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CONSTRAINED NULL CONTROLLABILITY FOR DISTRIBUTED SYSTEMS AND APPLICATIONS TO HYPERBOLIC-LIKE EQUATIONS

Larbi Berrahmoune*

Abstract. We consider linear control systems of the form $y'(t) = Ay(t) + Bu(t)$ on a Hilbert space Y. We suppose that the control operator B is bounded from the control space U to a larger extrapolation space containing Y. The aim is to study the null controllability in the case where the control u is constrained to lie in a bounded subset $\Gamma \subset U$. We obtain local constrained controllability properties. When $(e^{tA})_{t\in\mathbb{R}}$ is a group of isometries, we establish necessary conditions and sufficient ones for global constrained controllability. Moreover, when the constraint set Γ contains the origin in its interior, the local constrained property turns out to be equivalent to a dual observability inequality of L^1 type with respect to the time variable. In this setting, the study is focused on hyperbolic-like systems which can be reduced to a second order evolution equation. Furthermore, we treat the problem of determining a steering control for general constraint set Γ in nonsmooth convex analysis context. In the case where Γ contains the origin in its interior, a steering control can be obtained by minimizing a convenient smooth convex functional. Applications to the wave equation and Euler-Bernoulli beams are presented.

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1. Introduction, preliminary results

1.1. Problem formulation and reference to the literature

In this paper, we are concerned with the question of null controllability for linear infinite dimensional distributed control systems such as

$$\begin{cases} y'(t) = Ay(t) + Bu(t), \\ y(0) = y_0. \end{cases} \tag{1.1}$$

Our concern will focus on the situation where the control u is constrained to take on values in a preassigned subset Γ of the control space U and the following issues will be treated: (a) Does there exist a control u steering the system to the origin at some time $T > 0$? (b) Assuming that such a control exists, how can it be characterized and what properties it has? (c) Otherwise, can we characterize an appropriate control which steers

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ANLIMAD, Département de Mathématiques, ENS, Université Mohammed V, BP 5118 Rabat, Morocco.

* Corresponding author: l3berrahmoune@gmail.com

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the system (1.1) as close as possible to 0 at time T? Unlike the unconstrained controllability which has been well understood, few works have been devoted to the realistic case of constrained controllability. This problem is important since all real world applications involve actuators with amplitude and rate limitations so that the control is constrained to take on values in a prespecified bounded subset. Classical results are by now well known in the finite dimensional case. They depend on the assumptions on Γ and they assert that controllability is equivalent, in a certain sense, to unconstrained controllability plus an additional condition due to the constraint [9, 19, 20]. A major advance on the subject has been made in [6, 7, 32] where another point of view is presented. The approach may already be found in a similar form in [16]. It is based on a finite dimensional selection theorem [13] and it amounts to an optimization problem. The resulting constrained controllability conditions can be reduced to integral conditions in terms of the nontrivial solutions of the adjoint equation. In infinite dimension there are pioneering attacks on the constrained controllability problem for special control systems. See, for instance, the references [14, 25, 26] which are concerned with the wave equation with distributed control. In [14] the theory of sine and cosine operators is applied to show that any state in a suitably defined state space of a system governed by a second order differential equation in a Hilbert space may be controlled to the origin in finite time by means of controls whose values range in the unit ball of the control space. In [25] it is established that if there is a fixed time $T_0 > 0$, such that an initial state in the controlled space may be steered to the origin within $[0, T_0]$ by means of an unconstrained control, it may also be transferred to the origin at finite time T by a control u such that $\|u\|_{L^p(0,T;U)}$ is arbitrarily small. From this result, global controllability properties have been deduced for the wave equation with respect to appropriate Sobolev spaces even if one applies controls subject to rather arbitrary pointwise constraints [26]. In the abstract setting, constrained controllability of infinite dimensional systems has been treated, to the best of our knowledge, first in [33]. In this reference, the necessary and sufficient conditions for local controllability which have been obtained can be considered as an extension to the infinite-dimensional case of classical results, proved in [9, 19]. We mention also the approach developed in [1, 27] based on the results established in [32] for multivariable systems. Furthermore, an equivalence between local constrained null controllability and unconstrained null controllability is shown in [10] by adapting the general theory of observation and control developed in [12]. While the references above deal with state and control spaces which are supposed to be reflexive Banach spaces, some attention has been addressed to the delicate question where this assumption does not hold. See, for instance, [35]. Moreover, in the monograph [15], similar subtle questions on constrained controllability are treated in the context of time optimal and norm optimal problems for parabolic-like systems. In this setting, it is clearly pointed out the fact that some bounds on the control lead unavoidably to non-reflexive Banach spaces. Note that in all these references, it is supposed that the control operator B is bounded. Our aim is to treat the constrained null controllability with eventually unbounded control operator in a general Hilbert space framework. As we shall see in the applications, such systems with B unbounded appear naturally when we model boundary control or control exercised through a lower dimensional manifold (curve, point) in the interior of the domain. Beside the treatment of the unbounded control operator case, the paper presents the following main contributions.

(i) The problem of constrained null controllability is presented in natural abstract framework which encompasses various types of systems (multivariable and distributed systems).
(ii) The constraint control set Γ is general and the usual assumption that $0 \in \Gamma$ is dispensed with.
(iii) Under the assumption that Γ contains 0 in its interior, we recover the fact that the constrained controllability can be reduced to some kind of observability relative to the uncontrolled adjoint system. We shall also see that for hyperbolic-like systems, this observability is equivalent to the classical observability characterizing the unconstrained controllability provided that the control operator B is bounded. On the contrary, we shall see that when the control operator B is unbounded, the observability characterizing the local constrained controllability fails to be true even when the unconstrained controllability is verified.
(iv) Exploiting the properties dealing with the constrained null controllability, we present some methods characterizing the appropriate control solution of one of the problems (a), (b), and (c) stated above. By using nonsmooth convex analysis tools, we shall see that for general constraint set Γ, such a control satisfies a maximum principle property.

(v) In the case where Γ contains 0 in its interior, we present a variational approach in order to build the steering control. The latter is obtained by minimizing an appropriate convex functional in the context of smooth convex analysis. This approach has already been used when the control operator is bounded (see [8]). The novelty in the present paper is the fact that our convex functionals recover both the ones which have been implicitly used in the constrained control setting and the quadratic ones as well. Recall that the minimization of the latter leads to the unconstrained steering control by the so-called Hilbert Uniqueness Method (HUM) [24, 37].

The plan of the paper is as follows. In the remaining part of this section, we introduce our general abstract framework. The notion of admissible unbounded control operator B is given. The corresponding notion of admissible observation operator which is needed for its adjoint B^* is deduced by duality. We recall also the main results on unconstrained null controllability given in observability inequality form and we precise the various notions relative to the constrained null controllability. Moreover, we present a variational characterization of any control steering the system from a given initial state to the origin. In Section 2 we treat problem (a) and we extend the existing results to general constraint set Γ with unbounded admissible control operator. When the origin is in the interior of Γ, it turns out that the constrained controllability is equivalent to an appropriate observability inequality. This issue is focused on second order conservative systems. The third section deals with the application of the results obtained in the preceding section to specific partial differential equations (PDEs) including wave equations with distributed and boundary control, and an Euler-Bernoulli model subject to a pointwise actuator. The problems (b) and (c) are addressed in the fourth section. For general constraint set Γ, by nonsmooth analysis tools, the steering control will be characterized either by a maximum principle, or by a min-max strategy. As for the case where Γ contains 0 in its interior, we shall obtain the steering control by minimizing an appropriate convex functional in smooth analysis context. Finally, the section is ended by applying the results to the PDEs presented in Section 3.

1.2. Basic concepts and auxiliary results

Let Y, U be two Hilbert spaces denoting the state space and the control space respectively. Also, let A be the infinitesimal generator of a linear C_0-semigroup on Y denoted by $(e^{tA})_{t\geq 0}$. Finally, let B denote the linear control operator which may be unbounded in the sense that it is bounded from U into a space larger than the state space Y as follows. We define the space Y_1 to be $D(A)$ with the graph norm $\|.\|_1$ given by

$$\|y\|_1 = \|(\beta I - A)y\|_Y , \tag{1.2}$$

for some $\beta \in \rho(A)$ and the space Y_{-1} to be the completion of Y with respect to the norm

$$\|y\|_{-1} = \left\|(\beta I - A)^{-1}y\right\|_Y . \tag{1.3}$$

Using the graph norm topology in $D(A^*)$, we can as well define the analogue spaces $(Y')_1$ and $(Y')_{-1}$. It is well known that

$$D(A)' = (Y_1)' = (Y)'_{-1},\ (Y_{-1})' = (Y')_1 = D(A^*).$$

Exploiting these facts, we note that the duality product $\langle .,.\rangle_{Y',Y}$ can be expressed by the duality product $\langle .,.\rangle_{(Y')_1,Y_{-1}}$ as follows

$$\langle y', y\rangle_{Y',Y} = \left\langle(\overline{\beta} I - A^*)^{-1}y', (\beta I - A)y\right\rangle_{(Y')_1,Y_{-1}} ,\ y\in Y,\ y'\in Y',$$

and

$$\langle y', y\rangle_{(Y')_1, Y_{-1}} = \left\langle (\overline{\beta}I - A^*)y', (\beta I - A)^{-1}y\right\rangle_{Y', Y}, \; y \in Y_{-1}, \; y' \in (Y')_1,$$

for some $\beta \in \rho(A)$. It is clear that both duality pairings coincide when $y \in Y_1$ and $y' \in (Y)'_1$ so that they both will be denoted by $\langle ., . \rangle$ throughout this paper. We consider the abstract control system (1.1) where $u \in L^2_{\text{loc}}(0, \infty; U)$, $y_0 \in Y$ and B is bounded from U to Y_{-1}. Considering formally the solution given by the variation of constant formula

$$y(t) = e^{tA}y_0 + \int_0^t e^{(t-s)A}Bu(s)\mathrm{d}s, \tag{1.4}$$

we introduce for any $T > 0$ the operator L_T defined by

$$L_T u = \int_0^T e^{(T-t)A}Bu(t)\mathrm{d}s. \tag{1.5}$$

It is clear that $L_T \in \mathcal{L}(L^2(0, T; U), D(A^*)')$. Throughout this paper, we suppose that the control operator B is admissible in the sense that it satisfies for some (and hence any) $T > 0$ and for any $u \in L^2(0, T; U)$ one of the following equivalent admissibility conditions

$$L_T u = \int_0^T e^{(T-t)A}Bu(t)\mathrm{d}t \in Y, \tag{1.6}$$

$$\|L_T u\|_Y = \left\| \int_0^T e^{(T-t)A}Bu(t)\mathrm{d}t \right\|_Y \leq C_T \|u\|_{L^2(0,T;U)}, \tag{1.7}$$

for some positive constant C_T. It is well known that (1.6) or (1.7) ensures the existence of a unique solution for (1.1) given by (1.2) in the class $y \in C(0, T; Y)$ [34]. On the other hand, identifying U with its dual, we have $B^* \in \mathcal{L}((Y')_1, U)$. Then, considering B^* as observation operator with respect to the space Y', we introduce the dual admissibility condition

$$\int_0^T \left\| B^* e^{tA^*}\varphi_0 \right\|_U^2 \mathrm{d}t \leq C_T \|\varphi_0\|_{Y'}^2 \text{ for all } \varphi_0 \in D(A^*), \tag{1.8}$$

for some positive constant C_T. Note that the expression

$$z(t) = B^* e^{tA^*}\varphi_0, \; t > 0, \tag{1.9}$$

is well defined for $\varphi_0 \in D(A^*)$. The admissibility condition (1.8) implies that for any $T > 0$, the expression in (1.9) considered as a function of the variable φ_0, can be extended to a linear continuous operator from Y' to $L^2(0, T; U)$. It is well known that the admissibility of B as control operator with respect to Y is equivalent to the dual admissibility of B^* as observation operator in Y' [34, 36]. Moreover, it is easy to see that the adjoint operator $L_T^* : D(A^*) \to L^2(0; T; U)$ is given by

$$L_T^*\varphi_0(t) = B^* e^{(T-t)A^*}\varphi_0, \;\; 0 < t < T. \tag{1.10}$$

Then, (1.8) means that L_T^* admits a continuous extension, still denoted by L_T^*, from Y' to $L^2(0, T; U)$.

In particular, given a prescribed time $T > 0$, we shall be concerned with the final state

$$y_u(T) = e^{TA}y_0 + \int_0^T e^{(T-t)A}Bu(t)\mathrm{d}t \tag{1.11}$$

and the following space of reachable states from y_0 in time T

$$R(T, y_0) = \left\{y_u(T) \,|\, u \in L^2(0, T; U)\right\}. \tag{1.12}$$

Then the notion of unconstrained null controllability is defined as follows.

Definition 1.1. The system (1.1) (or the pair (A, B)) is null controllable in time T if given any initial state y_0, the set of reachable states $R(T, y_0)$ contains 0.

Combining the admissibility of the operators B and B^* as control and observation operators respectively, we state the following variational formulation of the fact that the control $u \in L^2_{\mathrm{loc}}(0, \infty; U)$ steers the system to the origin.

Proposition 1.2. *The origin is reached from y_0 in time T by a control $u \in L^2(0, T; U)$ if and only if,*

$$\int_0^T \left\langle u(t), B^*e^{(T-t)A^*}\varphi_0\right\rangle_U \mathrm{d}t + \left\langle e^{TA^*}\varphi_0, y_0\right\rangle = 0 \textit{ for all } \varphi_0 \in Y'. \tag{1.13}$$

Proof. Beside the controlled equation (1.1), let $\varphi_0 \in D(A^*)$ and consider $\varphi(t) = e^{(T-t)A^*}\varphi_0$ solution of the adjoint equation

$$\varphi'(t) + A^*\varphi(t) = 0,\ \varphi(T) = \varphi_0.$$

Clearly, $\varphi(t) \in D(A^*)$ for all $t \geq 0$ so that we can multiply (1.1) by $\varphi(t)$ by using the extended duality product $\langle ., .\rangle$. We obtain by integration by parts

$$\int_0^T \left\langle Bu(t), e^{(T-t)A^*}\varphi_0\right\rangle \mathrm{d}t + \left\langle e^{TA^*}\varphi_0, y_0\right\rangle = 0$$

$$= \int_0^T \left\langle u(t), B^*e^{(T-t)A^*}\varphi_0\right\rangle_U \mathrm{d}t + \left\langle e^{TA^*}\varphi_0, y_0\right\rangle.$$

Hence, (1.13) holds for all $\varphi_0 \in D(A^*)$. Then exploiting the admissibility condition (1.8) and using a density argument, we conclude that (1.13) is verified for all $\varphi_0 \in Y'$. □

The following remarks precise the fact that the notion of controllability can be related to an observability inequality. We shall obtain a similar property in the context of constrained control.

Remark 1.3. It is well-known that null controllability is equivalent to the following observability inequality (see [34], Thm. 11.2.1, p. 357)

$$\int_0^T \left\|B^*e^{(T-t)A^*}\varphi_0\right\|_U^2 \mathrm{d}t \geq c_2 \left\|e^{TA^*}\varphi_0\right\|_{Y'}^2 \text{ for all } \varphi_0 \in Y', \tag{1.14}$$

for some positive constant c_2. When $(e^{tA})_{t\in\mathbb{R}}$ is a group of isometries, then (1.14) is equivalent to the following inequalities

$$\int_0^T \left\| B^* e^{(T-t)A^*} \varphi_0 \right\|_U^2 \mathrm{d}t \geq c_2 \left\| \varphi_0 \right\|_{Y'}^2 \text{ for all } \varphi_0 \in Y', \tag{1.15}$$

$$\inf_{\|\varphi_0\|=1} \int_0^T \left\| B^* e^{(T-t)A^*} \varphi_0 \right\|_U^2 \mathrm{d}t \geq c_2. \tag{1.16}$$

Let us precise the framework of constrained controllability related to the prespecified subset $\Gamma \subset U$ introduced above. Let $\widetilde{\Gamma}$ denote the set of admissible control functions u in $L^2_{\mathrm{loc}}(0,\infty;U)$ such that $u(t) \in \Gamma$ almost everywhere in $(0,\infty)$. Also, $\widetilde{\Gamma}_T$ will denote the restriction of $\widetilde{\Gamma}$ to the interval $[0,T]$. Any control function in $\widetilde{\Gamma}$ or $\widetilde{\Gamma}_T$ will be referred to as Γ-admissible control. We introduce the following set of reachable states with Γ-admissible controls

$$R_\Gamma(T,y_0) = \left\{ y_u(T) \,|\, u \in \widetilde{\Gamma}_T \right\}. \tag{1.17}$$

Then we define the constrained null controllability notions as follows.

