AN EPIPERIMETRIC INEQUALITY FOR THE LOWER DIMENSIONAL OBSTACLE PROBLEM

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Abstract. In this paper we give a proof of an epiperimetric inequality in the setting of the lower dimensional obstacle problem. The inequality was introduced by Weiss [*Invent. Math.* **138** (1999) 23–50) for the classical obstacle problem and has striking consequences concerning the regularity of the free-boundary. Our proof follows the approach of Focardi and Spadaro [*Adv. Differ. Equ.* **21** (2015) 153–200] which uses an homogeneity approach and a Γ-convergence analysis.

1. INTRODUCTION

The obstacle problem consists in finding the minimizer of a suitable energy among all functions, with fixed boundary data, constrained to lie above a given obstacle. The obstacle can live in the whole domain or on a surface of codimension one, these cases are denoted by the classical obstacle and the lower dimensional obstacle (or the thin obstacle) respectively. In this paper we analyse a particular case of a lower dimensional obstacle where the obstacle is laid in a hyperplane of the domain and the energies are the weighted versions of Dirichlet energy. The motivation for studying lower dimensional obstacle problems has roots in many applications. There are examples in physics, mechanics, biology and financial mathematics and many prime examples can be found in [4, 12, 14, 16, 18, 27, 40, 41, 45, 46].

In this paper we consider the energy

$$\mathcal{E}(v) := \int_{B_1^+} |\nabla v|^2 \, x_n^a \, \mathrm{d}x, \tag{1.1}$$

and minimize $\mathcal{E}$ among all functions in the class of admissible functions

$$\mathfrak{A}_g := \left\{ v \in H^1(B_1^+, \mu_a) \, : \, v \geq 0 \, on \, B_1', v = g \, on \, (\partial B_1)^+ \right\}, \tag{1.2}$$

where $H^1(A, \mu_a)$ is the weighted Sobolev Space and μ_a is the measure $\mu_a := |x_n|^a \, \mathcal{L}^n \llcorner B_1$ with $a \in (-1, 1)$. In what follows we will extend automatically every functions in $\mathfrak{A}_g$ by even symmetry with respect to $\{x_n = 0\}$ and for convenience we will indicate any points $x \in \mathbb{R}^n$ as $x = (\widehat{x}, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$.

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By the direct method of calculus of variations it easy to prove the existence and uniqueness of the minimum of (1.1) on $\mathfrak{A}_g$. Let $u := \min_{\mathfrak{A}_g} \mathcal{E}$, we note that u satisfies the following Euler–Lagrange equations:

$$
\begin{cases}
u(\widehat{x}, 0) \geq 0, & \widehat{x} \in B_1' \\
u(\widehat{x}, x_n) = u(\widehat{x}, -x_n), & \\
\operatorname{div}(|x_n|^a \nabla u(\widehat{x}, x_n)) = 0, & x \in B_1 \setminus \{(\widehat{x}, 0) : u(\widehat{x}, 0) = 0\} \\
\operatorname{div}(|x_n|^a \nabla u(\widehat{x}, x_n)) \leq 0, & x \in B_1 \text{ in distributional sense.}
\end{cases}
\tag{1.3}
$$

We denote by $\Gamma(u) := \partial\{(\widehat{x}, 0) \in B_1' : u(\widehat{x}, 0) = 0\} \cap B_1'$ the free-boundary of u.

In order to establish the regularity of the solution u and its free-boundary a fundamental tool is the Almgren frequency type function (see [3] for $a = 0$). For all points $x_0 \in \Gamma(u)$:

$$
N_a^{x_0}(r, u) := \frac{r \int_{B_r(x_0)} |\nabla u|^2 \, d\mu_a}{\int_{\partial B_r(x_0)} u^2 \, |x_n|^a \, d\mathcal{H}^{n-1}}.
\tag{1.4}
$$

Caffarelli and Silvestre [13] proved the monotonicity of function $r \mapsto N_a^{x_0}(r, u)$ and some of its properties such as, the property of being constant over all homogeneous functions; Caffarelli *et al.* [15] established the property of the frequency function of being bigger than $1 + s$, where $s := \frac{1-a}{2}$ is the exponent of the fractional Laplacian of the trace, on $\mathbb{R}^{n-1} \times \{0\}$, of a global solution of (1.3) (see Sect. 2).

In the particular case $s = 1/2$, Giaquinta and Modica [35] proved $u \in Lip(B_1)$ moreover Athanasopoulos and Caffarelli [2] proved that $u \in C^{1, \frac{1}{2}}(B_1^{\pm} \cup B_1')$. More recently Rüland and Shi [47] proved the optimal regularity $C^{1, \min\{\frac{1}{2}, \beta\}}(B_1^{\pm} \cup B_1')$ in the case of $C^{0, \beta}$ coefficients.

In the general case $s \in (0, 1)$ Caffarelli *et al.* [15] proved the optimal regularity of the solution: $\nabla_{\widehat{x}} u$ is one-sided C^s, or rather $\nabla_{\widehat{x}} u \in C^s(B_1^{\pm} \cup B_1')$, and the weighted normal derivative $|x_n|^a \partial_n u$ is $C^a(B_1^{\pm} \cup B_1')$ for all $0 < a < 1 - s$.

Thanks to the monotonicity of the functional (1.4) it is possible to define the frequency of u in x_0 as $N_a^{x_0}(0^+, u) := \lim_{r \to 0^+} N_a^{x_0}(r, u)$ and to distinguish the points in $\Gamma(u)$ respect to their frequencies. The free-boundary $\Gamma(u)$ can be split as:

$$
\Gamma(u) = Reg(u) \cup Sing(u) \cup Other(u).
$$

The points of the subset $Reg(u)$ are called *regular points*, they are the points of the free-boundary with least frequency *i.e.* $1 + s$; we will denote $Reg(u)$ as $\Gamma_{1+s}(u)$. In the case $s = 1/2$ the regularity of $\Gamma_{1/2}$ was proved by Athanasopoulos *et al.* [3], while Focardi and Spadaro [23], Garofalo *et al.* [31] (case of variable Lipschitz coefficients) and Ruland and Shi [47] (case of variable Hölder coefficients) gave alternative proofs of the regularity using an epiperimetric inequality (see Thm. 1.1).

In the general case $s \in (0, 1)$ Caffarelli *et al.* [15] proved that $\Gamma_{1+s}(u)$ is locally a $C^{1, \alpha}$ $(n - 1)$-submanifold and Garofalo *et al.* [32] gave a proof of this result (following the approach in [31]) in the case with drift in the subcritical regime.

The points of the subset $Sing(u)$ are called singular points and are the points of the free-boundary with frequency $2m$ with $m \in \mathbb{N}$, equivalently their contact sets have density zero with respect to $\mathcal{H}^n$. In the case $s = 1/2$ Garofalo and Petrosyan [28] prove that $Sing(u)$ is contained in a countable union of C^1 submanifold. Very recently Garofalo and Ros-Oton [29] extended the result in [28] for $s \in (0, 1)$.

The subset $Other(u)$ is the complement of $Reg(u) \cup Sing(u)$ in $\Gamma(u)$. Recently Focardi and Spadaro [24, 24A] gave a complete description of the subset $Sing(u) \cup Other(u)$ up to a set of $\mathcal{H}^{n-2}$-measure zero. This result is new also in the framework of the Signorini problem, *i.e.* in the case $s = 1/2$, and it is obtained by a combination of analytical and geometric measure theory arguments. The goal of this paper is to give an alternative proof of the regularity of $\Gamma_{1+s}(u)$ given by Caffarelli *et al.* [15]. Our proof use an epiperimetric inequality and its

consequences. We extend the result proved by Focardi and Spadaro [23] in the case $s \in (0, 1)$. The two authors outline the presence in their proof of two competing variational principles that contribute to the achievement of proof.

In order to enunciate the epiperimetric inequality we introduce a sequence of rescaled functions $u_{x_0,r} = \frac{u(x_0+rx)}{r^{1+s}}$ and an auxiliary energy "à la Weiss"

$$W_{1+s}^{x_0}(r,u) := \frac{1}{r^{n+1}} \int_{B_r(x_0)} |\nabla u_r|^2 \, \mathrm{d}\mu_a - \frac{1+s}{r^{n+1}} \int_{\partial B_r(x_0)} |u_r|^2 \, |x_n|^a \, \mathrm{d}\mathcal{H}^{n-1}, \tag{1.5}$$

which is the sum of a volume energy and a boundary energy. We note that $1 + s$, the frequency of points of the free-boundary examined, is the exponent of the scaling factor of sequence $u_{x_0,r}$ (see Eq. (4.1)) and the coefficient of boundary energy. The existence of blow-ups is a consequence of a gradient estimate of rescaled function in $L^2(B_1, \mu_a)$; reasoning by contradiction, thanks to properties of the frequency and the optimal regularity of the solution we prove the $(1 + s)$-homogeneity of blow-ups. So, according to a result of classification by Caffarelli *et al.* [15] we state the result of the classification of $(1 + s)$-homogeneous global solutions of the fractional obstacle, which constitute the following closed cone

$$\mathfrak{H}_{1+s} := \{\lambda \, h_e \; : \; e \in \mathbb{S}^{n-2}, \lambda \in [0, +\infty)\} \subset H^1_{loc}(\mathbb{R}^n, \mu_a),$$

with

$$h_e(x) := \left(s^{-1}\widehat{x} \cdot e - \sqrt{(\widehat{x} \cdot e)^2 + x_n^2}\right) \left(\sqrt{(\widehat{x} \cdot e)^2 + x_n^2} + \widehat{x} \cdot e\right)^s.$$

The key result presented in this paper is an alternative proof (a first proof, with an extra hypothesis of closeness, was given by Garofalo *et al.* [32] which used it to give a proof of regularity of the free boundary) of a Weiss' epiperimetric inequality for the fractional obstacle problem (*cf.* [49], Thm. 1).

Theorem 1.1 (Epiperimetric inequality). *Let $\underline{0} \in \Gamma_{1+s}(u)$. There exists a dimensional constant $\kappa \in (0, 1)$ such that if $c \in H^1(B_1, \mu_a)$ is a $(1 + s)$-homogeneous function for which $c \geq 0$ on B_1' and $c(\widehat{x}, x_n) = c(\widehat{x}, -x_n)$ then*

$$\inf_{v \in \mathfrak{A}_c} W_{1+s}^0(1, v) \leq (1 - \kappa)W_{1+s}^0(1, c).$$

Taking the epiperimetric inequality into account, Weiss proved this result in [49] in the classical obstacle case. Recently Garofalo *et al.* [32] proved a similar epiperimetric inequality, with an extra hypothesis of closeness to a fixed blow-up. The assumption of closeness to a fixed blow-up in the version of the epiperimetric inequality established by Garofalo *et al.* [31] in the case $s = 1/2$ and by Garofalo *et al.* [32] in the case $s \in (1/2, 1)$ is not restrictive for the application to smoothness for the subset of regular points. On the other hand the formulation in this paper is stronger since we prove that any $(1 + s)$-homogeneous function satisfies the epiperimetric inequality without any closeness restriction. Moreover, rather than employing Almgren's and homogeneous rescalings as done in [31, 32], as a minor consequence of the stronger formulation proved in the paper only homogeneous rescalings are used to show smoothness of the subset of regular points (*cf.* Sect. 4). Finally, in case $s = 1/2$, the stronger formulation due to Focardi and Spadaro [23] has been used by Rüland and Shi [47] to show optimal regularity of solutions of thin obstacle problems with matrix coefficients in Hölder spaces (*cf.* [47], Cor. 1).

Taking the epiperimetric inequality proof into account, by a contradiction argument we obtain a quasi-minimality condition for a sequence of auxiliary functionals. Using a Γ-convergence argument we inspect the Γ-limits of the sequence of auxiliary energies and analyse their minimizer that represents the directions along which the epiperimetric inequality may fail. Using a variational method we obtain that such minimizers show in the same time contradictory relationship with the cone $\mathfrak{H}_{1+s}$.

The epiperimetric inequality is a key ingredient to deduce the following estimate of the decay of energy:

$$W_{1+s}^{x_0}(r, u) \leq C\, r^{\gamma}, \tag{1.6}$$

where C and γ are positive constants. Thanks to the decay estimate (1.6) we prove a property of nondegeneration of solutions, from which we deduce that the blow-ups are nonzero. Proceeding as in [23] we can prove the uniqueness of blow-ups and the regularity of $\Gamma_{1+s}(u)$; we state this results in Proposition 6.6 and Theorem 7.1 respectively and do not prove them because they follow by the epiperimetric inequality and its consequences as in [23], Propositions 4.8 and 4.10.

What follows is a summary of the structure of this paper: in Section 3 we introduce the frequency and its properties and define $\Gamma_{1+s}(u)$ the subset of free-boundary with low frequency. In Section 4 we prove the existence and $(1+s)$-homogeneity of blow-ups in the points in $\Gamma_{1+s}(u)$ and in Section 5, thanks to a result by [15], we characterize the $(1+s)$-homogeneous global solution of the fractional obstacle problem. Section 6 is devoted to establish the epiperimetric inequality and its consequences in the framework of the regularity of the free-boundary, a decay estimate of an auxiliary energy, the nondegeneracy of the solution and the uniqueness of the blow-ups. In Section 7 we state the regularity of $\Gamma_{1+s}(u)$.

2. PRELIMINARY RESULTS

Let $u \in \min_{\mathfrak{A}_g} \mathcal{E}$; we denote by $\Lambda(u)$ its coincidence set, $\Lambda(u) := \{\widehat{x} \in B_1' \ : \ u(\widehat{x}, 0) = 0\}$, and by $\Gamma(u)$ its free-boundary $\Gamma(u) := \partial \Lambda(u)$ in B_1' topology.

Caffarelli and Silvestre [13] showed that the Euler-Lagrange equations of u (1.3) are equivalent to the following equations:

$$\begin{cases} u(\widehat{x}, 0) \geq 0 & \widehat{x} \in B_1' \\ \operatorname{div}(x_n^a \, \nabla u(\widehat{x}, x_n)) = 0 & x_n > 0 \\ \lim_{x_n \to 0^+} x_n^a \partial_n u(\widehat{x}, x_n) = 0 & u(\widehat{x}, 0) > 0 \\ \lim_{x_n \to 0^+} x_n^a \partial_n u(\widehat{x}, x_n) \leq 0 & \widehat{x} \in B_1', \end{cases} \tag{2.1}$$

which are related to the study of the classical obstacle problem in $\mathbb{R}^{n-1}$ for fractional Laplacian $(-\Delta)^s$ with $s \in (0, 1)$, where $a = 1 - 2s$. In particular, for all v solution of $\operatorname{div}(x_n^a \, \nabla v(\widehat{x}, x_n)) = 0$ on B_1^+, with an appropriate extension to the whole $\mathbb{R}^n$, there exists the limit $\lim_{x_n \to 0^+} x_n^a \partial_n v(\widehat{x}, x_n)$ and $\lim_{x_n \to 0^+} x_n^a \partial_n v(\widehat{x}, x_n) = C(-\Delta)^s f(\widehat{x})$ with f the trace of v on $\mathbb{R}^{n-1} \times \{0\}$ and C a constant depending on n and s (cf. [13]).

For $x_n > 0$, $u(\widehat{x}, x_n)$ is smooth so the second condition in (2.1) holds in the classical sense, while the third and fourth condition in (2.1) hold in the weak sense. By Silvestre [48] $u(\widehat{x}, 0) \in C^{1,\alpha}$ with $\alpha < s$, in particular if $-a < \alpha < s$ the limit $\lim_{x_n \to 0^+} x_n^a \partial_n u(\widehat{x}, x_n)$ can be considered in the classical sense.

The function u, can be extended by simmetry $u(\widehat{x}, x_n) = u(\widehat{x}, -x_n)$. So, as shown in [13] we can rewrite the problem (2.1) as (1.3).

In order to simplify the notation, we introduce the following symbol:

$$R_a(\psi) := \lim_{\varepsilon \to 0^+} \varepsilon^a \partial_n \psi(\widehat{x}, \varepsilon) \tag{2.2}$$

for all functions ψ which are solutions for

$$\begin{cases} \psi(\widehat{x}, x_n) = \psi(\widehat{x}, -x_n) \\ L_a(\psi) := \operatorname{div}(|x_n|^a \, \nabla \psi(\widehat{x}, x_n)) = 0 & \{x_n \neq 0\}. \end{cases} \tag{2.3}$$

In what follows, we shall state a uniform estimate on the solution u, so we report a quantitative result stated in [24, 24A], Theorem 2.1.

