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# A NOTE ON RELAXATION WITH CONSTRAINTS ON THE DETERMINANT

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**Abstract.** We consider multiple integrals of the Calculus of Variations of the form $E(u) = \int W(x, u(x), Du(x))\, \mathrm{d}x$ where $W$ is a Carathéodory function finite on matrices satisfying an orientation preserving or an incompressibility constraint of the type, $\det Du > 0$ or $\det Du = 1$, respectively. Under suitable growth and lower semicontinuity assumptions in the $u$ variable we prove that the functional $\int W^{qc}(x, u(x), Du(x))\, \mathrm{d}x$ is an upper bound for the relaxation of $E$ and coincides with the relaxation if the quasiconvex envelope $W^{qc}$ of $W$ is polyconvex and satisfies $p$ growth from below for $p$ bigger then the ambient dimension. Our result generalises a previous one by Conti and Dolzmann [*Arch. Rational Mech. Anal.* **2**17 (2015) 413–437] relative to the case where $W$ depends only on the gradient variable.

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## 1. Introduction

In a recent paper [2] Conti and Dolzmann proved an interesting relaxation result concerning functionals of the type

$$E(u) = \int_\Omega W(Du(x))\, \mathrm{d}x,$$

where the function $W : \mathbb{R}^{n\times n} \to [0, +\infty]$ is finite and continuous on the set of $n \times n$ matrices with positive determinant $\mathbb{R}^{n\times n}_+ := \{F \in \mathbb{R}^{n\times n} : \det M > 0\}$ and $W \equiv +\infty$ elsewhere. Under suitable assumptions on the behaviour of $W(F)$ as $|F| \to +\infty$ or $\det F \to 0$, they prove that the $L^1$ relaxation of $E$ on $W^{1,p}(\Omega; \mathbb{R}^n)$ is given by

$$E^*(u) = \int_\Omega W^{qc}(Du(x))\, \mathrm{d}x,$$

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provided that the quasiconvex envelope

$$W^{qc}(F) = \inf \left\{ \fint_{B_1} W(D\varphi(x))\, \mathrm{d}x : \varphi \in W^{1,\infty}(B_1, \mathbb{R}^n),\ \varphi(x) = Fx \text{ for } x \in \partial B_1 \right\},$$

is indeed a polyconvex function. As far as we know, this is the first relaxation result where the energy functional takes into account the orientation preserving constraint $\det Du > 0$. The importance of such a constraint is evident in the theory of nonlinear elasticity where it is assumed as a replacement of the more complicated requirement of the injectivity of the deformation $u$ along with the condition that the potential energy diverges as the determinant of $u$ is positive and converges to zero. The authors also prove a similar result in the case that $\det Du = 1$ is assumed to hold almost everywhere.

In this note we extend the previous result to the case where $W$ is a Carathéodory function depending also on $x$ and $u$. More precisely, as in [2], we first prove that the $L^1$ relaxation of $E$ on $W^{1,p}(\Omega; \mathbb{R}^n)$ can be bounded from above as follows

$$E^*(u) \le \int_\Omega W^{qc}(x, u(x), Du(x))\, \mathrm{d}x, \tag{1.1}$$

provided that $W$ is controlled by $|u|^p + |F|^p + \theta(F)$ for some $p \ge 1$ and $\theta : \mathbb{R}^{n\times n}_+ \to [0, +\infty)$ is a continuous function satisfying the following sub-multiplicative inequality (see [1])

$$\theta(FG) \le C_0(1 + \theta(F))(1 + \theta(G)), \quad \forall F, G \in \mathbb{R}^{n\times n}_+.$$

To prove (1.1) one has to construct a sequence of $W^{1,p}(\Omega; \mathbb{R}^n)$ functions $u_j$ converging to $u$ in $L^1(\Omega; \mathbb{R}^n)$ and such that

$$\limsup_j \int_\Omega W(x, u_j(x), Du_j(x))\, \mathrm{d}x \le \int_\Omega W^{qc}(x, u(x), Du(x))\, \mathrm{d}x.$$

Such a sequence cannot be obtained by adding small variations to $u$ since this would not preserve the determinant constraint. Instead, following [2], one has to construct $u_j$ by composing $u$ with inner variations $\varphi_j$ having positive determinant. Note that, if $W$ were continuous in the gradient variable uniformly with respect to $(x, u)$ the proof of (1.1) would go exactly as in [2]. Therefore in our case the idea is to try to reduce to this case by finding two compact sets $K \subset \Omega$ and $H \subset \mathbb{R}^n$ such that $|\Omega \setminus K|$ is small, $W$ is continuous in $K \times H \times \mathbb{R}^{n\times n}_+$, $u|_K$ is continuous and $u(K) \subset H$. Then the upper-bound could be proved by constructing an approximating sequence $u_j$ converging uniformly to $u$ in $K$ and such that $u_j(K) \subset H$. However if $u_j = u \circ \varphi_j$ the last inclusion does not hold. Therefore, in order to restore this kind of argument, one needs to rely on more delicate density estimates (see Lem. 3.2).

Finally, the matching lower bound required to complete the proof of the relaxation formula follows by standard lower-semicontinuity results under the assumptions that $p \ge n$, $W^{qc}$ is polyconvex and $\theta(F) \ge \eta(\det F)$ for some convex function $\eta : \mathbb{R} \to (0, +\infty]$ with $\lim_{t\to 0^+} \eta(t) = +\infty$.

## 2. Setting of the problem and preliminary results

We denote by $\Omega \subset \mathbb{R}^n$ an open bounded set with Lipschitz boundary. Given a measurable set $E \subset \mathbb{R}^n$ we denote by $|E|$ its $n$-dimensional Lebesgue measure. For $x \in \mathbb{R}^n$ and $r > 0$ we denote by $B_r(x)$ the open ball of radius $r$ centred at $x$ and we set $B_r := B_r(0)$. Given $f \in L^1_{loc}(\mathbb{R}^n)$ we define its precise representative at $x$ to be $f(x)$ at every Lebesgue point and 0 otherwise. Given a measurable set $E \subset \mathbb{R}^n$ we say that $E$ has density one at $x \in \mathbb{R}^n$ if $x$ is a Lebesgue point for the characteristic function $\chi_E$ of $E$. Throughout the paper we shall denote by $C$ a positive constant whose value may change from line to line.

We assume that $W : \Omega \times \mathbb{R}^n \times \mathbb{R}^{n\times n}_+ \to [0, +\infty)$ is a *Carathéodory function*, that is such that for a.e. $x \in \Omega$ $W(x, \cdot, \cdot)$ is continuous and for all $(u, F) \in \mathbb{R}^n \times \mathbb{R}^{n\times n}_+$ $W(\cdot, u, F)$ is measurable. We assume that $W$ satisfies the following set of assumptions: there exist $C_0 > 0$ and $p \geq 1$ such that

$$\frac{1}{C_0}\left(|u|^p + |F|^p + \theta(F)\right) \leq W(x, u, F) \leq C_0\left(1 + |u|^p + |F|^p + \theta(F)\right), \tag{2.1}$$

where $\theta : \mathbb{R}^{n\times n}_+ \to [0, +\infty)$ is a continuous function such that for all $F, G \in \mathbb{R}^{n\times n}_+$

$$\theta(FG) \leq C_0(1 + \theta(F))(1 + \theta(G)). \tag{2.2}$$

In what follows we will make use of a characterisation of Charathéodory functions due to Scorza-Dragoni (see [3], Chap. VIII, Sect. 1.3).