Definition 1.4. *The system (1.1) is Γ-null controllable in time T at y_0 if there exists a control $u \in \widetilde{\Gamma}_T$ such that $y_u(T) = 0$.*

Definition 1.5. *The system (1.1) is locally Γ-null controllable in time T if there exists an open subset $V \subset Y$, containing the origin, such that (1.1) is Γ-null controllable in time T at any $y_0 \in V$.*

Following the framework used in the finite dimensional systems literature, we can enlarge the notion of Γ-null controllability by varying the final time T.

Definition 1.6. *The system (1.1) is said to be globally Γ-null controllable in finite time if for any initial state $y_0 \in Y$, there exists a final time $T = T(y_0)$ in which the system (1.1) is Γ-null controllable at y_0.*

Let us mention the following results which are available in the finite dimensional systems literature in the case where Γ is bounded and contains 0. The system (1.1) is locally Γ-null controllable in time T if, and only if, the pair (A,B) is null controllable. The system (1.1) is globally Γ-null controllable in finite time if the pair (A,B) is null controllable and the eigenvalues of the matrix A has negative real part (see, for instance, [20]).

2. Null controllability with constraint

2.1. Measurable selection theorem

In order to describe our necessary and sufficient conditions for Γ-null controllability, we make use of the support functional $S_\Gamma : U \to \mathbb{R}$ on Γ which for any $u \in U$ is given by

$$S_\Gamma(u) = \sup_{v\in\Gamma} \langle u,v \rangle_U . \qquad (2.1)2.1)$$

Taking into account the admissibility condition (1.8), the function

$$t \to h(t) = S_\Gamma\left(B^* e^{(T-t)A^*} \varphi_0 \right) \tag{2.2}$$

is well defined in $L^1_{\mathrm{loc}}(0,\infty)$ for any $\varphi_0 \in Y'$. Moreover, we shall use a general version of the selection theorem which will be valid in the infinite dimensional context. Note that the selection theorem stated in [13] deals with finite dimensional spaces and it has been exploited for constrained controllabilty purposes in [32]. Many Versions of the selection theorem are disseminated in various references. See, for instance, [2, 4, 17].

Lemma 2.1. *Let Γ be a closed bounded convex subset of U. Then, for any $\varphi_0 \in Y'$, there exists a measurable function $\widehat{v}$ such that $\widehat{v} \in \widetilde{\Gamma}_T$ and*

$$\int_0^T h(t)\mathrm{d}t = \int_0^T \left\langle \widehat{v}(t), B^* e^{(T-t)A^*}\varphi_0 \right\rangle_U \mathrm{d}t. \tag{2.3}$$

Proof. Let us consider the functional $g : (0,\infty) \times \Gamma \to \mathbb{R}$ given by

$$g(t,u) = \left\langle u, B^* e^{(T-t)A^*}\varphi_0 \right\rangle_U. \tag{2.4}$$

We introduce also the constant set-valued function $f : t \geq 0 \rightsquigarrow \Gamma$ given by

$$f(t) = \Gamma. \tag{2.5}$$

Clearly, f is measurable with closed nonempty images. For each $u \in U$, the real valued function $t \to g(t,u)$ is measurable. Furthermore, by the admissibility condition (1.8) we have after extension to Y', $B^* e^{(T-t)A^*}\varphi_0 \in U$ for almost all $t > 0$ so that the function $u \to g(t,u)$ is continuous on Γ. Hence, the function g is a Caratheodory map. On the other hand, the function h introduced in (2.2) is a measurable function. Since Γ is weakly compact, we have for almost all $0 < t < T$

$$h(t) = \left\langle u_t, B^* e^{(T-t)A^*}\varphi_0 \right\rangle_U, \tag{2.6}$$

for some $u_t \in \Gamma$. Thus, $h(t) \in g(t, f(t))$ for almost all $t \geq 0$. By using Filippov's theorem in [4], page 316, it follows that there exists a measurable function $\widehat{v} : (0,\infty) \to \Gamma$ such that $\widehat{v}(t) \in f(t)$ almost everywhere, and

$$h(t) = g(t, \widehat{v}(t)) \text{ a.e on } (0,T). \tag{2.7}$$

Obviously, $\widehat{v} \in \widetilde{\Gamma}_T$ and we have

$$\begin{aligned} \int_0^T h(t)\mathrm{d}t &= \int_0^T S_\Gamma \left(B^* e^{(T-t)A^*}\varphi_0\right) \mathrm{d}t \\ &= \int_0^T \left\langle \widehat{v}(t), B^* e^{(T-t)A^*}\varphi_0 \right\rangle_U \mathrm{d}t. \end{aligned}$$

This completes the proof of the lemma. □

2.2. Null controllability under general constraint set

The measurable selection theorem formulated in Lemma 2.1 enables us to establish the following Γ-null controllability result under general constraint set Γ. Our formulation is based on the functional $J_\Gamma : Y' \to \mathbb{R}$ defined by

$$J_\Gamma(\varphi_0) = \int_0^T S_\Gamma\left(B^* e^{(T-t)A^*}\varphi_0\right) \mathrm{d}t + \left\langle e^{TA^*}\varphi_0, y_0 \right\rangle. \tag{2.8}$$

Let us anticipate by noting that this functional and some of its variants will play a crucial role for the steering control question treated in the fourth section below.

Theorem 2.2. *Let Γ be a closed bounded convex subset of U. Suppose that $Bu = 0$ for some $u \in \Gamma$. Then, the system (1.1) is Γ-null controllable at y_0 if and only if, for some $T = T(y_0) > 0$,*

$$J_\Gamma(\varphi_0) \geq 0 \tag{2.9}$$

for all $\varphi_0 \in Y'$.

Proof. From the admissibility condition (1.8), it is easy to deduce that $R_\Gamma(T, y_0)$ is a closed bounded convex subset of Y. On the other hand, the system (1.1) is Γ-null controllable at y_0 in time T if and only if $0 \in R_\Gamma(T, y_0)$. Then we use a corollary of Hahn-Banach separation theorem stated in terms of inequality between appropriate support functionals. Let $S_R : Y' \to \mathbb{R}$ and $S_0 : Y' \to \mathbb{R}$ denote the support functionals associated to $R_\Gamma(T, y_0)$ and $\{0\}$ respectively. Reducing the problem to the inclusion $\{0\} \subset R_\Gamma(T, y_0)$ and using Proposition of [11], page 43, we obtain that the system (1.1) is Γ-null controllable at y_0 in time T if and only if, for any $\varphi_0 \in Y'$

$$S_R(\varphi_0) \geq S_0(\varphi_0). \tag{2.10}$$

This inequality means that

$$\left\langle \varphi_0, e^{TA} y_0 \right\rangle + \sup_{u \in \widetilde{\Gamma}_T} \left\{ \left\langle \varphi_0, \int_0^T e^{(T-t)A} Bu(t) \mathrm{d}t \right\rangle \right\} \geq 0,$$

and by using the admissibility condition (1.8), this amounts to

$$\left\langle \varphi_0, e^{TA} y_0 \right\rangle + \sup_{u \in \widetilde{\Gamma}_T} \left\{ \int_0^T \left\langle B^* e^{(T-t)A^*} \varphi_0, u(t) \right\rangle_U \mathrm{d}t \right\} \geq 0. \tag{2.11}$$

Consider the functional $\mathcal{F}_T : L^2(0, T; U) \to \mathbb{R}$ given by

$$\mathcal{F}_T(u) := \int_0^T \left\langle B^* e^{(T-t)A^*} \varphi_0, u(t) \right\rangle_U \mathrm{d}t. \tag{2.12}$$

The admissibility condition (1.8) implies that $\mathcal{F}_T$ is linear continuous on $L^2(0, T; U)$. Since $\widetilde{\Gamma}_T$ is weakly compact in $L^2(0, T; U)$, $\mathcal{F}_T$ attains its supremum on $\widetilde{\Gamma}_T$ so that

$$\sup_{u \in \widetilde{\Gamma}_T} \left\{ \int_0^T \left\langle B^* e^{(T-t)A^*} \varphi_0, u(t) \right\rangle_U \mathrm{d}t \right\} = \int_0^T \left\langle B^* e^{(T-t)A^*} \varphi_0, u^0(t) \right\rangle_U \mathrm{d}t \tag{2.13}$$

for some $u^0 \in \widetilde{\Gamma}_T$. Furthermore, clearly we have

$$\int_0^T S_\Gamma \left(B^* e^{(T-t)A^*} \varphi_0 \right) \mathrm{d}t \geq \int_0^T \left\langle B^* e^{(T-t)A^*} \varphi_0, u^0(t) \right\rangle_U \mathrm{d}t.$$

On the other hand, Lemma 2.1 yields for some $\widehat{v} \in \widetilde{\Gamma}_T$

$$\begin{aligned} \textstyle\int_0^T S_\Gamma \left(B^* e^{(T-t)A^*} \varphi_0 \right) \mathrm{d}t &= \textstyle\int_0^T \left\langle \widehat{v}(t), B^* e^{(T-t)A^*} \varphi_0 \right\rangle_U \mathrm{d}t \\ &\leq \textstyle\int_0^T \left\langle B^* e^{(T-t)A^*} \varphi_0, u^0(t) \right\rangle_U \mathrm{d}t, \end{aligned}$$

so that

$$\int_0^T S_\Gamma\left(B^* e^{(T-t)A^*}\varphi_0\right)\mathrm{d}t = \sup_{u\in\widetilde{\Gamma}_T}\left\{\left\langle \varphi_0, \int_0^T e^{(T-t)A}Bu(t)\mathrm{d}t\right\rangle\right\}. \tag{2.14}$$

This completes the proof of the theorem. □

Remark 2.3.

(i) Given any subset $\Lambda \in Y'$ containing the origin in its interior, it is easy to see by a positive homogeneity argument that the system (1.1) is Γ-null controllable at y_0 if and only if,

$$\inf_{\varphi_0\in\Lambda} J_\Gamma(\varphi_0) = 0.$$

(ii) By virtue of (1.10), the characterization property in Theorem 2.2 can be expressed by

$$\int_0^T S_\Gamma\left(L_T^*\varphi_0\right)\mathrm{d}t + \left\langle e^{TA^*}\varphi_0, y_0\right\rangle \geq 0 \text{ for all } \varphi_0 \in Y'. \tag{2.15}$$

This formulation will turn out to be important for second order systems in Section 2.4.

Theorem 2.4. *Suppose that the assumptions of Theorem 2.2 hold. Then the system (1.1) is Γ-null controllable at y_0 in time $T > 0$ provided that*

$$\inf_{\|\varphi_0\|_{Y'}=1}\int_0^T S_\Gamma\left(B^* e^{(T-t)A^*}\varphi_0\right)\mathrm{d}t \geq \left\|e^{TA}y_0\right\|_Y. \tag{2.16}$$

Moreover, if in addition $(e^{tA})_{t\in\mathbb{R}}$ is a unitary group on Y, then the system (1.1) is globally Γ-null controllable provided that

$$\sup_{T>0}\inf_{\|\varphi_0\|_{Y'}=1}\int_0^T S_\Gamma\left(B^* e^{(T-t)A^*}\varphi_0\right)\mathrm{d}t = \infty. \tag{2.17}$$

Proof. Suppose that (2.16) holds. Then for any φ_0 such that $\|\varphi_0\|_{Y'} = 1$, we have

$$\int_0^T S_\Gamma\left(B^* e^{(T-t)A^*}\varphi_0\right)\mathrm{d}t \geq \left\|e^{TA}y_0\right\|_Y \|\varphi_0\|_{Y'}$$

$$\geq \left\langle \varphi_0, e^{TA}y_0\right\rangle.$$

It follows from Theorem 2.2 that the system is Γ-null controllable at y_0 in time $T > 0$. On the other hand, if $(e^{tA})_{t\in\mathbb{R}}$ is a unitary group, then the equality (2.17) implies that for any initial state $y_0 \in Y$, there exists $T = T(y_0)$ such that

$$\inf_{\|\varphi_0\|=1}\int_0^T S_\Gamma\left(B^* e^{(T-t)A^*}\varphi_0\right)\mathrm{d}t \geq \left\|e^{TA}y_0\right\|_Y = \|y\|_Y,$$

so that y_0 can be steered to the origin by a Γ-admissible control. This completes the proof of the corollary. □

Theorem 2.5. *Suppose that the assumptions of Theorem 2.2 hold. Then the system (1.1) is locally Γ-null controllable in time $T > 0$ if and only if, for some $c > 0$*

$$\int_0^T S_\Gamma \left(B^* e^{tA^*} \varphi_0 \right) \mathrm{d}t \geq c \left\| e^{TA^*} \varphi_0 \right\|_{Y'} \tag{2.18}$$

for all $\varphi_0 \in Y'$.

Proof. From Theorem 2.2, the system (1.1) is locally Γ-null controllable in time $T > 0$ if and only if, for some $r > 0$ and any initial state satisfying $\|y_0\|_Y \leq r$, we have

$$\int_0^T S_\Gamma \left(B^* e^{(T-t)A^*} \varphi_0 \right) \mathrm{d}t + \left\langle \varphi_0, e^{TA} y_0 \right\rangle \geq 0, \ \forall \varphi_0 \in Y',$$

or, equivalently, for any $\varphi_0 \in Y'$

$$\int_0^T S_\Gamma \left(B^* e^{(T-t)A^*} \varphi_0 \right) \mathrm{d}t \geq \left\langle e^{TA^*} \varphi_0, y_0 \right\rangle$$

for all $y_0 \in Y$ satisfying $\|y_0\|_Y \leq r$. Clearly, this amounts to

$$\int_0^T S_\Gamma \left(B^* e^{tA^*} \varphi_0 \right) \mathrm{d}t \geq r \left\| e^{TA^*} \varphi_0 \right\|_{Y'} .$$

□

Remark 2.6. From the proof, it turns out that the constant c in (2.18) can be interpreted as the radius r of a closed ball in Y centered at the origin and contained in the set of initial states which can be steered to the origin in time T by Γ-admissible controls.

Theorem 2.7. *Suppose that the assumptions of Theorem 2.2 hold. Assume also that $(e^{tA})_{t\in\mathbb{R}}$ is a unitary group on Y. If the system (1.1) is globally Γ-null controllable, then*

$$\int_0^\infty S_\Gamma \left(B^* e^{-tA^*} \varphi_0 \right) \mathrm{d}t = \infty \tag{2.19}$$

for all $\varphi_0 \in Y'$, $\varphi_0 \neq 0$.

Proof. By contradiction, suppose that

$$\int_0^\infty S_\Gamma \left(B^* e^{-tA^*} \varphi_0 \right) \mathrm{d}t = c < \infty$$

for some $\varphi_0 \in Y'$ with $\varphi_0 \neq 0$. Let $\xi_0 \in Y$ such that

$$\langle \varphi_0, \xi_0 \rangle = \|\varphi_0\|^2_{Y'} ,$$

and consider $y_0 = -\frac{2c}{\|\varphi_0\|^2}\xi_0$ as initial condition. Let $T > 0$ be fixed and consider

$$\begin{aligned} J_\Gamma(e^{-TA^*}\varphi_0) &= \int_0^T S_\Gamma\left(B^* e^{(T-t)A^*}(e^{-TA^*}\varphi_0)\right)\mathrm{d}t + \left\langle e^{TA^*}(e^{-TA^*}\varphi_0), y_0\right\rangle \\ &= \int_0^T S_\Gamma\left(B^* e^{-tA^*}\varphi_0\right)\mathrm{d}t + \langle \varphi_0, y_0\rangle \\ &= \int_0^T S_\Gamma\left(B^* e^{-tA^*}\varphi_0\right)\mathrm{d}t - 2c \\ &\le c - 2c \le -c \end{aligned}$$

so that

$$J_\Gamma(e^{-TA^*}\varphi_0) \le -c < 0.$$

Thus the system (1.1) is not Γ-null controllable at y_0 in time T. This completes the proof of the theorem. □

Remark 2.8. Unlike the case of multivariable systems where the equalities

$$\inf_{\|\varphi_0\|=1} \int_0^\infty S_\Gamma\left(B^* e^{-tA^*}\varphi_0\right)\mathrm{d}t = \infty, \tag{2.20}$$

$$\int_0^\infty S_\Gamma\left(B^* e^{-tA^*}\varphi_0\right)\mathrm{d}t = \infty,\ \forall \varphi_0 \in Y',\ \varphi_0 \neq 0, \tag{2.21}$$

are equivalent, it turns out that the equivalence fails for distributed systems. Counter examples illustrating this fact can be found in [27].

