

## BACKWARD SDES AND INFINITE HORIZON STOCHASTIC OPTIMAL CONTROL\*

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**Abstract.** We study an optimal control problem on infinite horizon for a controlled stochastic differential equation driven by Brownian motion, with a discounted reward functional. The equation may have memory or delay effects in the coefficients, both with respect to state and control, and the noise can be degenerate. We prove that the value, *i.e.* the supremum of the reward functional over all admissible controls, can be represented by the solution of an associated backward stochastic differential equation (BSDE) driven by the Brownian motion and an auxiliary independent Poisson process and having a sign constraint on jumps.

In the Markovian case when the coefficients depend only on the present values of the state and the control, we prove that the BSDE can be used to construct the solution, in the sense of viscosity theory, to the corresponding Hamilton-Jacobi-Bellman partial differential equation of elliptic type on the whole space, so that it provides us with a Feynman-Kac representation in this fully nonlinear context.

The method of proof consists in showing that the value of the original problem is the same as the value of an auxiliary optimal control problem (called randomized), where the control process is replaced by a fixed pure jump process and maximization is taken over a class of absolutely continuous changes of measures which affect the stochastic intensity of the jump process but leave the law of the driving Brownian motion unchanged.

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### 1. INTRODUCTION

Let us consider a classical optimal control problem with infinite horizon for a stochastic equation in  $\mathbb{R}^n$  of the form

$$X_t^\alpha = x + \int_0^t b(X_s^\alpha, \alpha_s) ds + \int_0^t \sigma(X_s^\alpha, \alpha_s) dW_s, \quad t \geq 0, \quad (1.1)$$

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starting at a point  $x \in \mathbb{R}^n$ , with discounted reward functional

$$J(x, \alpha) = \mathbb{E} \left[ \int_0^\infty e^{-\beta t} f(X_t^\alpha, \alpha_t) dt \right].$$

Here a  $W$  is a  $d$ -dimensional Brownian motion defined in some probability space,  $b$  and  $\sigma$  are given coefficients with values in  $\mathbb{R}^n$  and  $\mathbb{R}^{n \times d}$  respectively,  $f$  is a real function representing the running cost rate,  $\beta > 0$  is a discount factor. The control process  $\alpha$  is a stochastic process taking values in a metric space  $A$  and progressive with respect to the completed filtration  $\mathbb{F}^W$  generated by the Brownian motion; the class of such controls is denoted by  $\mathcal{A}$ . The aim is to maximize  $J(x, \alpha)$  over  $\mathcal{A}$  and to characterize the value function

$$v(x) = \sup_{\alpha \in \mathcal{A}} J(x, \alpha).$$

It is well known (standard references for stochastic optimal control problems are [5, 11, 20], as well as the more recent books [12, 21, 28, 32], where the viscosity solutions approach is also treated) that, under natural assumptions, the value function is well defined and it is a viscosity solution to the Hamilton-Jacobi-Bellman (HJB) equation, which is the following elliptic partial differential equation on the whole space:

$$\beta v(x) - \sup_{a \in A} [\mathcal{L}^a v(x) + f(x, a)] = 0, \quad x \in \mathbb{R}^n, \quad (1.2)$$

where  $\mathcal{L}^a$  is the Kolmogorov operator depending on the control parameter  $a$ :

$$\mathcal{L}^a v(x) = \langle b(x, a), D_x v(x) \rangle + \frac{1}{2} \text{tr} [\sigma(x, a) \sigma^\top(x, a) D_x^2 v(x)].$$

Thus, when a uniqueness result holds, the HJB equation completely characterizes the value function.

It is the purpose of this paper to provide a different representation of the value function, based on backward stochastic differential equations (BSDEs). BSDEs are used since long to represent value functions of stochastic optimal control problems and more generally solutions to partial differential equations of parabolic and elliptic type. Besides their intrinsic interest, some motivations are the fact that they usually allow to extend the obtained results beyond the Markovian case and they often admit efficient numerical approximations, which makes them a competitive tool in comparison with other more common numerical methods for partial differential equations. However the classical results relating BSDEs with partial differential equations, see for instance [22–25], only cover cases when the equation is semilinear, *i.e.* it takes the form (in the elliptic case)

$$\beta v(x) - \mathcal{L}v(x) + \psi(x, v(x), D_x v(x) \sigma(x)) = 0, \quad x \in \mathbb{R}^n, \quad (1.3)$$

where  $\mathcal{L}$  is the linear Kolmogorov operator associated to (uncontrolled) coefficients  $b(x)$ ,  $\sigma(x)$ , and the nonlinear term  $\psi$  depends on the gradient  $D_x v(x)$  only through the product  $D_x v(x) \sigma(x)$ . The general HJB equation (1.2), being fully nonlinear, cannot be cast in the form (1.3) except in special cases. The corresponding optimal control problems also have a special form, and in particular the occurrence of a diffusion coefficient  $\sigma(x, a)$  depending on a control parameter  $a \in A$  cannot be allowed.

To overcome this difficulty, new methods have recently been developed. We mention the theory of second order BSDEs [30] and the theory of  $G$ -expectations [27], that both allow a probabilistic representation of solutions to classes of fully nonlinear equations.

In this paper we follow a different approach, based on a method that we call *randomization of control*. It has been introduced in [6] and then successfully applied to several stochastic optimization problems, including impulse control, optimal switching, optimal stopping; see [1–3, 9, 10, 14, 17], and especially [16] for a systematic application to a generalized class of HJB equations. In the proofs of our results we will especially follow [4, 13]

that deal with the optimal control problem with finite horizon and, in the Markovian case, with the corresponding parabolic HJB equation.

The general idea of the randomization method is as follows. By enlarging the original probability space if necessary, we consider an independent Poisson random measure  $\mu(dt, da)$  on  $(0, \infty) \times A$ , with finite intensity measure  $\lambda(da)$ , and the corresponding  $A$ -valued process with piecewise constant trajectories, denoted by  $I$ . We formally replace the control process  $\alpha$  by  $I$ , so we solve the equation

$$X_t = x + \int_0^t b(X_s, I_s) ds + \int_0^t \sigma(X_s, I_s) dW_s, \quad t \geq 0. \quad (1.4)$$

Then we consider an auxiliary optimization problem, called randomized problem, which consists in optimizing among equivalent changes of probability measures which only affect the intensity measure of  $I$  but not the law of  $W$ . In the randomized problem, an admissible control is a bounded positive map  $\nu$  defined on  $\Omega \times (0, \infty)^? \times A$ , which is predictable with respect to the filtration  $\mathbb{F}^{W, \mu}$  generated by  $W$  and  $\mu$ . Given  $\nu$ , by means of an absolutely continuous change of probability measure of Girsanov type we construct a probability  $\mathbb{P}^\nu$  such that the compensator of  $\mu$  is given by  $\nu_t(a)\lambda(da)dt$  and  $W$  remains a Brownian motion under  $\mathbb{P}^\nu$ . Then we introduce another reward functional and the corresponding value function

$$J^\mathcal{R}(x, \nu) = \mathbb{E}^\nu \left[ \int_0^\infty e^{-\beta t} f(X_t, I_t) dt \right], \quad v^\mathcal{R}(x) = \sup_\nu J^\mathcal{R}(x, \nu),$$

where  $\mathbb{E}^\nu$  denotes the expectation under  $\mathbb{P}^\nu$ . Some technical issues arise in connection with the use of Girsanov transformation on the whole time halfline, so the precise formulation is slightly more involved: see Section 3 below and especially Remark 3.1 for more details. Building on our previous results in [4, 13] we prove that the two value functions  $v$  and  $v^\mathcal{R}$  coincide: see Theorem 4.1.

Next we prove that the function  $v^\mathcal{R}$  can be represented by means of the following infinite horizon backward stochastic differential equation (BSDE) with constraint: for all  $0 \leq t \leq T < \infty$ ,

$$\begin{cases} Y_t = Y_T - \beta \int_t^T Y_s ds + \int_t^T f(X_s, I_s) ds + K_T - K_t - \int_t^T Z_s dW_s - \int_t^T \int_A U_s(a) \mu(ds, da), \\ U_t(a) \leq 0. \end{cases} \quad (1.5)$$

In this equation the unknown process is a quadruple  $(Y_t, Z_t, U_t(a), K_t)$  where  $Y$  is càdlàg adapted (with respect to  $\mathbb{F}^{W, \mu}$ ),  $Z$  is  $\mathbb{R}^d$ -valued progressive,  $K$  is increasing predictable,  $U$  is a predictable random field, all satisfying appropriate integrability conditions. The constraint on  $U$  can be seen as a constraint of nonpositivity on the totally inaccessible jumps of  $Y$ . We prove that the BSDE (1.5) admits a unique minimal solution in a suitable sense and that it represents the value function in the sense that  $Y_0 = v^\mathcal{R}(x)$  and so also  $Y_0 = v(x)$ , so that we obtain the desired representation of the value function for our original control problem (see Thm. 4.2). In addition, we prove that the solution to the BSDE satisfies a sort of recursive formula (Eq. (4.5)) which is a version of the dynamic programming principle in the setting of the randomized control problem. We exploit this functional equality to prove that the value function  $v$  is a viscosity solution to the HJB equation (1.2), see Theorem 5.8. Therefore the equality  $Y_0 = v(x)$  can also be seen as a fully nonlinear Feynman-Kac representation for the solution to (1.2). This approach allows to circumvent the difficulties related to a rigorous proof of the classical dynamic programming principle which might be lengthy and, in some versions, may require the use of nontrivial measurability arguments. On the contrary, we do not deal with uniqueness results for the HJB equation, which are classical and are known to hold under suitable assumptions, see Remark 5.9.

We stress that none of our results requires nondegeneracy assumptions on the noise, so no requirements are imposed on the diffusion coefficient  $\sigma$  except Lipschitz conditions and some boundedness and continuity assumptions. This is a common feature shared with the BSDEs approach to semilinear HJB equations of the form (1.3), and it is indeed one of the main motivations for the introduction of the randomization method.

As mentioned before, the use of BSDEs often allows for efficient numerical treatment. This is also the case for the constrained BSDEs of the form (1.5), at least in the finite horizon case: see [18, 19].

Another advantage of our technique is that we are able to generalize all the previous results (except the ones on the HJB equation) to the non Markovian case when the coefficients of the controlled equation exhibit memory effects, *i.e.* for an equation of the form

$$X_t^\alpha = x + \int_0^t b_s(X^\alpha, \alpha) ds + \int_0^t \sigma_s(X^\alpha, \alpha) dW_s, \quad t \geq 0, \quad (1.6)$$

where, at any time  $t$ , the value of the coefficients  $b_t(X^\alpha, \alpha)$  and  $\sigma_t(X^\alpha, \alpha)$  may depend on the entire past trajectory of the state  $(X_s^\alpha)_{s \in [0, t]}$  as well as the control process  $(\alpha_s)_{s \in [0, t]}$ . In fact, the previous results are formulated and proved directly in this generality, while the Markovian case is only addressed to deal with the HJB equation.

We finally mention that in the paper [7], co-authored by some of us, BSDEs of the form (1.5) have been introduced, as an intermediate technical step in the proofs, when dealing with HJB equations of ergodic type, namely of the form

$$\lambda - \sup_{a \in A} [\mathcal{L}^a v(x) + f(x, a)] = 0, \quad x \in \mathbb{R}^n,$$

where both the function  $v$  and the constant  $\lambda$  are unknown. However, the results in [7] impose strong restrictions on the coefficients  $b, \sigma, f$  and the space of control actions  $A$ . In particular, Lipschitz conditions were imposed on  $f$  and special dissipativity assumptions were imposed on  $b$  and  $\sigma$  in order to guarantee appropriate ergodicity properties. In the present paper these assumptions are dropped. In addition, the results in [7] depend in an essential way on the Markovianity of the stochastic system and cannot be applied to the controlled equation (1.6). As a result, we are led to a careful study of the growth rate of the solution  $X$  to the non Markovian equation (1.6) (compare Lem. 2.1) and we have to relate it to polynomial growth conditions imposed on  $f$  as well as an appropriate value for the discount factor  $\beta$ . This kind of difficulty is peculiar of infinite horizon control problems, while it does not arise in the finite horizon case. We also notice that the same difficulty is encountered in the study of the corresponding Hamilton-Jacobi-Bellman equation of elliptic type (as well as of the associated infinite horizon BSDE), for which again the growth conditions on  $f$  and the value of  $\beta$  play a crucial role in order to prove existence of a solution.

The plan of the paper is as follows. In Section 2 we formulate our assumptions for the general non Markovian framework and introduce the optimal control problem on infinite horizon, with special attention to the behaviour of the controlled system for large times, whereas in Section 3 we formulate the auxiliary randomized problem. In Section 4 we prove the equality of the values of these two problems, we introduce and study the well-posedness of the constrained BSDE (1.5) and we prove that it gives the desired representation of the values. Finally, in Section 5, we restrict to the Markovian case and prove that the solution to the BSDE provides us with a viscosity solution to the HJB equation (1.2).

## 2. FORMULATION OF THE INFINITE HORIZON OPTIMAL CONTROL PROBLEM

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a complete probability space, on which a  $d$ -dimensional Brownian motion  $W = (W_t)_{t \geq 0}$  is defined. Let  $\mathbb{F}^W = (\mathcal{F}_t^W)_{t \geq 0}$  denote the  $\mathbb{P}$ -completion of the filtration generated by  $W$ . Let  $A$  be a nonempty Borel space (namely,  $A$  is a topological space homeomorphic to a Borel subset of a Polish space) and denote by  $\mathcal{A}$  the set of  $\mathbb{F}^W$ -progressive processes  $\alpha: \Omega \times [0, \infty) \rightarrow A$ .  $A$  is the space of control actions and  $\mathcal{A}$  is the family of admissible control processes. Finally, we denote  $\mathcal{B}(A)$  the Borel  $\sigma$ -algebra of  $A$ .

Fix a deterministic point  $x_0 \in \mathbb{R}^n$ . For every  $\alpha \in \mathcal{A}$ , consider the controlled equation:

$$X_t^\alpha = x_0 + \int_0^t b_s(X^\alpha, \alpha) ds + \int_0^t \sigma_s(X^\alpha, \alpha) dW_s, \quad (2.1)$$

for all  $t \geq 0$ . The infinite horizon stochastic optimal control problem consists in maximizing over  $\alpha \in \mathcal{A}$  the gain functional

$$J(\alpha) = \mathbb{E} \left[ \int_0^\infty e^{-\beta t} f_t(X^\alpha, \alpha) dt \right].$$

The constant  $\beta > 0$  will be specified later. The coefficients  $b, \sigma, f$  are defined on  $[0, \infty) \times \mathbf{C}_n \times \mathbf{M}_A$  with values in  $\mathbb{R}^n, \mathbb{R}^{n \times d}, \mathbb{R}$ , respectively, where:

- $\mathbf{C}_n$  is the set of continuous trajectories  $x: [0, \infty) \rightarrow \mathbb{R}^n$ . We introduce the canonical filtration  $(\mathcal{C}_t^n)_{t \geq 0}$  and denote  $Prog(\mathbf{C}_n)$  the  $(\mathcal{C}_t^n)$ -progressive  $\sigma$ -algebra on  $[0, \infty) \times \mathbf{C}_n$ ;
- $\mathbf{M}_A$  is the set of Borel measurable trajectories  $a: [0, \infty) \rightarrow A$ . We introduce the canonical filtration  $(\mathcal{M}_t^A)_{t \in [0, T]}$  and denote  $Prog(\mathbf{C}_n \times \mathbf{M}_A)$  the  $(\mathcal{C}_t^n \otimes \mathcal{M}_t^A)$ -progressive  $\sigma$ -algebra on  $[0, \infty) \times \mathbf{C}_n \times \mathbf{M}_A$ .

In the present paper, we consider the two following alternative sets of assumptions on  $b, \sigma, f$ . Notice that **(A)** differs from **(A)'** only for points (iii) and (iv).

**(A)**

- (i) The functions  $b, \sigma, f$  are  $Prog(\mathbf{C}_n \times \mathbf{M}_A)$ -measurable.
- (ii) For every  $T > 0$ , if  $x_m, x \in \mathbf{C}_n, a_m, a \in \mathbf{M}_A, \sup_{t \in [0, T]} |x_m(t) - x(t)| \rightarrow 0, a_m(t) \rightarrow a(t)$  for  $dt$ -a.e.  $t \in [0, T]$  as  $m \rightarrow \infty$ , then we have

$$b_t(x_m, a_m) \rightarrow b_t(x, a), \quad \sigma_t(x_m, a_m) \rightarrow \sigma_t(x, a), \quad f_t(x_m, a_m) \rightarrow f_t(x, a), \quad \text{for } dt\text{-a.e. } t \in [0, T].$$

- (iii) For every  $T > 0$ , there exists a constant  $L_T \geq 0$  such that

$$\begin{aligned} |b_t(x, a) - b_t(x', a)| + |\sigma_t(x, a) - \sigma_t(x', a)| &\leq L_T \sup_{s \in [0, t]} |x(s) - x'(s)|, \\ |b_t(0, a)| + |\sigma_t(0, a)| &\leq L_T, \end{aligned}$$

for all  $t \in [0, T], x, x' \in \mathbf{C}^n, a \in \mathbf{M}_A$ .