**Theorem 2.1.** *[Scorza-Dragoni] Let $E \subset \mathbb{R}^m$ be a Borel set. A mapping $f : \Omega \times E \to [0, +\infty]$ is a Carathéodory function if and only if for all compact sets $K \subset \Omega$ and all $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subset K$ such that $|K \setminus K_\varepsilon| \leq \varepsilon$ for which the restriction of $f$ to $K_\varepsilon \times E$ is continuous.*

In the following we shall always assume that $W$ is extended outside $\mathbb{R}^n \times \mathbb{R}^{n\times n}_+$ by setting $W(x, u, F) := +\infty$ for all $(x, u) \in \Omega \times \mathbb{R}^n$ and $F$ such that $\det F \leq 0$.

Given $(x_0, u_0) \in \Omega \times \mathbb{R}^n$, we denote by $W^{qc} : \Omega \times \mathbb{R}^n \times \mathbb{R}^{n\times n} \to [0, +\infty]$ the *quasi-convex envelope* of $W$ defined as

$$W^{qc}(x_0, u_0, F) = \inf\left\{ \fint_{B_1} W(x_0, u_0, D\varphi(x))\,\mathrm{d}x : \varphi \in W^{1,\infty}(B_1, \mathbb{R}^n),\ \varphi(x) = Fx \text{ for } x \in \partial B_1 \right\}. \tag{2.3}$$

Note that if $\det F \leq 0$ then $W^{qc}(x_0, u_0, F) = +\infty$.

The next result is proved in Lemma 3.1 of [2] and generalises the continuity properties of the convolution operator between $L^p$ spaces.

**Lemma 2.2.** *Let $\psi \in W^{1,\infty}(B_r; \overline{B}_r)$, $g \in L^1(B_r)$, $f \in L^1(B_{2r}(x_0))$, for some $x_0 \in \mathbb{R}^n$ and $r > 0$. Then there exists a measurable set $E \subset B_r(x_0)$ of positive measure with the following property. For any $y_0 \in E$ the function*

$$\widetilde{f}(x) = f(\psi(x - y_0) + y_0)g(x - y_0)$$

*belongs to $L^1(B_r(y_0))$ and*

$$\|\widetilde{f}\|_{L^1(B_r(y_0))} \leq \frac{1}{|B_r|}\|f\|_{L^1(B_{2r}(x_0))}\|g\|_{L^1(B_r)}. \tag{2.4}$$

## 3. The orientation preserving case

In this section we state our main result in the orientation preserving case.

In the next lemma we prove some of the main properties of the quasiconvex envelope $W^{qc}$ of $W$.

**Lemma 3.1.** *Let $W$ satisfy the assumption* (2.1) *and* (2.2)*. Then there exists a Borel function $\widetilde{W}^{qc}:\Omega\times\mathbb{R}^n\times\mathbb{R}^{n\times n}_+\to[0,+\infty)$ such that for almost every $x\in\Omega$ $\widetilde{W}^{qc}(x,\cdot,\cdot)=W^{qc}(x,\cdot,\cdot)$,*

$$(u,F)\in\mathbb{R}^n\times\mathbb{R}^{n\times n}_+\mapsto\widetilde{W}^{qc}(x,u,F)\ \textit{is upper semicontinuous}$$

*and for almost every $x\in\Omega$, $u\in\mathbb{R}^n$ and $F\in\mathbb{R}^{n\times n}_+$ it holds*

$$\frac{1}{C_0}\left(|u|^p+|F|^p+\theta^{qc}(F)\right)\le\widetilde{W}^{qc}(x,u,F)\le C_0\left(1+|u|^p+|F|^p+\theta(F)\right),\tag{3.1}$$

*where $C_0$ is the constant in* (2.1)*. Assume moreover that for almost every $x_0\in\Omega$ and all $u_0\in\mathbb{R}^n$ there exists a modulus of continuity $\omega_0:(0,+\infty)\to(0,+\infty)$ such that for all $u\in\mathbb{R}^n$ and for all $F\in\mathbb{R}^{n\times n}_+$*

$$W(x_0,u_0,F)\le W(x_0,u,F)+\omega_0(|u-u_0|)(1+W(x_0,u,F)).\tag{3.2}$$

*Then $W^{qc}$ is a Carathéodory function and satisfies* (3.2)*.*

*Proof.* Since $W$ is a Carathéodory function, by Theorem 2.1 there exists an increasing sequence of compact sets $K_i\subset\Omega$ such that $|\Omega\setminus\bigcup_i K_i|=0$ and $W$ is continuous when restricted to $K_i\times\mathbb{R}^n\times\mathbb{R}^{n\times n}_+$. As a result, in order to prove the first part of the lemma, it is enough to show that the function $W^{qc}(x,u,F)\chi_{K_i}$ is upper semicontinuous on $\Omega\times R^n\times\mathbb{R}^{n\times n}_+$ for all $i\in\mathbb{N}$ and to set

$$\widetilde{W}^{qc}(x,u,F)=\begin{cases}W^{qc}(x,u,F) & \text{if } x\in\bigcup_{i=1}^\infty K_i\\ 0 & \text{otherwise.}\end{cases}$$

Let $(x_h,u_h,F_h)\to(x,u,F)$. Note that the upper semicontinuity of $W^{qc}\chi_{K_i}$ follows trivially if $x_h\notin K_i$ for $h$ large. Hence, without loss of generality we may assume that $x,x_h\in K_i$ for all $h\in\mathbb{N}$. Fix $\varphi\in W^{1,\infty}(B_1,\mathbb{R}^n)$ such that $\varphi(x)=Fx$ on $\partial B_1$. Without loss of generality we may assume that $\fint_{B_1}W(x,u,D\varphi(y))\,dy\le C$. Then, given $\gamma>0$ we can write

$$\begin{aligned}\fint_{B_1}&W(x_h,u_h,F_hF^{-1}D\varphi(y))\,dy-\fint_{B_1}W(x,u,D\varphi(y))\,dy\\&=\frac{1}{|B_1|}\int_{B_1\cap\{\det D\varphi<\gamma\}}\big(W(x_h,u_h,F_hF^{-1}D\varphi(y))-W(x,u,D\varphi(y))\big)\,dy\\&\quad+\frac{1}{|B_1|}\int_{B_1\cap\{\det D\varphi\ge\gamma\}}\big(W(x_h,u_h,F_hF^{-1}D\varphi(y))-W(x,u,D\varphi(y))\big)\,dy.\end{aligned}$$

Observe that the first integral on the right hand side is controlled by

$$C\int_{B_1\cap\{\det D\varphi<\gamma\}}(1+W(x,u,D\varphi(y)))\,dy,$$

hence it converges to zero, uniformly with respect to $h$ as $\gamma\to0$. The second integral tends to zero thanks to the uniform continuity of $W$ on the compact subsets of $K_i\times\mathbb{R}^n\times\mathbb{R}^{n\times n}_+$. Therefore, taking first the limsup as $h\to+\infty$ and then letting $\gamma\to0$ we have

$$\limsup_h W^{qc}(x_h,u_h,F_h)\le\limsup_h\fint_{B_1}W(x_h,u_h,F_hF^{-1}D\varphi(y))\,dy\le\fint_{B_1}W(x,u,D\varphi(y))\,dy.$$

