

DYNAMIC MODELS OF WASSERSTEIN-1-TYPE UNBALANCED TRANSPORT

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Abstract. We consider a class of convex optimization problems modelling temporal mass transport and mass change between two given mass distributions (the so-called dynamic formulation of unbalanced transport), where we focus on those models for which transport costs are proportional to transport distance. For those models we derive an equivalent, computationally more efficient static formulation, we perform a detailed analysis of the model optimizers and the associated optimal mass change and transport, and we examine which static models are generated by a corresponding equivalent dynamic one. Alongside we discuss thoroughly how the employed model formulations relate to other formulations found in the literature.

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1. INTRODUCTION

Optimal transport seeks the optimal way of transporting mass from a given initial distribution ρ_0 to a final distribution ρ_1 , both on some domain $\Omega \subset \mathbb{R}^n$. In Kantorovich's classical formulation, the transport is described by a transport plan or coupling π , where $\pi(x, y)$ is the amount of mass transported from x to y so that, formally, $\rho_0 = \pi(\cdot \times \Omega)$ and $\rho_1 = \pi(\Omega \times \cdot)$. Among all those couplings, the optimal one minimizes

$$\int_{\Omega \times \Omega} c(x, y) \, d\pi(x, y),$$

where $c(x, y)$ denotes the cost per mass unit for transport from x to y . Solving the above minimization problem is computationally very costly due to the high dimensionality of π . An equivalent convex formulation (originally for the case $c(x, y) = \|x - y\|^2$) in much lower dimensions is provided by the celebrated Benamou–Brenier formula [4], which describes the transport *via* a material flow on Ω during a time interval $[0, 1]$. For the special case $c(x, y) = \|x - y\|$, in which the total transport cost is known as the Wasserstein-1 or W_1 distance between ρ_0 and ρ_1 and in which the transport cost is proportional to the transport distance, one can even eliminate the time coordinate, reducing the problem dimensionality yet further.

Keywords and phrases: Wasserstein distance, unbalanced transport, convex optimization.

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1.1. Unbalanced optimal transport

In applications that need to quantify how close two mass distributions ρ_0 and ρ_1 are to each other, pure optimal transport does typically not suffice as a similarity measure since it requires ρ_0 and ρ_1 to have the same mass. Therefore, optimal transport models have recently been extended to the case of so-called unbalanced transport, where the masses are allowed to change during the transport. Early proposals for unbalanced transport problems can be found, for instance, in [3, 14].

A more systematic investigation started from dynamic formulations based on the Benamou–Brenier formula for the Wasserstein-2 distance by adding a source term to the mass conservation constraint and a suitable corresponding penalty to the energy functional. In [8, 10, 12] the source penalty was chosen to be the Fisher–Rao or Hellinger distance, which leads to the Wasserstein–Fisher–Rao (WFR) or Hellinger–Kantorovich (HK) distance. In [13, 16] variants of the total variation norm are studied as penalties. All models are careful to retain some form of 1-homogeneity (at least in space) to allow for spatially singular measures.

In [7, 12] equivalent expressions for the WFR/HK distance are derived, based on an extension of the “static” Kantorovich formulation. In [12] the mass change is modelled by relaxing the exact marginal constraints $\rho_0 = \pi(\cdot \times \Omega)$ and $\rho_1 = \pi(\Omega \times \cdot)$ to soft marginal constraints where one penalizes the deviation with suitable entropy functionals. In [7] transport is described by two so-called semi-couplings (γ_0, γ_1) with $\rho_0 = \gamma_0(\cdot \times \Omega)$ and $\rho_1 = \gamma_1(\Omega \times \cdot)$. Intuitively, $\gamma_0(x, y)$ describes the mass starting out at x with destination y , while $\gamma_1(x, y)$ is the mass arriving in y from x . The Kantorovich functional is adapted suitably. It is shown that for a family of dynamic unbalanced problems (beyond WFR/HK) one can find corresponding semi-coupling formulations. For the WFR/HK distance the static formulas given in [7, 12] are related *via* dualization and a change of variables ([7], Cor. 5.9).

Due to its special structure, unbalanced extensions of the Wasserstein-1 distance have attracted particular attention. Marginal constraint relaxations of Wasserstein-1 where the deviation of the π -marginals from ρ_0 and ρ_1 is penalized by the total variation norm are studied, for instance, in [11, 15]. This is closely related to the optimal partial transport problem studied in [6]. The article [16] cited above gives essentially a dynamic reformulation of this distance. It is observed that this extension leads to a modified form of the Kantorovich–Rubinstein formula. A family of more general unbalanced extensions of this formula (in a certain sense the most general family) is studied in [21].

Roughly speaking, the above discussion mentions three types of formulations for unbalanced transport problems: “dynamic” formulations based on the Benamou–Brenier formula, “static” semi-coupling extensions of the Kantorovich formulation, and unbalanced Wasserstein-1-type extensions of the Kantorovich–Rubinstein formula. In the following, we refer to these families by the shorthands (Dyn), (SC), and (W1T). *via* convex duality, each of these formulations can be expressed in a primal and a dual form, which we denote by a suffix (P) or (D). (By convention we refer to the measure formulation as primal, even though measures are identified with the topological dual of continuous functions.)

As discussed, it was observed that various unbalanced transport distances can be expressed in more than one formulation. In this article we study systematically the correspondence between unbalanced extensions of the Wasserstein-1 distance in the formulations (Dyn), (SC), and (W1T). A schematic relation between different (primal and dual) formulations, established correspondences, and new correspondences established in this article is shown in Figure 1. Precise definitions for all formulas are given throughout the article at the indicated positions. We return to a more in-depth discussion in Section 4.2, when the technical definitions have been established.

1.2. Outline and contribution

Restricting to a W_1 -type penalization of transport, this article aims at augmenting the picture of unbalanced transport shown in Figure 1 by several relations. The article is organized as follows.

