with $1 < p \leq 2$, even for linear elliptic equations with matrix $M(x)$ having discontinuous coefficients and singular drift $E(x)$; the solvability in $W_0^{1,p}(\Omega)$, with $p > 2$, can fail also with $E = 0$ (see Sect. 3.1 and [7]).

Note that, in the last case (Calderón–Zygmund theory), in our problem with $u \in W_0^{1,m^*}(\Omega)$ distributional solution of $-\operatorname{div}(M(x)\nabla u) = E(x) \cdot \nabla u + f$, the right hand side $E(x) \cdot \nabla u$ belongs to $L^r(\Omega)$, with $\frac{1}{r} = \frac{1}{N} + \frac{1}{m^*} = \frac{1}{m}$, so that we have a "maximal regularity" situation. On the other hand, it is clear (recall the case $E = 0$) that it is not possible to handle the case $m = 1$ in a distributional framework setting, since $E(x) \cdot \nabla u$ does not belong to $L^1(\Omega)$ (see [3] and Rem. 2.2).

In the papers [3, 16, 24], existence of solutions, in a suitable sense (nondistributional), of Dirichlet problems (even nonlinear) are studied in the extremal cases of right hand side measures; for this definition see also [1, 8].

4.1. Remarks about the uniqueness

The uniqueness of distributional solutions can fail even if $E = 0$ (see [31]). Nevertheless, following [18], it is possible to give a partial answer to the problem of uniqueness. We consider (joint with (2.1)) the Dirichlet problem

$$w_n \in W_0^{1,2}(\Omega): \ -\operatorname{div}(M(x)\nabla w_n) = \frac{E(x) \cdot \nabla w_n}{(1 + \frac{1}{n}|E|)(1 + \frac{1}{n}|w_n|)^2} + f_n, \tag{4.3}$$

where $\{f_n\}$ is any sequence of bounded functions converging to f in $L^m(\Omega)$. Repeating the proof of the previous theorem, we can say that w_n converges weakly (and in measure) to some w in $W_0^{1,m^*}(\Omega)$. Let $\psi_n \in W_0^{1,2}(\Omega)$ be the weak solution of

$$\int_\Omega M^*(x)\nabla\psi_n\nabla\phi = \int_\Omega \psi_n\, E(x)\nabla\phi + \int_\Omega \frac{(u_n - w_n)}{|(u_n - w_n)|}\, \frac{1}{1 + |(u_n - w_n)|}\, \chi_{\{u_n \neq w_n\}}\phi.$$

Now, we adapt the proof of Theorem 3.6 and we have

$$\int_\Omega \frac{|(u_n - w_n)|}{1 + |(u_n - w_n)|} = \int_\Omega \left[f_n - \frac{f(x)}{1 + \frac{1}{n}|f|} \right] \psi_n.$$

In the right hand side, $f_n - \frac{f(x)}{1+\frac{1}{n}|f|}$ converges to zero in $L^m(\Omega)$ and the sequence $\{\psi_n\}$ is bounded in $L^p(\Omega)$, for all $p < \infty$; thus, the right hand side converges to zero. In the left hand side, we use the convergence in measure of $u_n - w_n$ to $u - w$. So that we have

$$\int_\Omega \frac{|u - w|}{1 + |u - w|} = \lim \int_\Omega \frac{|(u_n - w_n)|}{1 + |(u_n - w_n)|} = 0,$$

that is $u = w$.

Thus, the above discussion proves the uniqueness of the solution obtained as the limit of approximate solutions (see [18]). □

5. G-convergence

We recall the definition of G-convergence for sequences of elliptic and bounded matrices.

Definition 5.1. Let $\{M_n\}$ be a sequence of matrices which satisfies

$$\alpha|\xi|^2 \le M_n(x)\,\xi\,\xi, \quad |M_n(x)| \le \beta, \quad \text{a.e. } x \in \Omega, \quad \forall\, \xi \in \mathbb{R}^N\,. \tag{5.1}$$