Taking the infimum over $\varphi$ the upper semicontinuity follows by the definition of quasiconvex envelope in (2.3). Again using definition (2.3) we have that inequality (3.1) follows from (2.1). Similarly, if $W$ satisfies (3.2), the same holds for $W^{qc}$. In order to prove that $W^{qc}$ is a Carathéodory function it is enough to show that for a.e. $x \in \Omega$ the function $(u, F) \mapsto W^{qc}(x, u, F)$ is lower semicontinuous. This property follows by combining (3.2) for $W^{qc}$ with the fact that for a.e. $x \in \Omega$ and for all $u \in \mathbb{R}^n$ the function $W^{q,c}(x, u, \cdot)$ is quasiconvex in $\mathbb{R}^{n\times n}_+$ hence continuous ([2], Lem. 3.4). □

In what follows we shall assume that the function $\theta$ satisfies

$$\theta(F) \leq C(1 + \theta^{qc}(F)), \tag{3.3}$$

for some $C > 0$. The next lemma provides the key ingredients to prove the upper-bound estimate on the energy in (3.16).

**Lemma 3.2.** *Let $W : \Omega \times \mathbb{R}^n \times \mathbb{R}^{n\times n}_+ \to [0, +\infty)$ be a Carathéodory function satisfying (2.1) and (2.2) and such that $W^{qc}$ is Carathéodory too. Let $K \subset \Omega$ be a compact set such that $W$ and $W^{qc}$ are both continuous in $K \times \mathbb{R}^n \times \mathbb{R}^{n\times n}_+$ and that $u|_K$ is continuous. Assume moreover that there exists $M > 1$ such that for a.e. $x \in K$*

$$|u(x)| + |Du(x)| \leq M, \quad \det Du(x) \geq \frac{1}{M}.$$

*Given $\varepsilon > 0$, for a.e. $x_0 \in K$ there exists $r_0 \in (0, 1)$ such that for all $0 < r < r_0$ there exist $y_0 \in B_r(x_0)$, $z \in W^{1,p}(B_{2r}(x_0); \mathbb{R}^n)$ with $z = u$ in $B_{2r}(x_0) \setminus B_r(y_0)$, such that*

$$\int_{B_r(y_0)} W(x, z(x), Dz(x))\, \mathrm{d}x \leq \int_{B_r(y_0)} (W^{qc}(x, u(x), Du(x)) + \varepsilon)\, \mathrm{d}x, \tag{3.4}$$

$$\int_{B_r(y_0)} |z(x) - u(x)|^p\, \mathrm{d}x \leq C_1 r^p \int_{B_r(y_0)} (1 + W(x, u(x), Du(x)))\, \mathrm{d}x, \tag{3.5}$$

*where $C_1 > 0$ is a constant depending only on $n, p, M$ and $C_0$.*

*Proof.* In what follows we will explicitly indicate the dependence of the constants on the various parameters by a subscript. Since $M$ is fixed, the dependence on $M$ will be not highlighted.

Let $x_0 \in K$ be a a Lebesgue point for $u$, $Du$ and $\theta(Du)$ where $K$ has density one. We set $u_0 = u(x_0)$ and $F = Du(x_0)$. Let $\varphi_\varepsilon \in W^{1,\infty}(B_1; \mathbb{R}^n)$ be such that $\varphi_\varepsilon(x) = Fx$ on $\partial B_1$ and

$$⨍_{B_1} W(x_0, u_0, D\varphi_\varepsilon(y))\, \mathrm{d}y \leq W^{qc}(x_0, u_0, F) + \varepsilon. \tag{3.6}$$

For $r > 0$ we set $\varphi_{\varepsilon,r}(x) = r\varphi_\varepsilon(\frac{x}{r})$. Clearly we have that

$$\|D\varphi_{\varepsilon,r}\|_{L^\infty(B_r)} \leq c_\varepsilon, \tag{3.7}$$

for some $c_\varepsilon > 0$ independent of $r$. Given $\delta > 0$ there exists $r_\delta > 0$ such that for all $0 < r < r_\delta$

$$⨍_{B_{2r}(x_0)} (|\chi_K(x) - 1| + |u(x) - u_0|^p + |Du(x) - F|^p + |\theta(Du(x)) - \theta(F)|)\, \mathrm{d}x \leq \delta. \tag{3.8}$$

We now apply Lemma 2.2 with $f(x) = |u(x) - u_0|^p + |Du(x) - F|^p + |\theta(Du(x)) - \theta(F)|)$, $g = 1 + \theta(F^{-1}D\varphi_{\varepsilon,r})$ and $\psi = F^{-1}\varphi_{\varepsilon,r}$. Note that by Theorem 1 from [1] $\psi(B_r) \subset \overline{B}_r$. Using the uniform bound (3.7) and the assumption (2.2) we have that $\|g\|_{L^1(B_r)} \leq C_\varepsilon |B_r|$. Therefore thanks to (2.4) we get the existence of $y_0 \in B_r$ such that

$$-\!\!\!\!\!\!\int_B (1 + \theta(Dv(x)))(|u(v(x)) - u_0|^p + |Du(v(x)) - F|^p + |\theta(Du(v(x))) - \theta(F)|) \,\mathrm{d}x \leq C_\varepsilon \delta, \tag{3.9}$$

where $B = B_r(y_0)$ and where we have set

$$v(x) = \begin{cases} F^{-1}\varphi_{\varepsilon,r}(x - y_0) + y_0 & \text{if } x \in B_r(y_0) \\ x & \text{if } x \in \Omega \setminus B_r(y_0). \end{cases}$$

Define $z \in W^{1,p}(\Omega;\mathbb{R}^n)$ as $z(x) = u(v(x))$ and choose $\gamma = \gamma_\varepsilon > 0$ such that

$$\int_{B_r \cap \{\det D\varphi_{\varepsilon,r} \leq \gamma\}} (1 + \theta(D\varphi_{\varepsilon,r}(x))) \,\mathrm{d}x \leq \frac{\varepsilon}{\tilde{C}_\varepsilon}|B_r|, \tag{3.10}$$

for all $r < 1$, where $\tilde{C}_\varepsilon > 0$ is a constant that will appear below. Note that this choice of $\gamma$ independent of $r$ is possible since $\varphi_{\varepsilon,r}(x) = r\varphi_\varepsilon(\frac{x}{r})$ and thanks to (3.6) and (2.1). In what follows, to shorten notation we set $\hat{\varphi}_{\varepsilon,r}(x) = \varphi_{\varepsilon,r}(x - y_0)$. We now split the difference of the two integrals in (3.4) as follows

$$\begin{aligned}
\int_B W(x, z(x), Dz(x)) - W^{qc}(x, u(x), Du(x)) &= \int_B W(x, z(x), Dz(x)) - W(x_0, u_0, Dz(x)) \\
&+ \int_B W(x_0, u_0, Dz(x)) - W(x_0, u_0, D\hat{\varphi}_{\varepsilon,r}(x)) + \int_B W(x_0, u_0, D\hat{\varphi}_{\varepsilon,r}(x)) - W^{qc}(x_0, u_0, F) \\
&+ \int_B W^{qc}(x_0, u_0, F) - W^{qc}(x_0, u_0, Du(x)) + \int_B W^{qc}(x_0, u_0, Du(x)) - W^{qc}(x, u(x), Du(x)) \\
&= N_1 + V_1 + V_2 + V_3 + N_2.
\end{aligned}$$