- Section 2: In Section 2.1 we introduce a family of *dynamic unbalanced transport problems*, (Dyn), where the penalty for transport is linear in its distance. This is the W_1 -type subset of the more general family of

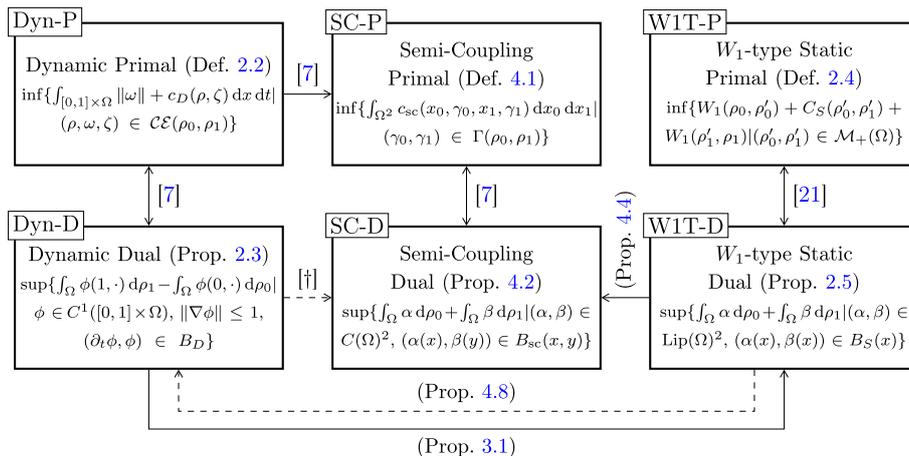


FIGURE 1. Relating different dynamic and static unbalanced transport formulations. A solid arrow $A \rightarrow B$ indicates that every problem of class (A) induces a corresponding formulation in class (B). A dashed link indicates that a correspondence has been established for a subset of class (A). The dashed link marked with [†] has been established for special cases: balanced transport for various cost functions, in particular for the Wasserstein-2 distance, is discussed in [5], the Wasserstein–Fisher–Rao (Hellinger–Kantorovich) distance is treated in [12]. The links marked with round brackets are established in this article. A more detailed discussion of the relation between the different formulations is given in Section 4.2.

transport problems studied in [7]. In Section 2.2 on the other hand, we introduce a family of *static W_1 -type unbalanced transport problems*, (W1T), based on generalizing the Kantorovich–Rubinstein formula. This is a subset of the family introduced in [21].

- Section 3.1: We establish for every (Dyn-D) problem a corresponding (W1T-D) problem such that the resulting optimal values are identical (Prop. 3.1). This includes explicit relations between feasible candidates (*e.g.* Lem. 3.11) and model parameters (Prop. 3.12).
- Sections 3.2 and 3.3: We examine the relation between primal optimizers of (W1T-P) and (Dyn-P). For any optimizer of (W1T-P) we construct in Section 3.2 an optimizer for (Dyn-P) (Prop. 3.15). These dynamic optimizers exhibit a very particular structure which is characteristic for W_1 -type transport problems: transport only occurs instantaneously at times 0 and 1, while in between only mass growth and shrinkage take place. In Section 3.3 we give a sufficient condition for the dynamic model which implies that any optimizer of (Dyn-P) is of this particular form (Cor. 3.21). Essentially, this temporal structure is the reason for (W1T-P) having a mass change penalty in between two Wasserstein-1 distances (as opposed to, for instance, a Wasserstein-1 distance between two mass change penalties, studied in [15], which has no equivalent formulation in (Dyn-P)).
- Section 3.4: We characterize minimizers of (W1T-P) models concerning the spatial relation between mass growth, shrinkage, and transport (Prop. 3.29). This provides an intuition of how the unbalanced transport operates and automatically implies a corresponding characterization of (Dyn-P) model minimizers. In particular, mass transport can neither occur into a region of previous or subsequent mass decrease nor out of a region of previous or subsequent mass increase. Moreover, we derive a model-dependent distance threshold (which may be infinite) beyond which no transport occurs (Prop. 3.31).
- Sections 4.1–4.2: We establish the equivalence of (W1T-D) models to corresponding (SC-D) models (Prop. 4.4). By the above results this implies a correspondence between (Dyn) and (SC) models. Using the particular W_1 -type structure, the relation between the corresponding model parameters is more explicit than the corresponding result in [7], but it should be noted that the latter covers more general transport problems.

- Section 4.3: We characterize precisely, which (W1T-D) models have an equivalent (Dyn-D) model (Prop. 4.8). In addition we provide corresponding simple sufficient as well as necessary conditions.
- Section 5.1: We perform a detailed analysis of the optimal unbalanced transport between two Dirac masses (Example 5.1). This provides information on maximum and minimum transport distances as well as on how the optimal mass changes depend on the previous or subsequent transport.
- Section 5.2: We provide novel examples of (Dyn-D) models by seeking the dynamic formulation of known (W1T-D) models (Tab. 1). Though most of these induce the same topology on the space of nonnegative measures (Prop. 5.8), they penalize mass changes differently. We also give a static counterexample for which no dynamic formulation exists (Rem. 5.7).

1.3. Setting and notation

Throughout the article, Ω denotes the closure of a fixed open bounded connected subset of \mathbb{R}^n (note that the results would also hold for an arbitrary compact metric length space Ω). We will interpret Ω as a metric space with the metric $d : \Omega \times \Omega \rightarrow [0, \infty)$ induced by shortest paths in Ω . The Euclidean norm on \mathbb{R}^n is indicated by $\|\cdot\|$.

For a metric space X we denote the set of continuous functions $f : X \rightarrow \mathbb{R}$ by $C(X)$ and the space of Lipschitz continuous functions with Lipschitz constant no larger than 1 by $\text{Lip}(X)$ (for $X = \Omega$, the Lipschitz constant is with respect to the metric d). If X has the structure of a differentiable manifold, then $C^1(X)$ denotes the set of continuously differentiable functions $f : X \rightarrow \mathbb{R}$. For the analogous function spaces of vector-valued functions into \mathbb{R}^n we write $C(X; \mathbb{R}^n)$, $\text{Lip}(X; \mathbb{R}^n)$, and $C^1(X; \mathbb{R}^n)$.