Corollary 2.9. *Suppose that the assumptions of Theorem 2.7 hold. If the system (1.1) is locally Γ-null controllable in some time $T > 0$, then it is globally Γ-null controllable.*

Proof. From Theorem 2.4, it is sufficient to establish

$$\sup_{T>0} \inf_{\|\varphi_0\|_{Y'}=1} \int_0^T S_\Gamma(B^* e^{(T-t)A^*}\varphi_0)\mathrm{d}t = \infty.$$

To this end, we consider for $k \in \mathbb{N}^*$ the decomposition

$$\int_0^{kT} S_\Gamma\left(B^* e^{(kT-t)A^*}\varphi_0\right)\mathrm{d}t = \sum_{j=0}^{k-1} \int_{jT}^{(j+1)T} S_\Gamma\left(B^* e^{(kT-t)A^*}\varphi_0\right)\mathrm{d}t. \tag{2.22}$$

By an elementary change of variables, we get for all $j = 0, ..., k-1$

$$\begin{aligned} \int_{jT}^{(j+1)T} S_\Gamma\left(B^* e^{(kT-t)A^*}\varphi_0\right)\mathrm{d}t &= \int_0^T S_\Gamma\left(B^* e^{((k-j)T-t)A^*}\varphi_0\right)\mathrm{d}t \\ &= \int_0^T S_\Gamma\left(B^* e^{(T-t)A^*}(e^{(k-1-j)TA^*}\varphi_0)\right)\mathrm{d}t, \end{aligned} \tag{2.23}$$

so that

$$\sup_{k\in\mathbb{N}^*} \inf_{\|\varphi_0\|_{Y'}=1} \int_0^{kT} S_\Gamma\left(B^* e^{(kT-t)A^*}\varphi_0\right)\mathrm{d}t \ge \sum_{j=0}^{k-1} \inf_{\|\varphi_0\|_{Y'}=1} \int_0^T S_\Gamma\left(B^* e^{(T-t)A^*}(e^{(k-1-j)TA^*}\varphi_0)\right)\mathrm{d}t.$$

On the other hand, since $(e^{tA^*})_{t\in\mathbb{R}}$ is a unitary group, we obtain for all $j = 0, ..., k-1$

$$\inf_{\|\varphi_0\|_{Y'}=1} \int_0^T S_\Gamma\left(B^* e^{(T-t)A^*}(e^{(k-1-j)TA^*}\varphi_0)\right) \mathrm{d}t$$
$$= \inf_{\left\|e^{(k-1-j)TA^*}\varphi_0\right\|_{Y'}=1} \int_0^T S_\Gamma\left(B^* e^{(T-t)A^*}(e^{(k-1-j)TA^*}\varphi_0)\right) \mathrm{d}t = \inf_{\|\varphi_0\|_{Y'}=1} \int_0^T S_\Gamma\left(B^* e^{(T-t)A^*}\varphi_0\right) \mathrm{d}t.$$

This gives by using (2.18)

$$\inf_{\|\varphi_0\|_{Y'}=1} \int_0^{kT} S_\Gamma\left(B^* e^{(kT-t)A^*}\varphi_0\right) \mathrm{d}t \geq k\left(\inf_{\|\varphi_0\|_{Y'}=1} \int_0^T S_\Gamma\left(B^* e^{(T-t)A^*}\varphi_0\right) \mathrm{d}t\right) \tag{2.24}$$

$$\geq kc.$$

Thus, (2.17) holds and the proof is complete. □

2.3. The case $0 \in \mathrm{int}(\Gamma)$

Let us examine more closely the special case which arises under the strengthened assumption $0 \in \mathrm{int}(\Gamma)$. Introducing for any $r > 0$ the closed ball

$$\Gamma_r = \{u \in U \mid \|u\|_U \leq r\}, \tag{2.25}$$

we can find two positive constants $0 < r < R$ such that

$$\Gamma_r \subset \Gamma \subset \Gamma_R, \tag{2.26}$$

so that

$$S_{\Gamma_r}(u) = r\|u\|_U \leq S_\Gamma(u) \leq S_{\Gamma_R}(u) = R\|u\|_U \tag{2.27}$$

for all $u \in U$. Then we get easily from Theorem 2.4 and Theorem 2.5 that for both local and global Γ-null controllability the structure of the set Γ will not matter. Hence, our Γ-null controllability problem can be reduced to a canonical one in which the constraint set coincides with the closed unit ball Γ_1. Moreover, it will be interesting to reformulate the results above in this situation. Indeed, in this case, the support function of the canonical constraint set Γ_1 has the more familiar expression

$$S_{\Gamma_1}(u) = \|u\|_U \tag{2.28}$$

so that the corresponding functional J_{Γ_1} is given by

$$J_{\Gamma_1}(\varphi_0) = \int_0^{T_0} \left\|B^* e^{(T_0-t)A^*}\varphi_0\right\|_U \mathrm{d}t + \left\langle e^{TA^*}\varphi_0, y_0\right\rangle. \tag{2.29}$$

Hence, the results concerned with Γ_1-null controllability can be stated as follows.

Corollary 2.10. *The system (1.1) is locally Γ_1-null controllable in time $T > 0$ if and only if, for some $c_1 > 0$*

$$\int_0^T \left\|B^* e^{tA^*}\varphi_0\right\|_U \mathrm{d}t \geq c_1 \left\|e^{TA^*}\varphi_0\right\|_{Y'} \tag{2.30}$$

for all $\varphi_0 \in Y'$. Moreover, if $(e^{tA})_{t\in\mathbb{R}}$ is a unitary group, then the system (1.1) is locally Γ_1-null controllable in time $T > 0$ if and only if, for some $c > 0$

$$\int_0^T \left\| B^* e^{tA^*} \varphi_0 \right\|_U \mathrm{d}t \geq c_1 \left\| \varphi_0 \right\|_{Y'} \tag{2.31}$$

for all $\varphi_0 \in Y'$.

Remark 2.11.

- The inequality (2.30) means that the set of initial states which can be steered (at time T) to the origin by Γ_1-admissible controls contains a ball centered at the origin with radius c_1.
- The inequality (2.30) is well known as $L^1(0,T)$-observability property for the system (1.1). Hence we conclude from Theorem 2.5 that for conservative systems, $L^1(0,T)$-observability and local Γ_1-null controllability in time T are equivalent.

It is clear by the use of Cauchy-Schwarz inequality that $L^1(0,T)$-observability implies $L^2(0,T)$-observability. 0n the other hand, it has been established in [38] that the converse is also true when the semigroup $(e^{tA})_{t\in\mathbb{R}}$ is unitary and the control operator B is bounded. The proof in this reference is based on optimality arguments based on HUM strategy. Here, we give another direct proof and reformulate the result in the context of constrained and unconstrained controllability.

Theorem 2.12. *Suppose that $(e^{tA})_{t\in\mathbb{R}}$ is a unitary group and the control operator B is bounded. Then the pair (A, B) is null controllable in time T if and only if, the system (1.1) is locally Γ_1-null controllable in time T.*

Proof. Taking into account Remarks 1.3 and 2.11, the proof can be reduced to see that when (1.16) holds true, then necessarily (2.31) is also verified. By contradiction, suppose that for some sequence $(\varphi_n)_n$ such that $\|\varphi_n\|_{Y'} = 1$, we have

$$\lim_{n\to\infty} \int_0^T \left\| B^* e^{(T-t)A^*} \varphi_n \right\|_U \mathrm{d}t = 0. \tag{2.32}$$

Consider on $(0,T)$ the sequence $(\alpha_n)_n$ of non-negative functions given by

$$\alpha_n(t) = \left\| B^* e^{(T-t)A^*} \varphi_n \right\|_U . \tag{2.33}$$

Since B is bounded, this sequence is also bounded on $(0,T)$ and (2.32) implies that $\alpha_n \to 0$ in $L^1(0,T)$ as $n \to \infty$. Hence there exists a subsequence, still denoted by $(\alpha_n)_n$, such that

$$\alpha_n(t) \to 0 \text{ a.e. on } (0,T). \tag{2.34}$$

The boundedness of the operator B yields by using the dominated convergence theorem that as $n \to \infty$, $\alpha_n \to 0$ in $L^2(0,T)$ so that

$$\lim_{n\to\infty} \int_0^T \left\| B^* e^{(T-t)A^*} \varphi_n \right\|_U^2 \mathrm{d}t = 0. \tag{2.35}$$

This contradicts the null controllability of the pair (A, B) characterized by the $L^2(0,T)$-observability inequality

$$\int_0^T \left\| B^* e^{(T-t)A^*} \varphi_n \right\|_U^2 \mathrm{d}t \geq c > 0 \text{ for all } n. \tag{2.36}$$

This completes the proof of the theorem. □

2.4. Constrained null controllability for second order conservative systems

Note at the outset that in the case where the semigroup $(e^{tA})_{t\in\mathbb{R}}$ is a group of isometries, we shall focus our study on the situation in which the system (1.1) can be recast as second order hyperbolic-like system. This choice is motivated by the fact that the admissibility property is available in some significant cases that we shall consider later. Let V, H be real Hilbert spaces such that $V \subset H$ with dense, continuous, and compact embedding. Let us introduce $\mathcal{A} : V \to V'$ the unique linear bounded operator characterized by

$$\langle \mathcal{A}v, w\rangle_{V',V} = \langle v, w\rangle_V \text{ for all } v, w \in V. \tag{2.37}$$

Recall that, by identifying H with its dual, we have $V \subset H \equiv H' \subset V'$ and

$$\langle v, w\rangle_{V',V} = \langle v, w\rangle_H \text{ for all } v \in V, w \in H. \tag{2.38}$$

Note also that if we consider the square root of $\mathcal{A}$, then we can obtain the following identifications

$$V \equiv D(\mathcal{A}^{\frac{1}{2}}),\ V' \equiv D(\mathcal{A}^{\frac{1}{2}})', \tag{2.39}$$

with respect to the pivot space H (see [34], Chap. 3). In what follows, the spaces $D(\mathcal{A}^{\frac{1}{2}})$ and $D(\mathcal{A})$ will be denoted by $H_{\frac{1}{2}}$ and H_1 respectively. Their corresponding dual spaces with respect to the pivot space $H = H_0$ will be denoted by $H_{-\frac{1}{2}}$ and H_{-1}. These notations are in accordance with the following explicit formulas characterizing H_α ($\alpha = 1, \frac{1}{2}, 0$) by using the spectral properties of $\mathcal{A}$ resulting from the compactness of the resolvent of $\mathcal{A}$ (see [34], Prop. 3.4.8, p. 84). More precisely, let us consider the set $\{\lambda_n\}_n = \{\omega_n^2\}_n$ of eigenvalues of $\mathcal{A}$ and denote by $\{\psi_n\}_n$ the corresponding orthonormal basis of eigenvectors in H. Then, for $\alpha = 1, \frac{1}{2}, 0$, we have the standard identifications

$$H_\alpha = \left\{ h \in H \mid \sum_n \lambda_n^{2\alpha} \left|\langle v, \phi_n\rangle\right|^2 < \infty \right\}, \tag{2.40}$$

with the norms

$$\|v\|_\alpha := \|v\|_{H_\alpha} = \left(\sum_n \lambda_n^{2\alpha} \left|\langle v, \phi_n\rangle\right|^2 \right)^{\frac{1}{2}}. \tag{2.41}$$

The corresponding dual spaces can be interpreted as the completion of H for the norms

$$\|v\|_{-\alpha} := \|v\|_{H_{-\alpha}} = \left(\sum_n \lambda_n^{-2\alpha} \left|\langle v, \phi_n\rangle\right|^2 \right)^{\frac{1}{2}}. \tag{2.42}$$

In the context of second order systems such as (2.43) below, we shall denote the duality pairing between H_α and $H_{-\alpha}$ and the scalar product in H by $\langle ., .\rangle$. Hence, the operator $\mathcal{A}$ viewed as unbounded operator on H_α can be extended to a bounded operator from H_α to $H_{\alpha-1}$. Let $\mathcal{U}$ be the control space and $\mathcal{B}$ denote the linear control operator which is supposed bounded from $\mathcal{U}$ to H_{-1}. Then as hyperbolic-like system, we consider the one given by

$$\begin{cases} z''(t) + \mathcal{A}z(t) = \mathcal{B}u(t), \\ z(0) = z_0,\ z'(0) = z_1. \end{cases} \tag{2.43}$$

Clearly, the system has a first order version similar to (1.1) if we set $U = \{0\} \times \mathcal{U}$, $B = \begin{pmatrix} 0 \\ \mathcal{B} \end{pmatrix}$, $y(t) = \{z(t), z'(t)\}$, and

$$A(\{x_0, x_1\}) = \{x_1, -\mathcal{A}x_0\}\,, \tag{2.44}$$

for $\{x_0, x_1\} \in D(A) \subset Y$ where the operator A and the state space Y have to be precised. To this end, the appropriate framework depends on the "degree of unboundedness" of the operator $\mathcal{B}$ in a sense which will be made precise below. To avoid some redundance, our concern will focus directly on the practical situation where the control actuation is subject to the canonical saturation constraint

$$\|u(t)\|_{\mathcal{U}} \leq 1 \text{ for all } t \geq 0. \tag{2.45}$$

Hence, we are led to take as subset of control constraint the unit ball

$$\mathcal{U}_s = \{u \in \mathcal{U} \mid \|u\|_{\mathcal{U}} \leq 1\}\,. \tag{2.46}$$

Let $\mathcal{L}_T$ denote the analogue of the operator L_T given by (1.10). In order to formulate the various $\mathcal{U}_s$-null controllability resulting from the abstract study elaborated in Sections 2.2 and 2.3, we shall explicit the adjoint $\mathcal{L}_T^*$ by exploiting the transposition method machinery developed in [23]. We consider for $\left\{\eta^0, \eta^1\right\}$ given in $H_{\frac{1}{2}} \times H_1$ the solution of

$$\begin{cases} \eta''(t) + \mathcal{A}\eta(t) = 0, \\ \eta(T) = -\eta^1,\ \eta'(T) = \eta^0. \end{cases} \tag{2.47}$$

Considering w solution to

$$\begin{cases} w''(t) + \mathcal{A}w(t) = \mathcal{B}u(t), \\ w(0) = 0,\ w'(0) = 0, \end{cases} \tag{2.48}$$

the operator $\mathcal{L}_T$ can be defined by

$$\mathcal{L}_T : u \in L^2(0, T; \mathcal{U}) \to \{w(T), w'(T)\} \in H_{-\frac{1}{2}} \times H_{-1}. \tag{2.49}$$

$\mathcal{L}_T$ is linear continuous and its adjoint $\mathcal{L}_T^* : H_{\frac{1}{2}} \times H_1 \to L^2(0, T; \mathcal{U})$ is given by

$$\begin{aligned} \left\langle \mathcal{L}_T u, \left\{\eta^0, \eta^1\right\}\right\rangle &= \left\langle \mathcal{L}_T^* \left\{\eta^0, \eta^1\right\}, u\right\rangle_{L^2(0,T;\mathcal{U})} \\ &= \left\langle w(T), \eta^0\right\rangle + \left\langle w'(T), \eta^1\right\rangle. \end{aligned}$$

In order to give a more explicit form of $\mathcal{L}_T^*$, we exploit the fact that the solution of (2.47) satisfies $\eta \in C(0, T; H_1) \cap C^1(0, T; H_{\frac{1}{2}})$. Then by using a density argument we can compute the duality product $\langle ., . \rangle$ of (2.47) with $w(t)$ to obtain by integrating by parts

$$\begin{aligned} \langle w(T), \eta'(T)\rangle - \langle w'(T), \eta(T)\rangle &= \int_0^T \langle \mathcal{B}u(t), \eta(t)\rangle \,\mathrm{d}t \\ &= \int_0^T \langle u(t), \mathcal{B}^*\eta(t)\rangle_{\mathcal{U}} \,\mathrm{d}t, \end{aligned}$$

so that

$$\mathcal{L}_T^* \left\{\eta^0, \eta^1\right\}(t) = \mathcal{B}^*\eta(t). \tag{2.50}$$

Hence, any extension of the operator $\mathcal{L}_T^*$ yields an admissibility property which in turn gives the appropriate state space where the controllability has to be studied. As for the analogue of the property (1.13), we can proceed by transposition for the systems (2.43) and (2.47) in similar way so that the system (2.43) is steered to rest by a control u in time T if and only if

$$\int_0^T \langle \mathcal{B}u(t), \eta(t)\rangle \,\mathrm{d}t - \langle z_0, \eta(0)\rangle + \langle z_1, \eta'(0)\rangle = 0 \tag{2.51}$$

for all $\left\{\eta^0, \eta^1\right\} \in H_{\frac{1}{2}} \times H_1$. Furthermore, by adapting Theorem 2.2 to the system (2.43) and using (2.15) in Remark 2.3, this system is $\mathcal{U}_s$-null controllable at $\{z_0, z_1\}$ in time T if and only if,

$$\int_0^T \|\mathcal{B}^*\eta(t)\|_{\mathcal{U}} \,\mathrm{d}t - \langle z_0, \eta(0)\rangle + \langle z_1, \eta'(0)\rangle \geq 0 \tag{2.52}$$

for all $\left\{\eta^0, \eta^1\right\} \in H_{\frac{1}{2}} \times H_1$. Having in mind the null controllability for systems modeled by the wave and Euler-Bernoulli beam equations, we shall consider various "degrees of unboundedness" for the control operator $\mathcal{B}$. To this classification will correspond appropriate state spaces with convenient admissibility assumptions. To be more precise, we shall consider the following representative classes of control operators: **(i)** $\mathcal{B} \in \mathcal{L}(\mathcal{U}, H)$, **(ii)** $\mathcal{B} \in \mathcal{L}(\mathcal{U}, H_{-\frac{1}{2}})$ with $\mathcal{B} \notin \mathcal{L}(\mathcal{U}, H)$, and **(iii)** $\mathcal{B} \in \mathcal{L}(\mathcal{U}, H_{-1})$ with $\mathcal{B} \notin \mathcal{L}(\mathcal{U}, H_{-\frac{1}{2}})$. We shall treat each item by formulating the appropriate admissibility condition, the unconstrained, and the constrained null controllability properties.