Note that the terms $V_i$ in the previous chain of equalities already appear in the proof of Lemma 3.2 in [2] and can be treated as therein. Thus we start by estimating the terms $N_i$ which are produced by the dependence of $W$ on $x$ and $u$. To this end we set for $\sigma \in (0,1)$

$$E_\sigma := \{x \in \Omega : |u(v(x)) - u_0|^p + |Du(v(x)) - F|^p \leq \sigma\}.$$

In order to estimate $N_1$ we split $B$ in four mutually disjoint subsets as follows. First, observe that from assumption on $u$ we have immediately that $|F| \leq M$ and $|F^{-1}| \leq c(n)M^n$, for some constant $c(n)$ depending only on the dimension. Then, using (2.1), (2.2), (3.9) and (3.10), it follows that there exists $\tilde{C}_\varepsilon > 0$ depending on $\varepsilon$ but not on $r$ and $\gamma$, such that

$$\begin{aligned}
&\int_{B\cap\{\det D\hat{\varphi}_{\varepsilon,r}\leq\gamma\}} W(x,z(x),Dz(x))-W(x_0,u_0,Dz(x))\,\mathrm{d}x\\
&\quad\leq C_\varepsilon\int_{B\cap\{\det D\hat{\varphi}_{\varepsilon,r}\leq\gamma\}}(1+|u(v(x))-u_0|^p+|Du(v(x))Dv(x)-FDv(x)|^p\\
&\qquad+(1+\theta(Du(v(x))))(1+\theta(Dv(x))))\,\mathrm{d}x\\
&\quad\leq C_\varepsilon\int_B|u(v(x))-u_0|^p+|Du(v(x))-F|^p+|\theta(Du(v(x)))-\theta(F)|\,\mathrm{d}x\\
&\qquad+\tilde{C}_\varepsilon\int_{B\cap\{\det D\hat{\varphi}_{\varepsilon,r}\leq\gamma\}}\big(1+\theta(D\hat{\varphi}_{\varepsilon,r}(x))\big)\,\mathrm{d}x\leq\tilde{C}_\varepsilon\delta|B|+\varepsilon|B|. \qquad (3.11)
\end{aligned}$$

Observe now that there exists $\sigma_\varepsilon$ depending only on $\gamma$ (hence on $\varepsilon$) such that if $0<\sigma<\sigma_\varepsilon$ on the set $B\cap K\cap\{\det D\hat{\varphi}_{\varepsilon,r}\geq\gamma\}\cap E_\sigma$ we have that $\det Dz\geq\gamma/2$, $|Dz|\leq C_\varepsilon M^n$, $|z|\leq 2M$. Therefore by the uniform continuity of $W$ on compact subsets of $K\times\mathbb{R}^n\times\mathbb{R}^{n\times n}_+$, there exists $r_\varepsilon$ such that for $0<r<r_\varepsilon$ and for $\sigma_\varepsilon$ sufficiently small

$$\int_{B\cap K\cap\{\det D\hat{\varphi}_{\varepsilon,r}\geq\gamma\}\cap E_\sigma}|W(x,z(x),Dz(x))-W(x_0,u_0,Dz(x))|\,\mathrm{d}x\leq\varepsilon|B|. \qquad (3.12)$$

Arguing as in the proof of (3.11) we have

$$\begin{aligned}
&\int_{B\cap K\cap\{\det D\hat{\varphi}_{\varepsilon,r}\geq\gamma\}\setminus E_\sigma}|W(x,z(x),Dz(x))-W(x_0,u_0,Dz(x))|\,\mathrm{d}x\\
&\leq C_\varepsilon\int_{B\setminus E_\sigma}(1+\theta(Dv(x)))(1+|u(v(x))-u_0|^p+|Du(v(x))-F|^p+|\theta(Du(v(x)))-\theta(F)|)\,\mathrm{d}x\\
&\leq\frac{C_\varepsilon}{\sigma}\int_{B\setminus E_\sigma}(1+\theta(Dv(x)))(\sigma+|u(v(x))-u_0|^p+|Du(v(x))-F|^p+|\theta(Du(v(x)))-\theta(F)|)\,\mathrm{d}x\\
&\leq\frac{C_\varepsilon}{\sigma}\delta|B|, \qquad (3.13)
\end{aligned}$$

where the last inequality follows from (3.9) and the fact that $\sigma<|u(v(x))-u_0|^p+|Du(v(x))-F|^p$ for $x\in\Omega\setminus E_\sigma$.

$$\begin{aligned}
&\int_{B\setminus K\cap\{\det D\hat{\varphi}_{\varepsilon,r}\geq\gamma\}}|W(x,z(x),Dz(x))-W(x_0,u_0,Dz(x))|\,\mathrm{d}x\\
&\leq C_\varepsilon\int_B(1+\theta(Dv(x)))(|u(v(x))-u_0|^p+|Du(v(x))-F|^p+|\theta(Du(v(x)))-\theta(F)|)\,\mathrm{d}x\\
&\quad+C\int_{B\setminus K\cap\{\det D\hat{\varphi}_{\varepsilon,r}\geq\gamma\}}(1+\theta(Dv(x)))\,\mathrm{d}x\leq C_\varepsilon\delta|B|+C_\varepsilon|B_{2r}(x_0)\setminus K|\leq C_\varepsilon\delta|B|, \qquad (3.14)
\end{aligned}$$

where in the last two inequalities we used first (3.9) and then (3.8). Combining the previous estimates, for all $0<r<\min\{r_\delta,r_\varepsilon\}$ and for all $0<\sigma<\sigma_\varepsilon$ we have that

$$N_1\leq|B|\left(C_\varepsilon\delta+2\varepsilon+\frac{C_\varepsilon}{\sigma}\delta\right).$$

In order to estimate $N_2$ we split it in two terms

$$N_2 = \int_{B\setminus K} W^{qc}(x_0, u_0, Du(x)) - W^{qc}(x, u(x), Du(x))\,\mathrm{d}x + \int_{B\cap K} W^{qc}(x_0, u_0, Du(x)) - W^{qc}(x, u(x), Du(x))\,\mathrm{d}x. \tag{3.15}$$