- (iv) The function  $f$  is bounded. We denote  $\|f\|_\infty := \sup_{t, x, a} |f_t(x, a)| < \infty$ .
- (v)  $\beta$  can be any strictly positive real number.

**(A)'**

- (i) The functions  $b, \sigma, f$  are  $Prog(\mathbf{C}_n \times \mathbf{M}_A)$ -measurable.
- (ii) For every  $T > 0$ , if  $x_m, x \in \mathbf{C}_n, a_m, a \in \mathbf{M}_A, \sup_{t \in [0, T]} |x_m(t) - x(t)| \rightarrow 0, a_m(t) \rightarrow a(t)$  for  $dt$ -a.e.  $t \in [0, T]$  as  $m \rightarrow \infty$ , then we have

$$b_t(x_m, a_m) \rightarrow b_t(x, a), \quad \sigma_t(x_m, a_m) \rightarrow \sigma_t(x, a), \quad f_t(x_m, a_m) \rightarrow f_t(x, a), \quad \text{for } dt\text{-a.e. } t \in [0, T].$$

- (iii) There exists a constant  $L \geq 0$  such that

$$\begin{aligned} |b_t(x, a) - b_t(x', a)| + |\sigma_t(x, a) - \sigma_t(x', a)| &\leq L \sup_{s \in [0, t]} |x(s) - x'(s)|, \\ |b_t(0, a)| + |\sigma_t(0, a)| &\leq L, \end{aligned}$$

for all  $t \geq 0, x, x' \in \mathbf{C}^n, a \in \mathbf{M}_A$ .

(iv) There exist constants  $M > 0$  and  $r > 0$  such that

$$|f_t(x, a)| \leq M(1 + \sup_{s \in [0, t]} |x(s)|^r),$$

for all  $t \geq 0$ ,  $x \in \mathbf{C}^n$ ,  $a \in \mathbf{M}_A$ .

(v)  $\beta > \bar{\beta}$ , where  $\bar{\beta}$  is a strictly positive real number such that

$$\mathbb{E} \left[ \sup_{s \in [0, t]} |X_s^\alpha|^r \right] \leq \bar{C} e^{\bar{\beta} t} (1 + |x_0|^r), \quad (2.2)$$

for some constant  $\bar{C} \geq 0$ , with  $\bar{\beta}$  and  $\bar{C}$  independent of  $t \geq 0$ ,  $\alpha \in \mathcal{A}$ ,  $x_0 \in \mathbb{R}^n$ . See Lemma 2.1.

Notice that under either  $(\mathbf{A})$ -(i)-(ii)-(iii) or  $(\mathbf{A}')$ -(i)-(ii)-(iii), there exists a unique  $\mathbb{F}^W$ -progressive continuous process  $X^\alpha = (X_t^\alpha)_{t \geq 0}$  solution to equation (2.1). Moreover, under  $(\mathbf{A})$ -(i)-(ii)-(iii), for every  $T > 0$  and  $p > 0$ , we have that there exists a constant  $C_{T,p} \geq 0$  such that

$$\mathbb{E} \left[ \sup_{t \in [0, T]} |X_t^\alpha|^p \right] \leq C_{T,p} (1 + |x_0|^p).$$

On the other hand, under  $(\mathbf{A}')$ -(i)-(ii)-(iii), we have that there exists  $\bar{\beta} \geq 0$  such that estimate (2.2) holds. This latter estimate follows from the next Lemma 2.1, where we prove a more general result needed later.

**Lemma 2.1.** *Suppose that assumption  $(\mathbf{A}')$ -(i)-(ii)-(iii) holds. Let  $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$  be a complete probability space and let  $\hat{\mathbb{F}} = (\hat{\mathcal{F}}_t)_{t \geq 0}$  be a filtration satisfying the usual conditions. Let also  $\hat{W} = (\hat{W}_t)_{t \geq 0}$  be a  $d$ -dimensional Brownian motion on the filtered space  $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}, \hat{\mathbb{F}})$ . Let  $\gamma: \hat{\Omega} \times [0, \infty) \rightarrow A$  be an  $\hat{\mathbb{F}}$ -progressive process. Finally, let  $\hat{X} = (\hat{X}_t)_{t \geq 0}$  be the unique continuous  $\hat{\mathbb{F}}$ -adapted process solution to the following equation*

$$\hat{X}_t = x_0 + \int_0^t b_s(\hat{X}, \gamma) ds + \int_0^t \sigma_s(\hat{X}, \gamma) d\hat{W}_s, \quad (2.3)$$

for all  $t \geq 0$ . Then, for every  $p > 0$ , there exist two constants  $\bar{C}_{p,L} \geq 0$  and  $\bar{\beta}_{p,L} \geq 0$  such that

$$\hat{\mathbb{E}} \left[ \sup_{s \in [0, T]} |\hat{X}_s|^p \middle| \hat{\mathcal{F}}_t \right] \leq \bar{C}_{p,L} e^{\bar{\beta}_{p,L} (T-t)} (1 + \sup_{s \in [0, t]} |\hat{X}_s|^p), \quad \hat{\mathbb{P}}\text{-a.s.} \quad (2.4)$$

for all  $t \geq 0$ ,  $T \geq t$ , with  $\bar{C}_{p,L}$  and  $\bar{\beta}_{p,L}$  depending only on  $p$  and the constant  $L$  appearing in assumption  $(\mathbf{A}')$ -(iii). When  $p = r$ , with  $r$  as in assumption  $(\mathbf{A}')$ -(iv), we denote  $\bar{C}_{r,L}$  and  $\bar{\beta}_{r,L}$  simply by  $\bar{C}$  and  $\bar{\beta}$ .

*Proof.* See Appendix. We remark that, to our knowledge, a proof of estimate (2.4) in the path-dependent case does not exist in the literature. On the other hand, for the non-path-dependent case we refer for instance to Theorem II.5.9 in [20]. Notice however that those proofs use in an essential way the fact that  $b$  and  $\sigma$  depends only on the present value of the process  $\hat{X}$ , so that they cannot be extended to the path-dependent case.  $\square$

We define the value of the control problem as

$$V = \sup_{\alpha \in \mathcal{A}} J(\alpha).$$

Now, for every  $T > 0$ , consider the finite horizon optimal control problem with value

$$V_T = \sup_{\alpha \in \mathcal{A}} J_T(\alpha),$$

where

$$J_T(\alpha) = \mathbb{E} \left[ \int_0^T e^{-\beta t} f_t(X^\alpha, \alpha) dt \right].$$

**Lemma 2.2.** *Under either (A) or (A)', we have*

$$V = \lim_{T \rightarrow \infty} V_T.$$

*Proof.* Assumption (A) holds. We have

$$|V - V_T| \leq \sup_{\alpha \in \mathcal{A}} \mathbb{E} \left[ \int_T^\infty e^{-\beta t} |f_t(X^\alpha, \alpha)| dt \right] \leq \|f\|_\infty \frac{e^{-\beta T}}{\beta} \xrightarrow{T \rightarrow \infty} 0.$$

Assumption (A)' holds. We have

$$\begin{aligned} |V - V_T| &\leq \sup_{\alpha \in \mathcal{A}} \mathbb{E} \left[ \int_T^\infty e^{-\beta t} |f_t(X^\alpha, \alpha)| dt \right] \leq M \frac{e^{-\beta T}}{\beta} + M \mathbb{E} \left[ \int_T^\infty e^{-\beta t} \sup_{s \in [0, t]} |X_s^\alpha|^r dt \right] \\ &\leq M \frac{e^{-\beta T}}{\beta} + M \int_T^\infty e^{-(\beta - \bar{\beta})t} \mathbb{E} \left[ e^{-\bar{\beta} t} \sup_{s \in [0, t]} |X_s^\alpha|^r \right] dt. \end{aligned}$$

Using (2.2), we obtain

$$|V - V_T| \leq M \frac{e^{-\beta T}}{\beta} + M \bar{C} (1 + |x_0|^r) \frac{e^{-(\beta - \bar{\beta})T}}{\beta - \bar{\beta}} \xrightarrow{T \rightarrow \infty} 0.$$

□

### 3. RANDOMIZED OPTIMAL CONTROL PROBLEM

In the present section we formulate the randomized infinite horizon optimal control problem. Firstly, we fix a finite positive measure  $\lambda$  on  $(A, \mathcal{B}(A))$  with full topological support. We also fix a deterministic point  $a_0$  in  $A$ .

Let  $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$  be a complete probability space on which a  $d$ -dimensional Brownian motion  $\bar{W} = (\bar{W}_t)_{t \geq 0}$  and a Poisson random measure  $\bar{\mu}$  on  $[0, \infty) \times A$  are defined. The Poisson random measure  $\bar{\mu} = \sum_{n \geq 1} \delta_{(\bar{T}_n, \bar{A}_n)}$  is associated with a marked point process  $(\bar{T}_n, \bar{A}_n)_{n \geq 1}$  on  $[0, \infty) \times A$ , where  $(\bar{T}_n)_{n \geq 1}$  is the sequence of jump times, while  $(\bar{A}_n)_{n \geq 1}$  is the sequence of  $A$ -valued marks. The compensator of  $\bar{\mu}$  is given by  $\lambda(da)dt$ . We denote by  $\bar{\mathbb{F}}^{W, \mu} = (\bar{\mathcal{F}}_t^{W, \mu})_{t \geq 0}$  the  $\bar{\mathbb{P}}$ -completion of the filtration generated by  $\bar{W}$  and  $\bar{\mu}$ , by  $\mathcal{P}_T(\bar{\mathbb{F}}^{W, \mu})$ ,  $T \in (0, \infty)$ , the predictable  $\sigma$ -algebra on  $[0, T] \times \bar{\Omega}$  associated with  $\bar{\mathbb{F}}^{W, \mu}$ , and by  $\mathcal{P}(\bar{\mathbb{F}}^{W, \mu})$  the predictable  $\sigma$ -algebra on  $[0, \infty) \times \bar{\Omega}$  associated with  $\bar{\mathbb{F}}^{W, \mu}$ .

We define the  $A$ -valued pure-jump process

$$\bar{I}_t = \sum_{n \geq 0} \bar{A}_n 1_{[\bar{T}_n, \bar{T}_{n+1})}(t), \quad \text{for all } t \geq 0, \quad (3.1)$$

with the convention  $\bar{T}_0 = 0$  and  $\bar{A}_0 = a_0$ , where  $a_0$  is the deterministic point fixed at the beginning of this section. We now consider the following equation:

$$\bar{X}_t = x_0 + \int_0^t b_s(\bar{X}, \bar{I}) ds + \int_0^t \sigma_s(\bar{X}, \bar{I}) d\bar{W}_s, \quad \text{for all } t \geq 0. \quad (3.2)$$

Under either **(A)** or **(A)'**, there exists a unique  $\bar{\mathbb{F}}^{W,\mu}$ -progressive continuous process  $\bar{X} = (\bar{X}_t)_{t \geq 0}$  solution to equation (3.2).

The set  $\bar{\mathcal{V}}$  of admissible controls for the randomized problem is given by all  $\mathcal{P}(\bar{\mathbb{F}}^{W,\mu}) \otimes \mathcal{B}(A)$ -measurable and bounded maps  $\bar{\nu}: \Omega \times \mathbb{R}_+ \times A \rightarrow (0, \infty)$ . We also define, for every  $n \in \mathbb{N} \setminus \{0\}$ , the set  $\bar{\mathcal{V}}_n := \{\bar{\nu} \in \bar{\mathcal{V}}: \bar{\nu} \text{ is bounded by } n\}$ . Notice that  $\bar{\mathcal{V}} = \cup_n \bar{\mathcal{V}}_n$ . For every  $\bar{\nu} \in \bar{\mathcal{V}}$ , we consider the corresponding Doléans-Dade exponential process

$$\begin{aligned} \kappa_t^{\bar{\nu}} &= \mathcal{E}_t \left( \int_0^t \int_A (\bar{\nu}_s(a) - 1) (\bar{\mu}(ds da) - \lambda(da) ds) \right) \\ &= \exp \left( \int_0^t \int_A (1 - \bar{\nu}_s(a)) \lambda(da) ds \right) \prod_{0 < \bar{T}_n \leq t} \bar{\nu}_{\bar{T}_n}(\bar{A}_n), \end{aligned} \quad (3.3)$$

for all  $t \geq 0$ . Notice that  $\kappa^{\bar{\nu}}$  is a  $(\bar{\mathbb{P}}, \bar{\mathbb{F}}^{W,\mu})$ -martingale, since  $\bar{\nu}$  is bounded. Then, for every  $T > 0$ , we define the probability  $\bar{\mathbb{P}}_T^{\bar{\nu}}(d\bar{\omega}) = \kappa_T^{\bar{\nu}}(\bar{\omega}) \bar{\mathbb{P}}(d\bar{\omega})$  on  $(\bar{\Omega}, \bar{\mathcal{F}}_T^{W,\mu})$ . For every  $T > 0$ , by Girsanov's theorem, under  $\bar{\mathbb{P}}_T^{\bar{\nu}}$  the  $(\bar{\mathcal{F}}_t^{W,\mu})_{t \in [0, T]}$ -compensator of  $\bar{\mu}$  on  $[0, T] \times A$  is  $\bar{\nu}_t(a) \lambda(da) dt$ , and  $(\bar{W}_t)_{t \in [0, T]}$  is still a Brownian motion on  $[0, T]$  under  $\bar{\mathbb{P}}_T^{\bar{\nu}}$ . Notice that, under **(A)**, for every  $T > 0$  and  $p > 0$ , we have that there exists a constant  $C_{T,p} \geq 0$  such that ( $\bar{\mathbb{E}}_T^{\bar{\nu}}$  denotes the  $\bar{\mathbb{P}}_T^{\bar{\nu}}$ -expectation)

$$\bar{\mathbb{E}}_T^{\bar{\nu}} \left[ \sup_{s \in [0, T]} |\bar{X}_s|^p \right] \leq C_{T,p} (1 + |x_0|^p).$$

On the other hand, let **(A)'** hold. Then, by Lemma 2.1 we know that there exists two positive constants  $\bar{C}$  and  $\bar{\beta}$  such that

$$\bar{\mathbb{E}}_T^{\bar{\nu}} \left[ \sup_{s \in [0, T]} |\bar{X}_s|^r \right] \leq \bar{C} e^{\bar{\beta} T} (1 + |x_0|^r),$$

with  $\bar{C}$  and  $\bar{\beta}$  independent of  $T > 0$ ,  $\bar{\nu} \in \bar{\mathcal{V}}$ ,  $x_0 \in \mathbb{R}^n$ . More generally, by Lemma 2.1 we have the following estimate:

$$\bar{\mathbb{E}}_T^{\bar{\nu}} \left[ \sup_{s \in [0, T]} |\bar{X}_s|^r \middle| \bar{\mathcal{F}}_t^{W,\mu} \right] \leq \bar{C} e^{\bar{\beta}(T-t)} (1 + \sup_{s \in [0, t]} |\bar{X}_s|^r), \quad \bar{\mathbb{P}}\text{-a.s.} \quad (3.4)$$

For all  $T > 0$ , we consider the finite horizon randomized control problem

$$V_T^{\mathcal{R}} = \sup_{\bar{\nu} \in \bar{\mathcal{V}}} J_T^{\mathcal{R}}(\bar{\nu}),$$

where

$$J_T^{\mathcal{R}}(\bar{\nu}) = \bar{\mathbb{E}}_T^{\bar{\nu}} \left[ \int_0^T e^{-\beta t} f_t(\bar{X}, \bar{I}) dt \right].$$

Finally, we define the value of the randomized control problem as follows:

$$V^{\mathcal{R}} := \lim_{T \rightarrow \infty} V_T^{\mathcal{R}} = \lim_{T \rightarrow \infty} \sup_{\bar{\nu} \in \bar{\mathcal{V}}} J_T^{\mathcal{R}}(\bar{\nu}). \quad (3.5)$$

Proceeding as in the proof of Lemma 2.2, it is easy to see that

$$\sup_{T, T' \geq S} |V_T^{\mathcal{R}} - V_{T'}^{\mathcal{R}}| \xrightarrow{S \rightarrow \infty} 0,$$

therefore the limit in (3.5) exists.

**Remark 3.1.** Assume that either **(A)** or **(A)'** holds, and suppose that  $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$  has a canonical representation. More precisely, consider the following sets:

- $\Omega'$  the set of continuous trajectories  $\omega': [0, \infty) \rightarrow \mathbb{R}^d$  satisfying  $\omega'(0) = 0$ . We denote  $\bar{W}$  the canonical process on  $\Omega'$ ,  $(\mathcal{F}_t^W)_{t \geq 0}$  the canonical filtration,  $\mathbb{P}'$  the Wiener measure on  $(\Omega', \mathcal{F}_\infty^W)$ ;
- $\Omega''$  is the set of double sequences  $\omega'' = (t_n, a_n)_{n \geq 1} \subset (0, \infty) \times A$  satisfying  $t_n < t_{n+1} \nearrow \infty$ . We denote  $(\bar{T}_n, \bar{A}_n)_{n \geq 1}$  the canonical marked point process,  $\bar{\mu} = \sum_{n \geq 1} \delta_{(\bar{T}_n, \bar{A}_n)}$  the associated random measure,  $(\mathcal{F}_t^\mu)_{t \geq 0}$  the filtration generated by  $\bar{\mu}$ , and  $\mathbb{P}''$  the unique probability on  $\mathcal{F}_\infty^\mu$  such that  $\bar{\mu}$  is a Poisson random measure with compensator  $\lambda(da)dt$ .