Now let X be a Borel measurable subset of a Euclidean space. By $\mathcal{M}_+(X)$ and $\mathcal{M}(X)$ we denote the space of nonnegative and of signed Radon measures (regular countably additive measures) on X , respectively. The subset of probability measures on X is denoted $\mathcal{P}(X)$. Given $\rho, \rho' \in \mathcal{M}(X)$, we indicate that ρ is absolutely continuous with respect to ρ' by $\rho \ll \rho'$, and we denote the corresponding Radon–Nikodym derivative by $\frac{d\rho}{d\rho'}$. Given a measurable subset $A \subset X$, the restriction of ρ to A is denoted $\rho \llcorner A$. For a measure $\pi \in \mathcal{M}(X \times X)$ its two marginals are denoted by $P_X\pi$ and $P_Y\pi$ and are defined for any Borel set A via

$$P_X\pi(A) = \pi(A \times X), \quad P_Y\pi(A) = \pi(X \times A).$$

Finally, the domain of a function $f : X \rightarrow [-\infty, \infty]$ is indicated as $\text{dom } f = \{x \in X \mid f(x) \notin \{-\infty, \infty\}\}$, the indicator function of a set $A \subset X$ is defined as $\iota_A(x) = 0$ if $x \in A$ and $\iota_A(x) = \infty$ otherwise, and the interior of a set $A \subset X$ is abbreviated as $\text{int } A$. For a normed vector space X we denote its topological dual by X^* . If $f : X \rightarrow [-\infty, \infty]$ is a proper function, then the Legendre–Fenchel conjugate of f is defined as $f^* : X^* \rightarrow (-\infty, \infty]$, $f^*(y) = \sup_{x \in X} \langle x, y \rangle_{X, X^*} - f(x)$. For a linear operator $A : X \rightarrow Y$ we denote its adjoint by $A^* : Y^* \rightarrow X^*$. As usual we identify the topological duals of $C(\Omega, \mathbb{R}^n)$ and $C([0, 1] \times \Omega, \mathbb{R}^n)$ with $\mathcal{M}(\Omega)^n$ and $\mathcal{M}([0, 1] \times \Omega)^n$.

2. REMINDER: MODELS FOR UNBALANCED OPTIMAL TRANSPORT

Here we recall different extensions of the classical Wasserstein-1 metric that allow for mass changes during the mass transport. In particular, we recapitulate the class of dynamic models from [7] as well as a class of static unbalanced optimal transport models from [21]. Establishing the equivalence of those two model classes belongs to the main aims of this work. We will use subscripts D and S to indicate dynamic and static formulations throughout; primal and dual energies are denoted P and D , respectively.

2.1. Dynamic W_1 -type models

In the dynamic model formulation we consider a time-varying measure ρ which moves at a flux ω and simultaneously changes mass at rate ζ . The relation between ρ , ω , and ζ is described by the following weak continuity equation.

Definition 2.1 (Weak continuity equation with source [7], Def. 4.1). For $(\rho_0, \rho_1) \in \mathcal{M}_+(\Omega)^2$ denote by $\mathcal{CE}(\rho_0, \rho_1)$ the affine subset of $\mathcal{M}([0, 1] \times \Omega)^{1+n+1}$ of triplets of measures (ρ, ω, ζ) satisfying the continuity equation $\partial_t \rho + \nabla \cdot \omega = \zeta$ in the distributional sense, interpolating between ρ_0 and ρ_1 and satisfying homogeneous Neumann boundary conditions. More precisely, we require

$$\int_{[0,1] \times \Omega} (\partial_t \phi) d\rho + \int_{[0,1] \times \Omega} (\nabla \phi) \cdot d\omega + \int_{[0,1] \times \Omega} \phi d\zeta = \int_{\Omega} \phi(1, \cdot) d\rho_1 - \int_{\Omega} \phi(0, \cdot) d\rho_0 \quad (2.1)$$

for all $\phi \in C^1([0, 1] \times \Omega)$.

This definition does not require that the map $t \mapsto \rho_t$, which takes t to the corresponding time-disintegration of ρ , is (weakly) continuous. Instantaneous movement of mass *via* ω is characteristic for the W_1 distance and the unbalanced extensions that we study (*cf.* Rem. 3.18). A Wasserstein-1 type model for unbalanced transport now penalizes the flux ω *via* its total variation as well as the mass change ζ *via* an infinitesimal cost c_D .

Definition 2.2 (Dynamic unbalanced W_1 -type model [7], Def. 4.2-3). Let $c_D : \mathbb{R}^2 \rightarrow [0, \infty]$ be lower semi-continuous, convex, 1-homogeneous, and let it satisfy

$$c_D(\rho, \zeta) \begin{cases} = +\infty & \text{if } \rho < 0, \\ = 0 & \text{if } \rho \geq 0, \zeta = 0, \\ > 0 & \text{else.} \end{cases} \quad (2.2)$$

The associated *dynamic cost functional* is defined as $P_D : \mathcal{M}([0, 1] \times \Omega)^{1+n+1} \rightarrow [0, \infty]$,

$$P_D(\rho, \omega, \zeta) = \int_{[0,1] \times \Omega} \left\| \frac{d\omega}{d\mu} \right\| + c_D \left(\frac{d\rho}{d\mu}, \frac{d\zeta}{d\mu} \right) d\mu, \quad (2.3)$$

where $\mu \in \mathcal{M}_+([0, 1] \times \Omega)$ is any measure such that $(\rho, \omega, \zeta) \ll \mu$. By the 1-homogeneity of $\|\cdot\|$ and c_D this definition does not depend on the choice of μ . The corresponding *primal dynamic unbalanced W_1 -type transport problem* for fixed marginals $(\rho_0, \rho_1) \in \mathcal{M}_+(\Omega)^2$ reads

$$W_D(\rho_0, \rho_1) = \inf \{ P_D(\rho, \omega, \zeta) \mid (\rho, \omega, \zeta) \in \mathcal{CE}(\rho_0, \rho_1) \}. \quad (2.4)$$

The cost W_D also admits an equivalent dual formulation.

Proposition 2.3 (Dynamic dual problem). *For a dynamic problem as in Definition 2.2 let $B_D \subset \mathbb{R}^2$ be the closed convex set characterized by $\iota_{B_D} = c_D^*$. Introduce $B = \{(\alpha, \beta, \gamma) \in \mathbb{R}^{1+n+1} \mid \|\beta\| \leq 1, (\alpha, \gamma) \in B_D\}$ and $D_D : C^1([0, 1] \times \Omega) \rightarrow \mathbb{R} \cup \{\infty\}$,*

$$D_D(\phi) = \begin{cases} \int_{\Omega} \phi(1, \cdot) d\rho_1 - \int_{\Omega} \phi(0, \cdot) d\rho_0 & \text{if } (\partial_t \phi(t, x), \nabla \phi(t, x), \phi(t, x)) \in B \\ & \text{for all } (t, x) \in [0, 1] \times \Omega, \\ -\infty & \text{else.} \end{cases} \quad (2.5)$$

Then

$$W_D(\rho_0, \rho_1) = \sup \{ D_D(\phi) \mid \phi \in C^1([0, 1] \times \Omega) \}. \quad (2.6)$$

Furthermore, minimizers of problem (2.4) exist if the infimum is finite.