The first integral in (3.15) is estimated by

$$C\int_{B\setminus K}(1 + |u(x) - u_0|^p + |Du(x) - F|^p + |\theta(Du(x) - \theta(F)|)\,\mathrm{d}x \le C\delta|B|$$

thanks to (3.8). We recall that on $K$ we have that $|u(x)| + |Du(x)| \le M$, that $\det Du(x) \ge \frac{1}{M}$ and that $u|_K$ is continuous. Therefore, by the uniform continuity of $W^{qc}$ on compact subsets of $K\times\mathbb{R}^n\times\mathbb{R}^{n\times n}_+$, the second integral can be estimated by $\varepsilon|B|$ provided $0 < r < r_\varepsilon$ for a suitable $r_\varepsilon$. In conclusion we have that, if $r < \min\{r_\varepsilon, r_\delta\}$

$$N_2 \le (C\delta + \varepsilon)|B|.$$

We now turn to the estimates of the $V_i$ terms. To estimate $V_1$ it is enough to split the integral into the three sets $B\cap\{\det\hat\varphi_{\varepsilon,r} \le \gamma\}$, $B\cap\{\det\hat\varphi_{\varepsilon,r} \ge \gamma\}\cap\{x\in\Omega : |Du(v(x)) - F|^p \le \sigma\}$ and $B\cap\{\det\hat\varphi_{\varepsilon,r} \ge \gamma\}\cap\{x\in\Omega : |Du(v(x)) - F|^p \ge \sigma\}$. The integral on the first set is estimated as in (3.11). The second one is estimated as in (3.12), using the continuity of $W(x_0, u_0, \cdot)$, and the third one is estimated inside and outside $K$ as in (3.13) and (3.14). In conclusion there exist $\sigma_\varepsilon$ and $r_\varepsilon$ such that if $0 < r < \min\{r_\varepsilon, r_\delta\}$ and $0 < \sigma < \sigma_\varepsilon$ we have

$$V_1 \le \left(C_\varepsilon\delta + 2\varepsilon + C_\varepsilon\frac{\delta}{\sigma}\right)|B|.$$

By (3.6)

$$V_2 \le \varepsilon|B|.$$

The term $V_3$ can be estimated by splitting $B$ into the three sets $B\cap K\cap\{x\in B : |Du(x) - F|^p \le \sigma\}$, $B\cap K\cap\{x\in B : |Du(x) - F|^p \ge \sigma\}$ and $B\setminus K$. Recalling that on $K$ it holds that $|Du(x)| \le M$, $\det Du \ge \frac{1}{M}$, by the uniform continuity of $W^{qc}(x_0, u_0, \cdot)$ on compact sets of $\mathbb{R}^{n\times n}_+$ we conclude that the integral on $B\cap K\cap\{x\in B : |Du(x) - F|^p \le \sigma\}$ is controlled by $\varepsilon|B|$ provided $\sigma < \sigma_\varepsilon$ for $\sigma_\varepsilon$ sufficiently small. The integral on the set $B\cap K\cap\{x\in B : |Du(x) - F|^p \ge \sigma\}$ can be treated as in (3.13) and hence estimated by $C\frac{\delta}{\sigma}|B|$ if $r < r_\delta$. Finally the integral over the set $B\setminus K$ is estimated by

$$C\int_{B\setminus K}(1 + |Du(x) - F|^p + |\theta(Du(x)) - \theta(F)|)\,\mathrm{d}x \le C\delta|B|,$$

for $0 < r < r_\delta$. On gathering together all the previous estimates, we eventually deduce that for $0 < r < \min\{r_\varepsilon, r_\delta\}$ and for $0 < \sigma < \sigma_\varepsilon$ we have

$$\int_B W(x, z(x), Dz(x)) - W^{qc}(x, u(x), Du(x))\,\mathrm{d}x \le \left(C_\varepsilon\delta + 6\varepsilon + C_\varepsilon\frac{\delta}{\sigma}\right)|B|.$$

On choosing $\sigma = \sqrt{\delta}$ and $\delta < \sigma_\varepsilon^2$ sufficiently small we get that there exists a radius $r_0$, ultimately depending only on $\varepsilon$, such that, if $r < r_0$ (3.4) holds with $\varepsilon$ replaced by $C\varepsilon$ for some constant $C$ independent of $\varepsilon$. Finally (3.5) is a consequence of Poincaré inequality and of (2.1). □

**Theorem 3.3.** *Let $W : \Omega \times \mathbb{R}^n \times \mathbb{R}^{n\times n}_+ \to [0, +\infty)$ be a Carathéodory function satisfying assumptions (2.1) and (2.2) and such that $W^{qc}$ is Carathéodory too. Assume that (3.3) holds.Then there exists a sequence $u_j \in W^{1,p}(\Omega;\mathbb{R}^n)$ such that $u_j - u \in W^{1,p}_0(\Omega;\mathbb{R}^n)$, $u_j \rightharpoonup u$ weakly in $W^{1,p}(\Omega;\mathbb{R}^n)$ and such that*

$$\limsup_j \int_\Omega W(x, u_j(x), Du_j(x))\,\mathrm{d}x \le \int_\Omega W^{qc}(x, u(x), Du(x))\,\mathrm{d}x. \tag{3.16}$$

*Proof.* Throughout the proof we may assume that $W(\cdot, u, Du) \in L^1(\Omega)$, otherwise by (3.3) and Lemma 3.1 also the right hand side of (3.16) equals $+\infty$.

Given $\varepsilon > 0$, to prove the theorem it is enough to construct a function $u_\varepsilon$ such that

$$\begin{aligned}
&\int_\Omega W(x, u_\varepsilon(x), Du_\varepsilon(x))\,\mathrm{d}x \le \int_\Omega W^{qc}(x, u(x), Du(x))\,\mathrm{d}x + C\varepsilon,\\
&\int_\Omega |u_\varepsilon(x) - u(x)|^p\,\mathrm{d}x \le \varepsilon\left(\int_\Omega 1 + W(x, u(x), Du(x))\,\mathrm{d}x\right).
\end{aligned}$$

Let $E := \{x \in \Omega : \, |u(x)| + |Du(x)| \le M, \, \det Du(x) \ge \frac{1}{M}\}$ where $M > 1$ is chosen so that $|\Omega \setminus E| < |\Omega|/8$. Since both $W$ and $W^{qc}$ are Carathéodory functions there exists a compact set $K \subset E$ with $|\Omega \setminus K| \le |\Omega|/8$ such that $W$, $W^{qc}$ are continuous in $K \times \mathbb{R}^n \times \mathbb{R}^{n\times n}_+$ and $u|_K$ is continuous too. By applying Lemma 3.2 there exists a null set $N$ such that for all $x \in K \setminus N$ there exists $r_x$ such that $C_1 r_x^p < \varepsilon$ with the property that for all $0 < r < r_x$ there exists $y \in B_r(x)$ and a function $z \in W^{1,p}(B_{2r}(x);\mathbb{R}^n)$, $z = u$ on $B_{2r}(x) \setminus B_r(y)$ and such that (3.4) and (3.5) hold.