The case (i) $\mathcal{B} \in \mathcal{L}(\mathcal{U}, H)$

In this case, we can choose as framework

$$Y = H_{\frac{1}{2}} \times H,\; D(A) = H_1 \times H_{\frac{1}{2}}. \tag{2.53}$$

In this setting, the control operator $\mathcal{B}$ is bounded so that the solution of (2.43) satisfies $w \in C(0, T; H_{\frac{1}{2}}) \cap C^1(0, T; H)$. Furthermore, the operator $\mathcal{L}_T$ defined by (2.49) is in fact linear continuous from $L^2(0, T; \mathcal{U})$ to $H_{\frac{1}{2}} \times H$. Here no additional admissibility assumption is needed since the adjoint $\mathcal{L}_T^*$ is readily continuous from $H_{-\frac{1}{2}} \times H$ to $L^2(0, T; \mathcal{U})$. We mention also the following observability inequalities characterizing the unconstrained and the local $\mathcal{U}_s$-null controllability in time $T > 0$ respectively

$$\int_0^T \|\mathcal{B}^*\eta(t)\|_{\mathcal{U}}^2 \,\mathrm{d}t \geq c_2 \left\|\left\{\eta^0, \eta^1\right\}\right\|_{H_{-\frac{1}{2}} \times H}^2, \tag{2.54}$$

$$\int_0^T \|\mathcal{B}^*\eta(t)\|_{\mathcal{U}} \,\mathrm{d}t \geq c_1 \left\|\left\{\eta^0, \eta^1\right\}\right\|_{H_{-\frac{1}{2}} \times H} \tag{2.55}$$

for all $\left\{\eta^0, \eta^1\right\} \in H_{-\frac{1}{2}} \times H$. Then, the controllability results established in the general framework when the control operator is bounded can be stated as follows.

Proposition 2.13. *The system (2.43) is locally $\mathcal{U}_s$-null controllable in time $T > 0$ if and only if one of the equivalent observability inequalities (2.54) and (2.55) holds. Moreover, the system (2.43) is globally $\mathcal{U}_s$-null controllable if it is locally $\mathcal{U}_s$-null controllable in some time $T > 0$.*

The case (ii) $\mathcal{B} \in \mathcal{L}(\mathcal{U}, H_{-\frac{1}{2}})$ **with** $\mathcal{B} \notin \mathcal{L}(\mathcal{U}, H)$
If we choose as framework

$$Y = H_{\frac{1}{2}} \times H,\ D(A) = H_1 \times H_{\frac{1}{2}}, \tag{2.56}$$

then the control operator $\mathcal{B}$ is unbounded so that, for all final data $\{\eta(T), \eta'(T)\} \in H_1 \times H_{\frac{1}{2}}$, the following admissibility condition

$$\int_0^T \|\mathcal{B}^*\eta(t)\|_{\mathcal{U}}^2 \,\mathrm{d}t \le C \,\|\{\eta(T), \eta'(T)\}\|_{H \times H_{-\frac{1}{2}}}^2 , \tag{2.57}$$

is needed. Taking into account (2.47) and (2.50), this estimation can be reformulated by

$$\int_0^T \|\mathcal{B}^*\eta(t)\|_{\mathcal{U}}^2 \,\mathrm{d}t \le C \left\|\left\{\eta^0, \eta^1\right\}\right\|_{H_{-\frac{1}{2}} \times H}^2 , \tag{2.58}$$

After extension, this enhances us to extend continuously the adjoint operator $\mathcal{L}_T^*$ from $H_{-\frac{1}{2}} \times H$ to $L^2(0, T; \mathcal{U})$. By duality, this guarantees the continuity of the operator $\mathcal{L}_T$ from $L^2(0, T; \mathcal{U})$ to $H_{\frac{1}{2}} \times H$. Moreover, by adapting the controllability results established in the general framework, we get:

- The system (2.43) is locally $\mathcal{U}_s$-null controllable in time T if and only if

$$\int_0^T \|\mathcal{B}^*\eta(t)\|_{\mathcal{U}} \,\mathrm{d}t \ge c_1 \left\|\left\{\eta^0, \eta^1\right\}\right\|_{H_{-\frac{1}{2}} \times H} \tag{2.59}$$

  for all $\left\{\eta^0, \eta^1\right\} \in H_{-\frac{1}{2}} \times H$.
- The system (2.43) is globally $\mathcal{U}_s$-null controllable provided that

$$\sup_{T>0} \inf \left\{ \int_0^T \|\mathcal{B}^*\eta(t)\|_{\mathcal{U}} \,\mathrm{d}t \mid \left\|\left\{\eta^0, \eta^1\right\}\right\|_{H_{-\frac{1}{2}} \times H} = 1 \right\} = \infty. \tag{2.60}$$

The case (iii) $\mathcal{B} \in \mathcal{L}(\mathcal{U}, H_{-1})$ **with** $\mathcal{B} \notin \mathcal{L}(\mathcal{U}, H_{-\frac{1}{2}})$
In this case, we can choose as framework

$$Y = H \times H_{-\frac{1}{2}},\ D(A) = H_{\frac{1}{2}} \times H. \tag{2.61}$$

Note that it corresponds to the situation where the control operator $\mathcal{B}$ is unbounded so that some kind of admissibility is needed. Let us consider the following admissibility condition

$$\int_0^T \|\mathcal{B}^*\eta(t)\|_{\mathcal{U}}^2 \,\mathrm{d}t \le C \,\|\{\eta(T), \eta'(T)\}\|_{H_{\frac{1}{2}} \times H}^2 \tag{2.62}$$

for all final data $\{\eta(T), \eta'(T)\} \in H_1 \times H_{\frac{1}{2}}$. By proceeding as in the previous case, this inequality can be reformulated as

$$\int_0^T \|\mathcal{B}^*\eta(t)\|_{\mathcal{U}}^2 \,\mathrm{d}t \le C \left\|\left\{\eta^0, \eta^1\right\}\right\|_{H \times H_{\frac{1}{2}}}^2 . \tag{2.63}$$

Hence, the adjoint operator $\mathcal{L}_T^*$ given by (2.47) and (2.50) can be extended continuously from $H \times H_{\frac{1}{2}}$ to $L^2(0,T;\mathcal{U})$. By duality we conclude that the system (2.43) is well-posed with respect to the state space $H \times H_{-\frac{1}{2}}$ ([34], Thm. 4.4.3, p. 127).

The controllability results established in the general framework can be stated as follows.

- The system (2.43) is locally $\mathcal{U}_s$-null controllable in time T if and only if

$$\int_0^T \|\mathcal{B}^*\eta(t)\|_{\mathcal{U}}\,\mathrm{d}t \geq c_1 \left\|\left\{\eta^0,\eta^1\right\}\right\|_{H\times H_{\frac{1}{2}}} \tag{2.64}$$

  for all $\left\{\eta^0,\eta^1\right\} \in H \times H_{\frac{1}{2}}$.
- The system (2.43) is globally $\mathcal{U}_s$-null controllable provided that

$$\sup_{T>0}\inf\left\{\int_0^T \|\mathcal{B}^*\eta(t)\|_{\mathcal{U}}\,\mathrm{d}t \mid \left\|\left\{\eta^0,\eta^1\right\}\right\|_{H\times H_{\frac{1}{2}}} = 1\right\} = \infty. \tag{2.65}$$

Remark 2.14. In all these cases above, by using a standard density argument, we deduce easily from Theorem 2.7 the following necessary condition for global $\mathcal{U}_s$-null controllability

$$\int_0^\infty \|\mathcal{B}^*\eta(t)\|_{\mathcal{U}}\,\mathrm{d}t = \infty \tag{2.66}$$

for all $\left\{\eta^0,\eta^1\right\} \in H_{\frac{1}{2}} \times H_1$, $\left\{\eta^0,\eta^1\right\} \neq \{0,0\}$.

3. Applications I

In this section, we present some applications of the abstract results stated above to systems governed by PDEs. In the unconstrained control case, it is well-known that the controllability properties depend in a very sensitive way on the class of equations under consideration [31, 37]. In particular, parabolic and hyperbolic equations behave in a significantly different way, because of their different behavior with respect to time reversal. Here, the applications we present concern hyperbolic-like systems and apply, essentially, to other models like, for instance, Schrödinger and plate equations. In what follows, $\Omega \subset \mathbb{R}^N$ is an open bounded domain with sufficiently smooth boundary $\partial\Omega$ and ω denotes a non-empty open subset of Ω with χ_ω as characteristic function.

Example 3.1. *Wave equation with internal distributed control*

We consider the wave equation given by

$$\begin{cases} z'' = \Delta z + u\chi_\omega \text{ in } \Omega\times(0,\infty), \\ z = 0 \text{ on } \partial\Omega\times(0,\infty), \\ z(x,0) = z_0(x),\ \frac{\partial z}{\partial t}(x,0) = z_1(x) \text{ in } \Omega. \end{cases} \tag{3.1}$$

This system has the form (2.43) if we set $H = L^2(\Omega)$, $\mathcal{U} = L^2(\Omega)$, and

$$\mathcal{A} = -\Delta,\ D(\mathcal{A}) = H^2(\Omega)\cap H_0^1(\Omega). \tag{3.2}$$

The control operator $\mathcal{B} : L^2(\Omega) \to L^2(\Omega)$ is given by

$$\mathcal{B}f = \chi_\omega f. \tag{3.3}$$

In this setting, we have $H_{\frac{1}{2}} = H_0^1(\Omega)$ and $H_{-\frac{1}{2}} = H^{-1}(\Omega)$. Considering the uncontrolled wave equation

$$\left\{ \begin{array}{l} \eta'' = \Delta\eta \text{ in } \Omega \times (0,\infty), \\ \eta = 0 \text{ on } \partial\Omega \times (0,\infty), \end{array} \right. \tag{3.4}$$

and due to the fact that $\mathcal{B}$ is bounded, the following observability conditions

$$\int_0^T \int_\omega |\eta(x,t)|^2 \,\mathrm{d}x\mathrm{d}t \geq c_2 \left(\|\eta(.,T)\|^2_{L^2(\Omega)} + \|\eta'(.,T)\|^2_{H^{-1}(\Omega)} \right), \tag{3.5}$$

$$\left(\int_0^T \int_\omega |\eta(x,t)| \,\mathrm{d}x\mathrm{d}t \right)^2 \geq c_2 \left(\|\eta(.,T)\|^2_{L^2(\Omega)} + \|\eta'(.,T)\|^2_{H^{-1}(\Omega)} \right), \tag{3.6}$$

are equivalent. Inequality (3.5) has been established for T large enough and ω satisfying the following condition: there exists $x_0 \in \mathbb{R}^N$ such that ω is a neighborhood of the closure of the set

$$\gamma(x_0) = \{x \in \partial\Omega \,|\, (x - x_0).\nu(x) > 0\}, \tag{3.7}$$

where $\nu(x)$ denotes the unit outward normal at $x \in \partial\Omega$. Moreover, it is well known that when the domain Ω satisfies the following "geometric control condition" (GCC): every ray of geometric optics propagating in Ω and being reflected on its boundary, enters the control region ω in a time less than T [21, 22]. As bounded control constraint subset, we consider

$$\Gamma_1^\omega = \left\{ \chi_\omega u \,|\, u \in L^2(\Omega) \text{ and } \int_\omega |u(x)|^2 \,\mathrm{d}x \leq 1 \right\}. \tag{3.8}$$

Then from Theorem 2.12 and Corollary 2.9 we get

Proposition 3.2. *Suppose that the data (ω, T) satisfy one of the conditions ensuring the observability inequality (3.5). Then the system (3.1) is both locally Γ_1^ω-null controllable in time T and globally Γ_1^ω-null controllable with respect to the state space $H_0^1(\Omega) \times L^2(\Omega)$.*

Example 3.3. *Euler-Bernoulli beam with pointwise actuator*

Let $0 < a < 1$; we consider the Euler-Bernoulli beam with spatial extent from $x = 0$ to $x = 1$ with displacement at position x and time t represented by $y(x,t)$ given by

$$\left\{ \begin{array}{l} z'' + \frac{\partial^4}{\partial x^4} z = u(t)\delta(x-a),\ 0 < x < 1,\ t > 0, \\ z(0,t) = \frac{\partial^2 z}{\partial x^2}(0,t) = \frac{\partial z}{\partial x}(1,t) = \frac{\partial^3 z}{\partial x^3}(1,t) = 0,\ t > 0, \\ z(x,0) = z_0(x),\ \frac{\partial z}{\partial t}(x,0) = z_1(x),\ 0 < x < 1. \end{array} \right. \tag{3.9}$$

This equation has the form (2.43) if we set $H = L^2(0,1)$ and

$$\mathcal{A} = \frac{d^4 v}{\mathrm{d}x^4}, D(\mathcal{A}) = \left\{ v \in H^4(0,1) \cap H_{\frac{1}{2}} \,|\, \frac{d^2 v}{\mathrm{d}x^2}(0) = \frac{d^3 v}{\mathrm{d}x^3}(1) = 0 \right\}. \tag{3.10}$$

We introduce the eigenvalues $\left\{\lambda_n = \omega_n^2\right\}_n$ of the operator $\mathcal{A}$ associated to the eigenfunctions $\{\phi_n\}_n$:

$$\lambda_n = \left(-\frac{\pi}{2} + n\pi \right)^4,\ n = 1, 2, ..., \tag{3.11}$$

$$\phi_n(x) = \sin\left(-\frac{\pi}{2} + n\pi\right) x,\ n = 1, 2, ... \tag{3.12}$$

Then $H_{\frac{1}{2}}$ is identified by

$$H_{\frac{1}{2}} = \left\{ v \in H^2(0,1) \mid v(0) = \frac{\mathrm{d}v}{\mathrm{d}x}(1) = 0 \right\}, \tag{3.13}$$

and its can topology can be defined by the norm

$$\|v\|^2_{H_{\frac{1}{2}}} = \sum_n \omega_n^2 \left|\langle v, \phi_n\rangle\right|^2.$$

The dual $H_{-\frac{1}{2}}$ can be characterized as the completion of $L^2(0,1)$ for the norm

$$\|v\|^2_{H_{-\frac{1}{2}}} = \sum_n \omega_n^{-2} \left|\langle v, \phi_n\rangle\right|^2.$$

Since $\delta(x-a) \in H_{-\frac{1}{2}}$, we can choose as appropriate control space $\mathcal{U} = \mathbb{R}$ with the control operator $\mathcal{B} : \mathbb{R} \to H_{-\frac{1}{2}}$ given by

$$\mathcal{B}u = u\delta(x-a). \tag{3.14}$$

The adjoint $\mathcal{B}^* : H_{\frac{1}{2}} \to \mathbb{R}$ is defined by

$$\mathcal{B}^* v = \langle \delta(x-a), v\rangle = v(a). \tag{3.15}$$

On the other hand, considering the solution of the uncontrolled equation

$$\left\{ \begin{array}{l} \eta'' + \frac{\partial^4 \eta}{\partial x^4} = 0,\ 0 < x < 1,\ t > 0, \\ \eta(0,t) = \frac{\partial^2 \eta}{\partial x^2}(0,t) = \frac{\partial \eta}{\partial x}(1,t) = \frac{\partial^3 \eta}{\partial x^3}(1,t) = 0,\ t > 0, \end{array} \right. \tag{3.16}$$

given by

$$\eta(.,t) = \sum_n \left[\langle \eta(.,T), \phi_n\rangle \cos\left(\omega_n (T-t)\right) + \omega_n^{-1} \langle \eta'(.,T), \phi_n\rangle \sin\left(\omega_n (T-t)\right)\right] \phi_n,$$

we have

$$\mathcal{B}^* \eta(.,t) = \eta(a,t)$$

$$= \sum_n \left[\langle \eta(.,T), \phi_n\rangle \cos\left(\omega_n (T-t)\right) + \omega_n^{-1} \langle \eta'(.,T), \phi_n\rangle \sin\left(\omega_n (T-t)\right)\right] \phi_n(a).$$

By using for any $T>0$ the generalized Ingham's inequalities [5], we get

$$\begin{aligned} &c_2 \sum_n \left(|\langle \eta(.,T), \phi_n\rangle|^2 + \omega_n^{-2} |\langle \eta'(.,T), \phi_n\rangle|^2 \right) |\phi_n(a)|^2 \le \int_0^T |\eta(a,t)|^2 \,\mathrm{d}t \\ &\le C_2 \sum_n \left(|\langle \eta(.,T), \phi_n\rangle|^2 + \omega_n^{-2} |\langle \eta'(.,T), \phi_n\rangle|^2 \right) |\phi_n(a)|^2 \\ &\le C_2 \sum_n |\langle \eta(.,T), \phi_n\rangle|^2 + \omega_n^{-2} |\langle \eta'(.,T), \phi_n\rangle|^2 . \end{aligned}$$

Hence, by comparing with the estimates (2.57)–(2.58) we conclude that the control operator $\mathcal{B}$ is admissible with respect to the state space $H_{\frac{1}{2}} \times L^2(0,1)$. Moreover, when a is a rational number with coprime factorization

$$a = \frac{a_1}{a_2}, \tag{3.17}$$

such that a_1 is odd, then, following Lemma 2.9 of [28], for some positive constant m

$$|\phi_n(a)| \ge m \text{ for all } n, \tag{3.18}$$

so that the system (3.9) is null controllable in time T. As for the constrained null controllability question, let $\Gamma_1^a = [-1,1]$ be the canonical constraint control set for the system (3.9). We note that we have at our disposal the following sharp L^1 version of Ingham inequality [18]

$$c_1 \sup_{n\in\mathbb{N}} \left(|\langle \eta(.,T), \phi_n\rangle| + \omega_n^{-1} |\langle \eta'(.,T), \phi_n\rangle| \right) |\phi_n(a)| \le \int_0^T |\eta(a,t)| \,\mathrm{d}t. \tag{3.19}$$

Hence, the analogue of the L^1 observability inequality (2.59) cannot occur. In other words, the system (3.9) is not locally Γ_1^a-null controllable with respect to the state space $H_{\frac{1}{2}} \times L^2(0,1)$ in any time $T>0$.