Proof. The proof is essentially identical to Theorem 2.1 of [8] and Proposition 4.2 of [7]. Problem (2.6) can be rewritten as

$$\sup_{\phi \in C^1([0,1] \times \Omega)} -F(A\phi) - G(-\phi)$$

with

$$\begin{aligned} F : C([0,1] \times \Omega; \mathbb{R}^{1+n+1}) &\rightarrow \mathbb{R} \cup \{\infty\}, & (\alpha, \beta, \gamma) &\mapsto \int_{[0,1] \times \Omega} \iota_B(\alpha(t, x), \beta(t, x), \gamma(t, x)) \, d(t, x), \\ G : C^1([0,1] \times \Omega) &\rightarrow \mathbb{R} \cup \{\infty\}, & \phi &\mapsto \int_{\Omega} \phi(1, \cdot) \, d\rho_1 - \int_{\Omega} \phi(0, \cdot) \, d\rho_0, \\ A : C^1([0,1] \times \Omega) &\rightarrow C([0,1] \times \Omega; \mathbb{R}^{1+n+1}), & \phi &\mapsto (\partial_t \phi, \nabla \phi, \phi). \end{aligned}$$

Note that F and G are convex and lower semi-continuous. A is linear and bounded. Further, there is some $\phi \in C^1([0,1] \times \Omega)$ such that $G(-\phi) < \infty$ and F is continuous at $A\phi$: for instance, let $(-a, c) \in \text{int } B_D$ with $a, c > 0$ and set $\phi(t, x) = c \exp(-\frac{a}{c}t)$, then $t \mapsto (\partial_t \phi(t, x), \phi(t, x))$ will move along a line from $(-a, c)$ towards $(0, 0)$, which can easily be shown to remain in the interior of B_D (the reader may refer to Lem. 3.2, which will show this calculation in detail).

Using the above, by virtue of the Fenchel-Rockafellar Theorem ([17], Thm. 3) one obtains

$$\sup_{\phi \in C^1([0,1] \times \Omega)} -F(A\phi) - G(-\phi) = \min_{(\rho, \omega, \zeta) \in \mathcal{M}_+([0,1] \times \Omega)^{1+n+1}} G^*(A^*(\rho, \omega, \zeta)) + F^*(\rho, \omega, \zeta),$$

where the minimizer on the right-hand side exists if it is finite. We find

$$\begin{aligned} G^*(A^*(\rho, \omega, \zeta)) &= \sup_{\phi \in C^1([0,1] \times \Omega)} \int_{[0,1] \times \Omega} \partial_t \phi \, d\rho + \int_{[0,1] \times \Omega} \nabla \phi \cdot d\omega \\ &\quad + \int_{[0,1] \times \Omega} \phi \, d\zeta - \int_{\Omega} \phi(1, \cdot) \, d\rho_1 + \int_{\Omega} \phi(0, \cdot) \, d\rho_0 \end{aligned}$$

which is the indicator function of $\mathcal{CE}(\rho_0, \rho_1)$. With Theorem 5 of [18] one obtains $F^* = P_D$. \square

2.2. Static W_1 -type models

A different class of unbalanced optimal transport models was introduced in [21] of which we here consider a particular subclass. It is based on an infimal convolution type extension of the Wasserstein-1 distance by a penalty $c_S(m_0, m_1)$ for changing a mass m_0 into m_1 .

Definition 2.4 (Static unbalanced W_1 -type model [21], Def. 2.21). A *local discrepancy* is a function $c_S : \mathbb{R}^2 \rightarrow [0, \infty]$ satisfying the following properties,

- (i) c_S is convex, 1-homogeneous, and lower semi-continuous,
- (ii) $c_S(m, m) = 0$ if $m \in [0, \infty)$ and $c_S(m_0, m_1) > 0$ if $m_0 \neq m_1$,
- (iii) $c_S(m_0, m_1) = \infty$ if $m_0 < 0$ or $m_1 < 0$.

A local discrepancy c_S induces a *discrepancy* $C_S : \mathcal{M}_+(\Omega)^2 \rightarrow [0, \infty]$ via

$$C_S(\rho_0, \rho_1) = \int_{\Omega} c_S \left(\frac{d\rho_0}{d\gamma}(x), \frac{d\rho_1}{d\gamma}(x) \right) \, d\gamma(x), \quad (2.7)$$

where $\gamma \in \mathcal{M}_+(\Omega)$ is any measure with $\rho_0, \rho_1 \ll \gamma$ (the functional is independent of the particular choice). C_S measures the cost of changing ρ_0 into ρ_1 by pointwise mass changes, where each mass change is penalized according to c_S . The associated *static cost functional* is defined as $P_S : \mathcal{M}_+(\Omega \times \Omega)^2 \rightarrow [0, \infty]$,

$$P_S(\pi_0, \pi_1) = \int_{\Omega \times \Omega} d(x, y) d\pi_0(x, y) + C_S(P_Y \pi_0, P_X \pi_1) + \int_{\Omega \times \Omega} d(x, y) d\pi_1(x, y), \quad (2.8)$$

and the corresponding *primal static unbalanced W_1 -type transport problem* reads

$$W_S(\rho_0, \rho_1) = \inf \{ P_S(\pi_0, \pi_1) \mid (\pi_0, \pi_1) \in \mathcal{M}_+(\Omega \times \Omega)^2, P_X \pi_0 = \rho_0, P_Y \pi_1 = \rho_1 \}. \quad (2.9)$$

The cost W_S also admits an equivalent dual formulation.