Now, let  $\bar{\Omega} = \Omega' \times \Omega''$ , let  $\bar{\mathcal{F}}$  be the completion of  $\mathcal{F}_\infty^W \otimes \mathcal{F}_\infty^\mu$  with respect to  $\mathbb{P}' \otimes \mathbb{P}''$  and let  $\bar{\mathbb{P}}$  be the extension of  $\mathbb{P}' \otimes \mathbb{P}''$  to  $\bar{\mathcal{F}}$ . Notice that  $\bar{W}$  and  $\bar{\mu}$  can be extended in a canonical way to  $\bar{\Omega}$ . We will denote these extensions by the same symbols. We also denote by  $\mathbb{F}^{W, \mu} = (\mathcal{F}_t^{W, \mu})_{t \geq 0}$  the filtration generated by  $\bar{W}$  and  $\bar{\mu}$ , and by  $\bar{\mathbb{F}}^{W, \mu} = (\bar{\mathcal{F}}_t^{W, \mu})_{t \geq 0}$  the  $\bar{\mathbb{P}}$ -completion of  $\mathbb{F}^{W, \mu}$ .

Recall that, for every  $T > 0$ ,  $\bar{\mathbb{P}}_T^{\bar{\nu}}$  is a probability on  $(\bar{\Omega}, \bar{\mathcal{F}}_T^{W, \mu})$ , then, in particular, on  $(\bar{\Omega}, \mathcal{F}_T^{W, \mu})$ . Moreover, the following consistency condition holds:  $\bar{\mathbb{P}}_T^{\bar{\nu}}$  coincides with  $\bar{\mathbb{P}}_t^{\bar{\nu}}$  on  $\mathcal{F}_t^{W, \mu}$ , whenever  $0 < t \leq T$ . Then, by Kolmogorov's extension theorem, we deduce that there exists a probability measure  $\bar{\mathbb{P}}^{\bar{\nu}}$  on  $(\bar{\Omega}, \mathcal{F}_\infty^{W, \mu})$  such that  $\bar{\mathbb{P}}^{\bar{\nu}}$  coincides with  $\bar{\mathbb{P}}_T^{\bar{\nu}}$  on  $\mathcal{F}_T^{W, \mu}$ , for all  $T > 0$ .

Notice that  $\bar{\mathbb{P}}^{\bar{\nu}}$  can be defined in a consistent way only on  $\mathcal{F}_\infty^{W, \mu}$  (rather than on  $\bar{\mathcal{F}}_\infty^{W, \mu}$ ). Indeed, since the martingale  $(\kappa_t^{\bar{\nu}})_{t \geq 0}$  is in general not uniformly integrable, it follows from Proposition VIII.1.1 in [29] that  $\bar{\mathbb{P}}^{\bar{\nu}}$  is in general not absolutely continuous with respect to  $\bar{\mathbb{P}}$  on  $\mathcal{F}_\infty^{W, \mu}$ . In particular, when this is the case, there exists some  $N \in \mathcal{F}_\infty^{W, \mu}$  such that  $N \notin \mathcal{F}_T^{W, \mu}$ , for all  $T > 0$ , and  $\bar{\mathbb{P}}(N) = 0$ , however  $\bar{\mathbb{P}}^{\bar{\nu}}(N) > 0$ . Since  $N \in \bar{\mathcal{F}}_T^{W, \mu}$  for all  $T > 0$ , we have  $\bar{\mathbb{P}}_T^{\bar{\nu}}(N) = 0$ , therefore we cannot extend  $\bar{\mathbb{P}}^{\bar{\nu}}$  to  $\bar{\mathcal{F}}_\infty^{W, \mu}$  without violating the consistency condition:  $\bar{\mathbb{P}}^{\bar{\nu}} = \bar{\mathbb{P}}_T^{\bar{\nu}}$  on  $\bar{\mathcal{F}}_T^{W, \mu}$ .

Now, notice that the process  $\bar{I}$  is given by (3.1) and hence it is  $\mathbb{F}^{W, \mu}$ -adapted. On the other hand, the process  $\bar{X}$ , solution to equation (3.2), is  $\bar{\mathbb{F}}^{W, \mu}$ -progressive and continuous, therefore it is  $\bar{\mathbb{F}}^{W, \mu}$ -predictable. By IV-78 in [8] it follows that there exists an  $\mathbb{F}^{W, \mu}$ -predictable process  $\hat{X}$ , such that  $\bar{X}$  and  $\hat{X}$  are  $\bar{\mathbb{P}}$ -indistinguishable. Then, we have the following representation for  $V^{\mathcal{R}}$ :

$$V^{\mathcal{R}} = \sup_{\bar{\nu} \in \bar{\mathcal{V}}} J^{\mathcal{R}}(\bar{\nu}), \quad (3.6)$$

where  $(\bar{\mathbb{E}}^{\bar{\nu}}$  denotes the  $\bar{\mathbb{P}}^{\bar{\nu}}$ -expectation)

$$J^{\mathcal{R}}(\bar{\nu}) := \bar{\mathbb{E}}^{\bar{\nu}} \left[ \int_0^\infty e^{-\beta t} f_t(\hat{X}, \bar{I}) dt \right]. \quad (3.7)$$

Let us prove formula (3.6). We begin by noting that, for all  $T > 0$ ,

$$J_T^{\mathcal{R}}(\bar{\nu}) = \bar{\mathbb{E}}_T^{\bar{\nu}} \left[ \int_0^T e^{-\beta t} f_t(\bar{X}, \bar{I}) dt \right] = \bar{\mathbb{E}}_T^{\bar{\nu}} \left[ \int_0^T e^{-\beta t} f_t(\hat{X}, \bar{I}) dt \right] = \bar{\mathbb{E}}^{\bar{\nu}} \left[ \int_0^T e^{-\beta t} f_t(\hat{X}, \bar{I}) dt \right].$$

Then, under either **(A)** or **(A)'**, proceeding along the same lines as in the proof of Lemma 2.2, we obtain

$$\sup_{\bar{\nu} \in \bar{\mathcal{V}}} |J^{\mathcal{R}}(\bar{\nu}) - J_T^{\mathcal{R}}(\bar{\nu})| \xrightarrow{T \rightarrow \infty} 0$$

and so

$$\lim_{T \rightarrow \infty} V_T^{\mathcal{R}} = \sup_{\bar{\nu} \in \bar{\mathcal{V}}} J^{\mathcal{R}}(\bar{\nu}).$$

Since, by definition,  $V^{\mathcal{R}} = \lim_{T \rightarrow \infty} V_T^{\mathcal{R}}$ , we conclude that formula (3.6) holds.

#### 4. IDENTIFICATION OF THE VALUES AND BACKWARD SDE REPRESENTATION

In the present section we prove that the original control problem and the randomized control problem have the same value, namely  $V = V^{\mathcal{R}}$ . We exploit this result in order to derive a backward stochastic differential equation representation for the value  $V$ .

**Theorem 4.1.** *Under either **(A)** or **(A)'**, we have*

$$V_T = V_T^{\mathcal{R}}, \tag{4.1}$$

for all  $T \in (0, \infty)$ , and also

$$V = V^{\mathcal{R}}. \tag{4.2}$$

*Proof.* We begin by noting that identity (4.2) is a straightforward consequence of identity (4.1), Lemma 2.2 and definition (3.5). On the other hand, for every  $T \in (0, \infty)$ , identity (4.1) is a direct consequence of Theorem 3.1 in [4].  $\square$

We now prove that  $V$  is related to the following infinite horizon backward stochastic differential equation with nonpositive jumps:

$$\begin{aligned} Y_t = & Y_T - \beta \int_t^T Y_s ds + \int_t^T f_s(\bar{X}, \bar{I}) ds + K_T - K_t \\ & - \int_t^T Z_s d\bar{W}_s - \int_t^T \int_A U_s(a) \bar{\mu}(ds, da), \quad 0 \leq t \leq T, \forall T \in (0, \infty), \bar{\mathbb{P}}\text{-a.s.} \end{aligned} \tag{4.3}$$

$$U_t(a) \leq 0, \quad dt \otimes d\bar{\mathbb{P}} \otimes \lambda(da)\text{-a.e. on } [0, \infty) \times \bar{\Omega} \times A. \tag{4.4}$$

Let us introduce some additional notations. Given  $T \in (0, \infty)$ , we denote:

- $\mathbf{S}^\infty$ , the family of real càdlàg  $\bar{\mathbb{F}}^{W, \mu}$ -adapted stochastic processes  $Y = (Y_t)_{t \geq 0}$  on  $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$  which are uniformly bounded.
- $\mathbf{S}^2(\mathbf{0}, \mathbf{T})$ , the family of real càdlàg  $\bar{\mathbb{F}}^{W, \mu}$ -adapted stochastic processes  $Y = (Y_t)_{0 \leq t \leq T}$  on  $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ , with

$$\|Y\|_{\mathbf{S}^2(\mathbf{0}, \mathbf{T})}^2 := \bar{\mathbb{E}} \left[ \sup_{0 \leq t \leq T} |Y_t|^2 \right] < \infty.$$

We set  $\mathbf{S}_{\text{loc}}^2 := \cap_{T > 0} \mathbf{S}^2(\mathbf{0}, \mathbf{T})$ .

- $\mathbf{L}^2(\mathbf{W}; \mathbf{0}, \mathbf{T})$ , the family of  $\mathbb{R}^d$ -valued  $\mathcal{P}_T(\bar{\mathbb{F}}^{W,\mu})$ -measurable stochastic processes  $Z = (Z_t)_{0 \leq t \leq T}$  on  $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ , with

$$\|Z\|_{\mathbf{L}^2(\mathbf{W}; \mathbf{0}, \mathbf{T})}^2 := \bar{\mathbb{E}} \left[ \int_0^T |Z_t|^2 dt \right] < \infty.$$

We set  $\mathbf{L}_{\text{loc}}^2(\mathbf{W}) := \cap_{T>0} \mathbf{L}^2(\mathbf{W}; \mathbf{0}, \mathbf{T})$ .

- $\mathbf{L}^2(\bar{\mu}; \mathbf{0}, \mathbf{T})$ , the family of  $\mathcal{P}_T(\bar{\mathbb{F}}^{W,\mu}) \otimes \mathcal{B}(A)$ -measurable maps  $U: [0, T] \times \bar{\Omega} \times A \rightarrow \mathbb{R}$  such that

$$\|U\|_{\mathbf{L}^2(\bar{\mu}; \mathbf{0}, \mathbf{T})}^2 := \bar{\mathbb{E}} \left[ \int_0^T \int_A |U_t(a)|^2 \lambda(da) dt \right] < \infty.$$

We set  $\mathbf{L}_{\text{loc}}^2(\bar{\mu}) := \cap_{T>0} \mathbf{L}^2(\bar{\mu}; \mathbf{0}, \mathbf{T})$ .

- $\mathbf{K}^2(\mathbf{0}, \mathbf{T})$ , the family of nondecreasing càdlàg  $\mathcal{P}_T(\bar{\mathbb{F}}^{W,\mu})$ -measurable stochastic processes  $K = (K_t)_{0 \leq t \leq T}$  such that  $\bar{\mathbb{E}}[|K_T|^2] < \infty$  and  $K_0 = 0$ . We set  $\mathbf{K}_{\text{loc}}^2 := \cap_{T>0} \mathbf{K}^2(\mathbf{0}, \mathbf{T})$ .

**Theorem 4.2.** *Under (A) (resp. (A')) there exists a unique quadruple  $(\bar{Y}, \bar{Z}, \bar{U}, \bar{K})$  in  $\mathbf{S}_{\text{loc}}^2 \times \mathbf{L}_{\text{loc}}^2(\mathbf{W}) \times \mathbf{L}_{\text{loc}}^2(\bar{\mu}) \times \mathbf{K}_{\text{loc}}^2$ , such that  $\bar{Y} \in \mathbf{S}^\infty$  (resp.  $|\bar{Y}_t| \leq C(1 + \sup_{s \in [0,t]} |\bar{X}_s|^r)$ , for all  $t \geq 0$ ,  $\bar{\mathbb{P}}$ -a.s., and for some positive constant  $C$ ), satisfying (4.3) and (4.4) which is minimal in the following sense: for any other quadruple  $(\hat{Y}, \hat{Z}, \hat{U}, \hat{K})$  in  $\mathbf{S}_{\text{loc}}^2 \times \mathbf{L}_{\text{loc}}^2(\mathbf{W}) \times \mathbf{L}_{\text{loc}}^2(\bar{\mu}) \times \mathbf{K}_{\text{loc}}^2$ , with  $\hat{Y} \in \mathbf{S}^\infty$  (resp.  $|\hat{Y}_t| \leq C(1 + \sup_{s \in [0,t]} |\bar{X}_s|^r)$ , for all  $t \geq 0$ ,  $\bar{\mathbb{P}}$ -a.s., and for some positive constant  $C$ ), satisfying (4.3) and (4.4) we have:*

$$\bar{Y}_t \leq \hat{Y}_t, \quad \text{for all } t \geq 0, \bar{\mathbb{P}}\text{-a.s.}$$

Under either (A) or (A'), we have  $V = \bar{Y}_0$   $\bar{\mathbb{P}}$ -a.s. Moreover, the following formula holds:

$$\bar{Y}_t = \text{ess sup}_{\bar{\nu} \in \bar{\mathcal{V}}} \bar{\mathbb{E}}_{\bar{\nu}}^{\bar{Y}} \left[ \int_t^\tau e^{-\beta(s-t)} f_s(\bar{X}, \bar{I}) ds + e^{-\beta(\tau-t)} \bar{Y}_\tau \middle| \bar{\mathcal{F}}_t^{W,\mu} \right], \quad \bar{\mathbb{P}}\text{-a.s.} \quad (4.5)$$

for all  $T > 0$ ,  $t \in [0, T]$ , and any  $\bar{\mathbb{F}}^{W,\mu}$ -stopping time  $\tau$  taking values in  $[t, T]$ .

*Proof.* For every  $T > 0$  and any  $n \in \mathbb{N}$ , consider the following penalized backward stochastic differential equation on  $[0, T]$  with zero terminal condition:

$$\begin{aligned} Y_t^{T,n} &= -\beta \int_t^T Y_s^{T,n} ds + \int_t^T f_s(\bar{X}, \bar{I}) ds + K_T^{T,n} - K_t^{T,n} \\ &\quad - \int_t^T Z_s^{T,n} d\bar{W}_s - \int_t^T \int_A U_s^{T,n}(a) \bar{\mu}(ds, da), \quad 0 \leq t \leq T, \bar{\mathbb{P}}\text{-a.s.} \end{aligned} \quad (4.6)$$

where

$$K_t^{T,n} = n \int_0^t \int_A (U_s^{T,n}(a))^+ \lambda(da) ds.$$

It is well-known (see [31], Lem. 2.4) that there exists a unique triple  $(Y^{T,n}, Z^{T,n}, U^{T,n})$  in  $\mathbf{S}^2(\mathbf{0}, \mathbf{T}) \times \mathbf{L}^2(\mathbf{W}; \mathbf{0}, \mathbf{T}) \times \mathbf{L}^2(\bar{\mu}; \mathbf{0}, \mathbf{T})$  satisfying (4.6). Our aim is to construct the quadruple  $(\bar{Y}, \bar{Z}, \bar{U}, \bar{K})$  starting from equation (4.6), first passing to the limit in (4.6) as  $T \rightarrow \infty$ , for any fixed  $n \in \mathbb{N}$ , and then sending  $n \rightarrow \infty$ . More precisely, we split the rest of the proof into five steps, which are organized as follows:

- In **Step I** we study the asymptotic behavior of the component  $Y^{T,n}$  of equation (4.6) as  $T \rightarrow \infty$ , for any fixed  $n \in \mathbb{N}$ .
- In **Step II** we study the asymptotic behavior of the components  $Z^{T,n}$  and  $U^{T,n}$  of equation (4.6) as  $T \rightarrow \infty$ , for any fixed  $n \in \mathbb{N}$ . Using the results of **Steps I** and **II**, we construct the solution  $(Y^n, Z^n, U^n)$  to equation (4.22).
- In **Step III** we study the asymptotic behavior of the component  $Y^n$  of equation (4.22) as  $n \rightarrow \infty$ ; moreover, we prove formula (4.5) and equality  $V = \bar{Y}_0$ ,  $\mathbb{P}$ -a.s.
- In **Step IV** we study the asymptotic behavior of the components  $Z^n$  and  $U^n$  of equation (4.22) as  $n \rightarrow \infty$ . Using the results of **Steps III** and **IV**, we construct the quadruple  $(\bar{Y}, \bar{Z}, \bar{U}, \bar{K})$  and we prove that it solves (4.3) and (4.4).
- In **Step V** we prove both minimality and uniqueness of the quadruple  $(\bar{Y}, \bar{Z}, \bar{U}, \bar{K})$ .