Theorem 2.1. *For every boundary datum $g \in H^1(B_1, \mu_a)$ that respects the condition of compatibility with the problem, i.e. $g(\hat{x}, x_n) = g(\hat{x}, -x_n)$ and $g(\hat{x}, 0) \geq 0$, there exists a unique solution u to the fractional obstacle problem* (1.3). *Moreover, $\partial_{x_i} u \in C^s(B_{1/2})$ for $i = 1, \ldots, n-1$ and $|x_n|^a \partial_{x_n} u \in C^\alpha(\overline{B_{1/2}^+})$ for all $0 < \alpha < 1 - s$, and*

$$\|u\|_{X_{s,\alpha}(\overline{B_{1/2}^+})} := \|u\|_{C^0(\overline{B_{1/2}^+})} + \|\nabla \hat{x} u\|_{C^s(\overline{B_{1/2}^+})} + \||x_n|^a \partial_{x_n} u\|_{C^\alpha(\overline{B_{1/2}^+})} \leq C\|u\|_{L^2(B_1^+, \mu_a)}, \tag{2.4}$$

with $X_{s,\alpha}(\overline{B_{1/2}^+}) := \left\{ v \in H^1(B_{1/2}) \ : \ v \in C^0(\overline{B_{1/2}^+}), \nabla \hat{x} v \in C^s(\overline{B_{1/2}^+}) \text{ and } |x_n|^a \partial_{x_n} v \in C^\alpha(\overline{B_{1/2}^+}) \right\}.$

Next, we state a version of the Divergence Theorem that will be used frequently in the paper.

Theorem 2.2 (Divergence Theorem). *Let $\varphi \in H^1(B_1, \mu_a)$ and ψ be a solution of* (2.3), *then*

$$\int_{B_1} \nabla \psi \cdot \nabla \varphi \, d\mu_a = \int_{\partial B_1} \varphi \, \nabla \psi \cdot x \, |x_n|^a \, d\mathcal{H}^{n-1} - 2 \int_{B_1'} \varphi R_a(\psi) \, d\mathcal{H}^{n-1}. \tag{2.5}$$

We conclude the paragraph stating some results related to weighted Sobolev spaces. We rewrite these results for our aims, but these also hold in more general conditions.

We state the analogous of the Banach-Alaoglu-Bourbaki Theorem (see [5], Thm. III.15) for which every bounded and closed set in $H^1(B_1, \mu_a)$ is relatively compact in the weak topology.

Theorem 2.3 (Banach-Alaoglu-Bourbaki Theorem [39], Thm. 1.31). *Let v_j be a bounded sequence in $H^1(B_1, \mu_a)$. Then there exists a subsequence v_{j_i} and a function $v \in H^1(B_1, \mu_a)$ such that $v_{j_i} \rightharpoonup v$ in $L^2(B_1, \mu_a)$ and $\nabla v_{j_i} \rightharpoonup \nabla v$ in $L^2(B_1, \mu_a; \mathbb{R}^n)$.*

Moreover in view of [38], Theorem 8.1, where Heinonen and Koskela obtained an analogous of the Rellich Theorem on Sobolev metric spaces, we can deduce that every bounded and closed set in $H^1(B_1, \mu_a)$ is relatively compact in $L^2(B_1, \mu_a)$.

Theorem 2.4 (Rellich Theorem [38], Thm. 8.1). *Let v_j be a bounded sequence in $H^1(B_1, \mu_a)$. Then there exists a subsequence v_{j_i} and a function $v \in H^1(B_1, \mu_a)$ such that $v_{j_i} \to v$ in $L^2(B_1, \mu_a)$*

Futhermore, we indicate two Theorems of compact Trace embedding. We are interested in the *trace* of functions in $H^1(B_1, \mu_a)$ on $L^2(B_1')$ and $L^2(\partial B_1, |x_n|^a \mathcal{H}^{n-1})$.

Theorem 2.5 (Trace Theorem [20], Thm. 3.4). *For all $a \in (-1, 1)$ there exists a compact operator $Tr : H^1(B_1^+, \mu_a) \to L^2(B_1')$ such that $Tr(u) = u$ for every $u \in C^\infty(\overline{B_1^+})$*

The Theorem of Trace embedding on $L^2(\partial B_1, |x_n|^a \mathcal{H}^{n-1})$ is similar to the Theorem of Trace embedding in the classical Sobolev spaces, for its proof we refer to [37], Section 3.7.

Theorem 2.6 (Trace Theorem). *For all $a \in (-1, 1)$ there exists a compact operator $Tr : H^1(B_1, \mu_a) \to L^2(\partial B_1, |x_n|^a \mathcal{H}^{n-1})$ such that $Tr(u) = u$ for every $u \in C^\infty(\overline{B_1})$*

3. FREQUENCY FORMULA

Let $x_0 \in \Gamma(u)$ and $r \in (0, 1 - |x_0|)$; let $N^{x_0}(r, u)$ be the frequency function defined by

$$N_a^{x_0}(r, u) := \frac{r \int_{B_r(x_0)} |\nabla u|^2 \, d\mu_a}{\int_{\partial B_r(x_0)} u^2 \, |x_n|^a \, d\mathcal{H}^{n-1}} \tag{3.1}$$

if $u|_{\partial B_r(x_0)} \not\equiv 0$. We recall the monotonicity result due to Caffarelli and Silvestre [13].

Theorem 3.1.   *(i) The frequency function $N_a^{x_0}(r, u)$ is monotone nondecreasing in the variable r for all $r \in (0, 1 - |x_0|)$.*

(ii) For all points $x_0 \in \Gamma(u)$ the function $N^{x_0}(r, u) = \lambda$ for all $r \in (0, 1 - |x_0|)$ if and only if $u(x_0 + \cdot)$ is λ-homogeneous.

(iii) If $u(x_0 + \cdot)$ is λ-homogeneous then $\lambda \geq 1 + s$.

(iv) $N_a^{x_0}(r, u) \geq 1 + s$ for all $x_0 \in \Gamma_u$ and $r \in (0, 1 - |x_0|)$.

Proof. As far as the proof of (i), (ii) and (iii) is concerned, we refer to [13], Theorem 6.1 and [15], Proposition 5.1. As regards the proof of (iv), see Remark 4.5.   □

Thanks to Theorem 3.1(i) it is possible to define the limit $N_a^{x_0}(0^+, u) := \lim_{r \to 0^+} N_a^{x_0}(r, u)$. We denote by $\Gamma_{1+s}(u)$ the subset of points of free-boundary with frequency $1 + s$:

$$\Gamma_{1+s}(u) := \{x_0 \in \Gamma_u \ : \ N_a^{x_0}(0^+, u) = 1 + s\}. \tag{3.2}$$

Note that from the monotonicity of the frequency and by the upper semicontinuity of the function $x \mapsto N_a^x(0^+, u)$[1] the set $\Gamma_{1+s} \subset \Gamma_u$ is open in the relative topology.

We introduce the notation:

$$D_a^{x_0}(r) = \int_{B_r(x_0)} |\nabla u|^2 \, \mathrm{d}\mu_a \qquad\qquad H_a^{x_0}(r) = \int_{\partial B_r(x_0)} u^2 \, |x_n|^a \, \mathrm{d}\mathcal{H}^{n-1}$$

and we can omit to write the point x_0 if $x_0 = \underline{0}$.

All functions $H_a^{x_0}(\cdot), D_a^{x_0}(\cdot)$ and $N_a^{x_0}(\cdot)$ are absolutely continuous functions of the radius, so they are differentiable a.e.

We prove two properties of $H_a^{x_0}(r)$ (see [1], Lem. 2, [23], Lem. A.2. for the case $a = 0$).

Lemma 3.2.

(i) The function

$$(0, 1 - |x_0|) \ni r \mapsto \frac{H_a^{x_0}(r)}{r^{n+2}} \tag{3.3}$$

is nondecreasing and in particular

$$H_a^{x_0}(r) \leq \frac{H_a^{x_0}(1 - |x_0|)}{(1 - |x_0|)^{n+2}} r^{n+2} \qquad\qquad \text{for all} \quad 0 < r < 1 - |x_0|. \tag{3.4}$$

(ii) Let $x_0 \in \Gamma_{1+s}$. For all $\varepsilon > 0$ there exists an $r_0(\varepsilon)$ such that

$$H_a^{x_0}(r) \geq \frac{H_a^{x_0}(r_0)}{r_0^{n+2+\varepsilon}} r^{n+2+\varepsilon} \qquad\qquad \text{for all} \quad 0 < r < r_0. \tag{3.5}$$

Proof.

(i) We proceed along a two-step argument. Let $x_0 \in \Gamma_{1+s}(u)$ we recall that $x_0 = (\widehat{x_0}, 0)$. Thanks to the Divergence Theorem and the third condition of (2.1) for which $u R_a(u) = 0$ in B_1' we can compute the

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[1]The function $N_a^{\cdot}(0^+, u)$ is the infimum on r of continuous functions $N_a^x(r, u)$.

derivative of $\frac{H_a^{x_0}(r)}{r^{n-2s}}$:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r^{n-2s}}H_a^{x_0}(r)\right) = \frac{2}{r^{n-2s}}\int_{B_r(x_0)}|\nabla u(x)|^2\,\mathrm{d}\mu_a. \tag{3.6}$$

Next, throught the equation (3.6), we compute the derivative of $\frac{H_a^{x_0}(r)}{r^{n+2}}$

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{H_a^{x_0}(r)}{r^{n+2}}\right) = 2\,r^{-n-3}\left(r\int_{B_r(x_0)}|\nabla u(x)|^2\,\mathrm{d}\mu_a - (1+s)\int_{\partial B_r(x_0)}u^2\,|x_n|^a\,\mathrm{d}\mathcal{H}^{n-1}\right), \tag{3.7}$$

then, according to item (i) in Theorem 3.1 and recalling that $x_0 \in \Gamma_{1+s}(u)$ we can deduce that $r^{-(n+2)}H_a^{x_0}(r)$ is nondecreasing.

(ii) Let $r_0 = r_0(\varepsilon)$ be a radius such that for all $r < r_0$ it holds $N_a^{x_0}(u) \le (1+s) + \varepsilon/2$. Then, thanks to (3.6), we obtain

$$N_a^{x_0}(r,u) = \frac{r}{2}\frac{\mathrm{d}}{\mathrm{d}r}\log\left(\frac{H_a^{x_0}(r)}{r^{n-2s}}\right) \le (1+s) + \varepsilon/2.$$

So, dividing to $\frac{r}{2}$ and integrating on (r, r_0) we have

$$H_a^{x_0}(r) \ge H_a^{x_0}(r_0)\left(\frac{r}{r_0}\right)^{n+2+\varepsilon}. \qquad \square$$

We now prove a version of the Rellich formula for weighted Sobolev spaces:

Proposition 3.3 (Rellich formula). *Let v be a solution of* (1.3). *Then it holds that:*

$$\int_{\partial B_r}|\nabla v|^2\,|x_n|^a\,\mathrm{d}\mathcal{H}^{n-1} = \frac{n-2+a}{r}\int_{B_r}|\nabla v|^2\,\mathrm{d}\mu_a + 2\int_{\partial B_r}\left(\langle\nabla v, \frac{x}{r}\rangle\right)^2|x_n|^a\,\mathrm{d}\mathcal{H}^{n-1}.$$

Proof. We apply the Divergence Theorem and the third condition of (2.1) for which $uR_a(u) = 0$ in B_1' and develop

$$\mathrm{div}\left(|\nabla v|^2\frac{x}{r}|x_n|^a - 2\left\langle\nabla v, \frac{x}{r}\right\rangle\nabla v\,|x_n|^a\right). \qquad \square$$

In view of Section 6 we compute the derivative of the volume and boundary energies.

Lemma 3.4. *The following formulae hold:*

(i) $(H_a^{x_0})'(r) = \frac{n-2s}{r}H_a^{x_0}(r) + 2\int_{\partial B_r(x_0)}u\nabla u\cdot\nu\,|x_n|^a\,\mathrm{d}\mathcal{H}^{n-1}$;

(ii) $(D_a^{x_0})'(r) = \frac{n-2+a}{r}D_a^{x_0}(r) + 2\int_{\partial B_r(x_0)}(\nabla u\cdot\nu)^2\,|x_n|^a\,\mathrm{d}\mathcal{H}^{n-1}$;

(iii) $D_a^{x_0}(r) = \int_{\partial B_r(x_0)}u\nabla u\cdot\nu\,|x_n|^a\,\mathrm{d}\mathcal{H}^{n-1}$.

Proof. (i) We can obtain the thesis observing that $\frac{\mathrm{d}}{\mathrm{d}r}u^2(x_0 + ry) = 2\,u(x_0 + ry)\nabla u(x_0 + ry)\cdot y$.
(ii) From Coarea and Rellich Formulae we obtain

$$(D_a^{x_0})'(r) = \int_{\partial B_r(x_0)}|\nabla u|^2\,\mathrm{d}\mu_a \overset{\text{Prop. 3.3}}{=} \frac{n-2+a}{r}D_a^{x_0}(r) + 2\int_{\partial B_r(x_0)}(\nabla u\cdot\nu)^2\,|x_n|^a\,\mathrm{d}\mathcal{H}^{n-1}.$$

(iii) In order to prove the formula, it is enough to apply the the Divergence Theorem and the third condition of (2.1) for which $uR_a(u) = 0$ in B'_1. □

4. The blow-up method: existence and $(1 + s)$-homogeneity of blow-ups

In order to study the properties of the free-boundary, we investigate the properties of the blow-up limits. Let $x_0 \in \Gamma_{1+s}(u)$, we shall consider a suitable sequence of rescaled functions of the solution u (called also $(1 + s)$-*homogeneous rescalings*):

$$u_{x_0,r}(x) := \frac{u(x_0 + rx)}{r^{1+s}}, \tag{4.1}$$

if $x_0 = \underline{0}$ we denote $u_r(x)$ in the place of $u_{\underline{0},r}(x)$. We do not take advantage of the Almgren type rescalings as done in [3, 15, 31, 32].

The first step in the analysis of blow-ups is to prove the existence of the limits of the sequence $(u_{x_0,r})_r$ for all $x_0 \in \Gamma_{1+s}(u)$. In order to prove their existence, we state the equiboundedness of $(u_{x_0,r})_r$ with respect to the $H^1(B_1, \mu_a)$-norm.