Set $\mathcal{F} := \{\overline{B}_{2r}(x) : \, x \in K \setminus N, \overline{B}_{2r}(x) \subset \Omega, \, 0 < r < r_x\}$. By Vitali-Besicovitch covering theorem there exists a sequence of pairwise disjoint balls $\overline{B}_{2r_j}(x_j) \in \mathcal{F}$ such that $|K \setminus \bigcup_j \overline{B}_{2r_j}(x_j)| = 0$. We denote by $y_j$ and $z_j$ the corresponding points and functions obtained via Lemma 3.2 applied to the ball $B_{2r_j}(x_j)$. We fix $m \in \mathbb{N}$ such that $|K \setminus \bigcup_{j=1}^m \overline{B}_{2r_j}(x_j)| \le |\Omega|/8$ and we set $C_1 := \bigcup_{j=1}^m \overline{B}_{r_j}(y_j)$. Denote now by $w_1 \in W^{1,p}(\Omega;\mathbb{R}^n)$ the function defined as follows

$$w_1(x) = \begin{cases} z_j(x) & x \in B_{r_j}(y_j) \\ u(x) & x \in \Omega \setminus C_1. \end{cases}$$

Note that by construction $w_1 = u$ on $\partial\Omega$ and

$$\int_{C_1} W(x, w_1(x), Dw_1(x))\,\mathrm{d}x \le \int_{C_1} \left(W^{qc}(x, u(x), Du(x)) + \varepsilon\right)\mathrm{d}x, \tag{3.17}$$

$$\int_{C_1} |w_1(x) - u(x)|^p\,\mathrm{d}x \le \varepsilon \int_{C_1} 1 + W(x, u(x), Du(x))\,\mathrm{d}x. \tag{3.18}$$

Finally we observe that

$$|\Omega \setminus C_1| \le |\Omega \setminus K| + \Big|K \setminus \bigcup_{j=1}^m \overline{B}_{2r_j}(x_j)\Big| + \Big|\bigcup_{j=1}^m \overline{B}_{2r_j}(x_j) \setminus C_1\Big| \le \frac{|\Omega|}{8} + \frac{|\Omega|}{8} + \frac{|\Omega|}{2} = \frac{3}{4}|\Omega|. \tag{3.19}$$

We now iterate the previous construction in $\Omega \setminus C_1$ thus finding a compact set $C_2 \subset \Omega \setminus C_1$ with

$$|\Omega \setminus C_1 \setminus C_2| \le \frac{3}{4}|\Omega \setminus C_1| \le \Big(\frac{3}{4}\Big)^2|\Omega|$$

and a function $w_2 \in W^{1,p}(\Omega \setminus C_1)$ with $w_2 = u$ on $\partial(\Omega \setminus C_1)$ satisfying (3.17) and (3.18) with $C_1$ replaced by $C_2$. Further iterating this argument $k$ times we eventually find compact sets $C_j \subset C_{j-1}$ for $j \in \{3, \dots, k\}$ with

$$|\Omega \setminus \bigcup_{j=1}^{k} C_j| \le \Big(\frac{3}{4}\Big)^k|\Omega|$$

and functions $w_j \in W^{1,p}(\Omega \setminus \bigcup_{i=1}^{j-1} C_i)$ with $w_j = u$ on $\partial(\Omega \setminus \bigcup_{i=1}^{j-1} C_i)$ satisfying (3.17) and (3.18) with $C_1$ replaced by $C_j$. Setting $u_\varepsilon(x) = w_j(x)$ for $x \in C_j$, and $u_\varepsilon(x) = u(x)$ for $x \in \Omega \setminus \bigcup_{j=1}^{k} C_j$ we have

$$\int_\Omega |u_\varepsilon(x) - u(x)|^p \,\mathrm{d}x \le \varepsilon \int_\Omega 1 + W(x, u(x), Du(x)) \,\mathrm{d}x$$

and

$$\begin{aligned}\int_\Omega W(x, u_\varepsilon(x), Du_\varepsilon(x)) \,\mathrm{d}x &= \sum_{j=1}^{k} \int_{C_j} W(x, w_j(x), Dw_j(x)) \,\mathrm{d}x + \int_{\Omega\setminus\bigcup_{j=1}^{k} C_j} W(x, u(x), Du(x)) \,\mathrm{d}x \\ &\le \int_\Omega \varepsilon + W^{qc}(x, u(x), Du(x)) \,\mathrm{d}x + \varepsilon,\end{aligned}$$

provided $k$ is chosen so large that the measure of $\Omega \setminus \bigcup_{j=1}^{k} C_j$ is sufficiently small. □

By combining the previous theorem with well-known lower semicontinuity results we obtain the following relaxation theorem.

**Theorem 3.4.** *Let $p \ge n$ and $W : \Omega \times \mathbb{R}^n \times \mathbb{R}^{n\times n}_+ \to [0, +\infty)$ be a Charathéodory function such that (2.1),(2.2), (3.2) and (3.3) hold. Assume moreover that there exists a convex function $\eta : (0, +\infty) \to (0, +\infty)$ with*

$$\lim_{t\to 0^+} \eta(t) = +\infty, \tag{3.20}$$

*such that for all $F \in \mathbb{R}^n \times \mathbb{R}^n_+$*

$$\theta(F) \ge \eta(\det F).$$

*Set $W := +\infty$ outside $\mathbb{R}^{n\times n}_+$ and define $W^{qc}$ as in (2.3). Finally set for all $u \in L^1(\Omega; \mathbb{R}^n)$*

$$E(u) = \int_\Omega W(x, u(x), Du(x)) \,\mathrm{d}x \qquad \text{and} \qquad E^*(u) = \int_\Omega W^{qc}(x, u(x), Du(x)) \,\mathrm{d}x.$$

*If for a.e. $x \in \Omega$ and for all $u \in \mathbb{R}^n$ the function $W^{qc}(x, u, \cdot)$ is polyconvex, then $E^*$ is the relaxation of $E$ with respect to the $L^1$ convergence, i.e.,*

$$E^*(u) = \inf\{\liminf_j E(u_j) : \, u_j \to u \text{ in } L^1(\Omega; \mathbb{R}^n)\}.$$

*Proof.* Thanks to Theorem 3.3 to prove the representation formula for $E^*$ it is enough to show that given $u_j$, $u$ with $u_j \to u$ in $L^1(\Omega;\mathbb{R}^n)$ then

$$\int_\Omega W^{qc}(x,u,Du)\,\mathrm{d}x \le \liminf_j \int_\Omega W^{qc}(x,u_j,Du_j)\,\mathrm{d}x. \tag{3.21}$$

To this end we may assume that the lim inf on the right hand side is actually a limit and that it is finite. Then, thanks to (3.1), (3.3) and recalling that $W^{qc}(x,u,F) = +\infty$ if $\det F \le 0$, the sequence $u_j$ is bounded in $W^{1,n}(\Omega;\mathbb{R}^n)$ and $\det Du_j(x) > 0$ for a.e. $x \in \Omega$. Therefore, by a well-known result in [5], we have that $\det Du_j$ is bounded in $L\log L(\Omega)$ and thus, up a to subsequence we may assume that $\det Du_j$ converges weakly in $L^1(\Omega)$ to $\det Du$ and that the same holds true for all the lower order minors. Observe also that

$$\int_\Omega \eta(\det Du(x))\,\mathrm{d}x < \infty,$$

hence from assumption (3.20) we have that $\det Du(x) > 0$ for a.e. $x \in \Omega$.

Recall that by assumption on $W^{qc}$ and by Lemma 3.1 we know that there exists a Carathéodory function $g : \Omega \times \mathbb{R}^n \times G$ where $G = \mathbb{R}^k \times (0,+\infty)$ and $k$ is the number of all minors of order $1 \le i \le (n-1)$ of an $n \times n$ matrix with $g(x,u,\cdot)$ convex for almost every $x \in \Omega$ and for all $u \in \mathbb{R}^n$. At this point (3.21) follows from well-known lower semicontinuity results (see for instance [4], Thm. 4.5). Note that this theorem is stated in the case $G = \mathbb{R}^N$. However it is easily checked that the same proof holds also in our setting with the only modification needed in the proof of Lemma 4.3 in [4] where the function $z^L$ must be replaced by

$$z^L = \begin{cases} z & \text{if } |z| < L \\ z_0 & \text{otherwise,} \end{cases}$$

for a fixed $z_0 \in G$. □

## 4. The incompressible case

In this section we consider the incompressible case.