Proposition 2.5 (Static dual problem ([21], Cor. 2.32)). *For a static problem as in Definition 2.4 let $B_S \subset \mathbb{R}^2$ be the closed convex set characterized by $\iota_{B_S} = c_S^*$. Introduce $D_S : C(\Omega)^2 \rightarrow \mathbb{R} \cup \{\infty\}$,*

$$D_S(\alpha, \beta) = \begin{cases} \int_{\Omega} \alpha d\rho_0 + \int_{\Omega} \beta d\rho_1 & \text{if } (\alpha, \beta) \in \text{Lip}(\Omega)^2, (\alpha(x), \beta(x)) \in B_S \text{ for all } x \in \Omega, \\ -\infty & \text{else,} \end{cases} \quad (2.10)$$

then

$$W_S(\rho_0, \rho_1) = \sup \{ D_S(\alpha, \beta) \mid (\alpha, \beta) \in C(\Omega)^2 \}. \quad (2.11)$$

Furthermore, minimizers of problem (2.9) exist if the infimum is finite. Finally, the sets B_S characterized by $\iota_{B_S} = c_S^*$ for some local discrepancy c_S are exactly the sets of the form

$$B_S = \{(\alpha, \beta) \in \mathbb{R}^2 : \beta \leq h_S(-\alpha)\} = \{(\alpha, \beta) \in \mathbb{R}^2 : \alpha \leq \bar{h}_S(-\beta)\}, \quad (2.12)$$

where $h_S : \mathbb{R} \rightarrow \mathbb{R} \cup \{-\infty\}$ (or equivalently $\bar{h}_S : \mathbb{R} \rightarrow \mathbb{R} \cup \{-\infty\}$) satisfies (we drop the indices)

1. h is concave, upper semi-continuous, and monotonically increasing,
2. $h(z) \leq z$ for $z \in \mathbb{R}$, $h(0) = 0$,
3. h is differentiable at 0 and $h'(0) = 1$.

Note that on their respective domains, $h_S = -\bar{h}_S^{-1}(-\cdot)$ and $\bar{h}_S = -h_S^{-1}(-\cdot)$.

3. EQUIVALENCE OF DYNAMIC AND STATIC PROBLEMS

We first show the equivalence between dynamic and static problems in their dual formulations, after which we shall turn to the primal interpretation.

3.1. Equivalence of optimal values

There are some structural similarities between the dynamic dual problem (2.6) and the static problem (2.11). If one identifies $\alpha = -\phi(0, \cdot)$ and $\beta = \phi(1, \cdot)$, then the finite part of (2.5) corresponds to the finite part of the objective in (2.10). The constraint $\|\nabla \phi(t, x)\| \leq 1$ guarantees that $\alpha, \beta \in \text{Lip}(\Omega)$. In this section we show that the constraint $(\partial_t \phi(t, x), \phi(t, x)) \in B_D$ for $(t, x) \in [0, 1] \times \Omega$ can be translated into a constraint $(\alpha(x), \beta(x)) \in B_S$ for $x \in \Omega$ for a suitable choice of B_S , such that problems (2.6) and (2.11) are in fact equivalent.

Proposition 3.1 (Equivalence between dynamic and static problems). *Consider the dynamic dual problem (2.6) and its characterizing set B_D . Set*

$$B_{S,\text{pre}} = \{(-\phi(0), \phi(1)) \mid \phi \in C^1([0, 1]), (\partial_t \phi(t), \phi(t)) \in B_D \text{ for all } t \in [0, 1]\}. \quad (3.1)$$

Then, by choosing

$$B_S = \overline{B_{S,\text{pre}}} + (-\infty, 0]^2 \quad (3.2)$$

for the static dual problem (2.11) one finds

$$W_D(\rho_0, \rho_1) = W_S(\rho_0, \rho_1).$$

The function h_S that characterizes B_S as described in Proposition 2.5 will be given in Proposition 3.12. A direct consequence of the proposition is that also the primal dynamic problem (2.4) is equivalent to a primal static problem (2.9). Here, the corresponding static mass change penalty c_S can be obtained from the dynamic mass change penalty c_D by first calculating the set B_D via Proposition 2.3, applying the above proposition to obtain the set B_S , and finally calculating c_S via Proposition 2.5. This will be done explicitly in Proposition 3.13.

The proof is divided into several auxiliary lemmas. The strategy is as follows: the constraints of (2.6) are $(\partial_t \phi(t, x), \phi(t, x)) \in B_D$ and $\|\nabla \phi(t, x)\| \leq 1$ for all $(t, x) \in [0, 1] \times \Omega$. Let us ignore the constraint on $\nabla \phi$ for now. Since ρ_1 is nonnegative, for fixed $\phi(0, \cdot)$ the function $\phi(1, \cdot)$ in (2.5) will want to be as large as the B_D -constraint allows. With the identification $(\alpha, \beta) = (-\phi(0, \cdot), \phi(1, \cdot))$ this yields an upper bound on β for fixed α and defines the set $B_{S,\text{pre}}$. This already implies $W_D \leq W_S$. Note that the set $B_{S,\text{pre}}$ does not necessarily satisfy the formal structural assumptions for B_S , as specified by (2.12), which is why some post-processing from the preliminary $B_{S,\text{pre}}$ to B_S is required. We show in the proof of Proposition 3.1, however, that replacing B_S with $B_{S,\text{pre}}$ does not change the optimal value of (2.11). From feasible (in the sense of $B_{S,\text{pre}}$) static dual variables (α, β) for (2.10) we then reconstruct a feasible dynamic dual variable ϕ for (2.5) such that $\phi(0, \cdot) = -\alpha$, $\phi(1, \cdot) = \beta$. The Lipschitz constraint on (α, β) suffices to imply $\|\nabla \phi(t, x)\| \leq 1$, thus establishing the converse inequality $W_S \leq W_D$. We now study the dynamic cost c_D and the corresponding set B_D in more detail.

Lemma 3.2 (Properties of B_D). *The set B_D from Proposition 2.3 satisfies*

$$B_D = \{(\psi, \phi) \in \mathbb{R}^2 : \psi \leq h_D(\phi)\},$$

where $h_D : \mathbb{R} \rightarrow \mathbb{R} \cup \{-\infty\}$ is concave, upper semi-continuous, nonpositive, increasing on $(-\infty, 0]$ and decreasing on $[0, \infty)$ with $h_D(0) = h'_D(0) = 0$ as well as $\overline{\text{dom } h_D} = [-c_D(0, -1), c_D(0, 1)]$.