Proposition 4.1 (Existence of blow-ups). *Let $u \in H^1(B_1, \mu_a)$ be the solution of (1.3) and let $x_0 \in \Gamma_{1+s}(u)$. Then for every sequence $r_k \downarrow 0$ there exists a subsequence $(r_{k_j})_j \subset (r_k)_k$ such that the rescaled functions $(u_{x_0,r_{k_j}})_j$ converge in $L^2(B_{1-|x_0|}, \mu_a)$.*

Proof. Since $x_0 \in \Gamma_{1+s}(u)$,

$$\|\nabla u_{x_0,r_k}\|^2_{L^2(B_1,\mu_a;\mathbb{R}^n)} = \frac{D_a^{x_0}(r_k)}{r_k^{n+1}} \overset{(3.1)}{=} \frac{H_a^{x_0}(r_k)}{N_a^{x_0}(r_k)\, r_k^{n+2}} \leq \frac{H_a^{x_0}(1 - |x_0|)}{(1 + s)\,(1 - |x_0|)^{n+2}} \tag{4.2}$$

where in the last inequality we used the inequality (3.4) and the Theorem 3.1(i). Due to Lemma 3.2(i), we have

$$\|u_{x_0,r_k}\|^2_{L^2(\partial B_1,|x_n|^a \mathcal{H}^{n-1})} = \frac{H_a^{x_0}(r_k)}{r_k^{n+2}} \overset{(3.4)}{\leq} \frac{H_a^{x_0}(1 - |x_0|)}{(1 - |x_0|)^{n+2}}.$$

So, according to the Poincaré inequality we have

$$\sup_k \|u_{x_0,r_k}\|_{L^2(B_1,\mu_a)} \leq C \left(\sup_k \|u_{x_0,r_k}\|_{L^2(\partial B_1,|x_n|^a\, \mathcal{H}^{n-1})} + \sup_k \|\nabla u_{x_0,r_k}\|_{L^2(B_1,\mu_a;\mathbb{R}^n)} \right) < \infty. \tag{4.3}$$

Therefore, thanks to Theorem 2.3 for every subsequence of radii $r_k \searrow 0$, there exists an extracted subsequence $r_{k_j} \searrow 0$ such that $u_{x_0,r_{k_j}} \to u_0$ in $L^2(B_{1-|x_0|}, \mu_a)$ as $j \to +\infty$. □

Remark 4.2. So, according to the quantitative estimate (2.4) and inequality (4.3)

$$\sup_k \|u_{x_0,r_k}\|_{X_{s,\alpha}(\overline{B_{1/2}})} \overset{(2.4)}{\leq} \sup_k \|u_{x_0,r_k}\|_{L^2(B_1,\mu_a)} < \infty. \tag{4.4}$$

In particular, in view of (4.4) we can easily deduce that

$$\|u\|_{L^\infty(B_r(x_0))} \leq Cr^{1+s}, \qquad \|\nabla_{\widehat{x}} u\|_{L^\infty(B_r(x_0);\mathbb{R}^n)} \leq Cr^s \quad \text{and} \quad \||x_n|^a \partial_{x_n} u\|_{L^\infty(B_r(x_0);\mathbb{R}^n)} \leq Cr^{1-s}. \tag{4.5}$$

Similarly to [49] we consider an energy "à la Weiss" used in [23] and [28] for fractional Laplacian (see [32] for a version in the fractional Laplacian problem with drift and [25, 33] for a version in the classical obstacle problem with quadratic energies with variable coefficients):

$$W_{1+s}^{x_0}(r,u) = \frac{1}{r^n+1} \int_{B_r(x_0)} |\nabla u|^2 |x_n|^a \, \mathrm{d}x - \frac{1+s}{r^{n+2}} \int_{\partial B_r(x_0)} u^2 |x_n|^a \, \mathrm{d}\mathcal{H}^{n-1}. \tag{4.6}$$

We note that

$$W_{1+s}^{x_0}(r,u) = \frac{H_a^{x_0}(r)}{r^{n+2}} (N_a^{x_0}(r,u) - (1+s)),$$

thus if $x_0 \in \Gamma_{1+s}(u)$ by (3.2) and Lemma 3.2 $\left(\text{which guarantees the boundedness of } \frac{H_a^{x_0}(r)}{r^{n+2}}\right)$ we have

$$\lim_{r \searrow 0} W_{1+s}^{x_0}(r,u) = 0$$

and due to Theorem 3.1, we obtain

$$W_{1+s}^{x_0}(r,u) \geq 0.$$

Moreover, the function $W_{1+s}^{x_0}(\cdot, u)$ satisfies a monotonicity formula in the same essence as Weiss' monotonicity formula proved in [49]. For a similar proof see [32], Theorem 3.5.

Proposition 4.3 (Weiss' monotonicity formula)**.** *Let $x_0 \in \Gamma_{1+s}(x_0)$ and u be a solution of problem* (2.1)*; then the function $r \mapsto W_{1+s}^{x_0}(r,u)$ is nondecreasing. In particular, the following formula holds:*

$$\frac{\mathrm{d}}{\mathrm{d}r} W_{1+s}^{x_0}(r,u) = \frac{2}{r} \int_{\partial B_1} (\nabla u_r \cdot \nu - (1+s)u_r)^2 |x_n|^a \, \mathrm{d}\mathcal{H}^{n-1}$$

Next, we prove the homogeneity property of blow-ups. We prove the result through properties of the frequency function and the optimal regularity of the solution. Proceeding as in [25], Proposition 4.2, and thanks to Proposition 4.3 it is possible to obtain the same result.

Proposition 4.4 (($1+s$)-homogeneity of blow-ups)**.** *Let $u \in H^1(B_1, \mu_a)$ be a solution of problem* (1.3)*. Let $x_0 \in \Gamma_{1+s}(u)$ and $(u_{x_0,r})_r$ be a sequence of rescaled functions. Then, for every sequence $(r_j)_j \downarrow 0$ there exists a subsequence $(r_{j_k})_k \subset (r_j)_j$ such that the sequence $(u_{x_0,r_{j_k}})_k$ converges in $X_{\beta,\alpha}(\mathbb{R}^n)$ (see* (2.4)*) for all $0 < \beta < s$ and $0 < \alpha < 1 - s$ to u_{x_0} a ($1+s$)-homogeneous function.*

Proof. In view of (4.4), thanks to the Ascoli-Arzelà Theorem there exists a subsequence (that we do not relabel) u_{x_0,r_k} and $u_{x_0} \in X_{s,\alpha}(\overline{B_{1/2}})$ such that $\|u_{x_0,r_k} - u_{x_0}\|_{X_{\beta,\alpha}(\overline{B_{1/2}})}$ converge to 0 for all $0 < \beta < s$ and $0 < \alpha < 1-s$. It is easy to prove that u_{x_0} is a solution of problem (1.3). In order to conclude the proof, we show that u_{x_0} is ($1+s$)-homogeneous.

We note that for every $\delta > 0$ we can fix $\rho > 0$ such that $N_a^{x_0}(\rho, u) \leq (1+s) + \delta$. So for $k >> 1$, for every $t \in (0,1)$ (such that $t\, r_k < \rho$)

$$\begin{aligned} N_a(t, u_{x_0, r_k}) &= N_a^{x_0}(t, u_{r_k}) = N_a^{x_0}(t\, r_k, u) - N_a^{x_0}(\rho, u) + N_a^{x_0}(\rho, u) \leq (1+s) + \delta, \\ N_a^{x_0}(t, u_{r_k}) &= N_a^{x_0}(t\, r_k, u) \geq 1+s, \end{aligned} \tag{4.7}$$

where we resort to Theorem 3.1. Now, from the convergence of u_{x_0,r_k} to u_{x_0} and thanks to the arbitrariness of δ, we obtain $N_a(t, u_{x_0}) \equiv 1 + s$; then, by Theorem 3.1(ii), u_{x_0} is ($1+s$)-homogeneous. $\square$

Remark 4.5. By proceeding in the same way, we can prove Theorem 3.1(iv) as well:

Proof of Theorem 3.1(iv). Let $x_0 \in \Gamma(u)$ and $\lambda = N_a^{x_0}(0^+, u)$. Then, if $r_k \searrow 0$ is a suitable sequence of radii, for all $\delta > 0$ we can fix $\rho > 0$ such that $N_a^{x_0}(\rho, u) \leq \lambda + \delta$. So, proceeding in much the same way as in (4.7), we deduce

$$\lambda \leq N_a(t, u_{x_0}, r_k) \leq \lambda + \delta,$$

thus, by the strong convergence of u_{x_0, r_k}, to its blow-up w_0 and by the arbitrariness of δ, we have $N_a(t, w_0) \equiv \lambda$. So, by the second item of Theorem 3.1 w_0 is λ-homogeneous and by Theorem 3.1(iii) $\lambda \geq 1 + s$. $\square$

5. Classification of the $(1+s)$-homogeneous global solutions

Let h_e be the function defined by

$$h_e(x) := \left(s^{-1} \widehat{x} \cdot e - \sqrt{(\widehat{x} \cdot e)^2 + x_n^2} \right) \left(\sqrt{(\widehat{x} \cdot e)^2 + x_n^2} + \widehat{x} \cdot e \right)^s. \tag{5.1}$$

From a simple calculation it is possible to prove the following properties:

(i) $h_e(\widehat{x}, x_n) = h_e(\widehat{x}, -x_n)$;
(ii) $h_e(x) \geq 0$ on $\{x_n = 0\}$ and $h_e = 0$ on $\{x_n = 0, \widehat{x} \cdot e \leq 0\}$;
(iii) $\partial_e h_e(x) = \frac{1-s^2}{s} \left(\sqrt{(\widehat{x} \cdot e)^2 + x_n^2} + \widehat{x} \cdot e \right)^s$;
(iv) $\partial_n h_e(x) = -(1+s)x_n \left(\sqrt{(\widehat{x} \cdot e)^2 + x_n^2} + \widehat{x} \cdot e \right)^{s-1}$;
(v) $\partial_{\tau\tau} h_e \geq 0$ for any vector $\tau \in \mathbb{S}^n \subset \mathbb{R}^{n-1} \times \{0\}$;
(vi) h_e is solution of (2.3);
(vii)

$$R_a h_e(\widehat{x}) = \begin{cases} 0 & \widehat{x} \cdot e \geq 0 \\ -(1+s)\left(2 \left| \widehat{x} \cdot e \right| \right)^{1-s} & \widehat{x} \cdot e < 0. \end{cases} \tag{5.2}$$

In particular, we obtain a complementarity property

$$h_e(\widehat{x}, x_n) R_a h_e(\widehat{x}) = 0 \qquad \text{on } \{x_n = 0\} \tag{5.3}$$

So h_e is a $(1+s)$-homogeneous solution of

$$\begin{cases} v(\widehat{x}, 0) \geq 0 & \widehat{x} \in \mathbb{R}^{n-1} \\ v(\widehat{x}, x_n) = v(\widehat{x}, -x_n) & \\ \mathrm{div}(|x_n|^a \nabla v(\widehat{x}, x_n)) = 0 & x \in \mathbb{R}^n \setminus \{(\widehat{x}, 0) : u(\widehat{x}, 0) = 0\} \\ \mathrm{div}(|x_n|^a \nabla v(\widehat{x}, x_n)) \leq 0 & x \in \mathbb{R}^n \text{ in distributional sense} \\ \partial_{\tau\tau} v \geq 0 & \text{for any vector } \tau \in \partial B_1'. \end{cases} \tag{5.4}$$

According to [15], Proposition 5.5, the function h_e is, up to a rotation and the product by scalar, the unique $(1+s)$-homogeneous, global solution of (5.4).

We consider the closed convex cone of $(1+s)$-homogeneous global solutions :

$$\mathfrak{H}_{1+s} := \{\lambda \, h_e \ : \ e \in \mathbb{S}^{n-2}, \lambda \in [0, +\infty)\} \subset H^1_{loc}(\mathbb{R}^n, \mu_a). \tag{5.5}$$

Caffarelli *et al.* [15] proved that $\mathfrak{H}_{1+s} \setminus \{\underline{0}\}$ is the set of blow-ups in the regular points of the free-boundary with lower frequency.

We note that $\mathfrak{H}_{1+s}$ is a closed cone in $H^1_{loc}(\mathbb{R}^n, \mu_a)$. The restriction

$$\mathfrak{H}_{1+s}|_{B_1} := \{v|_{B_1} \ : \ v \in \mathfrak{H}_{1+s}\} \subset H^1(B_1, \mu_a)$$

is a closed set, and $\mathfrak{H}_{1+s} \setminus \{0\}$ is parameterized by a $(n-1)$-manifold by the map

$$\begin{array}{ccc} \mathbb{S}^{n-2} \times (0, \infty) & \xrightarrow{\Phi} & \mathfrak{H}_{1+s} \setminus \{0\} \\ (e, \lambda) & \longmapsto & \lambda \, h_e. \end{array}$$

Next we can introduce the tangent plane to space $\mathfrak{H}_{1+s}$ in every point $\lambda \, h_e$ as

$$T_{\lambda \, h_e} \mathfrak{H}_{1+s} := \{d_{(e,\lambda)}\Phi(\xi, \alpha) \ : \ \xi \cdot e_n = \xi \cdot e = 0, \alpha \in \mathbb{R}\} \tag{5.6}$$

We compute the derivative of the map Φ in a point of $\mathbb{S}^{n-2} \times (0, \infty)$:

$$d_{(e,\lambda)}\Phi(\xi, \alpha) = \frac{\mathrm{d}}{\mathrm{d}t} h_{\sigma(t)}|_{t=0} \tag{5.7}$$

with $\sigma(t) = \frac{e+t\xi}{\|e+t\xi\|}$, a curve on $\mathbb{S}^{n-2}$ such that $\sigma(0) = e$ and $\sigma'(0) = \xi$. By (5.1) and (5.7) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} h_{\sigma(t)}\Big|_{t=0} = (s^{-1} - s)\, \widehat{x} \cdot \xi \left(\sqrt{(\widehat{x} \cdot e)^2 + x_n^2} + \widehat{x} \cdot e \right)^s.$$

Then, we can rewrite (5.6) as

$$T_{\lambda \, h_e} \mathfrak{H}_{1+s} := \{\alpha h_e + v_{e,\xi} \ : \ \xi \cdot e_n = \xi \cdot e = 0, \alpha \in \mathbb{R}\} \tag{5.8}$$

where the function $v_{e,\xi}$ is defined as follows:

$$v_{e,\xi} = \widehat{x} \cdot \xi \left(\sqrt{(\widehat{x} \cdot e)^2 + x_n^2} + \widehat{x} \cdot e \right)^s. \tag{5.9}$$

We highlight some properties of function $\psi \in \mathfrak{H}_{1+s}$. For all $\varphi \in H^1(B_1, \mu_a)$, integrating by parts, according to Theorem 2.2 and Euler's homogeneous function Theorem we obtain

$$\int_{B_1} \nabla\psi \cdot \nabla\varphi \mathrm{d}\mu_a = \int_{\partial B_1} \varphi \, \nabla\psi \cdot x \, |x_n|^a \, \mathrm{d}\mathcal{H}^{n-1} - 2\int_{B_1'} \varphi R_a(\psi) \, \mathrm{d}\mathcal{H}^{n-1}$$

$$= (1+s) \int_{\partial B_1} \varphi\psi \, |x_n|^a \, \mathrm{d}\mathcal{H}^{n-1} - 2\int_{B_1'} \varphi R_a(\psi) \, \mathrm{d}\mathcal{H}^{n-1}. \tag{5.10}$$

Remark 5.1. The first variation of functional $W^0_{1+s}(1, \cdot)$ in a point $\psi \in \mathfrak{H}_{1+s}$ along a direction $\varphi \in H^1(B_1, \mu_a)$ is[2]

$$\delta W^0_{1+s}(1, \psi)[\varphi] = 2 \int_{B_1} \nabla\psi \cdot \nabla\varphi \, \mathrm{d}\mu_a - 2(1+s) \int_{\partial B_1} \psi \, \varphi \, |x_n|^a \, \mathrm{d}\mathcal{H}^{n-1}.$$

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[2]The first variation is defined as $\delta W^0_{1+s}(1, \psi)[\varphi] := \lim_{t\to 0} \left(\frac{W^0_{1+s}(1, \psi+t\varphi) - W^0_{1+s}(1, \psi)}{t} \right)$

Then, by (5.10)

$$\delta W^0_{1+s}(1,\psi)[\varphi] = -4 \int_{B'_1} \varphi R_a(\psi)(\widehat{x}) \, \mathrm{d}\mathcal{H}^{n-1}, \tag{5.11}$$

by (5.3)

$$\delta W^0_{1+s}(1,\psi)[\psi] = 0, \tag{5.12}$$

so we can infer that

$$W^0_{1+s}(1,\psi) = \frac{1}{2}\delta W^0_{1+s}(1,\psi)[\psi] = 0 \qquad \forall \psi \in \mathfrak{H}_{1+s}. \tag{5.13}$$

6. The epiperimetric inequality and its consequences

In this section we prove an epiperimetric inequality for the points in $\Gamma_{1+s}(u)$, and its main consequences in the framework of the regularity of the free-boundary. In Section 6.1 we prove the epiperimetric inequality. In Section 6.2 we establish a decay estimate for adjusted boundary energy. In Sections 6.3 and 6.4 we state the nondegeneracy of the solution and the uniqueness of the blow-ups in $\Gamma_{1+s}(u)$ respectively.

6.1. Epiperimetric inequality

We now state the main result of this paper: the *epiperimetric inequality* "à la Weiss" in our setting. This result is a key ingredient in our approach to the decay of the boundary adjusted energy and to the uniqueness of blow-ups (see [23] for the classical case of Laplacian $s = 1/2$).

In this section we state and prove the epiperimetric inequality. For the convenience of readers, the proof will be split into several steps.