We denote by $\Sigma \subset \mathbb{R}^{n\times n}_+$ the set of $n \times n$ matrices $F$ with $\det F = 1$. In this section we will consider an integrand $W : \Omega \times \mathbb{R}^n \times \mathbb{R}^{n\times n}_+ \to [0,+\infty]$ such that $W(x,u,F) = +\infty$ if $F \notin \Sigma$. Moreover we will assume that $W$ restricted to $\Omega \times \mathbb{R}^n \times \Sigma$ is a Carathéodory function and that there exist $C_2 > 0$ and $p \ge 1$ such that for almost every $x \in \Omega$ and for all $(u,F) \in \mathbb{R}^n \times \Sigma$

$$\frac{1}{C_2}\left(|u|^p + |F|^p\right) \le W(x,u,F) \le C_2\left(1 + |u|^p + |F|^p\right). \tag{4.1}$$

We now observe that on defining $W^{qc}$ as in (2.3), we get that for almost every $x \in \Omega$ and for all $u \in \mathbb{R}^n$, $W^{qc}(x,u,F) = +\infty$ if and only if $F \notin \Sigma$. Indeed, if $F \notin \Sigma$ there exists no Lipschitz function $\varphi$ such that $\varphi(x) = Fx$ on $\partial B_1$ with $\det \varphi(x) \equiv 1$ for almost every $x \in B_1$. On the other hand, if $F \in \Sigma$, $W^{qc}(x,u,F) \le W(x,u,F)$.

The next lemma can be proved with the same arguments as in the proof of Lemma 3.1.

**Lemma 4.1.** *Let $W : \Omega \times \mathbb{R}^n \times \Sigma \to [0,+\infty)$ satisfy the assumption (4.1). Then there exists a Borel function $\widetilde{W}^{qc} : \Omega \times \mathbb{R}^n \times \Sigma \to [0,+\infty)$ such that for almost every $x \in \Omega$, $\widetilde{W}^{qc}(x,\cdot,\cdot) = W^{qc}(x,\cdot,\cdot)$,*

$$(u,F) \in \mathbb{R}^n \times \Sigma \mapsto \widetilde{W}^{qc}(x,u,F) \text{ is upper semicontinuous}$$

*and for almost every $x \in \Omega$, $u \in \mathbb{R}^n$ and $F \in \Sigma$ it holds*

$$\frac{1}{C_2}\left(|u|^p + |F|^p\right) \le W^{qc}(x,u,F) \le C_2\left(1 + |u|^p + |F|^p\right),$$

*where $C_2$ is the constant in (4.1). Assume moreover that for almost every $x_0 \in \Omega$ and all $u_0 \in \mathbb{R}^n$ there exists a modulus of continuity $\omega_0 : (0,+\infty) \to (0,+\infty)$ such that for all $u \in \mathbb{R}^n$ and for all $F \in \Sigma$*

$$W(x_0,u_0,F) \le W(x_0,u,F) + \omega_0(|u-u_0|)(1 + W(x_0,u,F)). \tag{4.2}$$

*Then $W^{qc}$ is a Carathéodory function and satisfies (4.2).*

The next Lemma is the analogous of Lemma 3.2, which in the incompressible case simplifies both in the statement and in the proof.

**Lemma 4.2.** *Let $W : \Omega \times \mathbb{R}^n \times \Sigma \to [0,+\infty)$ be a Carathéodory function satisfying (4.1) and such that $W^{qc}$ is Carathéodory too. Let $K \subset \Omega$ be a compact set such that $W$ and $W^{qc}$ are both continuous in $K \times \mathbb{R}^n \times \Sigma$ and that $u|_K$ is continuous. Assume moreover that there exists $M > 1$ such that for a.e. $x \in K$*

$$|u(x)| + |Du(x)| \le M.$$

*Given $\varepsilon > 0$, for a.e. $x_0 \in K$ there exists $r_0 \in (0,1)$ such that for all $0 < r < r_0$ there exist $y_0 \in B_r(x_0)$, $z \in W^{1,p}(B_{2r}(x_0);\mathbb{R}^n)$ with $z = u$ in $B_{2r}(x_0) \setminus B_r(y_0)$, such that*

$$\int_{B_r(y_0)} W(x,z(x),Dz(x))\,\mathrm{d}x \le \int_{B_r(y_0)} \left(W^{qc}(x,u(x),Du(x)) + \varepsilon\right)\mathrm{d}x, \tag{4.3}$$

$$\int_{B_r(y_0)} |z(x)-u(x)|^p\,\mathrm{d}x \le C_3 r^p \int_{B_r(y_0)} \left(1 + W(x,u(x),Du(x))\right)\mathrm{d}x,$$

*where $C_3 > 0$ is a constant depending only on $n, p$ and $C_2$.*

*Proof.* The proof goes as for Lemma 3.2 and it is actually simpler.

Let $x_0 \in K$ be a point where $K$ has density one and a Lebesgue point for $u$, $Du$ such that $F = Du(x_0) \in \Sigma$. We set $u_0 = u(x_0)$. Let $\varphi_\varepsilon \in W^{1,\infty}(B_1;\mathbb{R}^n)$ such that $\varphi_\varepsilon(x) = Fx$ on $\partial B_1$ and

$$⨍_{B_1} W(x_0,u_0,D\varphi_\varepsilon(y))\,\mathrm{d}y \le W^{qc}(x_0,u_0,F) + \varepsilon.$$

For $r > 0$ we set $\varphi_{\varepsilon,r}(x) = r\varphi_\varepsilon(\frac{x}{r})$. Clearly we have that

$$\|D\varphi_{\varepsilon,r}\|_{L^\infty(B_r)} \le c_\varepsilon,$$

for some $c_\varepsilon > 0$ independent of $r$. Given $\delta > 0$ there exists $r_\delta > 0$ such that for all $0 < r < r_\delta$

$$⨍_{B_r(x_0)} \left(|\chi_K(x) - 1| + |u(x)-u_0|^p + |Du(x)-F|^p\right)\mathrm{d}x \le \delta. \tag{4.4}$$

Set for $x \in B_r(x_0)$, $v(x) = F^{-1}\varphi_{\varepsilon,r}(x-x_0)+x_0$. Thanks to Theorem 1 of [1], $v(B_r(x_0)) \subset \overline{B}_r(x_0)$. Therefore there exists a constant $C_\varepsilon$ such that for all $0<r<r_\delta$

$$\fint_B |u(v(x))-u_0|^p + |Du(v(x))-F|^p \,\mathrm{d}x \le C_\varepsilon \delta, \tag{4.5}$$

where $B = B_r(x_0)$. We define $z \in W^{1,p}(\Omega;\mathbb{R}^n)$ as $z(x) = u(v(x))$ if $x \in B$ and as $z(x) = u(x)$ for $x \in \Omega \setminus B$. To shorten notation we set $\hat{\varphi}_{\varepsilon,r}(x) = \varphi_{\varepsilon,r}(x-x_0)$. We now split the difference of the two integrals in (4.3) as follows

$$\begin{aligned}
\int_B W(x,z(x),Dz(x)) - W^{qc}(x,u(x),Du(x)) &= \int_B W(x,z(x),Dz(x)) - W(x_0,u_0,Dz(x)) \\
&+ \int_B W(x_0,u_0,Dz(x)) - W(x_0,u_0,D\hat{\varphi}_{\varepsilon,r}(x)) + \int_B W(x_0,u_0,D\hat{\varphi}_{\varepsilon,r}(x)) - W^{qc}(x_0,u_0,F) \\
&+ \int_B W^{qc}(x_0,u_0,F) - W^{qc}(x_0,u_0,Du(x)) + \int_B W^{qc}(x_0,u_0,Du(x)) - W^{qc}(x,u(x),Du(x)) \\
&= P_1 + P_2 + P_3 + P_4 + P_5.
\end{aligned}$$