Example 3.4. *Wave equation with Dirichlet boundary control*

We consider the wave equation with boundary control given by

$$\left\{ \begin{array}{l} z'' = \Delta z \text{ in } \Omega \times (0,\infty), \\ z = \chi_\gamma u \text{ on } \partial\Omega \times (0,\infty), \\ z(x,0) = z_0(x),\ \frac{\partial z}{\partial t}(x,0) = z_1(x) \text{ in } \Omega. \end{array} \right. \tag{3.20}$$

where χ_γ denotes the characteristic function of a subset $\gamma \subset \partial\Omega$. We introduce also the Dirichlet map D defined by

$$D : L^2(\partial\Omega) \to L^2(\Omega),\ v \mapsto Dv = h, \tag{3.21}$$

$$\left\{ \begin{array}{l} \Delta h = 0 \text{ in } \Omega, \\ h = u \text{ on } \partial\Omega. \end{array} \right. \tag{3.22}$$

Then the system (3.20) can be modeled as an abstract second order equation similar to (2.43) and given by (see Prop. 10.9.1 of [34], p. 349)

$$z''(t) + \mathcal{A}z(t) = \mathcal{A}D\chi_\gamma u(t), \tag{3.23}$$

where $-\mathcal{A}$ is the Dirichlet Laplacian defined by (3.2). Hence, this system has the form (2.43) if we set $H = L^2(\Omega)$, $\mathcal{U} = L^2(\partial\Omega)$. In order to precise the appropriate state space, we recall from elliptic theory in [23], Chapter 2 that D is continuous so that the control operator can be defined as $\mathcal{B} : \mathcal{U} = L^2(\partial\Omega) \to H_{-1}$ and

$$\mathcal{B}u = \mathcal{A}D\chi_\gamma u. \tag{3.24}$$

In this setting, we have $H_{\frac{1}{2}} = H_0^1(\Omega)$ and $H_{-\frac{1}{2}} = H^{-1}(\Omega)$ and we are in the unboundedness case **(iii)**. In order to precise the appropriate admissibility condition, we consider the wave equations

$$\begin{cases} \eta'' = \Delta\eta \text{ in } \Omega \times (0,T), \\ \eta = 0 \text{ on } \partial\Omega \times (0,T), \\ \eta(x,T) = -\eta^1(x),\ \eta'(x,T) = \eta^0(x) \text{ in } \Omega, \end{cases} \tag{3.25}$$

$$\begin{cases} w'' = \Delta w \text{ in } \Omega \times (0,\infty), \\ w = \chi_\gamma u \text{ on } \partial\Omega \times (0,\infty), \\ w(x,0) = w'(x,0) = 0 \text{ in } \Omega. \end{cases} \tag{3.26}$$

It is well-known that the operator

$$\mathcal{L}_T : u \in L^2(0,T;L^2(\partial\Omega)) \to \{w(.,T), w'(.,T)\} \in L^2(\Omega) \times H^{-1}(\Omega),$$

is linear continuous [23], Chapter 3. Then the adjoint $\mathcal{L}_T^* : L^2(\Omega) \times H_0^1(\Omega) \to L^2(0,T;L^2(\partial\Omega))$ is defined by

$$\begin{aligned} \left\langle \mathcal{L}_T u, \left\{\eta^0,\eta^1\right\}\right\rangle &= \textstyle\int_{\partial\Omega\times(0,T)} \mathcal{L}_T^*\left\{\eta^0,\eta^1\right\} u(\sigma,t)\mathrm{d}\sigma\mathrm{d}t \\ &= \left\langle w(T),\eta^0\right\rangle + \left\langle w'(T),\eta^1\right\rangle. \end{aligned}$$

We consider for $\left\{\eta^0,\eta^1\right\}$ given in $L^2(\Omega) \times H_0^1(\Omega)$ the solution of (3.25). If we multiply (3.25) by w and if we apply the Green's formula, we obtain

$$\langle w(T), \eta'(T)\rangle - \langle w'(T), \eta(T)\rangle = \int_{\gamma\times(0,T)} u(\sigma,t)\frac{\partial\eta}{\partial\nu}(\sigma,t)\mathrm{d}\sigma\mathrm{d}t,$$

where $\frac{\partial}{\partial\nu}$ denotes the outward normal derivative to $\partial\Omega$. Hence,

$$\mathcal{L}_T^*\left\{\eta^0,\eta^1\right\}(t) = (\mathcal{A}D\chi_\gamma)^*\eta(t) = \chi_\gamma(.)\frac{\partial\eta}{\partial\nu}(.,t). \tag{3.27}$$

On the other hand, it can be shown by multiplier techniques that

$$\int_{\gamma\times(0,T)} \left|\frac{\partial\eta}{\partial\nu}(\sigma,t)\right|^2 \mathrm{d}\sigma\mathrm{d}t \le C\left\|\{\eta(T),\eta'(T)\}\right\|^2_{H_0^1(\Omega)\times L^2(\Omega)}$$

for all final data $\{\eta(.,T),\eta'(.,T)\} \in H_0^1(\Omega) \times L^2(\Omega)$ (see [22]). This inequality can be reformulated as

$$\int_{\gamma\times(0,T)} \left|\frac{\partial\eta}{\partial\nu}(\sigma,t)\right|^2 \mathrm{d}\sigma\mathrm{d}t \le C\left\|\left\{\eta^0,\eta^1\right\}\right\|^2_{L^2(\Omega)\times H_0^1(\Omega)}$$

for all final data $\left\{\eta^0, \eta^1\right\} \in L^2(\Omega) \times H_0^1(\Omega)$. Hence, by comparing with the estimates (2.62)–(2.63) we conclude that the control operator given by (3.24) is admissible with respect to the state space $L^2(\Omega) \times H^{-1}(\Omega)$. Moreover, the unconstrained null controllability is guaranteed by the observability inequality

$$\int_{\gamma\times(0,T)} \left|\frac{\partial\eta}{\partial\nu}(\sigma,t)\right|^2 \mathrm{d}\sigma\mathrm{d}t \geq c_2 \left\|\{\eta(T), \eta'(T)\}\right\|^2_{H_0^1(\Omega)\times L^2(\Omega)}. \tag{3.28}$$

It is well known that if there exists some $x_0 \in \mathbb{R}^N$ such that

$$\gamma \supset \{x \in \partial\Omega \,|\, (x - x_0).\nu(x) > 0\}, \tag{3.29}$$

then (3.28) holds for $T > 2 \sup\limits_{x\in\Omega} \|x - x_0\|$ [22]. As for the constrained null controllability question, let

$$\Gamma_1^\gamma = \left\{\chi_\gamma u \,|\, u \in L^2(\partial\Omega) \text{ and } \int_\gamma |u(\sigma)|^2 \,\mathrm{d}\sigma \leq 1\right\} \tag{3.30}$$

be the canonical constraint control set for the system (3.20). The admissibility inequality which means that $\frac{\partial\eta}{\partial\nu} \in L^2(\gamma \times (0,T))$ for finite energy solutions of the wave equation does not guarantee the analogue of (2.64) corresponding to the $L^1(0,T; L^2(\gamma))$ observability inequality. In [38], the one dimensional wave equation has been presented in order to confirm this fact by using the explicit expression of solutions obtained by the d'Alembert formula. We conclude that the system (3.20) is not locally Γ_1^γ-null controllable in any time $T > 0$. However, it can occur that the constrained local Γ_1^γ-null controllability holds for the energy subspace $H_0^1(\Omega) \times L^2(\Omega)$. To this end, we consider a larger open set Ω_γ containing both Ω and γ. Let $\omega_\gamma = \Omega_\gamma|\overline{\Omega}$ and suppose that the system defined by the wave equation on Ω_γ

$$\begin{cases} \widetilde{\eta}'' = \Delta\widetilde{\eta} \text{ in } \Omega_\gamma \times (0,\infty), \\ \widetilde{\eta} = 0 \text{ on } \partial\Omega_\gamma \times (0,\infty), \end{cases} \tag{3.31}$$

satisfies for some $T > 0$ the observability condition

$$\int_0^T \int_{\omega_\gamma} |\widetilde{\eta}(x,t)|^2 \,\mathrm{d}x\mathrm{d}t \geq c\left(\|\widetilde{\eta}(.,T)\|^2_{L^2(\Omega_\gamma)} + \|\widetilde{\eta}'(.,T)\|^2_{H^{-1}(\Omega_\gamma)}\right). \tag{3.32}$$

Then, we have

Proposition 3.5. *Suppose that the observability inequality (3.32) holds. Then the system (3.20) is locally Γ_1^γ-null controllable in time T with respect to the subspace $H_0^1(\Omega) \times L^2(\Omega) \subset L^2(\Omega) \times H^{-1}(\Omega)$ in the sense that there exists $\delta > 0$ such that for any initial state $\{z_0, z_1\} \in H_0^1(\Omega) \times L^2(\Omega)$ with*

$$\|\{z_0, z_1\}\|_{H_0^1(\Omega)\times L^2(\Omega)} < \delta, \tag{3.33}$$

there exists a Γ_1^γ-admissible control u such that the corresponding solution of (3.20) satisfies $z(.,T) = \frac{\partial z}{\partial t}(.,T) = 0$.

Proof. Let us introduce the controlled system

$$\begin{cases} \widetilde{z}'' = \Delta\widetilde{z} + \widetilde{u}\chi_{\omega_\gamma} \text{ in } \Omega_\gamma \times (0,\infty), \\ \widetilde{z} = 0 \text{ on } \partial\Omega_\gamma \times (0,\infty), \\ \widetilde{z}(x,0) = \widetilde{z}_0(x),\ \frac{\partial\widetilde{z}}{\partial t}(x,0) = \widetilde{z}_1(x) \text{ in } \Omega_\gamma, \end{cases} \tag{3.34}$$

and consider for any $r > 0$ the constraint control subset $\Gamma_r^{\omega_\gamma}$ given by

$$\Gamma_r^{\omega_\gamma} = \left\{ \chi_{\omega_\gamma} u \,|\, u \in L^2(\Omega_\gamma) \text{ and } \int_{\omega_\gamma} |u(x)|^2 \,\mathrm{d}x \le r^2 \right\}. \tag{3.35}$$

By virtue of Proposition 3.2 and following Section 2.3, it follows that the system (3.34) is locally $\Gamma_r^{\omega_\gamma}$-null controllable in time T for any $r > 0$. Hence there exists $\delta_r > 0$ such that for any initial state $\{\widetilde{z}_0, \widetilde{z}_1\} \in H_0^1(\Omega_\gamma) \times L^2(\Omega_\gamma)$ with

$$\|\{\widetilde{z}_0, \widetilde{z}_1\}\|_{H_0^1(\Omega_\gamma)\times L^2(\Omega_\gamma)} < \delta_r, \tag{3.36}$$

there exists a $\Gamma_r^{\omega_\gamma}$-admissible control $\widetilde{u}$ such that the corresponding solution of (3.34) satisfies $\widetilde{z}(.,T) = \frac{\partial \widetilde{z}}{\partial t}(.,T) = 0$. Then we extend any $\{z_0, z_1\} \in H_0^1(\Omega) \times L^2(\Omega)$ to $\{\widetilde{z}_0, \widetilde{z}_1\} \in H_0^1(\Omega_\gamma) \times L^2(\Omega_\gamma)$ by setting

$$\widetilde{z}_0 = \widetilde{z}_1 = 0 \text{ in } \omega_\gamma, \tag{3.37}$$

and we define for (3.20) the control

$$u(.,t) = \widetilde{z}(.,t)|\gamma. \tag{3.38}$$

Since $\widetilde{z}(.,t) \in H_0^1(\Omega_\gamma)$, we get by trace argument

$$\left(\int_\gamma |u(\sigma,t)|^2 \,\mathrm{d}\sigma\right)^{\frac{1}{2}} = \left(\int_\gamma |\widetilde{z}(\sigma,t)|^2 \,\mathrm{d}\sigma\right)^{\frac{1}{2}}$$

$$\le C \,\|\widetilde{z}(.,t)\|_{H_0^1(\Omega_\gamma)} \le C \,\|\{\widetilde{z}(t), \widetilde{z}'(t)\}\|_{H_0^1(\Omega_\gamma)\times L^2(\Omega_\gamma)}$$

$$\le \delta_r + C\,\|\widetilde{u}\|_{L^2(0,T;L^2(\Omega_\gamma))} \le \delta_r + Cr$$

for some generic positive constant C. On the other hand, it is easy to see that we can choose the correspondence between r and δ_r so that $\delta_r \searrow 0$ as $r \searrow 0$. Let $\widehat{r} > 0$ be such that

$$\delta_{\widehat{r}} + C\widehat{r} \le 1. \tag{3.39}$$

It follows from the inequalities above that the control given by (3.38) is Γ_1^γ-admissible and it steers the system (3.20) to the origin in time T provided that the initial conditions satisfy

$$\|\{z_0, z_1\}\|_{H_0^1(\Omega)\times L^2(\Omega)} < \delta_{\widehat{r}}. \tag{3.40}$$

This completes the proof of the proposition. □

4. Steering control

4.1. Preliminaries

Beside the results on constrained null controllability, it is important to see whether the appropriate control steering y_0 to the origin can be identified. To this end, we note that, assuming that such a control exists, a classical strategy has consisted of adopting a time optimal control method for both multivariable [20] and distributed systems [14, 15]. Here, we shall not adopt this approach. Instead, taking into account from the results stated in Section 2 the fact that the existence of the steering control depends closely on the initial state y_0 and the related final time $T = T(y_0)$, we shall formulate our problem in the following general form:

(P) Given a final time $T > 0$, characterize any control steering the system (1.1) as close as possible to 0 in time T .

We shall treat this problem by using a variational method based on minimizing a convex functional $J_c : Y' \to \mathbb{R}$ of the form

$$J_c(\varphi_0) := \int_0^T K\left(B^* e^{(T-t)A^*}\varphi_0\right) \mathrm{d}t + \left\langle e^{TA^*}\varphi_0, y_0\right\rangle, \tag{4.1}$$

where $K : U \to \mathbb{R}^+$ is convex and lower semicontinuous. Following the approach initiated in [32] for multivariable systems, a natural choice for a given general control constraint set Γ will consist of taking $K = S_\Gamma$ so that J_c coincides with the functional J_Γ already defined in (2.8). Note that an immediate difficulty arises because of the lack of differentiability of J_Γ resulting from the sup-operation present in the support function S_Γ. Here, using nonsmooth analysis tools, we shall extend the results in [32] to the case of distributed systems with unbounded control operator by minimizing J_Γ on the closed unit ball in Y'

$$\Phi_1 = \{\varphi_0 \in Y' \mid \|\varphi_0\|_{Y'} \le 1\}. \tag{4.2}$$

On the other hand, we shall establish that the smoothness question can be overcome provided that the functional K in (4.1) is differentiable with $K'(U) \subset \Gamma$. This issue has been treated in [8] in the case where the control operator B is supposed to be bounded. We shall see that the existence of such a functional is guaranteed in the canonical constraint case $\Gamma = \Gamma_1$. Furthermore, proceeding in such a way will enable us to recover even the results already established in the unconstrained control case by the HUM method. Below, we present some nonsmooth analysis notions which will be used with respect to the space Y'.