Theorem 6.1 (Epiperimetric inequality)**.** *There exists a dimensional constant $\kappa \in (0,1)$ such that if $c \in H^1(B_1, \mu_a)$ is a $(1+s)$-homogeneous function with $c \geq 0$ on B'_1 and $c(\widehat{x}, x_n) = c(\widehat{x}, -x_n)$ then*

$$\inf_{v \in \mathfrak{A}_c} W^0_{1+s}(v) \leq (1-\kappa)W^0_{1+s}(c). \tag{6.1}$$

Proof. Without loss of generality it is possible to suppose that there exists a constant $\delta > 0$ such that the function c satisfies the follows condition

$$\mathrm{dist}_{H^1(B_1,\mu_a)}(c, \mathfrak{H}_{1+s}) < \delta. \tag{6.2}$$

In fact, according to the $(1+s)$-homogeneity of c and recalling that $\mathfrak{H}_{1+s}$ is a cone, for all $\delta > 0$ there exists a constant $\gamma > 0$ such that

$$\mathrm{dist}_{H^1(B_1,\mu_a)}(\gamma c, \mathfrak{H}_{1+s}) < \delta.$$

We can observe that if $v \in \mathfrak{A}_{\gamma c}$ then $\gamma^{-1}v \in \mathfrak{A}_c$. So, if we prove inequality (6.1) for the function γc, or rather

$$\inf_{v \in \mathfrak{A}_{\gamma c}} W^0_{1+s}(1,v) \leq (1-\kappa)W^0_{1+s}(1,\gamma c),$$

then, thanks to $W_{1+s}^0(1, \gamma c) = \gamma^2 W_{1+s}^0(1, c)$ we infer

$$\inf_{w \in \mathfrak{A}_c} W_{1+s}^0(1, w) \le (1 - \kappa) W_{1+s}^0(1, c).$$

To simplify the notation we denote the functional $W_{1+s}^0(1, \cdot)$ by $\mathcal{G}(\cdot)$. We argue by contradiction. Let us suppose the existence of sequences of positive numbers $\kappa_j, \delta_j \downarrow 0$ and a sequence of $(1 + s)$-homogeneous functions $c_j \in H^1(B_1, \mu_a)$ with $c_j \ge 0$ on B_1' such that

$$\text{dist}_{H^1(B_1, \mu_a)}(c_j, \mathfrak{H}_{1+s}) = \delta_j, \tag{6.3}$$

$$(1 - \kappa)\mathcal{G}(c) \le \inf_{v \in \mathfrak{A}_c} \mathcal{G}(v). \tag{6.4}$$

In particular, fixing $h := h_{e_n}$, up to change of coordinate depending on j, we assume that there exists $\lambda_j \ge 0$ for which $\psi_j := \lambda_j h$ is the point satisfying the minimum distance between c_j and $\mathfrak{H}_{1+s}$, or rather

$$\|\psi_j - c_j\|_{H^1(B_1, \mu_a)} = \text{dist}_{H^1(B_1, \mu_a)}(c_j, \mathfrak{H}_{1+s}) = \delta_j, \qquad \forall j \in \mathbb{N}. \tag{6.5}$$

We split the proof into some intermediate steps.

Step 1: Auxiliary functionals. We can rewrite (6.4) and interpret this inequality as a condition of quasi-minimality for a sequence of new functionals.

Setting $j \in \mathbb{N}$, let $v \in \mathfrak{A}_{c_j}$, we use (5.11) (applied twice to ψ_j with test functions $c_j - \psi_j$ and $v - \psi_j$) and (5.13); we can rewrite (6.4):

$$(1 - \kappa_j) \left(\mathcal{G}(c_j) - \mathcal{G}(\psi_j) - \delta\mathcal{G}(\psi_j)[c_j - \psi_j] - 4 \int_{B_1'} (c_j - \psi_j) R_a(\psi_j) \, \mathrm{d}\mathcal{H}^{n-1} \right)$$

$$\le \mathcal{G}(v) - \mathcal{G}(\psi_j) - \delta\mathcal{G}(\psi_j)[v - \psi_j] - 4 \int_{B_1'} (v - \psi_j) R_a(\psi_j) \, \mathrm{d}\mathcal{H}^{n-1}. \tag{6.6}$$

We can observe that $\mathcal{G}(v_1) - \mathcal{G}(v_2) - \delta\mathcal{G}(v_2)[v_1 - v_2] = \mathcal{G}(v_1 - v_2)$, then for all $v \in \mathfrak{A}_{c_j}$ (6.6) can be rewritten as

$$(1 - \kappa_j) \left(\mathcal{G}(c_j - \psi_j) - 4 \int_{B_1'} (c_j - \psi_j) R_a(\psi_j) \, \mathrm{d}\mathcal{H}^{n-1} \right) \le \mathcal{G}(v - \psi_j) - 4 \int_{B_1'} (v - \psi_j) R_a(\psi_j) \, \mathrm{d}\mathcal{H}^{n-1}. \tag{6.7}$$

Next we define new sequences of functions

$$z_j := \frac{c_j - \psi_j}{\delta_j} \tag{6.8}$$

(recalling that $\psi_j = \lambda_j h$), positive numbers $\theta_j := \frac{\lambda_j}{\delta_j}$ and sets $\mathcal{B}_j := \{z \in z_j + H_0^1(B_1, \mu_a) : (z + \theta_j h)|_{B_1'} \ge 0\}$. Now we introduce a sequence of auxiliary functionals $\mathcal{G}_j : L^2(B_1, \mu_a) \to (-\infty, +\infty]$

$$\mathcal{G}_j(z) := \begin{cases} \int_{B_1} |\nabla z|^2 \, \mathrm{d}\mu_a - (1 + s) \int_{\partial B_1} z_j^2 \, |x_n|^a \, \mathrm{d}\mathcal{H}^{n-1} - 4\theta_j \int_{B_1'} z R_a(h) \, \mathrm{d}\mathcal{H}^{n-1} & \text{if } z \in \mathcal{B}_j \\ +\infty & \text{otherwise.} \end{cases} \tag{6.9}$$

We can observe that the second term in the formula above does not depend on z but only on its boundary datum $z|_{\partial B_1} = z_j|_{\partial B_1}$.

We can rewrite (6.7) with the new notation and obtain

$$(1 - \kappa_j)\left(\mathcal{G}(\delta_j z_j) - 4\delta_j \int_{B_1'} z_j R_a(\lambda_j h) \, d\mathcal{H}^{n-1}\right) \leq \mathcal{G}(\delta_j z) - 4\delta_j \int_{B_1'} z R_a(\lambda_j h) \, d\mathcal{H}^{n-1}$$

and dividing by δ_j^2 we obtain the condition of quasi-minimality for z_j with respect to $\mathcal{G}_j$:

$$(1 - \kappa_j)\mathcal{G}_j(z_j) \leq \mathcal{G}_j(z) \qquad \forall z \in L^2(B_1, \mu_a). \tag{6.10}$$

Therefore we note that by the very definitions of z_j and δ_j we have

$$\|z_j\|_{H^1(B_1, \mu_a)} = 1. \tag{6.11}$$

So, by the compactness of Sobolev embedding from $H^1(B_1, \mu_a)$ into the space $L^2(B_1, \mu_a)$ Theorem 2.3, the trace operator from $H^1(B_1, \mu_a)$ into the space $L^2(B_1')$ Theorem 2.5, and the trace operator from $H^1(B_1, \mu_a)$ into $L^2(\partial B_1, |x_n|^a \mathcal{H}^{n-1})$ Theorem 2.6, we may extract a subsequence (which we do not relabel) such that

(a) $(z_j)_{j \in \mathbb{N}}$ converges weakly in $H^1(B_1, \mu_a)$ to some z_∞;
(b) the sequences of traces $z_j|_{B_1'}$ and $z_j|_{\partial B_1}$ converge respectively in $L^2(B_1')$ and $L^2(\partial B_1, |x_n|^a \mathcal{H}^{n-1})$;
(c) θ_j has a limit $\theta \in [0, \infty]$.

Step 2: First property of $(\mathcal{G}_j)_{j \in \mathbb{N}}$. In this step we establish the equi-coercivity and some other properties of the family $(\mathcal{G}_j)_{j \in \mathbb{N}}$.

We observe that for all $w \in \mathcal{B}_j$, since $w|_{\partial B_1} = z_j|_{\partial B_1}$ and $hR_a(h)(\widehat{x}) = 0$, it holds that

$$- \int_{B_1'} w R_a(h)(\widehat{x}) \, d\mathcal{H}^{n-1} = - \int_{B_1'} (w + \theta_j h) R_a(h)(\widehat{x}) \, d\mathcal{H}^{n-1} + \theta_j \int_{B_1'} h R_a(h)(\widehat{x}) \, d\mathcal{H}^{n-1} \geq 0 \tag{6.12}$$

where we used (5.2) for which $R_a(h)(\widehat{x}) \leq 0$ and the condition $w \in \mathcal{B}_j$ for which $(w + \theta_j h)_{|B_1'} \geq 0$. Then from the definition of (6.9) we have

$$\int_{B_1} |\nabla w|^2 \, d\mu_a - (1 + s) \int_{\partial B_1} z_j^2 |x_n|^a \, d\mathcal{H}^{n-1} \leq \mathcal{G}_j(w). \tag{6.13}$$

This establishes the equi-coercivity of the sequence $\mathcal{G}_j$, in fact from (6.11), thanks to strong convergence of traces, we obtain

$$\liminf_{j \in \mathbb{N}} \mathcal{G}_j(z_j) \geq -(1 + s) \int_{\partial B_1} z_\infty^2 |x_n|^a \, d\mathcal{H}^{n-1} - 4\theta \int_{B_1'} z_\infty R_a(h) \, d\mathcal{H}^{n-1};$$

while if $\theta = +\infty$ from (6.11) and (6.13) we conclude that

$$\liminf_{j \in \mathbb{N}} \mathcal{G}_j(z_j) \geq -(1 + s) \int_{\partial B_1} z_\infty^2 |x_n|^a \, d\mathcal{H}^{n-1}.$$

Note that it is not restrictive (up to subsequence) to assume that $\mathcal{G}_j(z_j)$ has a limit in $(-\infty, +\infty]$. Finally we can observe that

$$\lim_{j \to \infty} \mathcal{G}_j(z_j) = +\infty \qquad \Longleftrightarrow \qquad \lim_{j \to \infty} \theta_j \int_{B_1'} z_j R_a(h) \, d\mathcal{H}^{n-1} = -\infty. \tag{6.14}$$

Step 3: Asymptotic analysis of $(\mathcal{G}_j)_{j \in \mathbb{N}}$. In this step we prove a result of Γ-convergence for the family of functionals $(\mathcal{G}_j)_{j \in \mathbb{N}}$.

We can distinguish three cases:

(1) If $\theta \in [0, +\infty)$, then $(z_\infty + \theta h)|_{B_1'} \geq 0$ and $\Gamma(L^2(B_1, \mu_a))\text{-}\lim \mathcal{G}_j = \mathcal{G}_\infty^{(1)}$ with

$$\mathcal{G}_\infty^{(1)}(z) := \begin{cases} \displaystyle\int_{B_1} |\nabla z|^2 \, d\mu_a - (1+s) \int_{\partial B_1} z_\infty^2 \, |x_n|^a \, d\mathcal{H}^{n-1} - 4\theta \int_{B_1'} z R_a(h) \, d\mathcal{H}^{n-1} \\ \hspace{8cm} \text{if } z \in \mathcal{B}_\infty^{(1)} \\ +\infty \hspace{7.2cm} \text{otherwise,} \end{cases}$$

where $\mathcal{B}_\infty^{(1)} := \{z \in z_\infty + H_0^1(B_1, \mu_a) \ : \ (z + \theta h)|_{B_1'} \geq 0\}$.

(2) If $\theta = +\infty$ and $\lim_j \mathcal{G}_j(z_j) < \infty$, then $z_\infty|_{B_1'^-} = 0$ (where $B_1'^- = B' \cap \{x_{n-1} \leq 0\}$) and $\Gamma(L^2(B_1, \mu_a))\text{-}\lim \mathcal{G}_j = \mathcal{G}_\infty^{(2)}$ with

$$\mathcal{G}_\infty^{(2)}(z) := \begin{cases} \displaystyle\int_{B_1} |\nabla z|^2 \, d\mu_a - (1+s) \int_{\partial B_1} z_\infty^2 \, |x_n|^a \, d\mathcal{H}^{n-1} & \text{if } z \in \mathcal{B}_\infty^{(2)} \\ +\infty & \text{otherwise,} \end{cases}$$

where $\mathcal{B}_\infty^{(2)} := \{z \in z_\infty + H_0^1(B_1, \mu_a) \ : \ z|_{B_1'^-} = 0\}$. We note that the third addendum of $\mathcal{G}_j$ is zero in $\mathcal{B}_\infty^{(2)}$, while if $z \in \mathcal{B}_j \setminus \mathcal{B}_\infty^{(2)}$ the sequence $\mathcal{G}_j(z)$ diverges; this heuristically justifies the choice of $\mathcal{G}_\infty^{(2)}(z)$ and $\mathcal{B}_\infty^{(2)}$.

(3) If $\theta = +\infty$ and $\lim_j \mathcal{G}_j(z_j) = +\infty$, then $\Gamma(L^2(B_1, \mu_a))\text{-}\lim \mathcal{G}_j = \mathcal{G}_\infty^{(3)}$ with

$$\mathcal{G}_\infty^{(3)}(z) = +\infty \qquad \text{on } L^2(B_1, \mu_a).$$

For the reader's convenience we recall the Definition of Γ-limit (see [17]); the equality $\Gamma(L^2(B_1, \mu_a))\text{-}\lim \mathcal{G}_j = \mathcal{G}_\infty^{(i)}$ with $i = 1, 2, 3$ is satisfied if the two following conditions hold:

(a) for all sequences $(w_j)_j \subset L^2(B_1, \mu_a)$ and $w \in L^2(B_1, \mu_a)$ such that $w_j \to w$ in $L^2(B_1, \mu_a)$ it holds

$$\liminf_j \mathcal{G}_j(w_j) \geq \mathcal{G}_j^{(i)}(w) \tag{6.15}$$

(b) for all $w \in L^2(B_1, \mu_a)$ there exists a sequence $(w_j)_j \subset L^2(B_1, \mu_a)$ such that $w_j \to w$ in $L^2(B_1, \mu_a)$ and

$$\limsup_j \mathcal{G}_j(w_j) \leq \mathcal{G}_j^{(i)}(w). \tag{6.16}$$

Proof of the Γ-convergence: Case (1).
(a) Without loss of generality we may suppose that $\liminf_j \mathcal{G}_j(w_j) = \lim_j \mathcal{G}_j(w_j) < +\infty$, then $w_j \in \mathcal{B}_j$ for all

$j \in \mathbb{N}$. Taking (6.13) into account, we deduce

$$\int_{B_1} |\nabla w_j|^2 \, \mathrm{d}\mu_a \leq \mathcal{G}_j(w_j) + (1+s) \int_{\partial B_1} w_j^2 \, |x_n|^a \, \mathrm{d}\mathcal{H}^{n-1} < +\infty,$$

then, since $w_j \to w$ in $L^2(B_1, \mu_a)$ we have $\sup_j \|w_j\|_{H^1(B_1,\mu_a)} < +\infty$, so from Theorem 2.3 $\nabla w_j \rightharpoonup \nabla w$ in $L^2(B_1, \mu_a)$. Then, thanks to Theorems 2.5 and 2.6, the respective traces converge in $L^2(B_1')$ and $L^2(\partial B_1, |x_n|^a \mathcal{H}^{n-1})$. Hence, we obtain $(w + \theta h)|_{B_1'} \geq 0$ and, in particular, since $w_j|_{B_1'} = z_j|_{B_1'}$ then $w|_{B_1'} = z_\infty|_{B_1'}$ and so $z_\infty \in \mathcal{B}_\infty^{(1)}$. At this point thanks to the convergence of traces of w_j and weak semicontinuity of the norm of the gradient in $L^2(B_1, \mu_a)$ we have (6.15).