Proof. Since c_D is one-homogeneous, B_D must be a closed convex set. Due to $c_D(\rho, \zeta) = \infty$ for $\rho < 0$ we have

$$\iota_{B_D}(\psi_1, \phi) = \sup_{\rho \geq 0, \zeta} \psi_1 \rho + \phi \zeta - c_D(\rho, \zeta) \leq \sup_{\rho \geq 0, \zeta} \psi_2 \rho + \phi \zeta - c_D(\rho, \zeta) = \iota_{B_D}(\psi_2, \phi)$$

for all $\psi_1 \leq \psi_2$ so that $(\psi, \phi) \in B_D$ implies $(-\infty, \psi] \times \{\phi\} \subset B_D$. Thus there exists a function h_D such that $(\psi, \phi) \in B_D \Leftrightarrow \psi \leq h_D(\phi)$. Concavity and upper semi-continuity of h_D follow from convexity and closedness of B_D . Now $c_D(\rho, 0) = 0$ for $\rho \geq 0$ implies

$$\iota_{B_D}(\psi, \phi) = \sup_{\rho \geq 0, \zeta} \psi \rho + \phi \zeta - c_D(\rho, \zeta) \geq \sup_{\rho \geq 0} \psi \rho + \phi \cdot 0 - c_D(\rho, 0) = \begin{cases} 0 & \text{if } \psi \leq 0 \\ \infty & \text{else} \end{cases}$$

so that $h_D \leq 0$. Furthermore, $\iota_{B_D}(0, 0) = \sup_{\rho, \zeta} -c_D(\rho, \zeta) = 0$ so that $h_D(0) = 0$. The monotonicity properties now follow from $h_D(0) = 0$, $h_D \leq 0$, and concavity. Next, by the assumptions on c_D it is its own convex envelope.

Therefore $c_D = \iota_{B_D}^*$ and in particular

$$0 < c_D(0, 1) = \sup\{\phi \mid h_D(\phi) > -\infty\}, \quad 0 < c_D(0, -1) = \sup\{\phi \mid h_D(-\phi) > -\infty\}.$$

Finally, note that the left and the right derivative of h_D in 0 exist due to concavity and are given by the monotone limits $h_D^{l/r}(0) = \lim_{\varepsilon \searrow 0} \frac{h_D(\mp\varepsilon)}{\mp\varepsilon}$. We show $h_D^r(0) = 0$ (the other equality follows analogously). For a contradiction assume the existence of $C > 0$ such that for all $\varepsilon > 0$ we have $\frac{h_D(\varepsilon)}{\varepsilon} < -C$. Thus, for any $\rho, \zeta > 0$ with $\zeta < C\rho$ we have

$$\begin{aligned} c_D(\rho, \zeta) &= \iota_{B_D}^*(\rho, \zeta) = \sup_{\phi, \psi \leq h_D(\phi)} \psi\rho + \phi\zeta = \sup_{\phi \geq 0, \psi \leq h_D(\phi)} \psi\rho + \phi\zeta \\ &\leq \sup_{\phi \geq 0, \psi \leq -C\phi} \psi\rho + \phi\zeta = \sup_{\phi \geq 0} \phi(\zeta - C\rho) = 0, \end{aligned}$$

which contradicts the positivity of c_D for $\rho, \zeta > 0$. \square

For an illustration of the sets B_D and B_S as well as the functions h_D and h_S see Figure 5.

Now we turn to the structure of B_S . As described above, for fixed $\phi(0, \cdot)$ the value of $\phi(1, \cdot)$ will intuitively try to be as large as the constraint $(\partial_t \phi, \phi) \in B_D$ allows. So, ignoring regularity, from Lemma 3.2 we infer that $\phi(1, x)$ will be given by the solution to the differential equation $\partial_t \phi(t, x) = h_D(\phi(t, x))$ for initial value $\phi(0, x)$. This upper bound is rigorously established in Lemma 3.4. We first consider the case where the function h_D introduced in Lemma 3.2 satisfies an additional technical assumption. Functions not satisfying this assumption will be treated *via* an extra smoothing argument in Lemma 3.6.

Assumption 3.3. Let $\bar{s} = \max\{z \in \mathbb{R} \mid h_D(z) = 0\}$. We assume that h_D is differentiable at \bar{s} (left-differentiable if $\bar{s} \in \partial(\text{dom } h_D)$).

Note that functions h_D with $\bar{s} = 0$ automatically satisfy Assumption 3.3 due to $h_D'(0) = 0$.

Lemma 3.4 (Properties of flow). *For $s > 0$ we define the flow of B_D as $F_s : \mathbb{R} \rightarrow [-\infty, \infty)$,*

$$\begin{aligned} F_s(z) &= \sup \{ \phi(s) \mid \phi \in C^1([0, s]), \phi(0) = z, (\partial_t \phi(t), \phi(t)) \in B_D \text{ for } t \in [0, s] \} \\ &= \sup \{ \phi(s) \mid \phi \in C^1([0, s]), \phi(0) = z, \partial_t \phi(t) \leq h_D(\phi(t)) \text{ for } t \in [0, s] \}, \end{aligned} \quad (3.3)$$

where the supremum of the empty set is $-\infty$. For $s = 0$ we set $F_0(z) = z - \iota_{\text{dom } h_D}(z)$. The flow satisfies the following properties for all $s \geq 0$.

(i) If $F_s(z) > -\infty$, the map $[0, s] \ni t \mapsto F_t(z)$ solves the initial value problem

$$\partial_t \phi(t) = h_D(\phi(t)) \text{ for } t \in [0, s], \quad \phi(0) = z. \quad (3.4)$$

(ii) If $F_s(z) > -\infty$, then for all $t \in [0, s]$ we have $F_t(z) > -\infty$ and $F_s(z) = F_t(F_{s-t}(z))$, and the map $[0, s] \ni t \mapsto F_t(z)$ is contained in $\{\phi \in C^1([0, s]) : (\partial_t \phi(t), \phi(t)) \in B_D \text{ for } t \in [0, s]\}$.

(iii) $F_s(z) \leq z$.

(iv) F_s is strictly increasing on its domain.

(v) F_s is concave.

(vi) For $z \in \text{dom } h_D \cap [0, \infty)$, $F_s(z) \geq 0$ with $F_s(z) > 0$ if $z > 0$.

(vii) For $z \in \text{dom } h_D \cap [0, \infty)$ the map $[0, \infty) \ni t \mapsto F_t(z)$ is convex.

(viii) F_s is differentiable in 0 with $F_s'(0) = 1$.

(ix) F_s is non-expansive on $\text{dom } h_D \cap [0, \infty)$.

(x) F_s is locally Lipschitz differentiable on $\text{int dom } h_D \cap [0, \infty) \setminus \bar{s}$ with $F_s'(z) = 1$ for $z < \bar{s}$ and $F_s'(z) = \frac{h_D(F_s(z))}{h_D(z)}$ for $z > \bar{s}$. Under Assumption 3.3 it is differentiable on all of $\text{int dom } h_D \cap [0, \infty)$.