**Step I.** *Convergence of  $(Y^{T,n})_{T>0}$ .* Applying Itô's formula to the process  $(e^{-\beta t} Y_t^{T,n})_{t \in [0, T]}$ , we obtain

$$\begin{aligned} e^{-\beta t} Y_t^{T,n} &= \int_t^T e^{-\beta s} f_s(\bar{X}, \bar{I}) ds + \int_t^T e^{-\beta s} dK_s^{T,n} \\ &\quad - \int_t^T e^{-\beta s} Z_s^{T,n} d\bar{W}_s - \int_t^T \int_A e^{-\beta s} U_s^{T,n}(a) \bar{\mu}(ds, da). \end{aligned} \quad (4.7)$$

Given  $\bar{\nu} \in \bar{\mathcal{V}}_n$ , notice that

$$\mathbb{E}_T^{\bar{\nu}} \left[ \int_t^T \int_A e^{-\beta s} U_s^{T,n}(a) \bar{\mu}(ds, da) \middle| \bar{\mathcal{F}}_t^{W,\mu} \right] = \mathbb{E}_T^{\bar{\nu}} \left[ \int_t^T \int_A e^{-\beta s} U_s^{T,n}(a) \bar{\nu}_s(a) \lambda(da) ds \middle| \bar{\mathcal{F}}_t^{W,\mu} \right]$$

and

$$\begin{aligned} \mathbb{E}_T^{\bar{\nu}} \left[ \int_t^T e^{-\beta s} dK_s^{T,n} \middle| \bar{\mathcal{F}}_t^{W,\mu} \right] &= \mathbb{E}_T^{\bar{\nu}} \left[ \int_t^T \int_A n e^{-\beta s} (U_s^{T,n}(a))^+ \lambda(da) ds \middle| \bar{\mathcal{F}}_t^{W,\mu} \right] \\ &\geq \mathbb{E}_T^{\bar{\nu}} \left[ \int_t^T \int_A e^{-\beta s} U_s^{T,n}(a) \bar{\nu}_s(a) \lambda(da) ds \middle| \bar{\mathcal{F}}_t^{W,\mu} \right], \end{aligned}$$

where we have used the numerical inequality  $nu^+ \geq \nu u$ , valid for any real numbers  $u$  and  $\nu$ , with  $\nu \in [0, n]$ . Then, taking the  $\mathbb{P}_T^{\bar{\nu}}$ -conditional expectation with respect to  $\bar{\mathcal{F}}_t^{W,\mu}$  in (4.7), we find

$$e^{-\beta t} Y_t^{T,n} \geq \mathbb{E}_T^{\bar{\nu}} \left[ \int_t^T e^{-\beta s} f_s(\bar{X}, \bar{I}) ds \middle| \bar{\mathcal{F}}_t^{W,\mu} \right].$$

Therefore

$$Y_t^{T,n} \geq \operatorname{ess\,sup}_{\bar{\nu} \in \bar{\mathcal{V}}_n} \mathbb{E}_T^{\bar{\nu}} \left[ \int_t^T e^{-\beta(s-t)} f_s(\bar{X}, \bar{I}) ds \middle| \bar{\mathcal{F}}_t^{W,\mu} \right]. \quad (4.8)$$

On the other hand, for every  $\varepsilon \in (0, 1)$ , let  $\bar{\nu}^{T,n,\varepsilon} \in \bar{\mathcal{V}}_n$  be given by

$$\bar{\nu}_t^{T,n,\varepsilon}(a) = n \mathbf{1}_{\{U_t^{T,n}(a) \geq 0\}} + \frac{\varepsilon}{T\lambda(A)} \mathbf{1}_{\{-1 \leq U_t^{T,n}(a) < 0\}} + \frac{\varepsilon}{T\lambda(A)U_t^{T,n}(a)} \mathbf{1}_{\{U_t^{T,n}(a) \leq -1\}}.$$

Taking the  $\bar{\mathbb{P}}^{\bar{\nu}^{T,n,\varepsilon}}$ -conditional expectation with respect to  $\bar{\mathcal{F}}_t^{W,\mu}$  in (4.7), we obtain

$$\begin{aligned} e^{-\beta t} Y_t^{T,n} &\leq \bar{\mathbb{E}}_T^{\bar{\nu}^{T,n,\varepsilon}} \left[ \int_t^T e^{-\beta s} f_s(\bar{X}, \bar{I}) ds \middle| \bar{\mathcal{F}}_t^{W,\mu} \right] + \varepsilon \\ &\leq \operatorname{ess\,sup}_{\bar{\nu} \in \bar{\mathcal{V}}_n} \bar{\mathbb{E}}_T^{\bar{\nu}} \left[ \int_t^T e^{-\beta s} f_s(\bar{X}, \bar{I}) ds \middle| \bar{\mathcal{F}}_t^{W,\mu} \right] + \varepsilon. \end{aligned}$$

By the arbitrariness of  $\varepsilon$ , and using also inequality (4.8), we conclude that

$$Y_t^{T,n} = \operatorname{ess\,sup}_{\bar{\nu} \in \bar{\mathcal{V}}_n} \bar{\mathbb{E}}_T^{\bar{\nu}} \left[ \int_t^T e^{-\beta(s-t)} f_s(\bar{X}, \bar{I}) ds \middle| \bar{\mathcal{F}}_t^{W,\mu} \right], \quad \bar{\mathbb{P}}\text{-a.s., for all } 0 \leq t \leq T. \quad (4.9)$$

Taking the absolute value of both sides, we obtain,

$$|Y_t^{T,n}| \leq \operatorname{ess\,sup}_{\bar{\nu} \in \bar{\mathcal{V}}_n} \bar{\mathbb{E}}_T^{\bar{\nu}} \left[ \int_t^T e^{-\beta(s-t)} |f_s(\bar{X}, \bar{I})| ds \middle| \bar{\mathcal{F}}_t^{W,\mu} \right], \quad \bar{\mathbb{P}}\text{-a.s., for all } 0 \leq t \leq T. \quad (4.10)$$

Moreover, for any  $T' > 0$ , from (4.9) we find

$$|Y_t^{T',n} - Y_t^{T,n}| \leq \operatorname{ess\,sup}_{\bar{\nu} \in \bar{\mathcal{V}}_n} \bar{\mathbb{E}}_{T \vee T'}^{\bar{\nu}} \left[ \int_{T \wedge T'}^{T \vee T'} e^{-\beta(s-t)} |f_s(\bar{X}, \bar{I})| ds \middle| \bar{\mathcal{F}}_t^{W,\mu} \right], \quad \bar{\mathbb{P}}\text{-a.s., for all } 0 \leq t \leq T \wedge T'. \quad (4.11)$$

Now, we distinguish two cases.

- *Assumption (A) holds.* Since  $f$  is bounded, from (4.10) we find

$$|Y_t^{T,n}| \leq \|f\|_\infty \frac{1 - e^{-\beta(T-t)}}{\beta} \leq \frac{\|f\|_\infty}{\beta},$$

$\bar{\mathbb{P}}$ -a.s., for all  $0 \leq t \leq T$ . Since  $(Y_t^{T,n})_{t \in [0,T]}$  is a càdlàg process, we obtain

$$|Y_t^{T,n}| \leq \frac{\|f\|_\infty}{\beta}, \quad (4.12)$$

for all  $0 \leq t \leq T$ ,  $\bar{\mathbb{P}}$ -a.s., so that  $Y^{T,n}$  is a uniformly bounded process. Proceeding in a similar way, we can deduce from estimate (4.11) that

$$|Y_t^{T',n} - Y_t^{T,n}| \leq \frac{\|f\|_\infty}{\beta} e^{-\beta(T \wedge T' - t)}, \quad \text{for all } 0 \leq t \leq T \wedge T', \quad \bar{\mathbb{P}}\text{-a.s.} \quad (4.13)$$

In particular, for any  $S \in (0, T \wedge T')$ , we have

$$\sup_{t \in [0,S]} |Y_t^{T',n} - Y_t^{T,n}| \leq \frac{\|f\|_\infty}{\beta} e^{-\beta(T \wedge T' - S)}, \quad \bar{\mathbb{P}}\text{-a.s.} \quad (4.14)$$

- *Assumption (A')* holds. By (4.10) we have

$$\begin{aligned} |Y_t^{T,n}| &\leq M \operatorname{ess\,sup}_{\bar{\nu} \in \bar{\mathcal{V}}_n} \bar{\mathbb{E}}_T^{\bar{\nu}} \left[ \int_t^T e^{-\beta(s-t)} \left( 1 + \sup_{u \in [0,s]} |\bar{X}_u|^r \right) ds \middle| \bar{\mathcal{F}}_t^{W,\mu} \right] \\ &\leq M \int_t^T e^{-\beta(s-t)} \left( 1 + \operatorname{ess\,sup}_{\bar{\nu} \in \bar{\mathcal{V}}_n} \bar{\mathbb{E}}_T^{\bar{\nu}} \left[ \sup_{u \in [0,s]} |\bar{X}_u|^r \middle| \bar{\mathcal{F}}_t^{W,\mu} \right] \right) ds, \end{aligned}$$

$\bar{\mathbb{P}}$ -a.s., for all  $0 \leq t \leq T$ . Using estimate (3.4), we find

$$\begin{aligned} |Y_t^{T,n}| &\leq M \int_t^T e^{-\beta(s-t)} \left( 1 + \bar{C} e^{\bar{\beta}(s-t)} \left( 1 + \sup_{u \in [0,t]} |\bar{X}_u|^r \right) \right) ds \\ &\leq 2M(1 + \bar{C}) \left( 1 + \sup_{u \in [0,t]} |\bar{X}_u|^r \right) \int_t^T e^{-(\beta - \bar{\beta})(s-t)} ds \\ &\leq \frac{2M(1 + \bar{C})}{\beta - \bar{\beta}} \left( 1 + \sup_{u \in [0,t]} |\bar{X}_u|^r \right), \end{aligned}$$

$\bar{\mathbb{P}}$ -a.s., for all  $0 \leq t \leq T$ . Since  $(Y_t^{T,n})_{t \in [0,T]}$  is a càdlàg process, we obtain

$$|Y_t^{T,n}| \leq \frac{2M(1 + \bar{C})}{\beta - \bar{\beta}} \left( 1 + \sup_{s \in [0,t]} |\bar{X}_s|^r \right), \quad (4.15)$$

for all  $0 \leq t \leq T$ ,  $\bar{\mathbb{P}}$ -a.s. In a similar way, starting from estimate (4.11) we can prove that

$$|Y_t^{T',n} - Y_t^{T,n}| \leq \frac{2M(1 + \bar{C})}{\beta - \bar{\beta}} \left( 1 + \sup_{s \in [0,t]} |\bar{X}_s|^r \right) e^{-(\beta - \bar{\beta})(T \wedge T' - t)}, \quad (4.16)$$

for all  $0 \leq t \leq T \wedge T'$ ,  $\bar{\mathbb{P}}$ -a.s. As a consequence, for any  $S \in (0, T \wedge T')$ ,

$$\sup_{t \in [0,S]} |Y_t^{T',n} - Y_t^{T,n}| \leq \frac{2M(1 + \bar{C})}{\beta - \bar{\beta}} \left( 1 + \sup_{t \in [0,S]} |\bar{X}_t|^r \right) e^{-(\beta - \bar{\beta})(T \wedge T' - S)}, \quad \bar{\mathbb{P}}\text{-a.s.} \quad (4.17)$$

Then, under either **(A)** or **(A')**, we see that for every  $n \in \mathbb{N}$  there exists a càdlàg process  $(Y_t^n)_{t \geq 0} \in \mathbf{S}_{\text{loc}}^2$  such that, for any  $S > 0$ ,  $Y^n$  is the  $\bar{\mathbb{P}}$ -a.s. uniform limit on  $[0, S]$  as  $T \rightarrow \infty$  of the sequence of càdlàg processes  $(Y^{T,n})_{T > S}$ . Moreover, under **(A)**, we have from (4.12)

$$|Y_t^n| \leq \frac{\|f\|_\infty}{\beta}, \quad \text{for all } t \geq 0, \bar{\mathbb{P}}\text{-a.s.} \quad (4.18)$$

In particular,  $Y^n \in \mathbf{S}^\infty$ . On the other hand, under **(A')**, we deduce from (4.15)

$$|Y_t^n| \leq \frac{2M(1 + \bar{C})}{\beta - \bar{\beta}} \left( 1 + \sup_{s \in [0,t]} |\bar{X}_s|^r \right), \quad \text{for all } t \geq 0, \bar{\mathbb{P}}\text{-a.s.} \quad (4.19)$$

Furthermore, under either **(A)** or **(A')**, from formula (4.9) we have

$$Y_t^n = \lim_{T \rightarrow \infty} \operatorname{ess\,sup}_{\bar{v} \in \bar{V}_n} \bar{\mathbb{E}}_T^{\bar{v}} \left[ \int_t^T e^{-\beta(s-t)} f_s(\bar{X}, \bar{I}) \, ds \middle| \bar{\mathcal{F}}_t^{W, \mu} \right], \quad \bar{\mathbb{P}}\text{-a.s., for all } t \geq 0. \quad (4.20)$$

**Step II.** *Convergence of  $(Z^{T,n}, U^{T,n})_{T>0}$ .* Take  $T' > 0$  and  $S \in (0, T \wedge T')$ . An application of Itô's formula to  $(e^{-\beta t} |Y_t^{T',n} - Y_t^{T,n}|)^2$  between 0 and  $S$ , yields (taking also the  $\bar{\mathbb{P}}$ -expectation)

$$\begin{aligned} & \bar{\mathbb{E}} \left[ \int_0^S e^{-2\beta s} |Z_s^{T',n} - Z_s^{T,n}|^2 \, ds \right] + \bar{\mathbb{E}} \left[ \int_0^S \int_A e^{-2\beta s} |U_s^{T',n}(a) - U_s^{T,n}(a)|^2 \lambda(da) \, ds \right] \\ &= \bar{\mathbb{E}} [e^{-2\beta S} |Y_S^{T',n} - Y_S^{T,n}|^2] - |Y_0^{T',n} - Y_0^{T,n}|^2 \\ & \quad + 2n \bar{\mathbb{E}} \left[ \int_0^S \int_A e^{-2\beta s} (Y_s^{T',n} - Y_s^{T,n}) ((U_s^{T',n}(a))^+ - (U_s^{T,n}(a))^+) \lambda(da) \, ds \right] \\ & \quad - 2 \bar{\mathbb{E}} \left[ \int_0^S \int_A e^{-2\beta s} (Y_s^{T',n} - Y_s^{T,n}) (U_s^{T',n}(a) - U_s^{T,n}(a)) \lambda(da) \, ds \right]. \end{aligned}$$

Then

$$\begin{aligned} & \bar{\mathbb{E}} \left[ \int_0^S e^{-2\beta s} |Z_s^{T',n} - Z_s^{T,n}|^2 \, ds \right] + \bar{\mathbb{E}} \left[ \int_0^S \int_A e^{-2\beta s} |U_s^{T',n}(a) - U_s^{T,n}(a)|^2 \lambda(da) \, ds \right] \\ & \leq \bar{\mathbb{E}} [e^{-2\beta S} |Y_S^{T',n} - Y_S^{T,n}|^2] + 2(n+1) \bar{\mathbb{E}} \left[ \int_0^S \int_A e^{-2\beta s} |Y_s^{T',n} - Y_s^{T,n}| |U_s^{T',n}(a) - U_s^{T,n}(a)| \lambda(da) \, ds \right] \\ & \leq \bar{\mathbb{E}} [e^{-2\beta S} |Y_S^{T',n} - Y_S^{T,n}|^2] + 2(n+1)^2 \lambda(A) \bar{\mathbb{E}} \left[ \int_0^S e^{-2\beta s} |Y_s^{T',n} - Y_s^{T,n}|^2 \, ds \right] \\ & \quad + \frac{1}{2} \bar{\mathbb{E}} \left[ \int_0^S \int_A e^{-2\beta s} |U_s^{T',n}(a) - U_s^{T,n}(a)|^2 \lambda(da) \, ds \right]. \end{aligned}$$

Therefore

$$\begin{aligned} & \frac{1}{2} e^{-2\beta S} \bar{\mathbb{E}} \left[ \int_0^S |Z_s^{T',n} - Z_s^{T,n}|^2 \, ds \right] + \frac{1}{2} e^{-2\beta S} \bar{\mathbb{E}} \left[ \int_0^S \int_A |U_s^{T',n}(a) - U_s^{T,n}(a)|^2 \lambda(da) \, ds \right] \\ & \leq \bar{\mathbb{E}} \left[ \int_0^S e^{-2\beta s} |Z_s^{T',n} - Z_s^{T,n}|^2 \, ds \right] + \frac{1}{2} \bar{\mathbb{E}} \left[ \int_0^S \int_A e^{-2\beta s} |U_s^{T',n}(a) - U_s^{T,n}(a)|^2 \lambda(da) \, ds \right] \\ & \leq \bar{\mathbb{E}} [e^{-2\beta S} |Y_S^{T',n} - Y_S^{T,n}|^2] + 2(n+1)^2 \lambda(A) \bar{\mathbb{E}} \left[ \int_0^S e^{-2\beta s} |Y_s^{T',n} - Y_s^{T,n}|^2 \, ds \right]. \end{aligned}$$

In conclusion, we find

$$\begin{aligned} & \bar{\mathbb{E}} \left[ \int_0^S |Z_s^{T',n} - Z_s^{T,n}|^2 \, ds \right] + \bar{\mathbb{E}} \left[ \int_0^S \int_A |U_s^{T',n}(a) - U_s^{T,n}(a)|^2 \lambda(da) \, ds \right] \\ & \leq 2 \bar{\mathbb{E}} [|Y_S^{T',n} - Y_S^{T,n}|^2] + 4(n+1)^2 \lambda(A) \bar{\mathbb{E}} \left[ \int_0^S e^{2\beta(S-s)} |Y_s^{T',n} - Y_s^{T,n}|^2 \, ds \right]. \end{aligned}$$