(b) We observe that it is sufficient to prove the inequality for $w \in \mathcal{B}_\infty^{(1)}$ with

$$\operatorname{supp}(w - z_\infty) \subset B_\rho \qquad \text{for some } \rho \in (0,1). \tag{6.17}$$

If we want to deal with the general case, we consider the function

$$w_t(x) = t^{1+s} \left(w\left(\frac{x}{t}\right) \chi_{B_1}\left(\frac{x}{t}\right) + z_\infty\left(\frac{x}{t}\right) \chi_{B_{1/t} \setminus \overline{B_1}}\left(\frac{x}{t}\right) \right) \quad \text{with } t < 1.$$

It is easy to prove that $w_t \in H^1(B_1, \mu_a)$ and $\operatorname{supp}(w_t - z_\infty) \subset B_t$; moreover, $w_t \to w$ in $H^1(B_1, \mu_a)$ (for a similar procedure see [34], Prop. 2.4.1, Chap. 2). If (6.16) holds for all w_t, resorting to a diagonalization argument we obtain (6.16) for w. Therefore for a Uryshon's type property it is sufficient to prove the following property: *fixing w as in (6.17), for all sub sequences $j_k \uparrow +\infty$ there exists an extract subsequence $j_{k_l} \uparrow +\infty$ and there exists $w_l \to w$ in $L^2(B_1, \mu_a)$ such that* [3]

$$\limsup_l \mathcal{G}_{j_{k_l}}(w_l) \leq \mathcal{G}_\infty^{(1)}(w). \tag{6.18}$$

Setting $r \in (\rho, 1)$ let $R := \frac{1+r}{2}$ and let $\varphi \in C_c^1(B_1)$ be a cut-off function such that

$$\varphi|_{B_r} \equiv 1, \qquad \varphi|_{B_1 \setminus \overline{B_R}} \equiv 0, \qquad \|\nabla \varphi\|_{L^\infty} \leq \frac{4}{1-r}.$$

We define

$$w_k^r := \varphi \left(w + (\theta - \theta_{j_k})h \right) + (1 - \varphi)z_{j_k} \tag{6.19}$$

and we verify that $w_k^r \in \mathcal{B}_{j_k}$. In fact $w \in \mathcal{B}_\infty^{(1)}$, $z_{j_k} \in \mathcal{B}_{j_k}$ and

$$w_k^r + \theta_{j_k}h = \varphi(w + \theta h) + (1 - \varphi)(z_{j_k} + \theta_{j_k}h) \geq 0.$$

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[3]Let us assume the property (6.18) and prove (6.16). Let us suppose by contradiction that there exists w such that

$$\Gamma - \limsup_j \mathcal{G}_j(w) > \mathcal{G}_\infty^{(1)}(w).$$

If $(w_j)_{j \in \mathbb{N}}$ is a sequence that achieves the Γ-lim sup, *i.e.* $\limsup_j \mathcal{G}_j(w_j) = \Gamma\text{-}\limsup_j \mathcal{G}_j(w_j)$, and j_k is a subsequence for which $\limsup_j \mathcal{G}_j(w_j) = \lim_k \mathcal{G}_{j_k}(w_{j_k})$, thanks to the property (6.18) there exists j_{k_l} such that

$$\lim_k \mathcal{G}_{j_k}(w_{j_k}) = \lim_l \mathcal{G}_{j_{k_l}}(w_{j_{k_l}}) \leq \mathcal{G}_\infty^{(1)}(w),$$

leading to a contradiction.

Therefore, since $\theta_{j_k} \to \theta \in [0, +\infty)$ we have $w_k^r \to \varphi w + (1 - \varphi) z_\infty$ in $L^2(B_1, \mu_a)$. Thanks to the convergence of traces of z_{j_k} in $L^2(B_1')$ it is enough to prove the upper bound inequality for the first addendum of $\mathcal{G}_j$ and $\mathcal{G}_\infty^{(1)}$ respectively. From (6.19), we can infer

$$\int_{B_1} |\nabla w_k^r|^2 \, \mathrm{d}\mu_a \leq \int_{B_r} |\nabla w + (\theta - \theta_{j_k}) \nabla h|^2 \, \mathrm{d}\mu_a + \underbrace{\int_{B_R \setminus \overline{B_r}} |\nabla w_k^r|^2 \, \mathrm{d}\mu_a}_{:=I_k} + \int_{B_1 \setminus \overline{B_R}} |\nabla z_{j_k}|^2 \, \mathrm{d}\mu_a. \tag{6.20}$$

Since $r > \rho$, from assumption (6.17), we estimate the term I_k as follows

$$\begin{aligned} I_k \leq &\, 3 \int_{B_R \setminus \overline{B_r}} \varphi^2 |\nabla w + (\theta - \theta_{j_k}) \nabla h|^2 \, \mathrm{d}\mu_a \\ &+ 3 \int_{B_R \setminus \overline{B_r}} (1 - \varphi)^2 |\nabla z_{j_k}|^2 \, \mathrm{d}\mu_a + 3 \int_{B_R \setminus \overline{B_r}} |\nabla \varphi|^2 |z_\infty - z_{j_k} + (\theta - \theta_{j_k}) \nabla h|^2 \, \mathrm{d}\mu_a \end{aligned}$$

So

$$\limsup_k \int_{B_1} |\nabla w_k^r|^2 \, \mathrm{d}\mu_a \leq \int_{B_r} |\nabla w|^2 \, \mathrm{d}\mu_a + 3 \int_{B_R \setminus \overline{B_r}} |\nabla w|^2 \, \mathrm{d}\mu_a + 4 \limsup_k \int_{B_1 \setminus \overline{B_r}} |\nabla z_{j_k}|^2 \, \mathrm{d}\mu_a \tag{6.21}$$

By the $(1 + s)$-homogeneity of z_{j_k}, we deduce

$$\begin{aligned} \int_{B_1 \setminus \overline{B_r}} |\nabla z_{j_k}|^2 \, \mathrm{d}\mu_a &= \int_r^1 \int_{\partial B_t} |\nabla z_{j_k}|^2 \, |x_n|^a \, \mathrm{d}\mathcal{H}^{n-1} \, \mathrm{d}t \\ &= \int_r^1 t^n \int_{\partial B_1} |\nabla z_{j_k}|^2 \, |x_n|^a \, \mathrm{d}\mathcal{H}^{n-1} \, \mathrm{d}t = \frac{1 - r^{n+1}}{n+1} \int_{\partial B_1} |\nabla z_{j_k}|^2 \, |x_n|^a \, \mathrm{d}\mathcal{H}^{n-1} \end{aligned}$$

which leads us to

$$\int_{\partial B_1} |\nabla z_{j_k}|^2 \, |x_n|^a \, \mathrm{d}\mathcal{H}^{n-1} = \frac{n+1}{1 - (1/2)^{n+1}} \int_{B_1 \setminus \overline{B_r}} |\nabla z_{j_k}|^2 \, \mathrm{d}\mu_a \overset{(6.11)}{\leq} 2(n+1)$$

in turn implying

$$\int_{B_1 \setminus \overline{B_r}} |\nabla z_{j_k}|^2 \, \mathrm{d}\mu_a \leq 2 \, (1 - r) \, (n + 1). \tag{6.22}$$

We apply this construction to a subsequence $r_l \uparrow 1$ and $R_l := \frac{1 + r_l}{2}$ and with a diagonal argument we obtain a subsequence $w_l \to w$ in $L^2(B_1, \mu_a)$. Thanks to (6.21) and (6.22)

$$\begin{aligned} \limsup_l \int_{B_1} |\nabla w_l|^2 \, \mathrm{d}\mu_a &\leq \int_{B_1} |\nabla w|^2 \, \mathrm{d}\mu_a + 3 \limsup_l \int_{B_{R_l} \setminus \overline{B_{r_l}}} |\nabla w|^2 \, \mathrm{d}\mu_a + 4 \limsup_l \int_{B_1 \setminus \overline{B_{r_l}}} |\nabla z_{j_l}|^2 \, \mathrm{d}\mu_a \\ &\leq \int_{B_1} |\nabla w|^2 \, \mathrm{d}\mu_a + \lim_l 8 \, (1 - r_l)(n + 1) = \int_{B_1} |\nabla w|^2 \, \mathrm{d}\mu_a, \end{aligned}$$

and this provides the conclusion.

Proof of the Γ-convergence: Case (2).

(a) Without loss of generality we assume that

$$\liminf_j \mathcal{G}_j(w_j) = \lim_j \mathcal{G}_j(w_j) < +\infty. \tag{6.23}$$

Let $w_j \to w$ in $L^2(B_1, \mu_a)$, since $w_j \in \mathcal{B}_j$ and (6.23), then $w \geq 0$ on $B_1'^{,-}$. From (6.12), we obtain

$$0 \leq -\theta_j \int_{B_1'} w_j R_a(h) \, \mathrm{d}\mathcal{H}^{n-1} \leq \mathcal{G}_j(w_j) + (1+s) \int_{\partial B_1} z_j^2 \, |x_n|^a \, \mathrm{d}\mathcal{H}^{n-1}$$

$$\leq \sup_j \left(\mathcal{G}_j(w_j) + (1+s) \int_{\partial B_1} z_j^2 \, |x_n|^a \, \mathrm{d}\mathcal{H}^{n-1} \right) < +\infty.$$

Then dividing by θ_j, the convergence of traces leads us to

$$\int_{B_1'} w R_a(h) \, \mathrm{d}\mathcal{H}^{n-1} = \lim_j \int_{B_1'} w_j R_a(h) \, \mathrm{d}\mathcal{H}^{n-1} = 0$$

From (5.2) we deduce that $w|_{B_1'^{,-}} = 0$, or rather $w \in \mathcal{B}_\infty^{(2)}$. In particular also $z_\infty \in \mathcal{B}_\infty^{(2)}$ because $\sup_j \mathcal{G}_j(z_j) < +\infty$. Then, according to the semicontinuity of the norm $H^1(B_1, \mu_a)$ with respect to weak convergence of gradient, the convergence of w_j in $L^2(B_1, \mu_a)$ and the convergence of traces in $L^2(\partial B_1, |x_n|^a \mathcal{H}^{n-1})$ we obtain the Γ-$\liminf$ inequality (6.15).

(b) Now we prove the inequality (6.16). With the same argument used in Case (1) we can consider the case of $w \in \mathcal{B}_\infty^{(2)}$ for which (6.17) holds and for all $j_k \uparrow +\infty$ we find a subsequence $j_{k_l} \uparrow +\infty$ and a sequence $w_l \to w$ in $L^2(B_1, \mu_a)$ such that

$$\limsup_l \mathcal{G}_{j_{k_l}}(w_l) \leq \mathcal{G}_\infty^{(2)}. \tag{6.24}$$

We introduce the positive Radon measures

$$\nu_k := |\nabla z_{j_k}|^2 \, |x_n|^a \, \mathcal{L}^n \llcorner B_1 - 4\theta_{j_k}(z_{j_k} + \theta_{j_k} h) \, R_a(h) \mathcal{H}^{n-1} \llcorner B_1'^{,-}.$$

Assuming that $k >> 1$, we obtain

$$\nu_k(B_1) = \mathcal{G}_{j_k}(z_{j_k}) + (1+s) \int_{\partial B_1} z_{j_k}^2 \, |x_n|^a \, \mathrm{d}\mathcal{H}^{n-1} \leq \sup_j \mathcal{G}_j(z_j) + C \sup_j \|z_j\|_{H^1(B_1, \mu_a)} < \infty,$$

which leads us to

$$\sup_k \nu_k(B_1) = \Lambda_0 < +\infty.$$

In order to prove $\nu_k(B_\rho) = \rho^{n+1}\nu(B_1)$ we observe that setting $\rho \in (0,1)$ by $(1+s)$-homogeneity of z_{j_k} we obtain

$$\int_{B_\rho} |\nabla z_{j_k}|^2 \, \mathrm{d}\mu_a = \int_0^\rho \mathrm{d}t \int_{\partial B_t} |\nabla z_{j_k}|^2 \, |x_n|^a \, \mathrm{d}\mathcal{H}^{n-1} \stackrel{x=ty}{=} \int_0^\rho t^{n-1} \int_{\partial B_1} |\nabla z_{j_k}(ty)|^2 \, |ty_n|^a \, \mathrm{d}\mathcal{H}^{n-1}(y) \, \mathrm{d}t$$

$$= \int_0^\rho t^n \int_{\partial B_1} |\nabla z_{j_k}(y)|^2 \, |y_n|^a \, \mathrm{d}\mathcal{H}^{n-1}(y) \, \mathrm{d}t = \frac{\rho^{n+1}}{n+1} \int_{\partial B_1} |\nabla z_{j_k}(y)|^2 \, |y_n|^a \, \mathrm{d}\mathcal{H}^{n-1}(y)$$

$$= \rho^{n+1} \int_0^1 t^n \, \mathrm{d}t \int_{\partial B_1} |\nabla z_{j_k}(y)|^2 \, |y_n|^a \, \mathrm{d}\mathcal{H}^{n-1}(y) \, \mathrm{d}t$$

$$\overset{ty=x}{=} \rho^{n+1} \int_0^1 \int_{\partial B_t} |\nabla z_{j_k}(x)|^2 \, |x_n|^a \, \mathrm{d}\mathcal{H}^{n-1}(y) \, \mathrm{d}t = \rho^{n+1} \int_{B_1} |\nabla z_{j_k}|^2 \, \mathrm{d}\mu_a,$$

and

$$\int_{B'_\rho} z_{j_k} \, R_a(h)(\widehat{x}) \mathrm{d}\mathcal{H}^{n-1} = \int_0^\rho \int_{\partial B'_t} z_{j_k} \, R_a(h)(\widehat{x}) \mathrm{d}\mathcal{H}^{n-2}$$

$$\overset{\widehat{x}=t\widehat{y}}{=} \int_0^\rho t^{n-2} \, \mathrm{d}t \int_{\partial B'_1} z_{j_k}(t\widehat{y},0) \lim_{\varepsilon \to 0} (t\varepsilon)^a \frac{\partial h}{\partial x_n}(t\widehat{y}, t\varepsilon) \mathrm{d}\mathcal{H}^{n-2}(\widehat{y})$$

$$= \frac{\rho^{n+1}}{n+1} \int_{\partial B'_1} z_{j_k}(\widehat{y},0) \, R_a(h)(\widehat{y}) \mathrm{d}\mathcal{H}^{n-2}(\widehat{y}) = \rho^{n+1} \int_{B'_1} z_{j_k} \, R_a(h)(\widehat{x}) \mathrm{d}\mathcal{H}^{n-1}$$

where in the last equality we did the previous calculus again in reverse order. Since $\nu_k(B_1) < \infty$ then $\nu_k(\partial B_\rho) = 0$ with $\rho \in (0,1) \setminus I$ where I is a set at the most countable. Thus

$$\nu_k(B_{\rho_1} \setminus B_{\rho_2}) \leq \Lambda_0(\rho_1^{n+1} - \rho_2^{n+1}) \leq c(n, \Lambda_0)(\rho_1 - \rho_2), \qquad (6.25)$$

for all $0 < \rho_1 \leq \rho_2 < 1$ such that $\rho_1, \rho_2 \in (0,1) \setminus I$. Repeating the argument in (6.17) we prove the Γ-lim sup inequality for function $w \in \mathcal{B}_\infty^{(2)}$ for which there exists some $\rho \in (0,1)$ such that $\{w \not\equiv z_\infty\} \subset\subset B_\rho$. We extend w on $\mathbb{R}^n$ as z_∞ in B_ρ^c and we indicate the extension by w again. We fix $\varepsilon > 0$ and introduce the following auxiliary tools.