Let $f : X \to \mathbb{R}$ be a given function, where X is a Hilbert space, and let $x \in X$. An element ζ of X' is called a subgradient of f at x (in the sense of convex analysis) if it satisfies the following subgradient inequality :

$$f(y) - f(x) \ge \langle \zeta, y - x, \rangle \text{ for all } y \in X. \tag{4.3}$$

The set of all subgradients of f at x is denoted by $\partial f(x)$, and referred to as the subdifferential of f at x. Let P be a convex subset of X. The tangent cone to P at a point $x \in P$, denoted $T_P(x)$, consists of all points $v \in X$ expressible in the form

$$v = \lim_{n\to\infty} \frac{x_n - x}{t_n},$$

where $(x_n)_n$ is a sequence in P converging to x and $(t_n)_n$ is a positive sequence decreasing to 0. The normal cone to P at $x \in P$, denoted $N_P(x)$, is the subset of the dual space X' defined by

$$N_P(x) = \{\zeta \in X' \mid \langle \zeta, v\rangle \le 0 \,\forall v \in T_P(x)\}. \tag{4.4}$$

It can be shown that the normal cone can be characterized by Proposition 4 of [3], page 168.

$$N_P(x) = \left\{\zeta \in X' \mid \langle \zeta, x\rangle = \sup_{v\in P} \langle \zeta, v\rangle\right\}. \tag{4.5}$$

The indicator function of P is the function $I_P : X \to \mathbb{R} \cup \{+\infty\}$ which has value 0 on P and $+\infty$ elsewhere. Let us note that when P is a convex subset of X, then $\partial I_P(x) = N_P(x)$; that is, the subdifferential of the indicator function is the normal cone [11], page 61.

4.2. Steering control in nonsmooth analysis setting

Even in the nonsmooth analysis setting, the minimum problem for J_Γ may be treated in view of generalized steepest descent scheme in the framework of subdifferential calculus. To this end, we state the following identification of the subdifferential of J_Γ.

Lemma 4.1. *For any $y_0 \in Y$, $T > 0$, the subdifferential of J_Γ at any $\varphi_0 \in Y'$ consists of $\psi_0^* \in Y$ such as*

$$\psi_0^* = e^{TA} y_0 + \int_0^T e^{(T-t)A} B u^*(t) \mathrm{d}t, \tag{4.6}$$

for some $u^* \in \widetilde{\Gamma}_T$ satisfying the maximum principle property

$$\max_{\mu \in \Gamma} \left\langle \mu, B^* e^{(T-t)A^*} \varphi_0 \right\rangle_U = \left\langle u^*(t), B^* e^{(T-t)A^*} \varphi_0 \right\rangle_U \quad a.e \text{ on } (0,T). \tag{4.7}$$

Proof. The lemma can be deduced from some properties of subdifferentials. Since J_Γ is the sum of finite convex functions, $\psi_0^* \in \partial J_\Gamma(\varphi_0)$ if and only if (see [11], Thm. 4.10, p. 63)

$$\psi_0^* \in \partial \left(\int_0^T S_\Gamma (B^* e^{(T-t)A^*} \varphi_0) \mathrm{d}t \right) + \partial \left(\langle \varphi_0, e^{TA} y_0 \rangle \right),$$

which is equivalent to

$$\psi_0^* - e^{TA} y_0 \in \partial \left(\int_0^T S_\Gamma (B^* e^{(T-t)A^*} \varphi_0) \mathrm{d}t \right). \tag{4.8}$$

In order to precise the subdifferential above, we introduce the functional $\mathcal{R} : L^2(0,T;U) \to \mathbb{R}$ given by

$$\mathcal{R}(u) = \int_0^T S_\Gamma(u(t)) \mathrm{d}t. \tag{4.9}$$

Then by applying the result in Theorem 23 of [30], page 62 it follows that $w \in \partial \mathcal{R}(u)$ if and only if,

$$w(t) \in \partial S_\Gamma(u(t)) \text{ a.e on } (0,T), \tag{4.10}$$

or, equivalently

$$S_\Gamma(v(t)) - S_\Gamma(u(t)) - \langle w(t), v(t) - u(t) \rangle \geq 0 \text{ a.e on } (0,T) \tag{4.11}$$

for all $v \in L^2(0,T;U)$. Moreover, exploiting the properties of the subgradient of a support function, (4.11) means also that for almost all $0 < t < T$, $w(t) \in \Gamma$ and $u(t)$ is normal to Γ at $w(t)$ in the sense of convex analysis so that (see [30], pp. 35–36)

$$\max_{\mu \in \Gamma} \langle \mu, u(t) \rangle_U = \langle w(t), u(t) \rangle_U \quad \text{a.e on } (0,T). \tag{4.12}$$

On the other hand, we note that

$$\begin{aligned} \int_0^T S_\Gamma(B^* e^{(T-t)A^*}\varphi_0)\mathrm{d}t &= \int_0^T S_\Gamma(L_T^*\varphi_0(t))\mathrm{d}t \\ &= \mathcal{R}\left(L_T^*\varphi_0\right), \end{aligned}$$

and by using Theorem 4.13 of [11], page 64, we obtain

$$\partial\left(\mathcal{R}(L_T^*\varphi_0)\right) = L_T \partial\mathcal{R}\left(L_T^*\varphi_0\right).$$

Hence, $\psi_0^* - e^{TA}y_0$ has the form

$$\psi_0^* - e^{TA}y_0 = L_T u^*,$$

for some $u^* \in \partial\mathcal{R}\left(L_T^*\varphi_0\right)$. Taking into account (4.12), u^* satisfies the maximum property (4.7). This completes the proof of the lemma. □

Remark 4.2.

(i) The assertion of Lemma 4.1 is similar to the one stated in [32]. However, the proof given in this reference is based on a direct determination of $\partial S_\Gamma(B^* e^{(T-t)A^*}\varphi_0)$ for a fixed time variable t in the interval $(0,T)$. Such a subdifferential may not be well defined when the control operator B is unbounded. The proof above is a generalization to the case of (eventually) unbounded admissible control operators.

(ii) In principle, the subdifferential of J_Γ is a multi-valued function. In the canonical constraint case $\Gamma = \Gamma_1$, the subdifferential of J_{Γ_1} at any $\varphi_0 \in Y'$ consists of $\psi_0^* \in Y$ satisfying (4.6) with

$$u^*(t) = \begin{cases} \dfrac{B^* e^{(T-t)A^*}\varphi_0}{\left\|B^* e^{(T-t)A^*}\varphi_0\right\|_U} \text{ if } B^* e^{(T-t)A^*}\varphi_0 \neq 0, \\ \Gamma_1 \text{ otherwise,} \end{cases} \tag{4.13}$$

for almost all $t \in (0,T)$. Hence, if

$$B^* e^{(T-t)A^*}\varphi_0 \neq 0 \text{ a.e on } (0,T),$$

then ψ_0^* is uniquely determined by (4.6), (4.13) and the subdifferential of J_{Γ_1} will correspond to the gradient $J'_{\Gamma_1}(\varphi_0)$.

The following theorem gives a strategy guaranteeing a control which steers the system (1.1) as close as possible to 0 at time T by minimizing J_Γ over the closed unit ball Φ_1 in Y'.

Theorem 4.3. *Suppose that $\widehat{\varphi}_0 \in Y'$ minimizes J_Γ on Φ_1. Then any solution $\widehat{u}$ to the problem* **(P)** *satisfies*

$$\max_{\mu\in\Gamma} \left\langle \mu, B^* e^{(T-t)A^*}\widehat{\varphi}_0 \right\rangle_U = \left\langle \widehat{u}(t), B^* e^{(T-t)A^*}\widehat{\varphi}_0 \right\rangle_U \quad a.e \text{ on } (0,T). \tag{4.14}$$

Moreover, the corresponding final state at time T given by

$$y_{\widehat{u}}(T) = e^{TA}y_0 + \int_0^T e^{(T-t)A}B\widehat{u}(t)\mathrm{d}t, \tag{4.15}$$

satisfies $y_{\widehat{u}}(T) \in \partial J_\Gamma(\widehat{\varphi}_0)$.

Proof. Let $F : L^2(0,T;U) \to \mathbb{R}$, $G : Y \to \mathbb{R}$ be given by

$$F(u) = \begin{cases} 0 \text{ if } u \in \widetilde{\Gamma}_T, \\ +\infty \text{ otherwise}, \end{cases} \tag{4.16}$$

$$G(y) = -\left\| e^{TA} y_0 + y \right\|_Y . \tag{4.17}$$

Then the problem **(P)** is equivalent to

$$\inf \left\{ \left\| e^{TA} y_0 + L_T(u) \right\|_Y \mid u \in \widetilde{\Gamma}_T \right\} = \inf \left\{ F(u) - G(L_T(u)) \mid u \in L^2(0,T;U) \right\}. \tag{4.18}$$

Written in this way, this problem is in the standard form for application of Rockafellar's extension of the duality theory in the sense of Fenchel (see [13], Chap. 3, [29]). The functionals F and G are, respectively, proper convex and concave functions. Moreover, for any $u \in \widetilde{\Gamma}_T$, F is finite at u and G is continuous at $L_T(u)$. Hence the problem (4.18) is "stably set" so that by applying Theorem 3 of [29], it can be shown that the problem

$$\min \left\{ J_\Gamma(\varphi_0) \mid \varphi_0 \in \Phi_1 \right\} \tag{4.19}$$

is dual to (4.18) in the following sense

$$\inf_{u \in L^2(0,T;U)} (F(u) - G(L_T(u)) + \min_{\varphi_0 \in \Phi_1} J_\Gamma(\varphi_0) = 0. \tag{4.20}$$

Furthermore, the "extremality condition" in [29] provides a necessary condition which must be satisfied by all solution pairs $\widehat{\varphi}_0$, solving (4.19) and $\widehat{u}$ solving (4.18). This extremality condition would read

$$L_T^*(\widehat{\varphi}_0) \in \partial F(\widehat{u}). \tag{4.21}$$

Since F coincides with the indicator function of $\widetilde{\Gamma}_T$, the condition (4.21) means that the normal cone of $\widetilde{\Gamma}_T$ at $\widehat{u}$, denoted by $N(\widehat{u})$, contains the function $L_T^*(\widehat{\varphi}_0)$. This amounts to

$$\int_0^T \left\langle \widehat{u}(t), B^* e^{(T-t)A^*} \widehat{\varphi}_0) \right\rangle_U \mathrm{d}t = \int_0^T \sup \left\{ \left\langle \mu, B^* e^{(T-t)A^*} \widehat{\varphi}_0) \right\rangle_U \mid \mu \in \Gamma \right\} \mathrm{d}t.$$

This is possible only if $\mu = \widehat{u}(t)$ achieves the supremum of $\left\langle \mu, B^* e^{(T-t)A^*} \widehat{\varphi}_0) \right\rangle$ for almost all $t \in [0,T]$. Equivalently, we must have (4.14). The remaining part of the proof is a ready consequence of Lemma 4.1. □

Remark 4.4.

- Since the initial state $y_0 \in Y$ satisfy $y_0 \neq 0$, whenever e^{TA} is one to one and the system (1.1) is Γ-null controllable at y_0 in time T, the null control cannot be a solution to the problem **(P)**. Moreover, from Theorem 2.2, we have

$$\min_{\varphi_0 \in Y'} J_\Gamma(\varphi_0) = \min_{\varphi_0 \in \Phi_1} J_\Gamma(\varphi_0) = 0.$$

  We note by using Theorem 4.3 that the obvious minimizer $\widehat{\varphi}_0 = 0$ is useless since it leads to a steering control function characterized by the trivial fact

$$\widehat{u}(t) \in \Gamma. \tag{4.22}$$

Hence, additional considerations, such as optimal time problem, are needed in order to build the steering control by using (4.14) (see [1]).

- On the contrary, if the system (1.1) is not Γ-null controllable at y_0 in time T, then any minimizer $\widehat{\varphi}_0$ would satisfy $\widehat{\varphi}_0 \neq 0$. In the same way, by exploiting other information such as minimum norm with respect to the control, it may be possible to obtain an appropriate control $\widehat{u}$ from (4.14). In principle, as noted in [32] in the case of multivariable systems, as T is increased, the corresponding control $\widehat{u}$ steers the system progressively closer to the origin.

In order to overcome the disadvantages mentioned above, we provide another alternative characterization of the steering controls by using a min-max strategy which has been treated in [32] for multivariable systems. Let $T > 0$, $y_0 \in Y$, and define the functional $\mathcal{M} : Y' \times \widetilde{\Gamma}_T \to \mathbb{R}$ by

$$\mathcal{M}(\varphi_0, u) = \int_0^T \left\langle B^* e^{(T-t)A^*} \varphi_0, u(t) \right\rangle_U \mathrm{d}t + \left\langle e^{TA^*} \varphi_0, y_0 \right\rangle. \tag{4.23}$$

The fact that the functional $\mathcal{M}$ is well defined is ensured by the admissibility condition (1.8). Then we have

Theorem 4.5. *The functional $\mathcal{M}$ possesses at least one saddle point $(\widehat{\varphi}_0, \widehat{u}) \in \Phi_1 \times \widetilde{\Gamma}_T$. Moreover, the control $\widehat{u}$ steers the system (1.1) to zero if and only if, $\mathcal{M}(\widehat{\varphi}_0, \widehat{u}) = 0$.*

Proof. We consider the set of reachable states from y_0 in time $T > 0$ defined by (1.12) and we introduce the functional $\mathcal{K} : \Phi_1 \times R_\Gamma(T; y_0) \to \mathbb{R}$ given by

$$\mathcal{K}(\varphi_0, y_u(T)) = \langle \varphi_0, y_u(T) \rangle . \tag{4.24}$$

We note that $\mathcal{K}$ satisfies the following conditions: (i) Φ_1 and $R_\Gamma(T; y_0)$ are convex, closed, and bounded. (ii) For all $\varphi_0 \in \Phi_1$, $\mathcal{K}(\varphi_0, .)$ is concave and upper semicontinuous. (iii) For all $y_u(T) \in R_\Gamma(T; y_0)$, $\mathcal{K}(., y_u(T))$ is convex and lower semicontinuous. By virtue of the result in Proposition 2.3 of [13], page 171, $\mathcal{K}$ possesses at least one saddle point on $\Phi_1 \times R_\Gamma(T; y_0)$. Moreover, clearly we have

$$\min_{\varphi_0 \in \Phi_1} \max_{u \in \widetilde{\Gamma}_T} \mathcal{M}(\varphi_0, u) = \min_{\varphi_0 \in \Phi_1} \max_{y_u(T) \in R_\Gamma(T; y_0)} \mathcal{K}(\varphi_0, y_u(T)).$$

Furthermore,

$$\max_{u \in \widetilde{\Gamma}_T} \min_{\varphi_0 \in \Phi_1} \mathcal{M}(\varphi_0, u) = \max_{y_u(T) \in R_\Gamma(T; y_0)} \min_{\varphi_0 \in \Phi_1} \mathcal{K}(\varphi_0, y_u(T))$$

Hence, $\mathcal{M}$ will also possess at least one saddle point $(\widehat{\varphi}_0, \widehat{u}) \in \Phi_1 \times \widetilde{\Gamma}_T$. To prove the last part of the theorem, we exploited the fact that

$$\mathcal{M}(\widehat{\varphi}_0, \widehat{u}) = \min_{\varphi_0 \in \Phi_1} \max_{u \in \widetilde{\Gamma}_T} \mathcal{M}(\varphi_0, u),$$

and we use a measurable selection argument, as in the proof of Theorem 2.2, to obtain

$$\min_{\varphi_0 \in \Phi_1} \max_{u \in \widetilde{\Gamma}_T} \mathcal{M}(\varphi_0, u) = \min_{\varphi_0 \in \Phi_1} J_\Gamma(\varphi_0),$$

so that

$$\mathcal{M}(\widehat{\varphi}_0, \widehat{u}) = \min_{\varphi_0 \in \Phi_1} J_\Gamma(\varphi_0).$$

On the other hand, from Remark 2.3, we deduce that y_0 can be steered to zero at time T if and only if

$$\min_{\varphi_0 \in \Phi_1} J_\Gamma(\varphi_0) = 0 = \mathcal{M}(\widehat{\varphi}_0, \widehat{u}).$$

Then the completion of the proof is reduced to see whether

$$\min_{\varphi_0 \in \Phi_1} J_\Gamma(\varphi_0) = 0 \;\Rightarrow\; y_{\widehat{u}}(T) = 0.$$

To this end, we note that

$$0 = \mathcal{M}(\widehat{\varphi}_0, \widehat{u}) \leq \mathcal{M}(\varphi_0, \widehat{u}) \;\text{ for all } \varphi_0 \in \Phi_1,$$

Or equivalently

$$0 \leq \left\langle \varphi_0, e^{TA} y_0 + \int_0^T e^{(T-t)A} B\widehat{u}(t)\mathrm{d}t \right\rangle \;\text{ for all } \varphi_0 \in \Phi_1.$$

Hence,

$$0 \leq \langle \varphi_0, y_{\widehat{u}}(T) \rangle \;\text{ for all } \varphi_0 \in \Phi_1.$$

Since 0 is an interior point of the closed unit ball Φ_1 in Y', this yields

$$y_{\widehat{u}}(T) = 0,$$

so that $\widehat{u}$ is an appropriate steering control. This completes the proof of the theorem. □

4.3. Steering control in smooth analysis setting

In this subsection, the systems under study are supposed conservative. Considering (4.1), we mention at the outset that when K is a quadratic form Q such as

$$Q(u) = \frac{1}{2} \|u\|_U^2 , \tag{4.25}$$

then the corresponding functional J_c, denoted by J_q, corresponds to

$$J_q(\varphi_0) = \frac{1}{2} \int_0^T \left\| B^* e^{(T-t)A^*} \varphi_0 \right\|_U^2 \mathrm{d}t + \left\langle e^{TA^*} \varphi_0, y_0 \right\rangle , \tag{4.26}$$

and the minimizer of J_q gives the appropriate unconstrained steering control by virtue of a generalized version of the HUM method developed in PDEs context [21, 22]. It is well known that this control is of minimal $L^2(0, T; U)$-norm whenever the system (1.1) is null controllable. As for the constrained Γ-null controllability, we mention the results established in [8] in the case where the control operator B is bounded and K is locally Lipschitz differentiable with $K'(U) \subset \Gamma$. Below we extend this result to the unbounded control operator case.