Using either (4.13) under **(A)** or (4.16) under **(A')**, we deduce that

$$\mathbb{E} \left[ \int_0^S |Z_s^{T',n} - Z_s^{T,n}|^2 ds \right] + \mathbb{E} \left[ \int_0^S \int_A |U_s^{T',n}(a) - U_s^{T,n}(a)|^2 \lambda(da) ds \right] \xrightarrow{T, T' \rightarrow \infty} 0.$$

In other words, for any  $S > 0$ , the sequence  $(Z^{T,n}, U^{T,n})_{T>S}$  is a Cauchy sequence in the Hilbert space  $\mathbf{L}^2(\mathbf{W}; \mathbf{0}, \mathbf{S}) \times \mathbf{L}^2(\bar{\mu}; \mathbf{0}, \mathbf{S})$ . It follows that there exists  $(Z^n, U^n) \in \mathbf{L}_{\text{loc}}^2(\mathbf{W}) \times \mathbf{L}_{\text{loc}}^2(\bar{\mu})$  such that

$$\mathbb{E} \left[ \int_0^S |Z_s^{T,n} - Z_s^n|^2 ds \right] + \mathbb{E} \left[ \int_0^S \int_A |U_s^{T,n}(a) - U_s^n(a)|^2 \lambda(da) ds \right] \xrightarrow{T \rightarrow \infty} 0. \quad (4.21)$$

Now, take  $S \in (0, T)$  and consider equation (4.6) between  $t \in [0, S]$  and  $S$ :

$$\begin{aligned} Y_t^{T,n} &= Y_S^{T,n} - \beta \int_t^S Y_s^{T,n} ds + \int_t^S f_s(\bar{X}, \bar{I}) ds + n \int_t^S \int_A (U_s^{T,n}(a))^+ \lambda(da) ds \\ &\quad - \int_t^S Z_s^{T,n} d\bar{W}_s - \int_t^S \int_A U_s^{T,n}(a) \bar{\mu}(ds, da). \end{aligned}$$

Letting  $T \rightarrow \infty$ , using either (4.14) under **(A)** or (4.17) under **(A')**, and also (4.21), we obtain

$$\begin{aligned} Y_t^n &= Y_S^n - \beta \int_t^S Y_s^n ds + \int_t^S f_s(\bar{X}, \bar{I}) ds + n \int_t^S \int_A (U_s^n(a))^+ \lambda(da) ds \\ &\quad - \int_t^S Z_s^n d\bar{W}_s - \int_t^S \int_A U_s^n(a) \bar{\mu}(ds, da), \quad 0 \leq t \leq S, \bar{\mathbb{P}}\text{-a.s.} \end{aligned} \quad (4.22)$$

Since  $S$  is arbitrary in (4.22), we conclude that  $(Y^n, Z^n, U^n)$  is a solution to the above infinite horizon backward stochastic differential equation.

**Step III.** *Convergence of  $(Y^n)_{n \in \mathbb{N}}$ , proofs of formula (4.5) and equality  $V = \bar{Y}_0$   $\bar{\mathbb{P}}$ -a.s.* Recalling that  $\bar{V}_n \subset \bar{V}_{n+1}$ , by formula (4.20) we see that  $(Y_t^n)_{n \in \mathbb{N}}$  is an increasing sequence, for all  $t \geq 0$ . In particular, we have  $Y_t^0 \leq Y_t^1 \leq \dots \leq Y_t^n \leq \dots$ ,  $\bar{\mathbb{P}}$ -a.s., for all  $t \geq 0$ . Since  $Y^n$ , for every  $n \in \mathbb{N}$ , is a càdlàg process, we deduce that  $Y_t^0 \leq Y_t^1 \leq \dots \leq Y_t^n \leq \dots$ , for all  $t \geq 0$ ,  $\bar{\mathbb{P}}$ -a.s. Therefore, there exists an  $\bar{\mathbb{F}}^{W, \mu}$ -adapted process  $(\bar{Y}_t)_{t \geq 0}$  such that  $Y_t^n$  converges pointwise increasingly to  $\bar{Y}_t$ , for all  $t \geq 0$ ,  $\bar{\mathbb{P}}$ -a.s. Moreover, under **(A)** we have, using estimate (4.18),

$$|\bar{Y}_t| \leq \frac{\|f\|_\infty}{\beta}, \quad \text{for all } t \geq 0, \bar{\mathbb{P}}\text{-a.s.} \quad (4.23)$$

Therefore  $\bar{Y}$  is uniformly bounded. On the other hand, under **(A')** we obtain, using estimate (4.19),

$$|\bar{Y}_t| \leq \frac{2M(1 + \bar{C})}{\beta - \bar{\beta}} \left( 1 + \sup_{s \in [0, t]} |\bar{X}_s|^r \right), \quad \text{for all } t \geq 0, \bar{\mathbb{P}}\text{-a.s.} \quad (4.24)$$

Let us now prove formula (4.5). Fix  $T > 0$ ,  $t \in [0, T]$ , and an  $\bar{\mathbb{F}}^{W, \mu}$ -stopping time  $\tau$  taking values in  $[t, T]$ . We begin noting that, considering equation (4.22) written between  $t$  and  $\tau$ , and proceeding along the same lines as in the proof of formula (4.9), taking into account that the terminal condition is now given by  $Y_\tau^n$ , we can prove

that

$$Y_t^n = \operatorname{ess\,sup}_{\bar{\nu} \in \bar{\mathcal{V}}_n} \bar{\mathbb{E}}_T^{\bar{\nu}} \left[ \int_t^\tau e^{-\beta(s-t)} f_s(\bar{X}, \bar{I}) \, ds + e^{-\beta(\tau-t)} Y_\tau^n \middle| \bar{\mathcal{F}}_t^{W, \mu} \right], \quad \bar{\mathbb{P}}\text{-a.s.} \quad (4.25)$$

Since  $\bar{\mathcal{V}}_n \subset \bar{\mathcal{V}}$  and  $Y_\tau^n \leq \bar{Y}_\tau$ ,  $\bar{\mathbb{P}}$ -a.s., we get

$$Y_t^n \leq \operatorname{ess\,sup}_{\bar{\nu} \in \bar{\mathcal{V}}} \bar{\mathbb{E}}_T^{\bar{\nu}} \left[ \int_t^\tau e^{-\beta(s-t)} f_s(\bar{X}, \bar{I}) \, ds + e^{-\beta(\tau-t)} \bar{Y}_\tau \middle| \bar{\mathcal{F}}_t^{W, \mu} \right], \quad \bar{\mathbb{P}}\text{-a.s.}$$

Recalling that  $Y_t^n \nearrow \bar{Y}_t$ ,  $\bar{\mathbb{P}}$ -a.s., we obtain the inequality

$$\bar{Y}_t \leq \operatorname{ess\,sup}_{\bar{\nu} \in \bar{\mathcal{V}}} \bar{\mathbb{E}}_T^{\bar{\nu}} \left[ \int_t^\tau e^{-\beta(s-t)} f_s(\bar{X}, \bar{I}) \, ds + e^{-\beta(\tau-t)} \bar{Y}_\tau \middle| \bar{\mathcal{F}}_t^{W, \mu} \right], \quad \bar{\mathbb{P}}\text{-a.s.} \quad (4.26)$$

On the other hand, let  $n, m \in \mathbb{N}$ , with  $n \geq m$ , then

$$\begin{aligned} \bar{Y}_t &\geq Y_t^n = \operatorname{ess\,sup}_{\bar{\nu} \in \bar{\mathcal{V}}_n} \bar{\mathbb{E}}_T^{\bar{\nu}} \left[ \int_t^\tau e^{-\beta(s-t)} f_s(\bar{X}, \bar{I}) \, ds + e^{-\beta(\tau-t)} Y_\tau^n \middle| \bar{\mathcal{F}}_t^{W, \mu} \right] \\ &\geq \operatorname{ess\,sup}_{\bar{\nu} \in \bar{\mathcal{V}}_n} \bar{\mathbb{E}}_T^{\bar{\nu}} \left[ \int_t^\tau e^{-\beta(s-t)} f_s(\bar{X}, \bar{I}) \, ds + e^{-\beta(\tau-t)} Y_\tau^m \middle| \bar{\mathcal{F}}_t^{W, \mu} \right], \quad \bar{\mathbb{P}}\text{-a.s.} \end{aligned}$$

Taking the supremum over  $n \in \{m, m+1, \dots\}$ , we find

$$\bar{Y}_t \geq \operatorname{ess\,sup}_{\bar{\nu} \in \bar{\mathcal{V}}} \bar{\mathbb{E}}_T^{\bar{\nu}} \left[ \int_t^\tau e^{-\beta(s-t)} f_s(\bar{X}, \bar{I}) \, ds + e^{-\beta(\tau-t)} Y_\tau^m \middle| \bar{\mathcal{F}}_t^{W, \mu} \right], \quad \bar{\mathbb{P}}\text{-a.s.}$$

In particular, we have

$$\bar{Y}_t \geq \bar{\mathbb{E}}_T^{\bar{\nu}} \left[ \int_t^\tau e^{-\beta(s-t)} f_s(\bar{X}, \bar{I}) \, ds + e^{-\beta(\tau-t)} Y_\tau^m \middle| \bar{\mathcal{F}}_t^{W, \mu} \right], \quad \bar{\mathbb{P}}\text{-a.s.}$$

for all  $\bar{\nu} \in \bar{\mathcal{V}}$ ,  $m \in \mathbb{N}$ . Taking the limit as  $m \rightarrow \infty$ , and afterwards the  $\operatorname{ess\,sup}_{\bar{\nu} \in \bar{\mathcal{V}}}$ , we conclude that

$$\bar{Y}_t \geq \operatorname{ess\,sup}_{\bar{\nu} \in \bar{\mathcal{V}}} \bar{\mathbb{E}}_T^{\bar{\nu}} \left[ \int_t^\tau e^{-\beta(s-t)} f_s(\bar{X}, \bar{I}) \, ds + e^{-\beta(\tau-t)} \bar{Y}_\tau \middle| \bar{\mathcal{F}}_t^{W, \mu} \right], \quad \bar{\mathbb{P}}\text{-a.s.}$$

which, together with (4.26), gives formula (4.5).

Let us now prove equality  $V = \bar{Y}_0$ ,  $\bar{\mathbb{P}}$ -a.s. By either (4.23) or (4.24), we see that

$$\operatorname{ess\,sup}_{\bar{\nu} \in \bar{\mathcal{V}}} \bar{\mathbb{E}}_T^{\bar{\nu}} [e^{-\beta(T-t)} |\bar{Y}_T| \middle| \bar{\mathcal{F}}_t^{W, \mu}] \xrightarrow[\bar{\mathbb{P}}\text{-a.s.}]{T \rightarrow \infty} 0.$$

Then, from (4.5) with  $\tau = T$ , we obtain

$$\bar{Y}_t = \lim_{T \rightarrow \infty} \operatorname{ess\,sup}_{\bar{\nu} \in \bar{\mathcal{V}}} \bar{\mathbb{E}}_T^{\bar{\nu}} \left[ \int_t^T e^{-\beta(s-t)} f_s(\bar{X}, \bar{I}) \, ds \middle| \bar{\mathcal{F}}_t^{W, \mu} \right], \quad \bar{\mathbb{P}}\text{-a.s.} \quad (4.27)$$

In particular, taking  $t = 0$  in (4.27), we see that  $V = \bar{Y}_0$ ,  $\bar{\mathbb{P}}$ -a.s.

**Step IV.** *Convergence of  $(Z^n, U^n)_{n \in \mathbb{N}}$ .* For every  $T > 0$ , consider the penalized infinite horizon backward stochastic differential equation (4.22) as an equation on  $[0, T]$  with terminal condition  $Y_T^n$ :

$$\begin{aligned} Y_t^n &= Y_T^n - \beta \int_t^T Y_s^n ds + \int_t^T f_s(\bar{X}, \bar{I}) ds + n \int_t^T \int_A (U_s^n(a))^+ \lambda(da) ds \\ &\quad - \int_t^T Z_s^n d\bar{W}_s - \int_t^T \int_A U_s^n(a) \bar{\mu}(ds, da), \quad 0 \leq t \leq T, \bar{\mathbb{P}}\text{-a.s.} \end{aligned} \quad (4.28)$$

Since the above equation is on the finite horizon  $[0, T]$ , we can proceed as in [16] in order to study the asymptotic behavior of  $(Z_{|[0,T]}^n, U_{|[0,T]}^n)_{n \in \mathbb{N}}$ . More precisely, define

$$K_t^n = n \int_0^t \int_A (U_s^n(a))^+ \lambda(da) ds, \quad \text{for all } t \geq 0.$$

Then, proceeding as in the proof of Lemma 2.3, we can prove the following estimate for the triplet  $(Z_{|[0,T]}^n, U_{|[0,T]}^n, K_{|[0,T]}^n)$ :

$$\begin{aligned} &\bar{\mathbb{E}} \left[ \int_0^T |Z_s^n|^2 ds \right] + \bar{\mathbb{E}} \left[ \int_0^T \int_A |U_s^n(a)|^2 \lambda(da) ds \right] + \bar{\mathbb{E}} [K_T^n^2] \\ &\leq C_T \left( \bar{\mathbb{E}} \left[ \sup_{s \in [0, T]} |Y_s^n|^2 \right] + \bar{\mathbb{E}} \left[ \int_0^T |f_s(\bar{X}, \bar{I})|^2 ds \right] \right), \quad \text{for every } n \in \mathbb{N}, \end{aligned} \quad (4.29)$$

for some positive constant  $C_T$ , possibly depending on  $T$ , but independent of  $n$ . Since, for every  $n$ ,  $Y^n$  satisfies either (4.18) or (4.19), it follows that (4.29) provides an estimate for  $(Z_{|[0,T]}^n, U_{|[0,T]}^n, K_{|[0,T]}^n)$  which is uniform with respect to  $n$ .

By (4.29), it is quite easy to prove (proceeding as in [16], Thm. 2.1) that the sequence  $(Z_{|[0,T]}^n, U_{|[0,T]}^n)_{n \in \mathbb{N}}$  weakly converges in  $\mathbf{L}^2(\mathbf{W}; \mathbf{0}, \mathbf{T}) \times \mathbf{L}^2(\bar{\mu}; \mathbf{0}, \mathbf{T})$  to some pair  $(\bar{Z}^T, \bar{U}^T)$ . It is also straightforward to see that there exists a unique pair  $(\bar{Z}, \bar{U})$  in  $\mathbf{L}_{\text{loc}}^2(\mathbf{W}) \times \mathbf{L}_{\text{loc}}^2(\bar{\mu})$ , whose restriction to  $[0, T]$  is equal to  $(\bar{Z}^T, \bar{U}^T)$ . Similarly, for every  $s \in [0, T]$ ,  $(K_s^n)_{n \in \mathbb{N}}$  weakly converges in  $L^2(\Omega, \bar{\mathcal{F}}_s^{W, \mu}, \bar{\mathbb{P}})$  to  $\bar{K}_s$ , for some nondecreasing predictable process  $\bar{K}$  such that  $\bar{K}_s \in L^2(\Omega, \bar{\mathcal{F}}_s^{W, \mu}, \bar{\mathbb{P}})$ , for any  $s \geq 0$ . Then, recalling from **Step III** that  $Y^n$  converges to  $\bar{Y}$ , we can pass to the limit in (4.28) and obtain equation (4.3) for the quadruple  $(\bar{Y}, \bar{Z}, \bar{U}, \bar{K})$ . Finally, relying on Lemma 2.2 in [26], we can prove that both  $\bar{Y}$  and  $\bar{K}$  admit a càdlàg version, hence  $\bar{Y} \in \mathbf{S}_{\text{loc}}^2$  and  $\bar{K} \in \mathbf{K}_{\text{loc}}^2$ .

It remains to prove the jump constraint (4.4). To this end, for any  $T > 0$ , we introduce the functional  $F_T: \mathbf{L}^2(\bar{\mu}; \mathbf{0}, \mathbf{T}) \rightarrow \mathbb{R}$  given by

$$F_T(U) = \bar{\mathbb{E}} \left[ \int_0^T \int_A (U_s(a))^+ \lambda(da) ds \right]^2, \quad \text{for every } U \in \mathbf{L}^2(\bar{\mu}; \mathbf{0}, \mathbf{T}).$$

Since  $F_T(U_{|[0,T]}^n) = \bar{\mathbb{E}}[K_T^n^2]/n^2$ , by estimate (4.29) it follows that  $F_T(U_{|[0,T]}^n) \rightarrow 0$  as  $n \rightarrow \infty$ . Notice that  $F_T$  is convex and strongly continuous on  $\mathbf{L}^2(\bar{\mu}; \mathbf{0}, \mathbf{T})$ , then  $F_T$  is weakly lower semicontinuous. Hence

$$F_T(\bar{U}_{|[0,T]}) \leq \liminf_{n \rightarrow \infty} F_T(U_{|[0,T]}^n) = 0.$$

From the arbitrariness of  $T$ , we deduce the validity of the jump constraint (4.4).