Due to the definition of $H^1(B_1, \mu_a)$ as $\overline{C^\infty(B_1)}^{\|\cdot\|_{H^1(B_1,\mu_a)}}$ (*cf.* [39], Sect. 1.9 and Lem. 1.15) there exists a function $v_\delta \in C^\infty(B_1)$ such that

$$\|v^\delta - w\|_{H^1(B_1,\mu_a)} < \delta(\varepsilon) \qquad \text{with } \delta(\varepsilon) = o(\varepsilon). \qquad (6.26)$$

Let $w^\varepsilon(x) := w(x - 3\varepsilon e_{n-1})$ be the translated function along the direction e_{n-1}. Since $w \in \mathcal{B}_\infty^{(2)}$, we observe that

$$w^\varepsilon(x) = 0 \quad \Longleftrightarrow \quad x - 3\varepsilon e_{n-1} \in \{(\widehat{x},0) \, : \, x_{n-1} \leq 0\} \quad \Longleftrightarrow \quad x \in \{(\widehat{x},0) \, : \, x_{n-1} \leq 3\varepsilon\}.$$

Let I_σ be the set defined as

$$I_\sigma = \{x \in B_1 \, : \, \mathrm{dist}(x, B_1^{'-}) < \sigma\} \qquad (6.27)$$

Let ϕ_ε and χ_ε be two cut-off functions such that

$$\phi_\varepsilon \in C_c^\infty(I_{3\varepsilon}), \quad \phi_{\varepsilon \, |I_{2\varepsilon}} \equiv 1, \quad \|\nabla \phi_\varepsilon\|_{L^\infty(B_1)} \leq \frac{C}{\varepsilon}$$

$$\chi_\varepsilon \in C_c^\infty(B_{1-\varepsilon}), \quad \chi_{\varepsilon \, |B_{1-2\varepsilon}} \equiv 1, \quad \|\nabla \chi_\varepsilon\|_{L^\infty(B_1)} \leq \frac{C}{\varepsilon}. \qquad (6.28)$$

For all $0 < \varepsilon << 1$ we build the sequence of functions

$$w_k^{(\varepsilon)} := \chi_\varepsilon(\phi_\varepsilon w^\varepsilon + (1 - \phi_\varepsilon)v^\delta) + (1 - \chi_\varepsilon)z_{j_k}.$$

Then we can at once infer

$$w_k^{(\varepsilon)} \in z_{j_k} + W_0^{1,2}(B_1)$$

and since we can write

$$w_k^{(\varepsilon)} + \theta_{j_k} h := \chi_\varepsilon(\phi_\varepsilon(v_\delta + \theta_{j_k} h) + (1 - \phi_\varepsilon)(w^\tau + \theta_{j_k} h)) + (1 - \chi_\varepsilon)(z_{j_k} + \theta_{j_k} h),$$

we prove that $w_k^{(\varepsilon)} \in \mathcal{B}_{j_k}$: $w_k^{(\varepsilon)}$ is a convex combination of functions v_δ, w^τ and z_{j_k} with boundary data as z_{j_k} and every addendum is bigger than $-\theta_{j_k} h$ restricted to B_1'. In fact

   (i) by definition $z_{j_k} + \theta_{j_k} h \geq 0$ in B_1';
  (ii) if $x \in \mathrm{supp}(\phi_\varepsilon) \cap B_1'$ then $x_{n-1} < 3\varepsilon$. Thus $w^\varepsilon(x) = 0$ then $\phi_\varepsilon(x)(w^\tau(x) + \theta_{j_k} h(x)) = \phi_\varepsilon(x)\theta_{j_k} h(x) \geq 0$;
 (iii) if $x \in \mathrm{supp}(1 - \phi_\varepsilon) \cap B_1'$ then $x_{n-1} \geq 2\varepsilon$, so $h(\widehat{x}, 0) > 0$ and as $\theta_{j_k} \to +\infty$ $v^\delta(x) + \theta_{j_k} h(x) \geq -\|v_\delta\|_{L^\infty(B_1)} + \theta_{j_k} h(x) \geq 0$ for $k > k_\delta$.

So $w_k^{(\varepsilon)} \in \mathcal{B}_{j_k}$ for $k > k_\delta$.
Next, consider,

$$J_k^\varepsilon := -4\theta_{j_k} \int_{B_1'} w_k^{(\varepsilon)} R_a(h) \, d\mathcal{H}^{n-1}$$

$$I_k^\varepsilon := \int_{B_1} |\nabla w_k^{(\varepsilon)}|^2 \, d\mu_a,$$

respectively the trace term and the volume term of the energy of $w_k^{(\varepsilon)}$. By definition we have

$$J_k^\varepsilon \leq -4\theta_{j_k} \int_{B_{1-\varepsilon}'} (\phi_\varepsilon w^\varepsilon + (1 - \phi_\varepsilon)v^\delta) R_a(h) \, d\mathcal{H}^{n-1} - 4\theta_{j_k} \int_{B_1' \setminus B_{1-2\varepsilon}'} z_{j_k} R_a(h) d\mathcal{H}^{n-1} = J_k^{(1)} + J_k^{(2)}.$$

According to (i), (5.3) and (6.25) we deduce

$$0 \leq \sup_k J_k^{(2)} \leq \sup_k \nu_k(B_1 \setminus B_{1-2\varepsilon}) \leq C\, 2\varepsilon. \tag{6.29}$$

Instead, due to (ii), the function $w_{|B_1' \cap G_{3\varepsilon}}^\varepsilon = 0$ and from definitions of $I_{2\varepsilon}$ and h we have $R_a(h)_{|B_{1-\varepsilon}' \setminus I_{2\varepsilon}} = 0$. From this we infer

$$0 \leq J_k^{(1)} \leq -4\theta_{j_k} \left(\int_{B_{1-\varepsilon}' \cap I_{3\varepsilon}} w^\varepsilon R_a(h) d\mathcal{H}^{n-1} + \int_{B_{1-\varepsilon}' \setminus I_{3\varepsilon}} v^\delta R_a(h) d\mathcal{H}^{n-1} \right) = 0. \tag{6.30}$$

Putting (6.29) and (6.30) together yields

$$\limsup_{k \to \infty} J_k^\varepsilon \leq C\varepsilon. \tag{6.31}$$

In order to estimate the functional I_k^ε we observe that

$$I_k^\varepsilon \leq \int_{B_{1-2\varepsilon}} |\nabla(\phi_\varepsilon w^\varepsilon + (1-\phi_\varepsilon)\, v^\delta)|^2 \, \mathrm{d}\mu_a + c \int_{B_{1-\varepsilon}\setminus B_{1-2\varepsilon}} |\nabla(\phi_\varepsilon w^\varepsilon + (1-\phi_\varepsilon)v^\delta)|^2 \, \mathrm{d}\mu_a$$

$$+ c\int_{B_1\setminus B_{1-2\varepsilon}} |\nabla z_{j_k}|^2 \, \mathrm{d}\mu_a + \frac{c}{\varepsilon^2} \int_{B_{1-\varepsilon}\setminus B_{1-2\varepsilon}} |(\phi_\varepsilon w^\varepsilon + (1-\phi_\varepsilon)v^\delta - z_{j_k}|^2 \, \mathrm{d}\mu_a = I_k^{(1)} + I_k^{(2)} + I_k^{(3)} + I_k^{(4)}.$$

We estimate the four addenda separately. From condition (6.25), we can infer

$$\sup_k I_k^{(3)} \leq \sup_k \nu_k(B_1 \setminus B_{1-2\varepsilon}) C \varepsilon. \tag{6.32}$$

We now estimate the first term; recalling that $\phi_{\varepsilon | I_{3\varepsilon}^c} = 0$

$$I_k^{(1)} = \int_{B_{1-2\varepsilon}\setminus I_{3\varepsilon}} |\nabla v^\delta|^2 \, \mathrm{d}\mu_a + \int_{B_{1-2\varepsilon}\cap I_{3\varepsilon}} \left|\nabla\left(\phi_\varepsilon(w^\varepsilon - v^\delta)\right)\nabla v^\delta\right|^2 \, \mathrm{d}\mu_a$$

$$\leq \int_{B_{1-2\varepsilon}\setminus I_{3\varepsilon}} |\nabla v^\delta|^2 \, \mathrm{d}\mu_a + c\int_{B_{1-2\varepsilon}\cap I_{3\varepsilon}} |\nabla v^\delta|^2 \, \mathrm{d}\mu_a$$

$$+ \int_{B_{1-2\varepsilon}\cap I_{3\varepsilon}} |\nabla(w^\varepsilon - v^\delta)|^2 \, \mathrm{d}\mu_a + \frac{c}{\varepsilon^2}\int_{B_{1-2\varepsilon}\cap I_{3\varepsilon}} |v^\delta - w^\varepsilon|^2 \, \mathrm{d}\mu_a$$

$$\leq \int_{B_{1-2\varepsilon}\setminus I_{3\varepsilon}} |\nabla v^\delta|^2 \, \mathrm{d}\mu_a + c\int_{B_{1-2\varepsilon}\cap I_{3\varepsilon}} |\nabla(v^\delta - \nabla w)|^2 \, \mathrm{d}\mu_a$$

$$+ c\int_{B_{1-2\varepsilon}\cap I_{3\varepsilon}} |\nabla w|^2 \, \mathrm{d}\mu_a + \frac{c}{\varepsilon^2}\int_{B_{1-2\varepsilon}\cap I_{3\varepsilon}} (|v^\delta - w|^2 + |w - w^\varepsilon|^2) \, \mathrm{d}\mu_a. \tag{6.33}$$

Taking the last addendum above into account, we notice that for all φ smooth functions and $\tau > 0$

$$|\varphi(x - \tau e_{n-1}) - \varphi(x)| \leq \tau \int_0^1 |\nabla\varphi|(x - \tau t e_{n-1}) \, \mathrm{d}t.$$

Then, by a simple application of Fubini's theorem we deduce

$$\frac{c}{\varepsilon^2}\int_{B_{1-2\varepsilon}\cap I_{3\varepsilon}} |\varphi(x - \tau e_{n-1}) - \varphi(x)|^2 \, \mathrm{d}\mu_a \leq c\frac{\tau^2}{\varepsilon^2}\int_{(B_{1-2\varepsilon}\cap I_{3\varepsilon})+[0,\tau]e_{n-1}} |\nabla\varphi|^2 \, \mathrm{d}\mu_a$$

where $(B_{1-2\varepsilon} \cap I_{3\varepsilon}) + [0,\tau]e_{n-1}$ denotes the Minkowski sum between sets. So, thanks to a density argument and for $\tau = 3\varepsilon$ we infer

$$\frac{c}{\varepsilon^2}\int_{G_{2\varepsilon}\setminus G_{3\varepsilon}} |w - w^\varepsilon|^2 \, \mathrm{d}\mu_a \leq c\int_{(B_{1-2\varepsilon}\cap I_{3\varepsilon})+[0,\tau]e_{n-1}} |\nabla w|^2 \, \mathrm{d}\mu_a.$$

So, from (6.33), according to (6.26), the continuity of translation in L^2 and the absolute continuity of the integral, and observing that $\mathcal{L}^n((B_{1-2\varepsilon} \cap I_{3\varepsilon}) + [0,\tau]e_{n-1}) = O(\varepsilon)$ we obtain

$$I_k^{(1)} \leq \int_{B_{1-2\varepsilon}\setminus I_{3\varepsilon}} |\nabla v_\varepsilon|^2 \, \mathrm{d}\mu_a + O(\varepsilon). \tag{6.34}$$

Reasoning in the same way as in the estimate of $I_k^{(1)}$ we obtain

$$I_k^{(2)} \leq O(\varepsilon) \tag{6.35}$$

Since $\mathrm{supp}\phi_\varepsilon \subset G_{2\varepsilon}$ and recalling that by condition (6.17), if we choose ε sufficiently small such that $\rho < 1 - 5\varepsilon$, $\mathrm{supp}(w^{3\varepsilon} - z_\infty^{3\varepsilon}) \subset B_{1-2\varepsilon}$, we obtain

$$
\begin{aligned}
I_k^{(4)} &\leq \frac{c}{\varepsilon^2} \int_{B_{1-\varepsilon}\setminus B_{1-2\varepsilon}} |\phi_\varepsilon(w^\varepsilon - v^\delta)|^2 \, \mathrm{d}\mu_a + \frac{c}{\varepsilon^2} \int_{B_{1-\varepsilon}\setminus B_{1-2\varepsilon}} |v^\delta - z_{j_k}|^2 \, \mathrm{d}\mu_a \\
&\leq \frac{c}{\varepsilon^2} \int_{B_{1-\varepsilon}\setminus B_{1-2\varepsilon}} (|w^\varepsilon - w|^2 + |w - v^\delta|^2 + |w - z_{j_k}|^2) \, \mathrm{d}\mu_a
\end{aligned}
$$

So, proceeding as in estimate of $I_k^{(1)}$ and recalling that $\mathrm{supp}(w - z_\infty) \subset B_\rho$ for ε sufficiently small we deduce

$$\limsup_{k\to\infty} I_k^{(4)} \leq \limsup_{k\to\infty} \frac{c}{\varepsilon^2} \int_{B_{1-\varepsilon}\setminus B_{1-2\varepsilon}} |z_\infty - z_{j_k}|^2 \, \mathrm{d}\mu_a + O(\varepsilon) \leq O(\varepsilon). \tag{6.36}$$

Then putting together estimates in (6.32)–(6.36) leads to

$$\limsup_{k\to\infty} I_k^\varepsilon \leq \int_{B_{1-2\varepsilon}\setminus I_{3\varepsilon}} |\nabla v_\varepsilon|^2 \, \mathrm{d}\mu_a + O(\varepsilon).$$

So, since

$$w_k^{(\varepsilon)} \xrightarrow{k\to\infty} \chi_\varepsilon(\phi_\varepsilon v_{\delta_\varepsilon} + (\phi_\varepsilon)w^{3\varepsilon}) + (1-\chi_\varepsilon)z_\infty =: w^{(\varepsilon)} \qquad \text{in } L^2(B_1, \mu_a)$$

and

$$w^{(\varepsilon)} \xrightarrow{\varepsilon\to 0} w \qquad \text{in } L^2(B_1, \mu_a),$$

we conclude by the lower semicontinuity of the Γ-$\limsup$

$$
\begin{aligned}
\Gamma-\limsup_{k\to\infty} \mathcal{G}_{j_k}(w) &\leq \liminf_{\varepsilon\to 0} \left(\Gamma-\limsup_{k\to\infty} \mathcal{G}_{j_k}(w^{(\varepsilon)}) \right) \\
&\leq \limsup_{\varepsilon\to 0} \left(\limsup_{k\to\infty} (I_k^\varepsilon + J_k^\varepsilon) \right) \leq \int_{B_1} |\nabla w|^2 \, \mathrm{d}\mu_a,
\end{aligned}
$$

that provides the thesis.

Proof of the Γ-convergence: Case (3).
(a) From (6.10), we immediately have

$$\liminf_j \mathcal{G}_j(w_j) \geq \liminf_j (1 - \kappa_j)\mathcal{G}_j(z_j) = +\infty = \mathcal{G}_\infty^{(3)}.$$

(b) This is trivial, in fact $\limsup_j \mathcal{G}_j(w_j) \leq +\infty = \mathcal{G}_\infty^{(3)}$.

Step 4: Improving the convergence of $(z_j)_j \in \mathbb{N}$) if $\lim_j \mathcal{G}_j(z_j) < +\infty$. Using a standard result of Γ-convergence we show that $z_j \to z_\infty$ in $H^1(B_1, \mu_a)$.

For equi-coercivity of $\mathcal{G}_j$ seen in (6.13), [15], Lemma 2.10 (a version of Poincaré inequality for weighted Sobolev spaces) and $\|z_j\|_{H^1(B_1, \mu_a)} = 1$ we have

$$\|w\|_{H^1(B_1, \mu_a)} \le C\sqrt{\mathcal{G}_j(w) + 1},$$

so every minimizing sequence converges weakly in $H^1(B_1, \mu_a)$ and thanks to Theorem 2.4 converges strongly in $L^2(B_1, \mu_a)$. Since $\mathcal{G}_j$ is semicontinuous with respect to weak topology of $H^1(B_1, \mu_a)$ there exists ζ_j minimizer of $\mathcal{G}_j$. Taking into account [17], Theorem 7.8, with $i = 1, 2$ there exists $\zeta_\infty \in H^1(B_1, \mu_a)$ such that

$$\zeta_j \to \zeta_\infty, \qquad \text{in } L^2(B_1, \mu_a) \tag{6.37}$$

$$\mathcal{G}_j(\zeta_j) \to \mathcal{G}_\infty^{(i)}(\zeta_\infty), \tag{6.38}$$

$$\zeta_\infty \text{ is the unique minimizer of } \mathcal{G}_\infty^{(i)}, \tag{6.39}$$

where due to (6.39) we have used the strict convexity of $\mathcal{G}_\infty^{(i)}$. Therefore using the strong convergence of traces in $L^2(\partial B_1, |x_n|^a \mathcal{H}^{n-1})$ and $L^2(B_1')$, then from the estimates

$$\mathcal{G}_j(\zeta_j) \le \mathcal{G}_j(z_j) \le \sup_j \mathcal{G}_j(z_j) < \infty, \tag{6.40}$$

and (6.39) we obtain

$$\int_{B_1} |\nabla \zeta_j|^2 \, d\mu_a \to \int_{B_1} |\nabla \zeta_\infty|^2 \, d\mu_a,$$

which implies $\zeta_j \to \zeta_\infty$ in $H^1(B_1, \mu_a)$. According to (6.10) and (6.40), z_j is an almost minimizer of $\mathcal{G}_j$ in the following sense

$$0 \le \mathcal{G}_j(z_j) - \mathcal{G}_j(\zeta_j) \le \kappa_j \mathcal{G}_j(z_j) \le \kappa_j \sup_j \mathcal{G}_j(z_j).$$

Since $\kappa_j \downarrow 0$ and $z_j \rightharpoonup z_\infty$ in $H^1(B_1, \mu_a)$, (6.38) and Step 3 yield that

$$\mathcal{G}_\infty^{(i)}(z_\infty) \le \liminf_j \mathcal{G}_j(z_j) = \lim_j \mathcal{G}_j(\zeta_j) = \mathcal{G}_\infty^{(i)}(\zeta_\infty), \tag{6.41}$$

with $i = 1, 2$. From (6.10), we infer

$$\mathcal{G}_j(z_j) \le \frac{1}{1 - k_j} \mathcal{G}_j(\zeta_j);$$

from this, by (6.41) and by strong convergence of traces we obtain

$$\liminf_j \int_{B_1} |\nabla z_j|^2 \, d\mu_a = \int_{B_1} |\nabla z_\infty|^2 \, d\mu_a,$$

that with the weak convergence of in $H^1(B_1, \mu_a)$ proves the convergence

$$z_j \to z_\infty \qquad \text{in } H^1(B_1, \mu_a).$$

In particular

$$\|z_\infty\|_{H^1(B_1, \mu_a)} = 1. \tag{6.42}$$

Step 5: Case (1) cannot occur. We recall properties of z_∞:

(i) $\|z_\infty\|_{H^1(B_1, \mu_a)} = 1$;

(ii) z_∞ is $(1+s)$-homogeneous and even with respect to $\{x_n = 0\}$;

(iii) z_∞ is the unique minimizer of $\mathcal{G}_\infty^{(1)}$ with respect to its boundary data;

(iv) $z_\infty \in \mathcal{B}_\infty^{(1)} = \{z \in z_\infty + H_0^1(B_1, \mu_a) : (z + \theta h)|_{B_1'} \geq 0\}$.