We give the details of the estimates of $P_1$ and $P_5$, since the remaining terms are treated as in Lemma 3.2. Given $\sigma \in (0,1)$ we set

$$E_\sigma := \{x \in \Omega :\ |u(v(x))-u_0|^p + |Du(v(x))-F|^p \le \sigma\}.$$

In order to estimate $P_1$ we split $B$ in three mutually disjoint subsets as follows. Observe now that on the set $B \cap K \cap E_\sigma$ we have that $|Dz| \le C_\varepsilon M^{n-1}$, $|z| \le 2M$. Therefore, thanks to the uniform continuity of $W$ on compact subsets of $K \times \mathbb{R}^n \times \Sigma$ there exists $\sigma_\varepsilon$ and $r_\varepsilon$ such that for $0<\sigma<\sigma_\varepsilon$ and $0<r<r_\varepsilon$

$$\int_{B\cap K\cap E_\sigma} |W(x,z(x),Dz(x)) - W(x_0,u_0,Dz(x))| \,\mathrm{d}x \le \varepsilon |B|.$$

Arguing as in the proof of (3.11) we have

$$\begin{aligned}
&\int_{B\cap K\setminus E_\sigma} |W(x,z(x),Dz(x)) - W(x_0,u_0,Dz(x))| \,\mathrm{d}x \\
&\le C_\varepsilon \int_{B\setminus E_\sigma} 1 + |u(v(x))-u_0|^p + |Du(v(x))-F|^p \,\mathrm{d}x \\
&\le \frac{C_\varepsilon}{\sigma} \int_{B\setminus E_\sigma} \sigma + |u(v(x))-u_0|^p + |Du(v(x))-F|^p \,\mathrm{d}x \le \frac{C_\varepsilon}{\sigma}\delta|B|,
\end{aligned}$$

where the last inequality follows from (4.5) and from the fact that $\sigma < |u(v(x))-u_0|^p + |Du(v(x))-F|^p$ for $x \in \Omega \setminus E_\sigma$. Finally we estimate

$$\begin{aligned}
&\int_{B\setminus K} |W(x,z(x),Dz(x)) - W(x_0,u_0,Dz(x))| \,\mathrm{d}x \\
&\le C_\varepsilon \int_{B\setminus K} 1 + |u(v(x))-u_0|^p + |Du(v(x))-F|^p \,\mathrm{d}x \le C_\varepsilon \delta |B|,
\end{aligned}$$

where in the last two inequalities we used first (4.4) and then (4.5). In order to estimate $P_5$ we first split it in two terms

$$P_5 = \int_{B\setminus K} W^{qc}(x_0, u_0, Du(x)) - W^{qc}(x, u(x), Du(x))\,\mathrm{d}x \\ + \int_{B\cap K} W^{qc}(x_0, u_0, Du(x)) - W^{qc}(x, u(x), Du(x))\,\mathrm{d}x. \tag{4.6}$$

The first integral in (4.6) is estimated by

$$C\int_{B\setminus K} (1 + |u(x) - u_0|^p + |Du(x) - F|^p\,\mathrm{d}x \le C\delta|B|$$

thanks to (4.4). We recall that on $K$ we have that $|u(x)| + |Du| \le M$ and that $u|_K$ is continuous. Therefore, by the uniform continuity of $W^{qc}$ on compact sets of $K\times\mathbb{R}^n\times\Sigma$, the second integral can be estimated by $\varepsilon|B|$ provided $0 < r < r_\varepsilon$ for a suitable $r_\varepsilon$. Therefore we have that if $0 < r < \min\{r_\delta, r_\varepsilon\}$

$$P_5 \le (C\delta + \varepsilon)|B|.$$

Estimating the other terms as in Lemma 3.2 we conclude that if $0 < r < \min\{r_\delta, r_\varepsilon\}$ and $0 < \sigma < \sigma_\varepsilon$ we have

$$\int_B W(x, z(x), Dz(x)) - W^{qc}(x, u(x), Du(x))\,\mathrm{d}x \le \left(C_\varepsilon\delta + C\varepsilon + C_\varepsilon\frac{\delta}{\sigma}\right)|B|.$$

To conclude the proof we argue as in the final part of Lemma 3.2. □

By repeating the same construction as in Theorem 3.3 we get

**Theorem 4.3.** *Let $W : \Omega\times\mathbb{R}^n\times\Sigma \to [0,+\infty)$ be a Carathéodory function satisfying assumption (4.1) and such that $W^{qc}$ is Carathéodory too. Then there exists a sequence $u_j \in W^{1,p}(\Omega;\mathbb{R}^n)$ such that $u_j - u \in W_0^{1,p}(\Omega;\mathbb{R}^n)$, $u_j \rightharpoonup u$ weakly in $W^{1,p}(\Omega;\mathbb{R}^n)$ and such that*

$$\limsup_j \int_\Omega W(x, u_j(x), Du_j(x))\,\mathrm{d}x \le \int_\Omega W^{qc}(x, u(x), Du(x))\,\mathrm{d}x.$$

As in the previous section we can now give the following relaxation result in the incompressible case.

**Theorem 4.4.** *Let $p \ge n$ and $W : \Omega\times\mathbb{R}^n\times\Sigma \to [0,+\infty)$ be a Charathéodory function such that (4.1) and (4.2) hold. Define $W^{qc}$ as in (2.3) and extend $W$ and $W^{qc}$ to $+\infty$ outside $\Sigma$. Finally set for all $u \in L^1(\Omega;\mathbb{R}^n)$*

$$E(u) = \int_\Omega W(x, u(x), Du(x))\,\mathrm{d}x \qquad \textit{and} \qquad E^*(u) = \int_\Omega W^{qc}(x, u(x), Du(x))\,\mathrm{d}x.$$

*If for a.e. $x\in\Omega$ and for all $u\in\mathbb{R}^n$ the function $W^{qc}(x,u,\cdot)$ is polyconvex, then $E^*$ is the relaxation of $E$ with respect to the $L^1$ convergence, i.e.,*

$$E^*(u) = \inf\{\liminf_j E(u_j) :\ u_j \to u \text{ in } L^1(\Omega;\mathbb{R}^n)\}.$$

*The proof of the statement above is the same as that of Theorem 3.4, with the observation that, since $W^{qc}$ satisfies (4.2) and $W^{qc}(x,u,\cdot)$ is polyconvex, then $W^{qc}$ is a Charathéodory function.*

## References

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