**Step V. Minimality and uniqueness.** Let us prove that  $(\bar{Y}, \bar{Z}, \bar{U}, \bar{K})$  is a minimal solution to equation (4.3) and (4.4). Consider an arbitrary quadruple  $(\hat{Y}, \hat{Z}, \hat{U}, \hat{K}) \in \mathbf{S}_{\text{loc}}^2 \times \mathbf{L}_{\text{loc}}^2(\mathbf{W}) \times \mathbf{L}_{\text{loc}}^2(\bar{\mu}) \times \mathbf{K}_{\text{loc}}^2$  solution to equation (4.3) and (4.4), with either  $\hat{Y} \in \mathbf{S}^\infty$  under  $(\mathbf{A})$  or  $|\hat{Y}_t| \leq C(1 + \sup_{s \in [0, t]} |\bar{X}_s|^r)$ , for all  $t \geq 0$ ,  $\bar{\mathbb{P}}$ -a.s., and for some positive constant  $C$ , under  $(\mathbf{A}')$ . In particular, we have, for any  $0 \leq t \leq T$ ,

$$\begin{aligned} \hat{Y}_t &+ \int_t^T e^{-\beta(s-t)} \hat{Z}_s d\bar{W}_s + \int_t^T \int_A e^{-\beta(s-t)} \hat{U}_s(a) \bar{\mu}(ds, da) \\ &= e^{-\beta(T-t)} \hat{Y}_T + \int_t^T e^{-\beta(s-t)} f_s(\bar{X}, \bar{I}) ds + \int_t^T e^{-\beta(s-t)} d\hat{K}_s. \end{aligned}$$

Given  $\bar{\nu} \in \bar{\mathcal{V}}$ , taking the  $\bar{\mathbb{P}}_T^{\bar{\nu}}$ -conditional expectation with respect to  $\bar{\mathcal{F}}_t^{W, \mu}$ , we obtain

$$\begin{aligned} \hat{Y}_t &\geq \hat{Y}_t + \bar{\mathbb{E}}_T^{\bar{\nu}} \left[ \int_t^T \int_A e^{-\beta(s-t)} \hat{U}_s(a) \lambda(da) ds \middle| \bar{\mathcal{F}}_t^{W, \mu} \right] \\ &= \bar{\mathbb{E}}_T^{\bar{\nu}} [e^{-\beta(T-t)} \hat{Y}_T | \bar{\mathcal{F}}_t^{W, \mu}] + \bar{\mathbb{E}}_T^{\bar{\nu}} \left[ \int_t^T e^{-\beta(s-t)} f_s(\bar{X}, \bar{I}) ds \middle| \bar{\mathcal{F}}_t^{W, \mu} \right] + \bar{\mathbb{E}}_T^{\bar{\nu}} \left[ \int_t^T e^{-\beta(s-t)} d\hat{K}_s \middle| \bar{\mathcal{F}}_t^{W, \mu} \right] \\ &\geq \bar{\mathbb{E}}_T^{\bar{\nu}} [e^{-\beta(T-t)} \hat{Y}_T | \bar{\mathcal{F}}_t^{W, \mu}] + \bar{\mathbb{E}}_T^{\bar{\nu}} \left[ \int_t^T e^{-\beta(s-t)} f_s(\bar{X}, \bar{I}) ds \middle| \bar{\mathcal{F}}_t^{W, \mu} \right]. \end{aligned}$$

From the arbitrariness of  $\bar{\nu}$ , we get

$$\hat{Y}_t \geq \text{ess sup}_{\bar{\nu} \in \bar{\mathcal{V}}} \bar{\mathbb{E}}_T^{\bar{\nu}} \left[ \int_t^T e^{-\beta(s-t)} f_s(\bar{X}, \bar{I}) ds + e^{-\beta(T-t)} \hat{Y}_T \middle| \bar{\mathcal{F}}_t^{W, \mu} \right].$$

Using the bounds satisfied by  $\hat{Y}$  under  $(\mathbf{A})$  or  $(\mathbf{A}')$ , and estimate (3.4), we see that

$$\text{ess sup}_{\bar{\nu} \in \bar{\mathcal{V}}} \bar{\mathbb{E}}_T^{\bar{\nu}} [e^{-\beta(T-t)} |\hat{Y}_T| | \bar{\mathcal{F}}_t^{W, \mu}] \xrightarrow[T \rightarrow \infty]{\bar{\mathbb{P}}\text{-a.s.}} 0.$$

This implies that

$$\begin{aligned} \hat{Y}_t &\geq \liminf_{T \rightarrow \infty} \text{ess sup}_{\bar{\nu} \in \bar{\mathcal{V}}} \bar{\mathbb{E}}_T^{\bar{\nu}} \left[ \int_t^T e^{-\beta(s-t)} f_s(\bar{X}, \bar{I}) ds + e^{-\beta(T-t)} \hat{Y}_T \middle| \bar{\mathcal{F}}_t^{W, \mu} \right] \\ &= \liminf_{T \rightarrow \infty} \text{ess sup}_{\bar{\nu} \in \bar{\mathcal{V}}} \bar{\mathbb{E}}_T^{\bar{\nu}} \left[ \int_t^T e^{-\beta(s-t)} f_s(\bar{X}, \bar{I}) ds \middle| \bar{\mathcal{F}}_t^{W, \mu} \right] \\ &\geq \lim_{T \rightarrow \infty} \text{ess sup}_{\bar{\nu} \in \bar{\mathcal{V}}_n} \bar{\mathbb{E}}_T^{\bar{\nu}} \left[ \int_t^T e^{-\beta(s-t)} f_s(\bar{X}, \bar{I}) ds \middle| \bar{\mathcal{F}}_t^{W, \mu} \right] = Y_t^n, \quad \bar{\mathbb{P}}\text{-a.s.} \end{aligned}$$

Letting  $n \rightarrow \infty$ , we end up with  $\hat{Y}_t \geq \bar{Y}_t$ ,  $\bar{\mathbb{P}}$ -a.s., for all  $t \geq 0$ , which yields the minimality of  $\bar{Y}$ .

Finally, let us prove the uniqueness of  $(\bar{Y}, \bar{Z}, \bar{U}, \bar{K})$ . Since  $(\bar{Y}, \bar{Z}, \bar{U}, \bar{K})$  is a minimal solution to equation (4.3) and (4.4), by definition, the component  $\bar{Y}$  is uniquely determined. In order to prove the uniqueness of the other components  $(\bar{Z}, \bar{U}, \bar{K})$ , we can proceed as in Remark 2.1 in [16] reasoning as follows: first, we identify the Brownian part of  $\bar{Y}$ , that is  $\bar{Z}$ , and its finite variation part,  $(\bar{U}, \bar{K})$ ; afterwards, concerning the finite variation part, we identify the predictable part,  $\bar{K}$ , and the totally inaccessible part,  $\bar{U}$ .  $\square$

## 5. FEYNMAN-KAC REPRESENTATION FOR FULLY NON-LINEAR ELLIPTIC PDE

We now apply the results of the previous sections in order to determine a non-linear Feynman-Kac representation formula for a fully non-linear elliptic partial differential equation. To this end, we introduce a Markovian framework, taking non-path-dependent coefficients:  $b_t(x, a)$ ,  $\sigma_t(x, a)$ ,  $f_t(x, a)$  will depend on  $(x, a)$  only through its value at time  $t$ , namely  $(x(t), a(t))$ . More precisely, in the present section we suppose that  $b$ ,  $\sigma$ ,  $f$  are defined on  $\mathbb{R}^n \times A$ , with values respectively in  $\mathbb{R}^n$ ,  $\mathbb{R}^{n \times d}$ ,  $\mathbb{R}$ .

### 5.1. Infinite horizon optimal control problem

We consider the same probabilistic setting as in Section 2, characterized by the following objects:  $(\Omega, \mathcal{F}, \mathbb{P})$ ,  $W = (W_t)_{t \geq 0}$ ,  $\mathbb{F}^W = (\mathcal{F}_t^W)_{t \geq 0}$ ,  $\mathcal{A}$ .

For every  $x \in \mathbb{R}^n$  and  $\alpha \in \mathcal{A}$ , we consider the following Markovian controlled stochastic differential equation:

$$X_t^{x, \alpha} = x + \int_0^t b(X_s^{x, \alpha}, \alpha_s) ds + \int_0^t \sigma(X_s^{x, \alpha}, \alpha_s) dW_s, \quad (5.1)$$

for all  $t \geq 0$ . We then define the value function  $v: \mathbb{R}^n \rightarrow \mathbb{R}$  of the corresponding infinite horizon stochastic optimal control problem as follows:

$$v(x) = \sup_{\alpha \in \mathcal{A}} J(x, \alpha), \quad \text{for all } x \in \mathbb{R}^n, \quad (5.2)$$

where the gain functional is given by

$$J(x, \alpha) = \mathbb{E} \left[ \int_0^\infty e^{-\beta t} f(X_t^{x, \alpha}, \alpha_t) dt \right].$$

On the positive constant  $\beta$  and on the coefficients  $b$ ,  $\sigma$ ,  $f$  we impose the same sets of assumptions (either **(A)** or **(A')**) as for the path-dependent case. In the present Markovian framework those assumptions read as follows.

#### **(A<sub>Markov</sub>)**

- (i) The functions  $b$ ,  $\sigma$ ,  $f$  are continuous.
- (iii) There exists a constant  $L \geq 0$  such that

$$\begin{aligned} |b(x, a) - b(x', a)| + |\sigma(x, a) - \sigma(x', a)| &\leq L|x - x'|, \\ |b(0, a)| + |\sigma(0, a)| &\leq L, \end{aligned}$$

for all  $x, x' \in \mathbb{R}^n$ ,  $a \in A$ .

- (iv) There exist constants  $M > 0$  and  $r \geq 0$  such that

$$|f(x, a)| \leq M(1 + |x|^r),$$

for all  $x \in \mathbb{R}^n$ ,  $a \in A$ .

- (v)  $\beta > \bar{\beta}$ , where  $\bar{\beta}$  is zero when the constant  $r$  at point (iv) is zero; otherwise, when  $r > 0$ ,  $\bar{\beta}$  is a strictly positive real number such that

$$\mathbb{E} \left[ \sup_{s \in [0, t]} |X_s^{x, \alpha}|^r \right] \leq \bar{C} e^{\bar{\beta} t} (1 + |x|^r), \quad (5.3)$$

for some constant  $\bar{C} \geq 0$ , with  $\bar{\beta}$  and  $\bar{C}$  independent of  $t \geq 0$ ,  $\alpha \in \mathcal{A}$ ,  $x \in \mathbb{R}^n$ . See Lemma 2.1.

**Remark 5.1.** Notice that, in the present Markovian framework, as the coefficients are independent of the time variable, we can unify **(A)** and **(A')** into a single set of assumptions **(A<sub>Markov</sub>)**. As a matter of fact, in this case, if we include in **(A')** the case  $r = 0$ , with corresponding  $\bar{\beta} = 0$ , then **(A')** coincides with **(A)**; while, in the path-dependent case, **(A')** is in general stronger than **(A)**, as the constant  $L$  in **(A')**-(iii) is taken independent of  $T$ .

It is well-known that under **(A<sub>Markov</sub>)**-(i)-(iii), for every  $x \in \mathbb{R}^n$  and  $\alpha \in \mathcal{A}$ , there exists a unique  $\mathbb{F}^W$ -progressive continuous process  $X^{x,\alpha} = (X_t^{x,\alpha})_{t \geq 0}$  solution to equation (5.1). Moreover, under **(A<sub>Markov</sub>)**-(i)-(iii), by Lemma 2.1, we have, for every  $p > 0$ ,

$$\mathbb{E} \left[ \sup_{s \in [0,t]} |X_s^{x,\alpha}|^p \right] \leq \bar{C}_{p,L} e^{\bar{\beta}_{p,L} t} (1 + |x|^p),$$

for some constants  $\bar{C}_{p,L} \geq 0$  and  $\bar{\beta}_{p,L} \geq 0$ , independent of  $t \geq 0$ ,  $\alpha \in \mathcal{A}$ ,  $x \in \mathbb{R}^n$ , depending only on  $p > 0$  and the constant  $L$  appearing in **(A<sub>Markov</sub>)**-(iii). Finally, we notice that the value function  $v$  in (5.2) is well-defined by **(A<sub>Markov</sub>)**-(iv)-(v).

## 5.2. Randomized problem and backward SDE representation of $v(x)$

We consider the same probabilistic setting as in Section 3, with  $\lambda$ ,  $a_0$ ,  $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ ,  $\bar{W} = (\bar{W}_t)_{t \geq 0}$ ,  $\bar{\mu}$  on  $[0, \infty) \times A$  with compensator  $\lambda(da)dt$ ,  $\bar{\mathbb{F}}^{W,\mu} = (\bar{\mathcal{F}}_t^{W,\mu})_{t \geq 0}$ ,  $\bar{I}$  defined by (3.1),  $\bar{\mathcal{V}}$ , and the family of probability measures  $\bar{\mathbb{P}}_T^{\bar{\nu}}$ , with  $\bar{\nu} \in \bar{\mathcal{V}}$  and  $T > 0$ . Then, for every  $x \in \mathbb{R}^n$ , we consider the stochastic differential equation

$$\bar{X}_t^x = x + \int_0^t b(\bar{X}_s^x, \bar{I}_s) ds + \int_0^t \sigma(\bar{X}_s^x, \bar{I}_s) d\bar{W}_s, \quad \text{for all } t \geq 0. \quad (5.4)$$

It is well-known that under **(A<sub>Markov</sub>)**-(i)-(iii), for every  $x \in \mathbb{R}^n$ , there exists a unique  $\bar{\mathbb{F}}^{W,\mu}$ -progressive continuous process  $\bar{X}^x = (\bar{X}_t^x)_{t \geq 0}$  solution to equation (5.4). Furthermore, under **(A<sub>Markov</sub>)**-(i)-(iii), by Lemma 2.1, for every  $p > 0$ , there exist two non-negative constants  $\bar{C}_{p,L}$  and  $\bar{\beta}_{p,L}$  such that

$$\bar{\mathbb{E}}_T^{\bar{\nu}} \left[ \sup_{s \in [0,T]} |\bar{X}_s^x|^p \middle| \bar{\mathcal{F}}_t^{W,\mu} \right] \leq \bar{C}_{p,L} e^{\bar{\beta}_{p,L} t} \left( 1 + \sup_{s \in [0,t]} |\bar{X}_s^x|^p \right),$$

$\bar{\mathbb{P}}$ -a.s., for all  $t \in [0, T]$ ,  $T > 0$ ,  $\bar{\nu} \in \bar{\mathcal{V}}$ , where  $\bar{C}_{p,L}$  and  $\bar{\beta}_{p,L}$  are independent of  $t \in [0, T]$ ,  $T > 0$ ,  $\bar{\nu} \in \bar{\mathcal{V}}$ ,  $x \in \mathbb{R}^n$ , and depend only on  $p > 0$  and the constant  $L$  appearing in **(A<sub>Markov</sub>)**-(iii).

**Remark 5.2.** As we did in Lemma 2.1, when  $r > 0$  in assumption **(A<sub>Markov</sub>)**-(iv), we denote  $\bar{C}_{r,L}$  and  $\bar{\beta}_{r,L}$  simply by  $\bar{C}$  and  $\bar{\beta}$ .

We define the value function  $v^{\mathcal{R}}: \mathbb{R}^n \rightarrow \mathbb{R}$  of the randomized infinite horizon stochastic optimal control problem as follows:

$$v^{\mathcal{R}}(x) = \lim_{T \rightarrow \infty} \sup_{\bar{\nu} \in \bar{\mathcal{V}}} J_T^{\mathcal{R}}(x, \bar{\nu}), \quad \text{for all } x \in \mathbb{R}^n, \quad (5.5)$$

where, for every  $T > 0$ ,

$$J_T^{\mathcal{R}}(x, \bar{\nu}) = \bar{\mathbb{E}}_T^{\bar{\nu}} \left[ \int_0^T e^{-\beta t} f(\bar{X}_t^x, \bar{I}_t) dt \right].$$

From Section 3, we know that the limit in (5.5) exists and it is finite for every  $x \in \mathbb{R}^n$ .