These properties imply that

$$w_\infty := z_\infty + \theta h$$

is the minimizer of $\int_{B_1} |\nabla \cdot|^2 \, d\mu_a$ among all functions $w \in w_\infty + H_0^1(B_1, \mu_a)$ and $w|_{B_1'} \geq 0$ in the sense of the trace. So, w_∞ is the solution of the fractional obstacle problem. To prove this claim, for all $z \in \mathcal{B}_\infty^{(1)}$ we consider $w := z + \theta h$ and, recalling (5.11), we have

$$\mathcal{G}_\infty^{(1)}(z) = \int_{B_1} |\nabla w|^2 \, d\mu_a - \theta^2 \int_{B_1} |\nabla h|^2 \, d\mu_a - (1+s) \int_{\partial B_1} z_\infty^2 \, |x_n|^a \, d\mathcal{H}^{n-1}$$

$$- 2\theta \int_{B_1} \nabla w \cdot \nabla h \, d\mu_a - 4\theta \int_{B_1'} z \lim_{\varepsilon \to 0} \left(\varepsilon^a \frac{\partial h}{\partial x_n}(\widehat{x}, \varepsilon) \right) \, d\mathcal{H}^{n-1}$$

$$\overset{(5.11)}{=} \int_{B_1} |\nabla w|^2 \, d\mu_a - \theta^2 \int_{B_1} |\nabla h|^2 \, d\mu_a - (1+s) \int_{\partial B_1} z_\infty^2 \, |x_n|^a \, d\mathcal{H}^{n-1} - 2(1+s) \int_{\partial B_1} z_\infty \, h \, |x_n|^a \, d\mathcal{H}^{n-1}.$$

Since $\mathcal{G}_\infty^{(1)}(z_\infty) \leq \mathcal{G}_\infty^{(1)}(z)$ for all $z \in \mathcal{B}_\infty^{(1)}$ then

$$\int_{B_1} |\nabla w_\infty|^2 \, d\mu_a \leq \int_{B_1} |\nabla w|^2 \, d\mu_a \qquad \forall w \in w_\infty + H_0^1(B_1, \mu_a).$$

Using the $(1+s)$-homogeneity and [15], Proposition 5.5, the result of classification of global solutions, we deduce that $w_\infty = \lambda_\infty h_{\nu_\infty} \in \mathfrak{H}_{1+s}$ for some $\lambda_\infty \geq 0$ and $\nu_\infty \in \mathbb{S}^{n-2}$.

Thanks to (6.5) we have the contradiction: from $z_j \rightharpoonup z_\infty$ in $H^1(B_1, \mu_a)$ and (6.8) we have

$$\frac{c_j}{\delta_j} = \theta_j h + z_j \to \theta h + z_\infty \in \mathfrak{H}_{1+s} \qquad \text{in } H^1(B_1, \mu_a), \tag{6.43}$$

so for $j >> 1$

$$\text{dist}_{H^1(B_1, \mu_a)}(c_j, \mathfrak{H}_{1+s}) \leq \|c_j - \delta_j \lambda_\infty h_{\nu_\infty}\|_{H^1(B_1, \mu_a)} \overset{(6.43)}{=} o(\delta_j) < \delta_j = \text{dist}_{H^1(B_1, \mu_a)}(c_j, \mathfrak{H}_{1+s}),$$

where we have used that $\delta_j \lambda_\infty h_{\nu_\infty} \in \mathfrak{H}_{1+s}$.

Step 6: Case (3) cannot occur. To prove that Case (3) cannot occur, we conveniently scale the energies so as to get a nontrivial Γ-limit for the rescaled functionals ultimately leading to a contradiction.

By means (6.14), since $\lim_j \mathcal{G}_j(z_j) = +\infty$, we have

$$\gamma_j := -4\theta_j \int_{B_1'} z_j \, R_a(h)(\widehat{x}) \, \mathrm{d}\mathcal{H}^{n-1} \uparrow +\infty. \tag{6.44}$$

Moreover $z_j \to z_\infty$ in $L^2(B_1')$ and (6.12) give us

$$\lim_j \frac{\gamma_j}{\theta_j} = -4 \lim_j \int_{B_1'} z_j \, R_a(h) \, \mathrm{d}\mathcal{H}^{n-1} = -4 \int_{B_1'} z_\infty \, R_a(h) \, \mathrm{d}\mathcal{H}^{n-1} \in [0, \infty)$$

so

$$\theta_j \gamma_j^{-1/2} \uparrow +\infty. \tag{6.45}$$

Then we rescale the functional $\mathcal{G}_j$ dividing by γ_j. For all $z \in \mathcal{B}_j$ we consider $\gamma_j^{-1}\mathcal{G}_j(z)$ and we note that

$$\gamma_j^{-1}\mathcal{G}_j(z) = \widetilde{\mathcal{G}_j}(\gamma_j^{-1/2}z) \tag{6.46}$$

with

$$\widetilde{\mathcal{G}_j}(w) = \begin{cases} \displaystyle\int_{B_1} |\nabla w|^2 \, \mathrm{d}\mu_a - (1+s) \int_{\partial B_1} w^2 \, |x_n|^a \, \mathrm{d}\mathcal{H}^{n-1} - 4\frac{\theta_j}{\gamma_j^{1/2}} \int_{B_1'} w \, R_a(h) \, \mathrm{d}\mathcal{H}^{n-1} & w \in \widetilde{\mathcal{B}}_j \\ +\infty & \text{otherwise,} \end{cases}$$

where

$$\widetilde{\mathcal{B}}_j := \{w \in \gamma_j^{-1/2} z_j + H_0^1(B_1, \mu_a) \ : \ (w + \theta_j \gamma_j^{-1/2} h)|_{B_1'} \geq 0\}.$$

Setting $\widetilde{z}_j := \gamma_j^{-1/2} z_j$, due to (6.11) and $\gamma_j \uparrow +\infty$, we have $\widetilde{z}_j \to 0$ in $H^1(B_1, \mu_a)$. Moreover the condition (6.46) and the definition of γ_j (6.44) yield

$$\widetilde{\mathcal{G}_j}(\widetilde{z}_j) = \frac{\int_{B_1} |\nabla w|^2 \, \mathrm{d}\mu_a - (1+s) \int_{\partial B_1} w^2 \, |x_n|^a \, \mathrm{d}\mathcal{H}^{n-1}}{\gamma_j} + 1 = 1 + O(\gamma_j^{-1}). \tag{6.47}$$

Thanks to (6.46) we can rewrite the inequalities (6.10) as

$$(1 - \kappa_j)\widetilde{\mathcal{G}_j}(\widetilde{z}_j) \leq \widetilde{\mathcal{G}_j}(\widetilde{z}) \qquad \forall \widetilde{z} \in \widetilde{\mathcal{B}}_j.$$

In particular, by taking into consideration (6.45), $\widetilde{z}_j \to 0$ in $H^1(B_1, \mu_a)$, and (6.47) (in other words $\lim_j \widetilde{\mathcal{G}_j}(\widetilde{z}_j) < \infty$) we proceed as in Case (2) of Step 3 establishing that

$$\Gamma(L^2(B_1, \mu_a))\text{-}\lim_j \widetilde{\mathcal{G}_j} = \widetilde{\mathcal{G}_\infty},$$

with

$$\widetilde{\mathcal{G}_\infty}(\widetilde{z}) = \begin{cases} \displaystyle\int_{B_1} |\nabla \widetilde{z}|^2 \, \mathrm{d}\mu_a & w \in \widetilde{\mathcal{B}}_\infty \\ +\infty & \text{otherwise,} \end{cases}$$

where $\widetilde{\mathcal{B}}_\infty = \{\widetilde{z} \in H_0^1(B_1, \mu_a) : \widetilde{z}|_{B_1'} = 0\}$. From Step 4 and the convergence $\widetilde{z}_j \to 0$ in $H^1(B_1, \mu_a)$, the zero function turns out to be the unique minimizer of $\widetilde{\mathcal{G}_\infty}$ and $\lim_j \widetilde{\mathcal{G}_j}(\widetilde{z}_j) \to \widetilde{\mathcal{G}_\infty}(0) = 0$; this is in contradiction with (6.47).

To prove the theorem we have only to exclude Case (2) of Step 3. In what follows, we suppose the hypothesis of Case (2) of Step 3: $\theta = +\infty$ and $\lim_j \mathcal{G}_j(z_j) < +\infty$. In the following steps we exhibit further properties of the limit z_∞.

Step 7: An orthogonality condition. By evaluating that ψ_j is a point of minimal distance between c_j and $\mathfrak{H}_{1+s}$, we prove that z_∞ is orthogonal to the tangent space $T_h \mathfrak{H}_{1+s}$.

From the hypothesis $\theta = +\infty$ we deduce that $\lambda_j > 0$ for $j >> 1$. Therefore, by the condition of minimal distance (6.5), we deduce that for all $\nu \in \mathbb{S}^{n-2}$ and $\lambda \geq 0$,

$$\|c_j - \psi_j\|_{H^1(B_1, \mu_a)} \leq \|c_j - \lambda\, h_\nu\|_{H^1(B_1, \mu_a)},$$

and thanks to definition of z_j in (6.8) it holds

$$\delta_j \|z_j\|_{H^1(B_1, \mu_a)} \leq \|\psi_j - \lambda\, h_\nu + \delta_j z_j\|_{H^1(B_1, \mu_a)}$$

or in the same way

$$-\|\psi_j - \lambda\, h_\nu\|^2_{H^1(B_1, \mu_a)} \leq 2\delta_j \langle z_j, \psi_j - \lambda h_\nu\rangle_{H^1(B_1, \mu_a)}. \tag{6.48}$$

Now we suppose $(\lambda, \nu) \neq (\lambda_j, e_{n-1})$ and renormalizing (6.48) we obtain

$$-\|\psi_j - \lambda\, h_\nu\|_{H^1(B_1, \mu_a)} \leq 2\delta_j \langle z_j, \frac{\psi_j - \lambda h_\nu}{\|\psi_j - \lambda\, h_\nu\|_{H^1(B_1, \mu_a)}}\rangle_{H^1(B_1, \mu_a)}$$

and by passing to the limit $(\lambda, \nu) \to (\lambda_j, e_{n-1})$, reminding the definition of tangent space $T\mathfrak{H}_{1+s}$ in (5.6), we deduce

$$\langle z_j, \zeta\rangle \geq 0 \qquad \zeta \in T_{\psi_j} \mathfrak{H}_{1+s} = T_h \mathfrak{H}_{1+s}, \tag{6.49}$$

where we used $\lambda_j > 0$ in the computation of the tangent vector. By choosing the sequence $(\lambda, \nu) \to (\lambda_j, e_{n-1})$ such that $\lim \frac{\psi_j - \lambda h_\nu}{\|\psi_j - \lambda\, h_\nu\|_{H^1(B_1, \mu_a)}} = -\zeta$ we obtain $\langle z_j, \zeta\rangle \leq 0$ thus

$$\langle z_j, \zeta\rangle = 0 \qquad \zeta \in T_h \mathfrak{H}_{1+s}. \tag{6.50}$$

So, taking the limit $j \to +\infty$ we conclude

$$\langle z_\infty, \zeta\rangle = 0 \qquad \zeta \in T_h \mathfrak{H}_{1+s}. \tag{6.51}$$

Step 8: Identification of z_∞ in Case (2). There exist real constants $a_0, \ldots, a_{n-2}$ such that

$$z_\infty = a_0 h + \left(\sum_{i=1}^{n-2} a_i x_i \right) \left(\sqrt{x_{n-1}^2 + x_n^2} + x_{n-1} \right)^s, \tag{6.52}$$

so, in view of (5.8) and (5.9), $z_\infty \in T_h \mathfrak{H}_{1+s}$.

The stated property follows from [32], Lemma A.3 (for the reader's convenience we report a proof in the Appendix).

Step 9: Case (2) cannot occur. We use the results of Step 4, 7 and 8 to deduce a contradiction.

From (6.52) we deduce that $z_\infty \in T_h \mathfrak{H}_{1+s}$, by using it as a test function in (6.51), the condition of orthogonality of Step 7 implies

$$\langle z_\infty, \zeta \rangle = 0 \qquad \zeta \in T_h \mathfrak{H}_{1+s}.$$

Then we have $z_\infty = 0$ but this is in contradiction with (6.42).

In this way we exclude the occurrence of Case (2) of Step 3, thus providing the conclusion of the proof of the theorem. $\square$

In what follows we show some important consequences of epiperimetric inequality.

6.2. Decay of the boundary adjusted energy

The following proposition establishes a decay estimate for the boundary adjusted energy. In this connection the epiperimetric inequality allows us to estimate from below, up to a constant, the difference between the energy $W_{1+s}^0(1, \cdot)$ evaluated respectively in the $(1+s)$-homogeneous extension of $u_{r|\partial B_1}$ and in u_r with $W_{1+s}^0(1, u_r)$; in this way we obtain a differential inequality from which we deduce the decay estimate.

Proposition 6.2 (Decay of the boundary adjusted energy). *Let $x_0 \in \Gamma_{1+s}(u)$. There exists a constant $\gamma > 0$ for which the following property holds:*
for every compact set $K \subset B_1'$ there exists a positive constant $C > 0$ such that

$$W_{1+s}^{x_0}(r, u) \leq C \, r^\gamma, \tag{6.53}$$

for all radii $0 < r < \mathrm{dist}(K, \partial B_1)$ and for all $x_0 \in \Gamma_{1+s}(u) \cap K$.