The sequence $\{M_n\}$ is said to G-converge to a bounded, elliptic matrix $M_0(x)$ if, for every g in $W^{-1,2}(\Omega)$, the sequence $\{w_n\}$ of the unique solutions

$$w_n \in W_0^{1,2}(\Omega), \quad -\mathrm{div}(M_n(x)\nabla w_n) = g \ \text{ in } \Omega, \tag{5.2}$$

satisfies

$$w_n \rightharpoonup w_0 \text{ weakly in } W_0^{1,2}(\Omega),$$

where w_0 is the unique solution of

$$w_0 \in W_0^{1,2}(\Omega), \quad -\mathrm{div}(M_0(x)\nabla w_0) = g \ \text{ in } \Omega. \tag{5.3}$$

This notion of G-convergence was introduced in [32] by Spagnolo in the symmetric case. He proved the following compactness theorem: any sequence of symmetric matrices $M_n(x)$ which satisfies (5.1) admits a subsequence which G-converges to a matrix $M_0(x)$ of the same type. A relationship between G-convergence of differential operators and Γ(weak-$W_0^{1,2}(\Omega)$)-convergence can be found in [12]. The study in the case of nonsymmetric matrices is due to Murat and Tartar [29] involving the H-convergence, where the boundary condition on the Dirichlet problem is removed.

We consider the solutions u_n of the Dirichlet problems (3.2) with the matrices M_n (which G-converge to M_0):

$$u_n \in W_0^{1,2}(\Omega) : \int_\Omega M_n(x)\nabla u_n \nabla v = \int_\Omega v\, E(x)\cdot\nabla u_n + \int_\Omega f\, v,\ \forall\, v \in W_0^{1,2}(\Omega). \tag{5.4}$$

Theorem 5.2. *Assume (5.1), (1.2), $m \ge \frac{2N}{N+2}$, and the G-convergence of M_n to M_0. Then the weak solution $u_n \in W_0^{1,2}(\Omega)$ of (5.4) weakly converges in $W_0^{1,2}(\Omega)$ to the weak solution u_0 of*

$$u_0 \in W_0^{1,2}(\Omega) : \int_\Omega M_0(x)\nabla u_0 \nabla v = \int_\Omega v\, E(x)\cdot\nabla u_0 + \int_\Omega f\, v,\ \forall\, v \in W_0^{1,2}(\Omega).$$

Proof. The estimate (3.3) and (5.1) give an *a priori* bound for the sequence $\{u_n\}$; so that we can say that there exists $\{u_{n_k}\}$ which converges weakly to some $\tilde{u}$ in $W_0^{1,2}(\Omega)$.

We will use the following lemma (see *e.g.* [13]). Let $\{M_n\}$ be a sequence of matrices which satisfies (5.1), and that G-converges to some matrix M_0. Then for every function ϕ_0 belonging to $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, there exists a sequence $\{\phi_n\}$ in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ such that

$$\begin{aligned} \{-\text{div}(M_n^*(x)\nabla\phi_n)\} \text{ converges strongly to } -\text{div}(M_0^*(x)\nabla\phi_0) \text{ in } W^{-1,2}(\Omega), \\ |\phi_n| \le \|\phi_0\|_{L^\infty(\Omega)}, \\ \{\phi_n\} \text{ converges to } \phi_0 \text{ weakly in } W_0^{1,2}(\Omega) \text{ and almost everywhere.} \end{aligned} \tag{5.5}$$

Then, in (5.4), we use $v = \phi_n$ and we have

$$\int_\Omega M_{n_k}(x)\nabla u_{n_k}\nabla\phi_{n_k} = \int_\Omega \phi_{n_k}\, E(x)\cdot\nabla u_{n_k} + \int_\Omega f\,\phi_{n_k}$$

that is

$$\int_\Omega M_{n_k}^*(x)\nabla\phi_{n_k}\nabla u_{n_k} = \int_\Omega \phi_{n_k}\, E(x)\cdot\nabla u_{n_k} + \int_\Omega f\,\phi_{n_k}.$$

Thanks to (5.5), we can pass to the limit and we say that

$$\int_\Omega M_0(x)\nabla\tilde{u}\nabla\phi_0 = \int_\Omega M_0^*(x)\nabla\phi_0\nabla\tilde{u} = \int_\Omega \phi_0\, E(x)\cdot\nabla\tilde{u} + \int_\Omega f\,\phi_0,$$

for every function ϕ_0 belonging to $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$; by uniqueness, this implies that $\tilde{u} = u_0$.