Proof.

- (i) We first show that a solution to (3.4) exists when $F_s(z) > -\infty$. We will construct such a solution explicitly. Let

$$S_0 = h_D^{-1}(\{0\}), \quad S_+ = (\text{dom } h_D \cap [0, \infty)) \setminus S_0, \quad S_- = (\text{dom } h_D \cap (-\infty, 0]) \setminus S_0$$

be a partition of $\text{dom } h_D$ into three connected components. If $z \in S_0$ the solution to (3.4) is given by $\phi(t) = z$ for all $t \geq 0$. If $z \in S_+ \cup S_-$, let $\psi \in C^1([0, s])$ be a suitable candidate in (3.3). The function ψ must be nonincreasing due to $h_D \leq 0$. Without loss of generality we may assume that $\psi([0, s])$ is contained in either S_+ or S_- . Indeed, when $z \in S_-$, then $\psi(t) \leq \psi(0) = z$ must be contained in S_- for all $t \in [0, s]$ as well. On the other hand, when $z \in S_+$, we may pick $\psi(t) = \bar{s} + (z - \bar{s}) \cdot \exp\left(\frac{h_D(z)}{z - \bar{s}} t\right)$ where $\bar{s} = \max S_0$. This is feasible in (3.3) since $h_D(\bar{s}) = 0$ by upper semi-continuity and h_D is concave. Now for $y \in \mathbb{R}$ consider the integral

$$I_z(y) = \int_z^y \frac{1}{h_D(x)} dx.$$

This is well-defined and finite when $z, y \in S_+$ or when $z, y \in S_-$. The function I_z is strictly decreasing and thus invertible. Denote its inverse by ϕ_z . Due to $I_z(z) = 0$ and

$$\infty > I_z(\psi(s)) = \int_{\psi(0)}^{\psi(s)} \frac{1}{h_D(x)} dx = \int_0^s \frac{\psi'(t)}{h_D(\psi(t))} dt \geq s$$

the inverse ϕ_z is well-defined at least on $[0, s]$. Furthermore, $\phi_z(0) = z$ and

$$\phi'_z(t) = \frac{1}{\partial_y I_z(\phi_z(t))} = h_D(\phi_z(t))$$

so that $\phi = \phi_z$ indeed solves the initial value problem (3.4).

Now let $s > 0$ and $z \in \mathbb{R}$ with $F_s(z) > -\infty$ and denote the solution to (3.4) by ϕ_z . Since ϕ_z is feasible in (3.3), $F_s(z) \geq \phi_z(s)$. To show the reverse inequality, consider a competitor $\psi \in C^1([0, s])$ feasible for (3.3). If $z \in S_0$, then $\psi(s) \leq z = \phi_z(s)$ since ψ is decreasing. If on the other hand $z \in S_+ \cup S_-$, then as above we may assume $\psi([0, s])$ to be contained in either S_+ or S_- . Due to $I_z(\phi_z(s)) = s$ and $I_z(\psi(s)) \geq s$, the monotonicity of I_z implies $\psi(s) \leq I_z^{-1}(s) = \phi_z(s)$. Summarizing, $F_s(z) \leq \phi_z(s)$.

- (ii) This is a direct consequence of (i).
 (iii) If $F_s(z) = -\infty$ this holds trivially. Otherwise, $t \mapsto F_t(z)$ is nonincreasing on $t \in [0, s]$ due to (i) and $h_D \leq 0$ so that $F_s(z) \leq F_0(z) = z$.
 (iv) This is a standard property of solutions to scalar ordinary differential equations. Indeed, for $z \in S_0$ this follows from $F_s(z) = z$, and for $z_1, z_2 \in S_- \cup S_+$ with $z_1 < z_2$ assume $F_s(z_1) \geq F_s(z_2)$. Then with the above notation,

$$s = I_{z_1}(F_s(z_1)) = \int_{z_1}^{F_s(z_1)} \frac{1}{h_D(x)} dx < \int_{z_2}^{F_s(z_2)} \frac{1}{h_D(x)} dx = I_{z_2}(F_s(z_2)) = s,$$

which yields a contradiction.

- (v) Let $z_1, z_2 \in \text{dom } F_s$ and let ϕ_1, ϕ_2 be the corresponding solutions to (3.4). For any $\lambda \in [0, 1]$ let $z_\lambda = \lambda \cdot z_1 + (1 - \lambda) \cdot z_2$ and $\phi_\lambda = \lambda \cdot \phi_1 + (1 - \lambda) \cdot \phi_2$. By convexity of B_D one finds that $(\partial_t \phi_\lambda(t), \phi_\lambda(t)) \in B_D$ for $t \in [0, s]$. Hence ϕ_λ is feasible in (3.3) for $F_s(z_\lambda)$ which implies $F_s(z_\lambda) \geq \phi_\lambda(s) = \lambda \cdot F_s(z_1) + (1 - \lambda) \cdot F_s(z_2)$.

- (vi) The function $\psi(t) = \bar{s} + (z - \bar{s}) \cdot \exp\left(\frac{h_D(z)}{z - \bar{s}} t\right)$ from above is feasible in (3.3) so that $F_s(z) \geq \psi(s) \geq 0$ and $\psi(s) > 0$ if $z > 0$.
 By (i) and (vi), $t \mapsto F_t(z)$ is decreasing and nonnegative. Thus, since h_D is decreasing on $[0, \infty)$, the derivative $t \mapsto \partial_t F_t(z) = h_D(F_t(z))$ is increasing.
- (viii) We show that the right derivative of F_s in 0 is 1 (the argument for the left derivative is analogous). If $\inf S_+ > 0$ this follows directly from $F_s(z) = z$ on S_0 . If $\inf S_+ = 0$, since F_s is concave its right derivative in 0 is given by the monotone limit $\lim_{\varepsilon \searrow 0} \frac{F_s(\varepsilon)}{\varepsilon}$, which cannot exceed 1 due to $F_s(\varepsilon) \leq \varepsilon$. For a contradiction we assume that the right derivative is strictly smaller than 1, that is, there is some $C < 1$ with $F_s(\varepsilon) \leq C \cdot \varepsilon$ for all $\varepsilon > 0$. Since $F_s(\varepsilon) = \phi_\varepsilon(s)$ this implies

$$s = I_\varepsilon(\phi_\varepsilon(s)) \geq I_\varepsilon(C \cdot \varepsilon) = \int_\varepsilon^{C\varepsilon} \frac{1}{h_D(x)} dx \geq \frac{(C-1) \cdot \varepsilon}{h_D(\varepsilon)} = \frac{\varepsilon}{h_D(\varepsilon)} \cdot (C-1).$$

Since $h'_D(0) = 0$ the right-hand side diverges to ∞ as $\varepsilon \rightarrow 0$ which is a contradiction.