Proof. Let us assume $x_0 = \underline{0} \in \Gamma_{1+s}(u)$. Thanks to Lemma 3.4, we calculate the derivative of the boundary adjusted energy $W_{1+s}(\cdot, u)$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}r} W_{1+s}^0(r, u) &= -\frac{(n+1)}{r^{n+2}} D_a(r) + \frac{1}{r^{n+1}} D_a'(r) - \frac{(1+s)}{r^{n+2}} H_a'(r) + \frac{(1+s)(n+2)}{r^{n+3}} H_a(r) \\
&= -\frac{(n+1)}{r^{n+2}} D_a(r) + \frac{1}{r^{n+1}} D_a'(r) - \frac{(1+s)(n-2s)}{r^{n+3}} H_a(r) - \frac{2(1+s)}{r^{n+2}} D_a(r) + \frac{(1+s)(n+2)}{r^{n+3}} H_a(r) \\
&= -\frac{n+1}{r} W_{1+s}^0(r, u) - \frac{(1+s)(n+1)}{r^{n+3}} H_a(r) + \frac{1}{r^{n+1}} D_a'(r) - \frac{2(1+s)}{r^{n+2}} D_a(r) + \frac{2(1+s)^2}{r^{n+3}} H_a(r) \\
&= -\frac{n+1}{r} W_{1+s}^0(r, u) - \frac{(1+s)(n+1)}{r^{n+3}} H_a(r) + I.
\end{aligned} \tag{6.54}$$

According to Lemma 3.4 and to the definition of rescaled functions (4.1), we can write

$$
\begin{aligned}
I \; = \; & \frac{1}{r^{n+2}} \int_{\partial B_r} |\nabla u|^2 \, |x_n|^a \, \mathrm{d}\mathcal{H}^{n-1} + \frac{2(1+s)^2}{r^{n+3}} \int_{\partial B_r} u^2 \, |x_n|^a \, \mathrm{d}\mathcal{H}^{n-1} - \frac{2(1+s)}{r^{n+2}} \int_{\partial B_r} u \nabla u \cdot \frac{x}{r} \, |x_n|^a \, \mathrm{d}\mathcal{H}^{n-1} \\
\overset{x=ry}{=} \; & \frac{1}{r} \int_{\partial B_1} \left(|\nabla u_r|^2 + 2(1+s)^2 u_r^2 - 2(1+s) u_r \nabla u_r \cdot y \right) |y_n|^a \, \mathrm{d}\mathcal{H}^{n-1} \\
= \; & \frac{1}{r} \int_{\partial B_1} \left((\nabla u_r \cdot \nu - (1+s) u_r)^2 + |\nabla_\theta u_r|^2 + (1+s)^2 u_r^2 \right) |y_n|^a \, \mathrm{d}\mathcal{H}^{n-1} \qquad (6.55)
\end{aligned}
$$

where by $\nabla_\theta u_r$ we denote the differential of u_r in the tangent direction to ∂B_1. Let c_r be the $(1+s)$-homogeneous extension of $u_{r|\partial B_1}$

$$
c_r(x) := |x|^{1+s} u_r \left(\frac{x}{|x|} \right).
$$

Thus, according to $(1+s)$-homogeneity and by Euler's homogeneous function Theorem and recalling that $H_a(r) = r^{n+2} H_a(1)$ and $W_{1+s}^0(1, u_r) = W_{1+s}^0(r, u)$, by putting together the equations (6.54) and (6.55), we deduce

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}r} W_{1+s}^0(r, u) = \; & -\frac{n+1}{r} W_{1+s}^0(r, u) - \frac{(n+1)(1+s)}{r} \int_{\partial B_1} u_r^2 \, |x_n|^a \, \mathrm{d}\mathcal{H}^{n-1} \\
& + \frac{1}{r} \int_{\partial B_1} (\nabla u_r \cdot \nu - (1+s) u_r)^2 \, |x_n|^a \, \mathrm{d}\mathcal{H}^{n-1} + \frac{1}{r} \int_{\partial B_1} \left(|\nabla_\theta u_r|^2 + (1+s)^2 u_r^2 \right) |x_n|^a \, \mathrm{d}\mathcal{H}^{n-1} \\
= \; & -\frac{n+1}{r} W_{1+s}^0(r, u) + \frac{1}{r} \int_{\partial B_1} (\nabla u_r \cdot \nu - (1+s) u_r)^2 \, |x_n|^a \, \mathrm{d}\mathcal{H}^{n-1} \\
& + \frac{1}{r} \int_{\partial B_1} \left(|\nabla_\theta c_r|^2 - (1+s)(n-s) c_r^2 \right) |x_n|^a \, \mathrm{d}\mathcal{H}^{n-1} \\
= \; & -\frac{n+1}{r} W_{1+s}^0(r, u) + \frac{1}{r} \int_{\partial B_1} (\nabla u_r \cdot \nu - (1+s) u_r)^2 \, |x_n|^a \, \mathrm{d}\mathcal{H}^{n-1} \\
& + \frac{1}{r} \int_{\partial B_1} \left(|\nabla c_r|^2 - (1+s)(n+1) c_r^2 \right) |x_n|^a \, \mathrm{d}\mathcal{H}^{n-1} \\
= \; & \frac{n+1}{r} W_{1+s}^0(1, c_r) - \frac{n+1}{r} W_{1+s}^0(1, u_r) + \frac{1}{r} \int_{\partial B_1} (\nabla u_r \cdot \nu - (1+s) u_r)^2 \, |x_n|^a \, \mathrm{d}\mathcal{H}^{n-1}.
\end{aligned}
$$

So, by Proposition 4.3 we have

$$
\frac{\mathrm{d}}{\mathrm{d}r} W_{1+s}^0(r, u) = 2 \frac{n+1}{r} \left(W_{1+s}^0(1, c_r) - W_{1+s}^0(1, u_r) \right).
$$

Then, according to the epiperimetric inequality proved in Theorem 6.1, and recalling that u_r minimize $W_{1+s}^0(1, \cdot)$ we obtain

$$
\frac{\mathrm{d}}{\mathrm{d}r} W_{1+s}^0(r, u) \geq \frac{2(n+1)\kappa}{r(1-\kappa)} W_{1+s}^0(1, u_r) = \frac{2(n+1)\kappa}{r(1-\kappa)} W_{1+s}^0(r, u),
$$

and integrating this inequality in $(0, r_0)$ we have

$$W^0_{1+s}(r, u) \leq W^0_{1+s}(1, u)\, r^\gamma,$$

with $\gamma := \frac{2\,(n+1)\kappa}{r(1-\kappa)}$. $\qquad\qquad\square$

Remark 6.3. In order to prove the Proposition 6.2 the Weiss' monotonicity formula is not necessary.

6.3. Nondegeneracy of the solution

In order to deduce the nondegeneracy property of the solution we note that the inequality (3.5) is not enough. We state an improved version of (3.5); this is a consequence of epiperimetric inequality and decay estimate of energy above.

Proposition 6.4 (Nondegeneracy)**.** *Let* $u \in H^1(B_1, \mu_a)$ *be a solution of the problem* (1.3)*. Let us assume that* $\underline{0} \in \Gamma_{1+s}(u)$ *. Then there exists a constant* $H_0 > 0$ *for which*

$$H_a(r) \geq H_0\, r^{n+2} \qquad \forall\, 0 < r < 1. \tag{6.56}$$

Proof. For the proof of this result we refer to [23], Proposition 4.6. $\qquad\qquad\square$

By means of the nondegeneracy condition (6.56), for all $x_0 \in \Gamma_{1+s}(u)$, we deduce

$$\int_{\partial B_1} u^2_{x_0, r}\, |x_n|^a\, d\mathcal{H}^{n-1} \geq H_0,$$

and if $(u_{x_0, r_k})_{k \in \mathbb{N}}$ is a sequence that converges to u_0 in $L^2(B_1, \mu_a)$, a blow-up function in x_0, due to estimate (4.2) and the convergence of the traces in Theorem 2.6 we obtain the convergence of the traces of u_{x_0, r_k} on ∂B_1; thus

$$\int_{\partial B_1} u_0\, |x_n|^a\, d\mathcal{H}^{n-1} \geq H_0 > 0.$$

So we infer $u_0 \not\equiv 0$ for all u_0 blow-up functions in a point of $\Gamma_{1+s}(u)$.

So, in view of Propositions 6.2, 6.4 and [15], Proposition 5.5 we can deduce the following result of the classification of blow-ups.

Proposition 6.5 (Classification of blow-ups)**.** *Let* u *be a solution of the problem* (1.3)*. Let* u_0 *be a blow-up of* u *in point* $x_0 \in \Gamma_{1+s}(u)$*. Then there exist a constant* $\lambda > 0$ *and a vector* $e \in \mathbb{S}^{n-2}$ *such that* $u_0 = \lambda\, h_e$*.*

6.4. The blow-up method: uniqueness of blow-ups

By summarizing what we have been showing so far, due to estimate (4.2) and to Theorem 2.3, for all $x_0 \in \Gamma_{1+s}(u)$ and for all sequences $r_k \to 0$ there exists at least a subsequence (that we do not relabel in what follows) such that $u_{x_0, r_k} \rightharpoonup u_{x_0}$ in $H^1(B_1, \mu_a)$ for some nontrivial functions $u_{x_0} \in H^1(B_1, \mu_a)$. It is easy to prove that u_{x_0} is a solution of problem (1.3). Furthermore u_{x_0} is $(1 + s)$-homogeneous. According to Proposition 6.5, the result of the classification of blow-ups, we obtain $u_{x_0} \in \mathfrak{H}_{1+s}$.

With the next Proposition we prove that the blow-up is unique, *i.e.* for all $x_0 \in \Gamma_{1+s}(u)$ there exists a function u_{x_0} such that for all $r_k \to 0$ the sequence $(u_{x_0, r_k})_{k \in \mathbb{N}}$ converges to u_{x_0} in $L^2(B_1, \mu_a)$. This is again a consequence of epiperimetric inequality. In particular, the epiperimetric inequality provides an explicit rate of convergence of the rescaled function $u_{x_0, r}$.

The uniqueness of the blow-up was proved in [32], Proposition 5.2, follows the approach of [31]. Thanks to our version of epiperimetric inequality, following the same computations in [23], Proposition 4.8 (for more details see [34]), we can obtain a proof of next proposition using only $(1+s)$-homogeneous scaling (see (4.1)).

Proposition 6.6 ([32], Prop. 5.2)**.** *Let u be a solution of the problem* (1.3) *and let $K \subset\subset B'_1$. Then there exists a positive constant $C > 0$ such that for all $x_0 \in \Gamma_{1+s}(u) \cap K$ the following inequality holds:*

$$\int_{\partial B_1} |u_{x_0,r} - u_{x_0}|\, |x_n|^a \, \mathrm{d}\mathcal{H}^{n-1} \leq C\, r^{\frac{\gamma}{2}},$$

where $\gamma > 0$ is the constant defined in Proposition 6.2. In particular the blow-up is unique.

7. The regularity of the free-boundary

Thanks to the uniqueness of blow-ups following the proof of [23], Proposition 4.10 (for more details see [34]) it is possible to give a different proof of the $C^{1,\alpha}$ regularity of $\Gamma_{1+s}(u)$ originally proved by Caffarelli *et al.* [15], Theorem 7.7. The case of the obstacle problem with drift has been dealt with in [32], Theorem 1.2.

Theorem 7.1. *Let $u \in H^1(B_1, \mu_a)$ be a solution of the problem* (1.3)*. Then, there exists a constant $\alpha > 0$ such that for all $x_0 \in \Gamma_{1+s}(u)$ there exists a radius $r = r(x_0)$ for which $\Gamma_{1+s}(u) \cap B'_r(x_0)$ is a $C^{1,\alpha}$ regular $(n-2)$-submanifold in B'_1.*

Appendix A

In this Appendix we report a result of structure of a $(1+s)$-homogeneous solution of (A.1) due to Garofalo *et al.* [32], Lemma A.3.

Lemma A.1. *Let z_∞ be a $(1+s)$-homogeneous solution of*

$$\begin{cases} L_a z_\infty = 0 & B_1 \setminus B'^{-}_1 \\ z_\infty = 0 & B'^{-}_1, \end{cases} \tag{A.1}$$

even symmetric w.r.t $\{x_n = 0\}$. Then there exist real constants $a_0, \ldots, a_{n-2}$ such that

$$z_\infty = a_0 h + \left(\sum_{i=1}^{n-2} a_i x_i \right) \left(\sqrt{x_{n-1}^2 + x_n^2} + x_{n-1} \right)^s, \tag{A.2}$$

or rather $z_\infty \in T_h \mathfrak{H}_{1+s}$.

Proof. For all multi-indices $\alpha \in \mathbb{N}^{n-2}$ the derivative $\partial_\alpha z_\infty$ is the solution of

$$\begin{cases} L_a \partial_\alpha z_\infty = 0 & B_1 \setminus B'^{-}_1 \\ \partial_\alpha z_\infty = 0 & B'^{-}_1, \end{cases} \tag{A.3}$$

According to [19], Lemma 2.4.1 and [15], Proposition 2.3, the derivative $\partial_\alpha z_\infty$ is bounded in $B_{1/2}$, thanks to [19], Thms. 2.3.12 and 2.4.6, it is also continuous in $B_{1/2} \setminus \{x_{n-1} = x_n = 0\}$. We consider the second derivative $\partial_{ij} z_\infty$ with $i,j = 1, \ldots, n-2$: since z_∞ is $(1+s)$-homogeneous, the function $\partial_{ij} z_\infty$ is $(s-1)$-homogeneous; as $0 < s < 1$ from the boundedness of the derivative we deduce

$$\partial_{ij} z_\infty = 0 \qquad \text{in } B_1 \qquad \forall i,j = 1, \ldots, n-2. \tag{A.4}$$

The solution z_∞ is a smooth function in $B_{1/2}^+$ and $B_{1/2}^-$ because the coefficients of the strictly elliptic operator L_a are smooth in these domains. Thus, fixed x_{n-1} and x_n, we can write the first order Taylor polynomial of $z_\infty(\cdot, x_{n-1}, x_n)$ in $(\underline{0}', x_{n-1}, x_n)$

$$z_\infty(x', x_{n-1}, x_n) = c_0(x_{n-1}, x_n) + \sum_{i=1}^{n-2} c_i(x_{n-1}, x_n) x_i,$$

with $c_0(x_{n-1}, x_n) = z_\infty(\underline{0}', x_{n-1}, x_n)$ and $c_i(x_{n-1}, x_n) = \partial_i z_\infty(\underline{0}', x_{n-1}, x_n)$. By definition the function $c_0(x_{n-1}, x_n)$ is $(1+s)$-homogeneous and the functions $c_i(x_{n-1}, x_n)$ are s-homogeneous. Since z_∞ and $\partial_i z_\infty$ are continuous in $B_{1/2} \setminus \{x_{n-1} = x_n = 0\}$ the function $c_0(x_{n-1}, x_n)$ and $c_i(x_{n-1}, x_n)$ are continuous in $\mathcal{B}_{1/2} \setminus \{x_{n-1} = 0\}$ with $\mathcal{B}_{1/2} := \{(x_{n-1}, x_n) \in \mathbb{R}^2 \; : \; x_{n-1}^2 + x_n^2 < 1/4\}$. Thanks to homogeneity with positive degree $c_0(x_{n-1}, x_n)$ and $c_i(x_{n-1}, x_n)$ are continuous in $\mathcal{B}_{1/2}$.

Taking into account (A.4), for all $i = 1, \ldots, n-2$ we obtain

$$c_i(x_{n-1}, x_n) = \partial_i z_\infty(x', x_{n-1}, x_n)$$
$$c_0(x_{n-1}, x_n) = z_\infty(x', x_{n-1}, x_n),$$

thus $c_i, c_0 \in H^1(\mathcal{B}_{1/2}^\pm, |x_n|^a \mathcal{L}^2)$ and are solutions of (A.3) on $\mathcal{B}_{1/2}^\pm$. Since $c_i(x_{n-1}, x_n)$ is s-homogeneous there exist some constants $(\tilde{a}_i)_{i=1,\ldots,n-2}$ such that $c_i(x_{n-1}, 0) = \tilde{a}_i x_{n-1}^s$ when $x_{n-1} > 0$ and similarly since $c_0(x_{n-1}, x_n)$ is $(1+s)$-homogeneous, there exists a constant $\tilde{a}_0$ such that $c_0(x_{n-1}, 0) = \tilde{a}_0 x_{n-1}^s$ when $x_{n-1} > 0$.

We show that

$$c_i(x_{n-1}, x_n) = \frac{\tilde{a}_i}{2^s} \left(x_{n+1} + \sqrt{x_{n-1}^2 + x_n^2} \right)^s. \tag{A.5}$$

Passing to polar coordinates we can write $c_i(x_{n-1}, x_n) = d_i(r, \theta) = r^s \varphi_i(\theta)$. From $L_a c_i = 0$ we deduce that the function φ_i is the solution of the following second order ordinary differential equation

$$\begin{cases} \sin\theta \varphi_{\theta\theta} + a\cos\theta \varphi_\theta + (a(1+s)x + (1+s)^2)\sin\theta \varphi = 0 & in \ (0, \pi) \\ \varphi(0) = \frac{\tilde{a}_i}{2^s} \\ \varphi(\pi) = 0, \end{cases}$$

and so it has a unique solution. Resorting to a direct calculation, we can verify that the function

$$\varphi_i(\theta) = \frac{\tilde{a}_i}{2^s}(\cos\theta + 1)^s$$

is solution for all $\theta \in [0, \pi]$. So the function $c_i(x_{n-1}, x_n)$ satisfies (A.5).

By proceeding in the same way we prove that the function $c_0(x_{n-1}, x_n)$ can be written as

$$c_0(x_{n-1}, x_n) = \frac{\tilde{a}_0}{2^s(s-1)} \left(x_{n+1} + \sqrt{x_{n-1}^2 + x_n^2} \right)^s \left(x_{n+1} - \sqrt{x_{n-1}^2 + x_n^2} \right),$$

and this provides the conclusion to the proof of the step. $\qquad\square$

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