For every  $x \in \mathbb{R}^n$ , we now consider the following infinite horizon backward stochastic differential equation with nonpositive jumps:

$$\begin{aligned} Y_t &= Y_T - \beta \int_t^T Y_s \, ds + \int_t^T f(\bar{X}_s^x, \bar{I}_s) \, ds + K_T - K_t \\ &\quad - \int_t^T Z_s \, d\bar{W}_s - \int_t^T \int_A U_s(a) \bar{\mu}(ds, da), \quad 0 \leq t \leq T, \forall T \in (0, \infty), \bar{\mathbb{P}}\text{-a.s.} \end{aligned} \quad (5.6)$$

$$U_t(a) \leq 0, \quad dt \otimes d\bar{\mathbb{P}} \otimes \lambda(da)\text{-a.e. on } [0, \infty) \times \bar{\Omega} \times A. \quad (5.7)$$

**Lemma 5.3.** *Under  $(\mathbf{A}_{\text{Markov}})$ , for every  $x \in \mathbb{R}^n$ , there exists a unique quadruple  $(\bar{Y}^x, \bar{Z}^x, \bar{U}^x, \bar{K}^x)$  in  $\mathbf{S}_{\text{loc}}^2 \times \mathbf{L}_{\text{loc}}^2(\mathbf{W}) \times \mathbf{L}_{\text{loc}}^2(\bar{\mu}) \times \mathbf{K}_{\text{loc}}^2$ , such that  $\bar{Y}^x \in \mathbf{S}^\infty$  (resp.  $|\bar{Y}_t^x| \leq C(1 + \sup_{s \in [0, t]} |\bar{X}_s^x|^r)$ ), for all  $t \geq 0$ ,  $\bar{\mathbb{P}}\text{-a.s.}$ , and for some positive constant  $C$ ), satisfying (5.6) and (5.7) which is minimal in the following sense: for any other quadruple  $(\hat{Y}^x, \hat{Z}^x, \hat{U}^x, \hat{K}^x)$  in  $\mathbf{S}_{\text{loc}}^2 \times \mathbf{L}_{\text{loc}}^2(\mathbf{W}) \times \mathbf{L}_{\text{loc}}^2(\bar{\mu}) \times \mathbf{K}_{\text{loc}}^2$ , with  $\hat{Y}^x \in \mathbf{S}^\infty$  (resp.  $|\hat{Y}_t^x| \leq C(1 + \sup_{s \in [0, t]} |\bar{X}_s^x|^r)$ ), for all  $t \geq 0$ ,  $\bar{\mathbb{P}}\text{-a.s.}$ , and for some positive constant  $C$ ), satisfying (5.6) and (5.7) we have:*

$$\bar{Y}_t^x \leq \hat{Y}_t^x, \quad \text{for all } t \geq 0, \bar{\mathbb{P}}\text{-a.s.}$$

*Proof.* The result follows directly from Theorem 4.2.  $\square$

**Proposition 5.4.** *Suppose that  $(\mathbf{A}_{\text{Markov}})$  holds. Then, for every  $x \in \mathbb{R}^n$ , we have:*

- (i)  $v(x) = v^{\mathcal{R}}(x)$ ;
- (ii)  $v(x) = \bar{Y}_0^x$ ,  $\bar{\mathbb{P}}\text{-a.s.}$ , where  $(\bar{Y}^x, \bar{Z}^x, \bar{U}^x, \bar{K}^x)$  is the minimal solution to (5.6) and (5.7); moreover, we have ( $r$  is the constant appearing in  $(\mathbf{A}_{\text{Markov}})$ -(iv)):
- $r = 0$ :  $|v(x)| \leq \frac{\|f\|_\infty}{\beta}$ , for all  $x \in \mathbb{R}^n$ ;
- $r > 0$ :  $|v(x)| \leq \frac{2M(1+\bar{C})}{\beta-\bar{\beta}}(1 + |x|^r)$ , for all  $x \in \mathbb{R}^n$ ;
- (iii)  $v(\bar{X}_t^x) = \bar{Y}_t^x$ ,  $\bar{\mathbb{P}}\text{-a.s.}$ , for all  $t \geq 0$ ;
- (iv)  $v$  satisfies the following equality:

$$v(x) = \sup_{\bar{v} \in \bar{\mathcal{V}}} \bar{\mathbb{E}}_{\bar{v}}^v \left[ \int_0^\tau e^{-\beta s} f(\bar{X}_s^x, \bar{I}_s) \, ds + e^{-\beta \tau} v(\bar{X}_\tau^x) \right], \quad (5.8)$$

for all  $T \geq 0$  and any  $\bar{\mathbb{F}}^{W, \mu}$ -stopping time  $\tau$  taking values in  $[0, T]$ .

**Remark 5.5.** We refer to formula (5.8) as the *randomized dynamic programming principle* for  $v$ .

*Proof of Proposition 5.4.* Point (i) is a consequence of Theorem 4.1, while point (ii) follows from Theorem 4.2, Lemma 5.3, and either estimate (4.23) (case  $r = 0$ ) or estimate (4.24) (case  $r > 0$ ) with  $t = 0$ . Concerning identity (iii), for every  $n \in \mathbb{N}$  consider the penalized infinite horizon backward stochastic differential equation (4.22) with  $f_s(\bar{X}, \bar{I})$  replaced by  $f(\bar{X}_s^x, \bar{I}_s)$ , and denote by  $(\bar{Y}^{n,x}, \bar{Z}^{n,x}, \bar{U}^{n,x}) \in \mathbf{S}_{\text{loc}}^2 \times \mathbf{L}_{\text{loc}}^2(\mathbf{W}) \times \mathbf{L}_{\text{loc}}^2(\bar{\mu})$  its solution. Then, it is a standard result in the theory of backward stochastic differential equations that, for every  $n$ , there exists a function  $v_n: \mathbb{R}^n \rightarrow \mathbb{R}$ , with  $v_n(x) = \bar{Y}_0^{n,x}$ , satisfying  $v_n(\bar{X}_t^x) = \bar{Y}_t^{n,x}$ ,  $\bar{\mathbb{P}}\text{-a.s.}$ , for all  $t \geq 0$ . By step II of the proof of Theorem 4.2, we know that  $\bar{Y}_0^{n,x} = v_n(x)$  converges increasingly to  $\bar{Y}_0^x = v(x)$ . Then, letting  $n \rightarrow \infty$  in identity  $v_n(\bar{X}_t^x) = \bar{Y}_t^{n,x}$ , we deduce (iii). Finally, formula (5.8) follows from point (iii) and formula (4.5), with  $t = 0$ .  $\square$

### 5.3. Fully non-linear elliptic PDE

Consider the following Hamilton-Jacobi-Bellman fully non-linear elliptic partial differential equation on  $\mathbb{R}^n$ :

$$\beta u(x) - \sup_{a \in A} [\mathcal{L}^a u(x) + f(x, a)] = 0, \quad \text{for all } x \in \mathbb{R}^n, \quad (5.9)$$

where

$$\mathcal{L}^a v(x) = \langle b(x, a), D_x v(x) \rangle + \frac{1}{2} \text{tr}[\sigma(x, a) \sigma^\top(x, a) D_x^2 v(x)].$$

We now prove, by means of the randomized dynamic programming principle, that the value function  $v$  in (5.2) is a viscosity solution of equation (5.9). As explained in the introduction, this allows us to avoid the difficulties related to a rigorous proof of the classical dynamic programming principle, which often requires the use of nontrivial measurability arguments. In order to do this, we need an additional assumption.

(**A<sub>f</sub>**) The function  $f = f(x, a): \mathbb{R}^n \times A \rightarrow \mathbb{R}$  is continuous in  $x$  uniformly with respect to  $a$ .

**Remark 5.6.** Notice that under (**A<sub>Markov</sub>**) and (**A<sub>f</sub>**), for every  $\varphi \in C^2(\mathbb{R}^n)$ , the map

$$x \in \mathbb{R}^n \longmapsto \sup_{a \in A} [\mathcal{L}^a \varphi(x) + f(x, a)]$$

is continuous on  $\mathbb{R}^n$ .

**Lemma 5.7.** *Under (**A<sub>Markov</sub>**) and (**A<sub>f</sub>**), the value function  $v$  in (5.2) is continuous on  $\mathbb{R}^n$ .*

*Proof.* We begin noting that the continuity of  $v$  follows if we prove the following:

$$\text{The discounted reward functional } J(x, \alpha) \text{ is continuous in } x \text{ uniformly with respect to } \alpha \text{ in } \mathcal{A}. \quad (5.10)$$

In order to prove (5.10), we rewrite  $J(x, \alpha)$  as follows:

$$J(x, \alpha) = J^T(x, \alpha) + J_T(x, \alpha), \quad (5.11)$$

for every  $x \in \mathbb{R}^n$ ,  $\alpha \in \mathcal{A}$ ,  $T > 0$ , where

$$J^T(x, \alpha) = \mathbb{E} \left[ \int_0^T e^{-\beta t} f(X_t^{x, \alpha}, \alpha_t) dt \right], \quad J_T(x, \alpha) = \mathbb{E} \left[ \int_T^\infty e^{-\beta t} f(X_t^{x, \alpha}, \alpha_t) dt \right].$$

Now, observe that, by the polynomial growth condition on  $f$  in (**A<sub>Markov</sub>**)-(iv), by assumption (**A<sub>Markov</sub>**)-(v) which guarantees that  $\beta > \bar{\beta}$ , together with estimate (5.3), we have

$$|J_T(x, \alpha)| \leq \int_T^\infty M e^{-\beta t} \left( 1 + \bar{C} e^{\bar{\beta} t} (1 + |x|^r) \right) dt = \frac{M}{\beta} e^{-\beta T} + \frac{M}{\beta - \bar{\beta}} \bar{C} (1 + |x|^r) e^{-(\beta - \bar{\beta})T}.$$

So, in particular, for every  $x \in \mathbb{R}^n$ , when  $T \rightarrow +\infty$  the quantity  $J_T(x, \alpha)$  goes to zero uniformly with respect to  $\alpha$  in  $\mathcal{A}$  and uniformly with respect to  $x$  in any bounded set. As a consequence, from equality (5.11) we deduce that claim (5.10) follows if we prove that the functional  $J^T$  is continuous in  $x$  uniformly with respect to  $\alpha$  in  $\mathcal{A}$ . Notice that  $J^T$  is the reward functional of a finite horizon control problem, with horizon  $T$  and zero terminal condition. Then, under assumptions (**A<sub>Markov</sub>**) and (**A<sub>f</sub>**), the required continuity of  $J^T$  follows from Theorem 3.2.2 in [20].  $\square$

We can now state the main result of this section.

**Theorem 5.8.** *Under  $(\mathbf{A}_{\text{Markov}})$  and  $(\mathbf{A}_f)$ , the value function  $v$  in (5.2) is a continuous viscosity solution to (5.9), namely:*

- $v$  is a viscosity subsolution to (5.9):

$$\beta \varphi(x) - \sup_{a \in A} [\mathcal{L}^a \varphi(x) + f(x, a)] \leq 0,$$

for every  $x \in \mathbb{R}^n$  and  $\varphi \in C^2(\mathbb{R}^n)$  such that

$$\max_{y \in \mathbb{R}^n} (v(y) - \varphi(y)) = v(x) - \varphi(x) = 0. \quad (5.12)$$

- $v$  is a viscosity supersolution to (5.9):

$$\beta \varphi(x) - \sup_{a \in A} [\mathcal{L}^a \varphi(x) + f(x, a)] \geq 0,$$

for every  $x \in \mathbb{R}^n$  and  $\varphi \in C^2(\mathbb{R}^n)$  such that

$$\min_{y \in \mathbb{R}^n} (v(y) - \varphi(y)) = v(x) - \varphi(x) = 0. \quad (5.13)$$

**Remark 5.9.** We do not provide here a comparison principle (and hence a uniqueness result) for the fully non-linear elliptic partial differential equation (5.9), which can be found for instance in [15], Thm. 7.4, under stronger assumptions than  $(\mathbf{A}_{\text{Markov}})$ - $(\mathbf{A}_f)$ . However, we notice that, when a uniqueness result for equation (5.9) holds, Theorem 5.8 provides a non-linear Feynman-Kac representation formula for the unique viscosity solution (reported in Proposition 5.4-(ii)), expressed in terms of the class of infinite horizon backward stochastic differential equations with nonpositive jumps (5.6) and (5.7), with  $x \in \mathbb{R}^n$ .

*Proof.* Notice that the continuity of  $v$  follows from Lemma 5.7. We split the rest of the proof into two steps.

**Step I. Viscosity subsolution property.** Take  $x \in \mathbb{R}^n$  and  $\varphi \in C^2(\mathbb{R}^n)$  such that (5.12) holds. Without loss of generality, we suppose that the maximum in (5.12) is strict. As a matter of fact, if we prove the viscosity subsolution property for all  $\varphi$  for which the maximum in (5.12) is strict, then given a generic  $\varphi \in C^2(\mathbb{R}^n)$  satisfying (5.12), it is enough to consider the function  $\psi \in C^2(\mathbb{R}^n)$  given by  $\psi(y) = \varphi(y) + |y - x|^4$ , for all  $y \in \mathbb{R}^n$ , and to notice that  $\psi(x) = \varphi(x)$ ,  $D_x \psi(x) = D_x \varphi(x)$ ,  $D_x^2 \psi(x) = D_x^2 \varphi(x)$ ; moreover,  $\psi$  satisfies (5.12) and the maximum is strict.

We proceed by contradiction, assuming that

$$\beta \varphi(x) - \sup_{a \in A} [\mathcal{L}^a \varphi(x) + f(x, a)] =: 2\varepsilon_0 > 0.$$

By Remark 5.6, it follows that the map  $y \mapsto \beta \varphi(y) - \sup_{a \in A} [\mathcal{L}^a \varphi(y) + f(y, a)]$  is continuous on  $\mathbb{R}^n$ , therefore there exists  $\eta > 0$  such that

$$\beta \varphi(y) - \sup_{a \in A} [\mathcal{L}^a \varphi(y) + f(y, a)] \geq \varepsilon_0, \quad \text{for all } y \in \mathbb{R}^n \text{ satisfying } |y - x| \leq \eta.$$

Since the maximum in (5.12) is strict, we have

$$\max_{y \in \mathbb{R}^n : |y-x|=\eta} (v(y) - \varphi(y)) = -\delta < 0. \quad (5.14)$$

Fix  $\varepsilon \in (0, \min(\varepsilon_0, \delta\beta)]$ . Take  $T > 0$  and set

$$\tau := \inf \{t \geq 0: |\bar{X}_t^x - x| \geq \eta\}, \quad \theta := \tau \wedge T,$$

where we recall that  $\bar{X}^x = (\bar{X}_t^x)_{t \geq 0}$  is the solution to equation (5.4). Notice that  $\tau$  and  $\theta$  are  $\bar{\mathbb{P}}^{W, \mu}$ -stopping times, with  $\theta$  taking values in  $[0, T]$ . Then, by formula (5.8), it follows that there exists  $\bar{\nu} \in \bar{\mathcal{V}}$  such that

$$v(x) - \frac{\varepsilon}{2\beta} \leq \bar{\mathbb{E}}_T^{\bar{\nu}} \left[ \int_0^\theta e^{-\beta s} f(\bar{X}_s^x, \bar{I}_s) ds + e^{-\beta\theta} v(\bar{X}_\theta^x) \right].$$

By  $v \leq \varphi$ ,  $v(x) = \varphi(x)$ , as well as (5.14), we deduce that

$$-\frac{\varepsilon}{2\beta} \leq \bar{\mathbb{E}}_T^{\bar{\nu}} \left[ \int_0^\theta e^{-\beta s} f(\bar{X}_s^x, \bar{I}_s) ds + e^{-\beta\theta} \varphi(\bar{X}_\theta^x) - \varphi(x) - \delta e^{-\beta\theta} \mathbf{1}_{\{\tau \leq T\}} \right].$$

An application of Itô's formula to  $e^{-\beta s} \varphi(\bar{X}_s^x)$  between  $s = 0$  and  $s = \theta$ , yields

$$-\frac{\varepsilon}{2\beta} \leq -\bar{\mathbb{E}}_T^{\bar{\nu}} \left[ \int_0^\theta e^{-\beta s} (\beta \varphi(\bar{X}_s^x) - \mathcal{L}^{\bar{I}_s} \varphi(\bar{X}_s^x) - f(\bar{X}_s^x, \bar{I}_s)) ds + \delta e^{-\beta\theta} \mathbf{1}_{\{\tau \leq T\}} \right].$$

Now, we observe that, whenever  $0 \leq s \leq \theta$ ,

$$\beta \varphi(\bar{X}_s^x) - \mathcal{L}^{\bar{I}_s} \varphi(\bar{X}_s^x) - f(\bar{X}_s^x, \bar{I}_s) \geq \beta \varphi(\bar{X}_s^x) - \sup_{a \in A} [\mathcal{L}^a \varphi(\bar{X}_s^x) + f(\bar{X}_s^x, a)] \geq \varepsilon.$$

Therefore

$$\begin{aligned} -\frac{\varepsilon}{2\beta} &\leq -\bar{\mathbb{E}}_T^{\bar{\nu}} \left[ \int_0^\theta \varepsilon e^{-\beta s} ds + \delta e^{-\beta\theta} \mathbf{1}_{\{\tau \leq T\}} \right] \\ &= -\frac{\varepsilon}{\beta} + \bar{\mathbb{E}}_T^{\bar{\nu}} \left[ \left( \frac{\varepsilon}{\beta} - \delta \right) e^{-\beta\theta} \mathbf{1}_{\{\tau \leq T\}} + \frac{\varepsilon}{\beta} e^{-\beta\theta} \mathbf{1}_{\{\tau > T\}} \right] \\ &\leq -\frac{\varepsilon}{\beta} + \frac{\varepsilon}{\beta} \bar{\mathbb{E}}_T^{\bar{\nu}} [e^{-\beta\theta} \mathbf{1}_{\{\tau > T\}}] = -\frac{\varepsilon}{\beta} + \frac{\varepsilon}{\beta} \bar{\mathbb{E}}_T^{\bar{\nu}} [e^{-\beta T} \mathbf{1}_{\{\tau > T\}}] \\ &= -\frac{\varepsilon}{\beta} + \frac{\varepsilon}{\beta} e^{-\beta T} \bar{\mathbb{P}}_T^{\bar{\nu}}(\tau > T) \leq -\frac{\varepsilon}{\beta} + \frac{\varepsilon}{\beta} e^{-\beta T}. \end{aligned}$$

Letting  $T \rightarrow \infty$ , we find the contradiction  $-\varepsilon/(2\beta) \leq -\varepsilon/\beta$ .