Theorem 4.6. *Suppose that $(e^{tA})_{t\in\mathbb{R}}$ is a group of isometries and K is convex, differentiable and globally Lipschitz. Assume also that $K'(U) \subset \mathit{\Gamma}$ and*

$$\liminf_{\|\varphi_0\|_{Y'}\to\infty} \frac{\int_0^T K(B^* e^{(T-t)A^*}\varphi_0)\mathrm{d}t}{\|\varphi_0\|_{Y'}} = \delta > 0. \tag{4.27}$$

Then the system (1.1) is locally $\mathit{\Gamma}$-null controllable in time T. Moreover, the control steering y_0 to 0 is given by

$$u(t) = K'(B^* e^{(T-t)A^*}\widehat{\varphi}_0) \text{ a.e on } (0,T), \tag{4.28}$$

where $\widehat{\varphi}_0$ satisfies

$$J_c(\widehat{\varphi}_0) = \min_{\varphi_0\in Y'} J_c(\varphi_0). \tag{4.29}$$

Proof. The functional J_c is clearly convex. On the other hand, J_c is also differentiable. To this end, we introduce the functional K_c given by

$$K_c(\varphi_0) = \int_0^T K\left(B^* e^{(T-t)A^*}\varphi_0\right)\mathrm{d}t. \tag{4.30}$$

Let $\theta_0 \in Y'$, $h_0 > 0$, and consider for $0 < |h| \le h_0$ the expression

$$\frac{K_c(\varphi_0 + h\theta_0) - K_c(\varphi_0)}{h} = \int_0^T \frac{K\left(B^* e^{(T-t)A^*}(\varphi_0 + h\theta_0)\right) - K\left(B^* e^{(T-t)A^*}\varphi_0\right)}{h}\mathrm{d}t. \tag{4.31}$$

In order to explicit its limit as $h \to 0$, we introduce

$$\mathit{\Theta}_h(t) = \frac{K\left(B^* e^{(T_0-t)A^*}(\varphi_0 + h\theta_0)\right) - K\left(B^* e^{(T_0-t)A^*}\varphi_0\right)}{h}. \tag{4.32}$$

From the global Lipschitz property, we have for some positive constant m_K

$$|K(v) - K(w)| \le m_K \|v - w\|_U, \ \forall v, w \in U. \tag{4.33}$$

Thus, we obtain

$$|\mathit{\Theta}_h(t)| \le m_K \left\|B^* e^{(T-t)A^*}\theta_0\right\|_U \text{ a.e on } (0,T), 0 < |h| \le h_0. \tag{4.34}$$

Considering the function $L_T^*\theta_0$ given by

$$L_T^*\theta_0(t) = B^* e^{(T-t)A^*}\theta_0,$$

the admissibility condition (1.8) ensures that, after extension, we have $L_T^*\theta_0 \in L^2(0,T;U)$. Hence, the dominated convergence theorem yields

$$\lim_{h\to 0} \frac{K_c(\varphi_0 + h\theta_0) - K_c(\varphi_0)}{h} = \int_0^T \left\langle K'\left(B^* e^{(T-t)A^*}\varphi_0\right), B^* e^{(T-t)A^*}\theta_0\right\rangle_U \mathrm{d}t, \tag{4.35}$$

so that

$$J_c'(\varphi_0)\theta_0 = \int_0^T \left\langle K'\left(B^* e^{(T-t)A^*}\varphi_0\right), B^* e^{(T-t)A^*}\theta_0\right\rangle_U \mathrm{d}t + \left\langle e^{TA^*}\theta_0, y_0\right\rangle. \tag{4.36}$$

Moreover, if the initial state satisfies $\|y_0\|_Y < \delta$, then J_c is coercive, *i.e.*

$$\liminf_{\|\varphi_0\|\to\infty} J_c(\varphi_0) = \infty. \tag{4.37}$$

Indeed, since $(e^{tA})_{t\in\mathbb{R}}$ is a group of isometries, we get from (4.1)

$$\begin{aligned} \liminf_{\|\varphi_0\|_{Y'}\to\infty} \frac{J_c(\varphi_0)}{\|\varphi_0\|_{Y'}} &\geq \liminf_{\|\varphi_0\|_{Y'}\to\infty} \frac{\int_0^T K(B^* e^{(T-t)A^*}\varphi_0)\mathrm{d}t}{\|\varphi_0\|} - \|y_0\|_Y, \\ &\geq \delta - \|y_0\|_Y, \end{aligned} \tag{4.38}$$

so that

$$\liminf_{\|\varphi_0\|\to\infty} \frac{J_c(\varphi_0)}{\|\varphi_0\|_{Y'}} > 0. \tag{4.39}$$

Therefore, the functional J_c achieves a minimum $\widehat{\varphi}_0$ which is characterized by $J_c'(\widehat{\varphi}_0) = 0$. Thus

$$\langle J_c'(\widehat{\varphi}_0), \varphi_0\rangle = 0 \text{ for all } \varphi_0 \in Y'. \tag{4.40}$$

This yields for all $\varphi_0 \in Y'$,

$$\int_0^T \left\langle K'(B^* e^{(T-t)A^*}\widehat{\varphi}_0), B^* e^{(T-t)A^*}\varphi_0\right\rangle_U \mathrm{d}t + \left\langle e^{T_0 A^*}\varphi_0, y_0\right\rangle = 0. \tag{4.41}$$

Hence, from Proposition 1.2, we conclude that the control given by (4.28)–(4.29) steers the state y_0 to the origin. This completes the proof of the theorem. □

Despite the fact that condition (4.27) is difficult to check, it seems however an appropriate tool in presenting the problem of constrained null controllability in a unified way. Also, it enables us to include the unconstrained null controllability question. Here, we shall give more explicit conditions on K under which the coercivity condition (4.27) is ensured. The proof is quite similar to the one given in [8] in the case where the control operator B is bounded.

Theorem 4.7. *Suppose that $(e^{tA})_{t\in\mathbb{R}}$ is a group of isometries and K is convex, differentiable, globally Lipschitz and for some positive constants α, β*

$$K(u) \geq \alpha \inf(\|u\|_U, \|u\|_U^2) - \beta \textit{ for all } u \in U. \tag{4.42}$$

Moreover, suppose that the following observability condition holds

$$\inf_{\|\varphi_0\|_{Y'}=1} \int_0^T \inf\left(\left\|B^* e^{(T-t)A^*}\varphi_0\right\|_U^2, \left\|B^* e^{(T-t)A^*}\varphi_0\right\|_U\right) \mathrm{d}t \geq c > 0. \tag{4.43}$$

Assume also that $K'(U) \subset \Gamma$. Then the system (1.1) is locally Γ-null controllable in time T. Moreover, the control driving y_0 to the origin is given by (4.28)–(4.29).

Proof. Following Theorem 4.6, it is sufficient to prove that

$$\liminf_{\|\varphi_0\|_{Y'}\to\infty} \frac{\int_0^T K(B^* e^{(T-t)A^*}\varphi_0)\mathrm{d}t}{\|\varphi_0\|_{Y'}} > 0. \tag{4.44}$$

To this end, we introduce

$$I = \left\{ t \in (0,T) \mid \left\| B^* e^{(T-t)A^*}\varphi_0 \right\|_U \leq 1 \right\}, \tag{4.45}$$

$$J = \left\{ t \in (0,T) \mid \left\| B^* e^{(T-t)A^*}\varphi_0 \right\|_U > 1 \right\}, \tag{4.46}$$

and the function $\lambda : U \to \mathbb{R}^+$ defined by

$$\lambda(u) = \alpha \inf(\|u\|_U, \|u\|_U^2). \tag{4.47}$$

Then

$$\int_0^T K\left(B^* e^{(T-t)A^*}\varphi_0\right)\mathrm{d}t \geq \int_0^T \lambda\left(B^* e^{(T-t)A^*}\varphi_0\right)\mathrm{d}t - \beta T$$

$$\geq \alpha \int_I \left\| B^* e^{(T-t)A^*}\varphi_0 \right\|_U^2 \mathrm{d}t + \alpha \int_J \left\| B^* e^{(T-t)A^*}\varphi_0 \right\|_U \mathrm{d}t - \beta T$$

On the other hand, without loss of generality, we can assume that $\|\varphi_0\| > 1$. By setting

$$\widetilde{\varphi}_0 = \frac{\varphi_0}{\|\varphi_0\|_{Y'}}, \tag{4.48}$$

we get from above

$$\frac{\int_0^T K\left(B^* e^{(T-t)A^*}\varphi_0\right)\mathrm{d}t}{\|\varphi_0\|} \geq \alpha \|\varphi_0\|_{Y'} \int_I \left\| B^* e^{(T-t)A^*}\widetilde{\varphi}_0 \right\|_U^2 \mathrm{d}t$$

$$+\alpha \int_J \left\| B^* e^{(T-t)A^*}\widetilde{\varphi}_0 \right\|_U \mathrm{d}t - \frac{\beta T_0}{\|\varphi_0\|_{Y'}}.$$

From which we deduce easily

$$\frac{\int_0^T K\left(B^* e^{(T-t)A^*}\varphi_0\right)\mathrm{d}t}{\|\varphi_0\|_{Y'}} \geq \alpha \int_I \left\| B^* e^{(T-t)A^*}\widetilde{\varphi}_0 \right\|_U^2 \mathrm{d}t + \alpha \int_J \left\| B^* e^{(T-t)A^*}\widetilde{\varphi}_0 \right\|_U \mathrm{d}t - \frac{\beta T}{\|\varphi_0\|_{Y'}},$$

so that

$$\frac{\int_0^T \left(B^* e^{(T_0-t)A^*}\varphi_0\right)\mathrm{d}t}{\|\varphi_0\|_{Y'}} \geq \int_0^T \lambda\left(B^* e^{(T-t)A^*}\widetilde{\varphi}_0\right)\mathrm{d}t - \frac{\beta T}{\|\varphi_0\|_{Y'}}. \tag{4.49}$$

The proof of (4.44) can be reduced to

$$\liminf_{\|\varphi_0\|_{Y'}\to\infty} \int_0^T \lambda\left(B^* e^{(T-t)A^*}\widetilde{\varphi}_0\right) \mathrm{d}t > 0. \tag{4.50}$$

By contradiction, suppose that for some sequence $(\varphi_n)_n$, we have

$$\lim_{n\to\infty} \int_0^{T_0} \lambda(B^* e^{(T-t)A^*}\widetilde{\varphi}_n)\mathrm{d}t = 0, \tag{4.51}$$

where $\widetilde{\varphi}_n = \frac{\varphi_n}{\|\varphi_n\|_{Y'}}$. Consider on $(0,T)$ the sequence $(\alpha_n)_n$ of non-negative functions given by

$$\alpha_n(t) = \lambda\left(B^* e^{(T-t)A^*}\widetilde{\varphi}_n\right). \tag{4.52}$$

This sequence is non-negative on $(0,T)$ and (4.51) implies $\alpha_n \to 0$ in $L^1(0,T)$ as $n\to\infty$. This contradicts the observability condition (4.43) characterized by

$$\int_0^T \inf\left(\left\|B^* e^{(T-t)A^*}\widetilde{\varphi}_n\right\|_U^2, \left\|B^* e^{(T-t)A^*}\widetilde{\varphi}_n\right\|_U\right) \mathrm{d}t \geq c > 0 \text{ for all } n. \tag{4.53}$$

This completes the proof of the theorem. □

Remark 4.8. It is clear that the observability condition (4.43) is stronger than both L^1-observability and L^2-observability. Considering carefully the proof presented in [8], it can be easily seen that when the control operator B is bounded, these three observability conditions are equivalent.

Here our concern will focus on the practical situation where the control actuation is subject to the canonical constraint set $\mathit{\Gamma}_1$. The appropriate convex functional K has been introduced in [8] and we shall extend this choice to the unbounded control operator case. Let $\rho : \mathbb{R}^+ \to \mathbb{R}$ be a continuous function with $r\rho(r)$ nondecreasing and

$$\rho(s) > 0 \text{ for all } s \geq 0, \tag{4.54}$$

$$\rho(s) \leq \min\left(1, \frac{1}{s}\right) \text{ for all } s > 0. \tag{4.55}$$

Then we consider the functional $K_\rho : U \to \mathbb{R}^+$ given by

$$K_\rho(u) = \int_0^{\|u\|_U} s\rho(s)\mathrm{d}s. \tag{4.56}$$

The function $t \mapsto \int_0^t s\rho(s)\mathrm{d}s$ is clearly convex on $\mathbb{R}^+$. Thus K_ρ is convex on U. Note also that the associated functional J_ρ given by

$$J_\rho(\varphi_0) = \int_0^T K_\rho\left(B^* e^{(T_0-t)A^*}\varphi_0\right) \mathrm{d}t + \left\langle y_0, e^{TA^*}\varphi_0\right\rangle, \tag{4.57}$$

allows us to recover both the functionals J_{Γ_1} and J_q as degenerate cases. Indeed, while J_{Γ_1} corresponds to the situation where

$$s\rho(s) = 1,\ \forall s \geq 0,$$

the functional J_q can be interpreted as the limit case in which $\rho \equiv 1$.