Appendix A

In this section, we prove the existence of a bounded weak solution u_n of (2.1). We consider the Dirichlet problem

$$u_h \in W_0^{1,2}(\Omega):\ -\text{div}(M(x)\nabla u_h) = \frac{B(x)\cdot\nabla u_h}{(1+h|u_h|)^2} + b(x), \tag{A.1}$$

where $h \in \mathbb{R}^+$, under the assumptions $B \in (L^\infty(\Omega))^N$, $b \in L^\infty(\Omega)$ and (1.1). Due to the properties of the differential operator $L(v) = -\text{div}(M(x)\nabla v)$, (A.1) can be rewritten as a fixed point problem

$$u_h \in W_0^{1,2}(\Omega):\ u_h = L^{-1}\bigg(\frac{B(x)\cdot\nabla u_h}{(1+h|u_h|)^2} + b(x)\bigg). \tag{A.2}$$

Define the operator

$$T(v) = L^{-1}\bigg(\frac{B(x)\cdot\nabla v}{(1+h|v|)^2} + b(x)\bigg).$$

Since $T: W_0^{1,2}(\Omega) \to W_0^{1,2}(\Omega)$ is well defined and compact, in order to apply the Schaefer fixed point theorem ([30]; see also [22], Thm. 4, Sect. 9.2.2), we need to prove that the set

$$X = \{v \in W_0^{1,2}(\Omega) : v = t\,T(v),\ \text{for some } t \in [0,1]\}$$

is bounded, so that T has a fixed point.

If $v \in X$, then

$$L(v) = t\left(\frac{B(x)\cdot\nabla v}{(1+h|v|)^2} + b(x)\right) \tag{A.3}$$

and we use $\frac{v}{1+h|v|}$ as test function and we have

$$\alpha\int_\Omega \frac{|\nabla v|^2}{(1+h|v|)^2} \le \int_\Omega \frac{B(x)\cdot\nabla v}{(1+h|v|)^2}\frac{v}{1+h|v|} + \int_\Omega b(x)\frac{v}{1+h|v|}$$

$$\le \frac{1}{h}\int_\Omega \frac{B(x)\cdot\nabla v}{(1+h|v|)} + \frac{1}{h}\int_\Omega b(x)$$

where use used that $t\in[0,1]$, $\frac{1}{(1+h|v|)^2} \le \frac{1}{(1+h|v|)}$, $\frac{v}{1+h|v|} \le \frac{1}{h}$. Then we use the Young inequality ($0<r<\alpha$) and we obtain

$$(\alpha - r)\int_\Omega \frac{|\nabla v|^2}{(1+h|v|)^2} \le \frac{1}{h^2}\int_\Omega |B(x)|^2 + \frac{1}{h}\int_\Omega b(x),$$

that is

$$\int_\Omega \frac{|\nabla v|^2}{(1+h|v|)^2} \le C_1(1+h^2). \tag{A.4}$$

Now, we use v has test function in (A.3) and we have (use again $t\in[0,1]$, $\frac{1}{(1+h|v|)^2} \le \frac{1}{(1+h|v|)}$ and the Hölder inequality)

$$\alpha\int_\Omega |\nabla v|^2 \le \|B\|_{L^\infty}\int_\Omega \frac{\nabla v}{(1+h|v|)}|v| + \|b\|_{L^\infty}\int_\Omega |v|$$

$$\le \|B\|_{L^\infty}[C_1(1+h^2)]^{\frac12}\left[\int_\Omega |v|^2\right]^{\frac12} + \|b\|_{L^\infty}\int_\Omega |v|,$$

which proves that X is bounded (of course for h, B and b fixed). Thus, the problem (A.2) has a fixed point and (A.1) has a solution.

Now we prove that $u_h \in L^\infty(\Omega)$. We take $G_k(u_h)$ as test function in (A.1) (recall (A.4))

$$\alpha\int_\Omega |\nabla G_k(u_h)|^2 \le \|B\|_{L^\infty}[C_1(1+h^2)]^{\frac12}\left[\int_\Omega |G_k(u_h)|^2\right]^{\frac12} + \|b\|_{L^\infty}\int_\Omega |G_k(u_h)|.$$

Here, the Stampacchia method (see [33]) gives the boundedness of u_h (of course for h, B and b fixed).

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