**Step II.** *Viscosity supersolution property.* Take  $x \in \mathbb{R}^n$  and  $\varphi \in C^2(\mathbb{R}^n)$  such that (5.13) holds. Given  $\eta > 0$ , we define

$$\tau := \inf \{t \geq 0: |\bar{X}_t^x - x| \geq \eta\}, \quad \theta_m := \tau \wedge \frac{1}{m},$$

for every  $m \in \mathbb{N} \setminus \{0\}$ . Then, by formula (5.8), we obtain (recall that the map  $\bar{\nu} \equiv 1$  belongs to  $\bar{\mathcal{V}}$  and, for such  $\bar{\nu}$ , we have that  $\bar{\mathbb{P}}_T^{\bar{\nu}}$  is  $\bar{\mathbb{P}}$ , for any  $T > 0$ )

$$v(x) \geq \bar{\mathbb{E}} \left[ \int_0^{\theta_m} e^{-\beta s} f(\bar{X}_s^x, \bar{I}_s) ds + e^{-\beta\theta_m} v(\bar{X}_{\theta_m}^x) \right].$$

Since  $v \geq \varphi$  and  $v(x) = \varphi(x)$ , we obtain

$$0 \geq \mathbb{E} \left[ \int_0^{\theta_m} e^{-\beta s} f(\bar{X}_s^x, \bar{I}_s) ds + e^{-\beta \theta_m} \varphi(\bar{X}_{\theta_m}^x) - \varphi(x) \right].$$

Applying Itô's formula to  $e^{-\beta s} \varphi(\bar{X}_s^x)$  between  $s = 0$  and  $s = \theta_m$ , and dividing by  $1/m$ , we find

$$0 \geq -\mathbb{E} \left[ \frac{1}{1/m} \int_0^{\theta_m} e^{-\beta s} (\beta \varphi(\bar{X}_s^x) - \mathcal{L}^{\bar{I}_s} \varphi(\bar{X}_s^x) - f(\bar{X}_s^x, \bar{I}_s)) ds \right]. \quad (5.15)$$

Now, notice that, for  $\bar{\mathbb{P}}$ -a.e.  $\bar{\omega} \in \bar{\Omega}$ , there exists  $M(\bar{\omega}) \in \mathbb{N}$  such that  $\theta_m(\bar{\omega}) = 1/m$  for any  $m \geq M(\bar{\omega})$ . Moreover,  $\bar{I}_s$  converges  $\bar{\mathbb{P}}$ -a.s. to  $a_0$  as  $s \searrow 0$ . In addition, we recall the following standard result:

$$\mathbb{E} \left[ \sup_{s \in [0, T]} |\bar{X}_s^x - x| \right] \xrightarrow{T \rightarrow 0} 0.$$

Therefore, up to a subsequence, we have

$$\xi_m := \frac{1}{1/m} \int_0^{\theta_m} e^{-\beta s} (\beta \varphi(\bar{X}_s^x) - \mathcal{L}^{\bar{I}_s} \varphi(\bar{X}_s^x) - f(\bar{X}_s^x, \bar{I}_s)) ds \xrightarrow[m \rightarrow \infty]{\bar{\mathbb{P}}\text{-a.s.}} \beta \varphi(x) - \mathcal{L}^{a_0} \varphi(x) - f(x, a_0).$$

Furthermore, since the time integral appearing in the definition of  $\xi_m$  is performed over the interval  $[0, \theta_m]$ , it follows that there exists a non-negative  $\xi \in L^1(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$  such that  $|\xi_m| \leq \xi$ ,  $\bar{\mathbb{P}}$ -a.s., for every  $m \in \mathbb{N}$ . Then, by the Lebesgue dominated convergence theorem, sending  $m \rightarrow \infty$  in (5.15), we find

$$\beta \varphi(x) \geq \mathcal{L}^{a_0} \varphi(x) + f(x, a_0).$$

Recalling that the deterministic point  $a_0 \in A$  fixed at the beginning of Section 3 was arbitrary, we conclude that

$$\beta \varphi(x) \geq \sup_{a \in A} [\mathcal{L}^a \varphi(x) + f(x, a)].$$

□

## APPENDIX A. PROOF OF LEMMA 2.1

*Proof.* We begin proving estimate (2.4) for  $p \geq 2$ . Given  $t \geq 0$ ,  $A \in \hat{\mathcal{F}}_t$ ,  $T \geq t$ , we obtain, from equation (2.3),

$$\begin{aligned} \hat{\mathbb{E}} \left[ \sup_{s \in [0, T]} |\hat{X}_s|^p \mathbf{1}_A \right]^{1/p} &\leq \hat{\mathbb{E}} \left[ \sup_{s \in [0, t]} |\hat{X}_s|^p \mathbf{1}_A \right]^{1/p} + \hat{\mathbb{E}} \left[ \sup_{s \in [t, T]} \left| \int_t^s b_u(\hat{X}, \gamma) du \right|^p \mathbf{1}_A \right]^{1/p} \\ &\quad + \hat{\mathbb{E}} \left[ \sup_{s \in [t, T]} \left| \int_t^s \sigma_u(\hat{X}, \gamma) d\hat{W}_u \right|^p \mathbf{1}_A \right]^{1/p}. \end{aligned} \quad (\text{A.1})$$

Notice that

$$\hat{\mathbb{E}} \left[ \sup_{s \in [t, T]} \left| \int_t^s b_u(\hat{X}, \gamma) du \right|^p \mathbf{1}_A \right]^{1/p} \leq \int_t^T \hat{\mathbb{E}} [ |b_u(\hat{X}, \gamma)|^p \mathbf{1}_A ]^{1/p} du.$$

Then, by assumption **(A)'**-(iii), we obtain

$$\begin{aligned} \hat{\mathbb{E}} \left[ \sup_{s \in [t, T]} \left| \int_t^s b_u(\hat{X}, \gamma) du \right|^p 1_A \right]^{1/p} &\leq L \int_t^T \hat{\mathbb{E}} \left[ \left( 1 + \sup_{s \in [0, u]} |\hat{X}_s| \right)^p 1_A \right]^{1/p} du \\ &\leq L(T-t) \hat{\mathbb{E}}[1_A]^{1/p} + L \int_t^T \hat{\mathbb{E}} \left[ \sup_{s \in [0, u]} |\hat{X}_s|^p 1_A \right]^{1/p} du. \end{aligned} \quad (\text{A.2})$$

On the other hand, by the Burkholder-Davis-Gundy inequality,

$$\begin{aligned} \hat{\mathbb{E}} \left[ \sup_{s \in [t, T]} \left| \int_t^s \sigma_u(\hat{X}, \gamma) d\hat{W}_u \right|^p 1_A \right]^{1/p} &\leq C_p \left( \hat{\mathbb{E}} \left[ \left( \int_t^T |\sigma_u(\hat{X}, \gamma)|^2 du \right)^{p/2} 1_A \right] \right)^{1/p} \\ &= C_p \left\| \int_t^T |\sigma_u(\hat{X}, \gamma)|^2 1_A du \right\|_{L^{p/2}(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})}^{1/2} \\ &\leq C_p \left( \int_t^T \hat{\mathbb{E}}[|\sigma_u(\hat{X}, \gamma)|^p 1_A]^{2/p} du \right)^{1/2}, \end{aligned}$$

where  $C_p > 0$  is the constant of the Burkholder-Davis-Gundy inequality. By assumption **(A)'**-(iii), we find

$$\begin{aligned} \hat{\mathbb{E}} \left[ \sup_{s \in [t, T]} \left| \int_t^s \sigma_u(\hat{X}, \gamma) d\hat{W}_u \right|^p 1_A \right]^{1/p} &\leq C_p L \left( \int_t^T \hat{\mathbb{E}} \left[ \left( 1 + \sup_{s \in [0, u]} |\hat{X}_s| \right)^p 1_A \right]^{2/p} du \right)^{1/2} \\ &\leq C_p L(T-t) \hat{\mathbb{E}}[1_A]^{1/p} + C_p L \left( \int_t^T \hat{\mathbb{E}} \left[ \sup_{s \in [0, u]} |\hat{X}_s|^p 1_A \right]^{2/p} du \right)^{1/2}. \end{aligned}$$

Plugging this latter estimate and (A.2) into (A.1), we obtain

$$\begin{aligned} \hat{\mathbb{E}} \left[ \sup_{s \in [0, T]} |\hat{X}_s|^p 1_A \right]^{1/p} &\leq \hat{\mathbb{E}} \left[ \sup_{s \in [0, t]} |\hat{X}_s|^p 1_A \right]^{1/p} + L \int_t^T \hat{\mathbb{E}} \left[ \sup_{s \in [0, u]} |\hat{X}_s|^p 1_A \right]^{1/p} du \\ &\quad + (1 + C_p) L(T-t) \hat{\mathbb{E}}[1_A]^{1/p} + C_p L \left( \int_t^T \hat{\mathbb{E}} \left[ \sup_{s \in [0, u]} |\hat{X}_s|^p 1_A \right]^{2/p} du \right)^{1/2}. \end{aligned} \quad (\text{A.3})$$

Taking the square of both sides of (A.3), and using the inequality  $(x_1 + x_2 + x_3 + x_4)^2 \leq 4(x_1^2 + x_2^2 + x_3^2 + x_4^2)$ , valid for any  $x_1, x_2, x_3, x_4 \in \mathbb{R}$ , we find

$$\begin{aligned} \hat{\mathbb{E}} \left[ \sup_{s \in [0, T]} |\hat{X}_s|^p 1_A \right]^{2/p} &\leq 4 \hat{\mathbb{E}} \left[ \sup_{s \in [0, t]} |\hat{X}_s|^p 1_A \right]^{2/p} + 4L^2 \left( \int_t^T \hat{\mathbb{E}} \left[ \sup_{s \in [0, u]} |\hat{X}_s|^p 1_A \right]^{1/p} du \right)^2 \\ &\quad + 4(1 + C_p)^2 L^2 (T-t)^2 \hat{\mathbb{E}}[1_A]^{2/p} + 4C_p^2 L^2 \int_t^T \hat{\mathbb{E}} \left[ \sup_{s \in [0, u]} |\hat{X}_s|^p 1_A \right]^{2/p} du. \end{aligned} \quad (\text{A.4})$$

Now, consider the nonnegative function  $v: [t, \infty) \rightarrow [0, \infty)$  given by

$$v(T) := \hat{\mathbb{E}} \left[ \sup_{s \in [0, T]} |\hat{X}_s|^p 1_A \right]^{2/p}, \quad \text{for all } T \geq t.$$

Then, rewriting inequality (A.4) in terms of  $v$ , we obtain

$$\begin{aligned} v(T) &\leq 4 \hat{\mathbb{E}} \left[ \sup_{s \in [0, t]} |\hat{X}_s|^p 1_A \right]^{2/p} + 4(1 + C_p)^2 L^2 (T - t)^2 \hat{\mathbb{E}}[1_A]^{2/p} + 4L^2 \left( \int_t^T \sqrt{v(u)} \, du \right)^2 \\ &\quad + 4C_p^2 L^2 \int_t^T v(u) \, du. \end{aligned} \tag{A.5}$$

Multiplying both sides of (A.5) by  $e^{-(T-t)}$ :

$$\begin{aligned} e^{-(T-t)} v(T) &\leq 4 \hat{\mathbb{E}} \left[ \sup_{s \in [0, t]} |\hat{X}_s|^p 1_A \right]^{2/p} + 4(1 + C_p)^2 L^2 (T - t)^2 \hat{\mathbb{E}}[1_A]^{2/p} \\ &\quad + 4L^2 \left( \int_t^T e^{-(T-t)/2} \sqrt{v(u)} \, du \right)^2 + 4C_p^2 L^2 \int_t^T e^{-(T-t)} v(u) \, du, \end{aligned} \tag{A.6}$$

where we have used the inequality  $e^{-(T-t)} \leq 1$ . Now, notice that

$$\begin{aligned} \left( \int_t^T e^{-(T-t)/2} \sqrt{v(u)} \, du \right)^2 &= \left( \int_t^T e^{-(T-u)/2} (e^{-(u-t)} v(u))^{1/2} \, du \right)^2 \\ &\leq \int_t^T e^{-(T-u)} \, du \int_t^T e^{-(u-t)} v(u) \, du \leq \int_t^T e^{-(u-t)} v(u) \, du \end{aligned}$$

and

$$\int_t^T e^{-(T-t)} v(u) \, du \leq \int_t^T e^{-(u-t)} v(u) \, du.$$

Therefore, from inequality (A.6), we get

$$\begin{aligned} e^{-(T-t)} v(T) &\leq 4 \hat{\mathbb{E}} \left[ \sup_{s \in [0, t]} |\hat{X}_s|^p 1_A \right]^{2/p} + 4(1 + C_p)^2 L^2 (T - t)^2 \hat{\mathbb{E}}[1_A]^{2/p} \\ &\quad + 4(1 + C_p^2) L^2 \int_t^T e^{-(u-t)} v(u) \, du. \end{aligned}$$

An application of Gronwall's lemma to the function  $u \mapsto e^{-(u-t)} v(u)$ , yields

$$e^{-(T-t)} v(T) \leq 4 \left( \hat{\mathbb{E}} \left[ \sup_{s \in [0, t]} |\hat{X}_s|^p 1_A \right]^{2/p} + (1 + C_p)^2 L^2 (T - t)^2 \hat{\mathbb{E}}[1_A]^{2/p} \right) e^{4(1 + C_p^2) L^2 (T-t)}.$$

Then, noting that  $(T - t)^2 \leq e^{T-t}$ , we end up with

$$\hat{\mathbb{E}} \left[ \sup_{s \in [0, T]} |\hat{X}_s|^p 1_A \right]^{2/p} \leq 4 \left( \hat{\mathbb{E}} \left[ \sup_{s \in [0, t]} |\hat{X}_s|^p 1_A \right]^{2/p} + (1 + C_p)^2 L^2 \hat{\mathbb{E}}[1_A]^{2/p} \right) e^{(1+4(1+C_p^2)L^2)(T-t)},$$

from which we find (using the inequality  $(a + b)^{p/2} \leq 2^{p/2-1}(a^{p/2} + b^{p/2})$ , valid for any  $a, b \geq 0$ )

$$\begin{aligned} \hat{\mathbb{E}} \left[ \sup_{s \in [0, T]} |\hat{X}_s|^p 1_A \right] &\leq 2^{p+p/2-1} \left( \hat{\mathbb{E}} \left[ \sup_{s \in [0, t]} |\hat{X}_s|^p 1_A \right] + (1 + C_p)^p L^p \hat{\mathbb{E}}[1_A] \right) e^{\frac{p}{2}(1+4(1+C_p^2)L^2)(T-t)} \\ &\leq \bar{C}_{p,L} \hat{\mathbb{E}} \left[ \left( 1 + \sup_{s \in [0, t]} |\hat{X}_s|^p \right) 1_A \right] e^{\bar{\beta}_{p,L}(T-t)}, \end{aligned}$$

with (for  $p \geq 2$ )

$$\bar{C}_{p,L} := 2^{p+p/2-1} \max(1, (1 + C_p)^p L^p), \quad \bar{\beta}_{p,L} := \frac{p}{2}(1 + 4(1 + C_p^2)L^2).$$

From the arbitrariness of  $A \in \hat{\mathcal{F}}_t$ , we conclude

$$\hat{\mathbb{E}} \left[ \sup_{s \in [0, T]} |\hat{X}_s|^p \middle| \hat{\mathcal{F}}_t \right] \leq \bar{C}_{p,L} \left( 1 + \sup_{s \in [0, t]} |\hat{X}_s|^p \right) e^{\bar{\beta}_{p,L}(T-t)}, \quad \hat{\mathbb{P}}\text{-a.s.} \quad (\text{A.7})$$

which yields estimate (2.4) for  $p \geq 2$

Finally, when  $p \in (0, 2)$ , we have, by Jensen's inequality,

$$\hat{\mathbb{E}} \left[ \sup_{s \in [0, T]} |\hat{X}_s|^p \middle| \hat{\mathcal{F}}_t \right]^{2/p} \leq \hat{\mathbb{E}} \left[ \sup_{s \in [0, T]} |\hat{X}_s|^2 \middle| \hat{\mathcal{F}}_t \right].$$

Then, by estimate (A.7) with  $p = 2$ , we obtain

$$\begin{aligned} \hat{\mathbb{E}} \left[ \sup_{s \in [0, T]} |\hat{X}_s|^p \middle| \hat{\mathcal{F}}_t \right] &\leq \bar{C}_{2,L}^{p/2} \left( 1 + \sup_{s \in [0, t]} |\hat{X}_s|^2 \right)^{p/2} e^{\frac{p}{2}\bar{\beta}_{2,L}(T-t)} \\ &\leq \bar{C}_{2,L}^{p/2} \left( 1 + \sup_{s \in [0, t]} |\hat{X}_s|^p \right) e^{\frac{p}{2}\bar{\beta}_{2,L}(T-t)}, \end{aligned}$$

which yields estimate (2.4) for  $p \in (0, 2)$ , with

$$\bar{C}_{p,L} := \bar{C}_{2,L}^{p/2} = 2^p \max(1, (1 + C_2)^p L^p), \quad \bar{\beta}_{p,L} := \frac{p}{2}\bar{\beta}_{2,L} = \frac{p}{2}(1 + 4(1 + C_2^2)L^2).$$

□

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