Theorem 4.9. *Suppose that $(e^{tA})_{t\in\mathbb{R}}$ is a group of isometries and the observability condition (4.43) holds. Then the system (1.1) is locally Γ_1-null controllable in time T and the control driving y_0 to the origin is given by*

$$u(t) = \rho\left(\left\|B^* e^{(T-t)A^*}\widehat{\varphi}_0\right\|_U\right) B^* e^{(T-t)A^*}\widehat{\varphi}_0, \tag{4.58}$$

where $\widehat{\varphi}_0$ satisfies

$$J_\rho(\widehat{\varphi}_0) = \min_{\varphi_0\in Y'} J_\rho(\varphi_0). \tag{4.59}$$

Proof. The fact that the system (1.1) is locally Γ_1-null controllable results from the L^1-observability implied by (4.43). Moreover, it is easy to check

$$\left\langle K'_\rho(u), v\right\rangle_U = \langle \rho(\|u\|_U)u, v\rangle_U \text{ for all } u, v \in U, \tag{4.60}$$

so that (4.55) yields

$$\left\|K'_\rho(u)\right\|_U \leq 1 \text{ for all } u \in U. \tag{4.61}$$

Hence, $K'_\rho(U) \subset \Gamma_1$ holds true and the mean theorem implies that K_ρ is globally Lipschitz. Following Theorem 4.7, it is sufficient to see that K_ρ satisfies (4.42). The proof of this estimate can be found in [8]. □

Remark 4.10. For any $r > 0$ and considering the constraint control subset Γ_r given by (2.25), we can as well define an appropriate functional, denoted by K_{ρ^r}, and similar to K_ρ such that $K'_{\rho^r}(U) \subset \Gamma_r$. Indeed, let $\rho^r : \mathbb{R}^+ \to \mathbb{R}$ be a continuous function with $s\rho^r(s)$ nondecreasing and

$$\rho^r(s) > 0 \text{ for all } s \geq 0, \tag{4.62}$$

$$\rho^r(s) \leq \min\left(r, \frac{r}{s}\right) \text{ for all } s > 0. \tag{4.63}$$

Then the functional $K_{\rho^r} : U \to \mathbb{R}^+$ is given by

$$K_{\rho^r}(u) = \int_0^{\|u\|_U} s\rho^r(s)\mathrm{d}s. \tag{4.64}$$

4.4. Applications II

Example 4.11. *Wave equation with distributed control (continued).* We consider the wave equation (3.1) and we suppose that the data (ω, T) satisfy one of the conditions ensuring the equivalent observability inequalities (3.5)–(3.6). Following Theorem 4.7 and Remark 4.8, the steering control can be obtained by minimizing the analogue of the functional J_ρ defined by (4.56)–(4.57) in the context of smooth analysis. The appropriate functional, denoted by $\mathcal{J}_\rho$, is defined by $\mathcal{J}_\rho : H^{-1}(\Omega) \times L^2(\Omega) \to \mathbb{R}$ and

$$\mathcal{J}_\rho(\{\eta^0, \eta^1\}) = \int_0^T \Psi_\rho(\chi_\omega(.)\eta(.,t))\mathrm{d}t - \langle \eta'(.,0), z_0\rangle + \langle \eta(.,0), z_1\rangle \,, \tag{4.65}$$

where η is solution to (3.25) and the functional $\Psi_\rho : L^2(\Omega) \to \mathbb{R}$ is given by

$$\Psi_\rho(u) = \int_0^{\|u\|_{L^2(\Omega)}} s\rho(s)\mathrm{d}s. \tag{4.66}$$

Then by applying Proposition 3.2, we get either local Γ_1^ω-null controllability or global Γ_1^ω-null controllability in finite time. In both cases, the appropriate control steering the system (3.1) from $\{z_0, z_1\}$ to $(0,0)$ at time T has the form

$$u(x,t) = \rho\left(\|\chi_\omega(.)\widehat{\eta}(.,t)\|_{L^2(\Omega)}\right)\chi_\omega(x)\widehat{\eta}(x,t), \tag{4.67}$$

where $\widehat{\eta}$ is the solution of (3.25) with terminal data $\left\{\widehat{\eta}(.,T) = -\widehat{\eta}^1, \widehat{\eta}'(.,T) = \widehat{\eta}^0\right\} \in L^2(\Omega) \times H^{-1}(\Omega)$ satisfying

$$\mathcal{J}_\rho\left(\{\widehat{\eta}^0, \widehat{\eta}^1\}\right) = \min\left\{\mathcal{J}_\rho\left(\{\eta^0, \eta^1\}\right) \mid \{\eta^0, \eta^1\} \in H^{-1}(\Omega) \times L^2(\Omega)\right\}. \tag{4.68}$$

Example 4.12. *Euler-Bernoulli beam with pointwise actuator (continued).* We consider the Euler-Bernoulli beam model (3.9) and we suppose that the actuator position a is a rational number with coprime factorization $a = \frac{a_1}{a_2}$ such that a_1 is odd. It has been noticed that the lack of local Γ_1^a-null controllability is a consequence of the fact that the L^1-observability cannot occur. Hence, the determination of the steering control in the smooth analysis setting is precluded. Considering the nonsmooth analysis setting, we shall consider the two approaches based on Theorem 4.3 and Theorem 4.5.

Considering the first one, we introduce the analogue of the functional in (2.8) corresponding to the constraint control set $\Gamma_1^a = [-1,1]$ as follows. This functional, denoted by $\mathcal{J}_a$, is defined by $\mathcal{J}_a : H_{-\frac{1}{2}} \times H \to \mathbb{R}$ and

$$\mathcal{J}_a(\{\eta^0, \eta^1\}) = \int_0^T |\eta(a,t)|\,\mathrm{d}t - \langle \eta'(.,0), z_0\rangle + \langle \eta(.,0), z_1\rangle \,, \tag{4.69}$$

where η is the solution of (3.16) with $\eta(.,T) = -\eta^1$, $\eta'(.,T) = \eta^0$. It follows from Theorem 4.3 that any control u solution to the problem **(P)** with respect to the state space $H_{\frac{1}{2}} \times H$ would satisfy

$$u(t) = \begin{cases} \mathrm{sgn}(\widehat{\eta}(a,t)) \text{ if } \widehat{\eta}(a,t) \neq 0, \\ [-1,1] \text{ otherwise,} \end{cases} \tag{4.70}$$

for almost all $t \in (0,T)$, where sgn denotes the multi-valued function defined by

$$\mathrm{sgn}(s) = \begin{cases} 1 \text{ if } s > 0, \\ -1 \text{ if } s < 0, \\ [-1,1] \text{ if } s = 0, \end{cases} \tag{4.71}$$

and $\widehat{\eta}$ is the solution of (3.16) with terminal data $\left\{\widehat{\eta}(T,.) = -\widehat{\eta}^1, \widehat{\eta}'(T,.) = \widehat{\eta}^0\right\} \in H \times H_{-\frac{1}{2}}$ satisfying

$$\mathcal{J}_a\left(\{\widehat{\eta}^0, \widehat{\eta}^1\}\right) = \min\left\{\mathcal{J}_a\left(\{\eta^0, \eta^1\}\right) \mid \left\|\{\eta^0, \eta^1\}\right\|_{H_{-\frac{1}{2}} \times H} \leq 1\right\}. \tag{4.72}$$

Hence, whenever $\left\{\widehat{\eta}^0, \widehat{\eta}^1\right\} \neq \{0, 0\}$, the steering control is of bang bang type by virtue of the actuator position a and the use of generalized Ingham inequality [5].

As for the second approach, let $\widetilde{\Gamma}^a_{1,T}$ denote the set of Γ^a_1-admissible controls given by

$$\widetilde{\Gamma}^a_{1,T} = \left\{u \in L^2(0,T) \mid |u(t)| \leq 1 \text{ a.e on } (0,T)\right\}, \tag{4.73}$$

and denote by $\mathcal{M}_a$ the analogue of the functional in (4.23). Clearly $\mathcal{M}_a$ is given by $\mathcal{M}_a : \left(H_{-\frac{1}{2}} \times H\right) \times \widetilde{\Gamma}^a_{1,T} \to \mathbb{R}$ and

$$\mathcal{M}_a\left(\left\{\eta^0, \eta^1\right\}, u\right) = \int_0^T u(t)\eta(a,t)\mathrm{d}t - \langle \eta'(.,0), z_0\rangle + \langle \eta(.,0), z_1\rangle, \tag{4.74}$$

where η is defined by (3.16). Then by applying Theorem 4.5 we obtain that the functional $\mathcal{M}_a$ possesses at least one saddle point $\left(\left\{\widehat{\eta}^0, \widehat{\eta}^1\right\}, \widehat{u}\right)$ such that

$$\mathcal{M}_a\left(\left\{\widehat{\eta}^0, \widehat{\eta}^1\right\}, \widehat{u}\right) = \min\left\{\max_{u \in \widetilde{\Gamma}^a_{1,T}} \mathcal{M}_a\left(\left\{\eta^0, \eta^1\right\}, u\right) \mid \left\|\{\eta^0, \eta^1\}\right\|_{H_{-\frac{1}{2}} \times H} \leq 1\right\}.$$

Moreover, the control $\widehat{u}$ steers the system (3.9) to $\{0, 0\}$ at time T if and only if, $\mathcal{M}_a\left(\left\{\widehat{\eta}^0, \widehat{\eta}^1\right\}, \widehat{u}\right) = 0$.

Example 4.13. *Wave equation with Dirichlet boundary control (continued).* We consider the wave equation (3.20) and we suppose that the data (γ, T) satisfies sufficient conditions ensuring the observability inequality (3.28). We have already pointed out the lack of local Γ^γ_1-null controllability as a consequence of the fact that the L^1-observability cannot occur. In contrast to the Euler-Bernoulli model (3.9), we shall see that the determination of the steering control can be treated in both the nonsmooth analysis and the smooth analysis settings. Considering the nonsmooth analysis setting, we shall consider the two approaches based on Theorems 4.3 and 4.5.

Considering the first one, we define the analogue of the functional in (2.8) corresponding to the constraint control set Γ^γ_1 by $\mathcal{J}_\gamma : L^2(\Omega) \times H^1_0(\Omega) \to \mathbb{R}$ and

$$\mathcal{J}_\gamma(\{\eta^0, \eta^1\}) = \int_0^T \left(\int_\gamma \left|\frac{\partial \eta}{\partial \nu}(\sigma, t)\right| \mathrm{d}\sigma\right) \mathrm{d}t - \langle \eta'(.,0), z_0\rangle + \langle \eta(.,0), z_1\rangle, \tag{4.75}$$

where η is solution to (3.25). It follows from Theorem 4.3 that any control u solution to the problem **(P)** with respect to the state space $L^2(\Omega) \times H^{-1}(\Omega)$ would satisfy

$$u(.,t) = \begin{cases} \chi_\gamma(.) \dfrac{\frac{\partial \widehat{\eta}}{\partial \nu}(.,t)}{\left\|\chi_\gamma(.) \frac{\partial \widehat{\eta}}{\partial \nu}(.,t)\right\|_{L^2(\partial\Omega)}} & \text{if } \chi_\gamma(.) \dfrac{\partial \widehat{\eta}}{\partial \nu}(.,t) \neq 0, \\ \Gamma^\gamma_1 \text{ otherwise}, \end{cases} \tag{4.76}$$

for almost all $t \in (0,T)$, where $\widehat{\eta}$ is the solution of (3.25) with terminal data $\left\{\widehat{\eta}(T,.) = -\widehat{\eta}^1, \widehat{\eta}'(T,.) = \widehat{\eta}^0\right\} \in L^2(\Omega) \times H_0^1(\Omega)$ satisfying

$$\mathcal{J}_\gamma\left(\left\{\widehat{\eta}^0, \widehat{\eta}^1\right\}\right) = \min\left\{\mathcal{J}_\gamma\left(\left\{\eta^0, \eta^1\right\}\right) \mid \left\|\left\{\eta^0, \eta^1\right\}\right\|_{L^2(\Omega)\times H_0^1(\Omega)} \leq 1\right\}. \tag{4.77}$$

It turns out that, due to the finite velocity of propagation for the wave equation, the set of time instances t for which

$$\frac{\partial \widehat{\eta}}{\partial \nu}(.,t) = 0 \text{ on } \gamma,$$

is of positive measure. In other words, despite the fact that (4.76) gives information on the steering control u, this information cannot determine u uniquely.

As for the min-max approach, let $\widetilde{\Gamma}_{1,T}^{\gamma}$ denote the set of Γ_1^γ-admissible controls given by

$$\widetilde{\Gamma}_{1,T}^{\gamma} = \left\{\chi_\gamma u \,|\, u \in L^2(0,T;L^2(\partial\Omega)) \text{ and } \int_\gamma |u(\sigma,t)|^2 \,\mathrm{d}\sigma \leq 1 \text{ a.e on } (0,T)\right\}, \tag{4.78}$$

and denote by $\mathcal{M}_\gamma$ the analogue of the functional in (4.23). Clearly $\mathcal{M}_\gamma$ is given by $\mathcal{M}_\gamma : \left(L^2(\Omega) \times H_0^1(\Omega)\right) \times \widetilde{\Gamma}_{1,T}^{\gamma} \to \mathbb{R}$ and

$$\mathcal{M}_\gamma\left(\left\{\eta^0, \eta^1\right\}, u\right) = \int_{\gamma\times(0,T)} u(\sigma,t)\frac{\partial \eta}{\partial \nu}(\sigma,t)\mathrm{d}\sigma\mathrm{d}t - \langle \eta'(.,0), z_0\rangle + \langle \eta(.,0), z_1\rangle, \tag{4.79}$$

where η is defined by (3.25). Then by applying Theorem 4.5 we obtain that the functional $\mathcal{M}_\gamma$ possesses at least one saddle point $\left(\left\{\widehat{\eta}^0, \widehat{\eta}^1\right\}, \widehat{u}(.)\right)$ such that

$$\mathcal{M}_\gamma\left(\left\{\widehat{\eta}^0, \widehat{\eta}^1\right\}, \widehat{u}(.)\right) = \min\left\{\max_{u\in\widetilde{\Gamma}_{1,T}^{\gamma}} \mathcal{M}_\gamma\left(\left\{\eta^0, \eta^1\right\}, u\right) \mid \left\|\left\{\eta^0, \eta^1\right\}\right\|_{L^2(\Omega)\times H_0^1(\Omega)} \leq 1\right\}.$$

Moreover, the control $\widehat{u}$ steers the system (3.20) to $\{0,0\}$ at time T if and only if, $\mathcal{M}_\gamma\left(\left\{\widehat{\eta}^0, \widehat{\eta}^1\right\}, \widehat{u}\right) = 0$.

Finally, we shall exploit the results obtained in the internal distributed control case in order to build the steering control in the smooth convex analysis setting. Let us consider the system (3.20) and suppose that both Ω and γ are contained in a larger open set Ω_γ so that the data $(\omega_\gamma = \Omega_\gamma|\overline{\Omega}, T)$ fulfil the observability condition defined by (3.31)–(3.32). Let $\{\widetilde{z}_0, \widetilde{z}_1\} \in H_0^1(\Omega_\gamma) \times L^2(\Omega_\gamma)$ denote the extension of $\{z_0, z_1\} \in H_0^1(\Omega) \times L^2(\Omega)$ to Ω_γ given by (3.37). Then combining Proposition 3.5 and the form of the auxiliary internal steering control in (4.58)–(4.59), we conclude that there exists $\delta > 0$ such that for any initial state $\{z_0, z_1\} \in H_0^1(\Omega) \times L^2(\Omega)$ satisfying (3.33), there exists a Γ_1^γ-admissible steering control $\widehat{u}$ given by

$$\widehat{u}(.,t) = \widehat{z}(.,t)_{|\gamma}, \tag{4.80}$$

where $\widehat{z}$ is defined as follows.

- $\widehat{z}$ is solution to

$$\begin{cases} \widehat{z}'' = \Delta\widehat{z} + \widetilde{u}\chi_\omega \text{ in } \Omega_\gamma \times (0,\infty), \\ \widehat{z} = 0 \text{ on } \partial\Omega_\gamma \times (0,\infty), \\ \widehat{z}(x,0) = \widetilde{z}_0(x),\ \frac{\partial \widehat{z}}{\partial t}(x,0) = \widetilde{z}_1(x) \text{ in } \Omega_\gamma. \end{cases} \tag{4.81}$$

- $\widetilde{u}$ is given by

$$\widetilde{u}(x,t) = \rho^r \left(\left\| \chi_{\omega_\gamma}(.)\widehat{\eta}(.,t) \right\|_{L^2(\Omega_\gamma)} \right) \chi_{\omega_\gamma}(x)\widehat{\eta}(x,t), \tag{4.82}$$

- $\widehat{\eta}$ is the solution of (3.25) with terminal data $\left\{ \widehat{\eta}(.,T) = -\widehat{\eta}^1, \widehat{\eta}'(.,T) = \widehat{\eta}^0 \right\} \in L^2(\Omega_\gamma) \times H^{-1}(\Omega_\gamma)$ satisfying

$$\mathcal{J}^\gamma_{\rho^r}\left(\left\{ \widehat{\eta}^0, \widehat{\eta}^1 \right\} \right) = \min \left\{ \mathcal{J}^\gamma_{\rho^r}\left(\left\{ \eta^0, \eta^1 \right\} \right) \mid \left\{ \eta^0, \eta^1 \right\} \in H^{-1}(\Omega_\gamma) \times L^2(\Omega_\gamma) \right\}. \tag{4.83}$$

- The functional $\mathcal{J}^\gamma_{\rho^r}$ is defined by $\mathcal{J}^\gamma_{\rho^r} : H^{-1}(\Omega_\gamma) \times L^2(\Omega_\gamma) \to \mathbb{R}$ and

$$\mathcal{J}^\gamma_{\rho^r}(\{\eta^0, \eta^1\}) = \int_0^T \Psi_{\rho^r}(\chi_{\omega_\gamma}(.)\eta(.,t))\mathrm{d}t - \langle \eta'(.,0), z_0 \rangle + \langle \eta(.,0), z_1 \rangle, \tag{4.84}$$

  where η is the solution of (3.25) with $\eta(.,T) = -\eta^1$, $\eta'(.,T) = \eta^0$.
- The functional Ψ_{ρ^r} is given by $\Psi_{\rho^r} : L^2(\Omega_\gamma) \to \mathbb{R}$,

$$\Psi_{\rho^r}(u) = \int_0^{\|u\|_{L^2(\Omega_\gamma)}} s\rho^r(s)\mathrm{d}s, \tag{4.85}$$

  where ρ^r has been introduced in Remark 4.10.

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