- (ix) This follows from the concavity and monotonicity of F_s together with $F'_s(0) = 1$.
- (x) For $z \in \text{int } S_0$ the statement is trivial. For $z \in \text{int } S_+$ fix some arbitrary $a \in S_+$ with $a > z$. Then by (i) we have $z = \phi_a(I_a(z)) = F_{I_a(z)}(a)$ and thus by (ii) $F_s(z) = F_s(F_{I_a(z)}(a)) = F_{I_a(z)+s}(a) = \phi_a(I_a(z) + s)$. Taking the derivative we find

$$F'_s(z) = \frac{\phi'_a(I_a(z) + s)}{h_D(z)} = \frac{h_D(F_s(z))}{h_D(z)},$$

which is locally Lipschitz on $\text{int } S_+$ since the concave functions h_D and F_s are locally Lipschitz on their domain interiors. Finally, at $\bar{s} = \max S_0$ the left derivative is 1 due to $F_s(z) = z$ on S_0 . If $\bar{s} \in \text{int dom } h_D$, using Assumption 3.3 one shows that the right derivative is 1 as well in the same way as for (viii). □

Lemma 3.5 (Inverse flow). *On $F_s(\text{dom } F_s)$ let F_s^{inv} be the inverse of F_s (which is well-defined by Lem. 3.4(iv)). Let $\tilde{h}_D : z \mapsto h_D(-z)$ and let \tilde{F}_s be the flow (3.3) associated with \tilde{h}_D . Then*

$$F_s^{\text{inv}}(z) = -\tilde{F}_s(-z). \tag{3.5}$$

Proof. By Lemma 3.4(i) one finds for $z \in F_s(\text{dom } F_s)$ that $z = \phi(s)$ where ϕ solves the initial value problem

$$\partial_t \phi(t) = -h_D(\phi(t)) \text{ for } t \in [0, s], \quad \phi(0) = F_s^{\text{inv}}(z).$$

Consequently the function $\tilde{\phi}(t) = -\phi(s-t)$ solves the initial value problem

$$\partial_t \tilde{\phi}(t) = h_D(-\tilde{\phi}(t)) = \tilde{h}_D(\tilde{\phi}(t)) \text{ for } t \in [0, s], \quad \tilde{\phi}(0) = -z,$$

so that $F_s^{\text{inv}}(z) = \phi(0) = -\tilde{\phi}(s) = -\tilde{F}_s(-z)$. □

Relation (3.5) allows to translate results of Lemma 3.4 to F_s^{inv} . Throughout this section we will use that arguments involving F_s can be applied to F_s^{inv} in an analogous way.

The flow F_s will be crucial in constructing feasible candidates for the dynamic dual problem (2.6). For this we need differentiability of F_s on $\text{dom } h_D \cap [0, \infty)$ which requires the additional Assumption 3.3 (see Lem. 3.4(x)). Since this assumption may not always hold we will have to approximate h_D by a suitable sequence of functions.

Lemma 3.6 (Approximation of h_D). *Assume that h_D does not satisfy Assumption 3.3, that is, $0 < \bar{s} \in \text{int}(\text{dom } h_D)$ and the right derivative $h_D^r(\bar{s})$ of h_D at \bar{s} is finite but strictly negative, $h_D^r(\bar{s}) = -C < 0$ for some $C > 0$. Then for $\varepsilon \in (0, \bar{s}/2]$ the function*

$$h_{D,\varepsilon}(z) = \begin{cases} h_D(z) & \text{if } z \leq \bar{s} - \varepsilon \\ -\frac{C}{2\varepsilon}(z - \bar{s} + \varepsilon)^2 & \text{if } z \in [\bar{s} - \varepsilon, \bar{s}] \\ h_D(z) - \frac{C\varepsilon}{2} & \text{if } z \geq \bar{s} \end{cases}$$

possesses the same properties from Lemma 3.2 as h_D and satisfies Assumption 3.3. In addition, the set

$$B_{D,\varepsilon} = \{(\psi, \phi) \in \mathbb{R}^2 \mid \psi \leq h_{D,\varepsilon}(\phi)\}$$

satisfies $(1 - \varepsilon/\bar{s}) \cdot B_D \subset B_{D,\varepsilon} \subset B_D$.

Proof. The properties of $h_{D,\varepsilon}^\varepsilon$ are straightforward to check. Moreover, since $h_{D,\varepsilon}^\varepsilon(z) \leq h_D(z)$ one finds $B_{D,\varepsilon}^\varepsilon \subset B_D$. It remains to show $\lambda \cdot B_D \subset B_{D,\varepsilon}^\varepsilon$ for $\lambda = (1 - \varepsilon/\bar{s}) \in [1/2, 1)$ or equivalently $h_D(z/\lambda) \leq h_{D,\varepsilon}^\varepsilon(z)/\lambda$ for $z \in \mathbb{R}$. This holds true for $z \in (-\infty, \bar{s} - \varepsilon]$ since $h_{D,\varepsilon}^\varepsilon(z) = h_D(z)$ is nondecreasing and concave on this interval. For $z \in [\bar{s} - \varepsilon, \bar{s}] = [\lambda\bar{s}, \bar{s}]$ one finds

$$h_D(z/\lambda) \leq -C \cdot (z/\lambda - \bar{s}) \leq -C \frac{\lambda \cdot (z/\lambda - \bar{s})^2}{2 \cdot \varepsilon} = h_{D,\varepsilon}^\varepsilon(z)/\lambda.$$

Finally, for $z \in \text{dom } h_D = \text{dom } h_{D,\varepsilon}^\varepsilon$, $z \geq \bar{s}$ one has

$$\begin{aligned} h_D(z/\lambda) &\leq h_D(z) + h_D^r(z) \cdot (1/\lambda - 1) \cdot z = \frac{h_D(z)}{\lambda} + (1/\lambda - 1) \cdot (h_D^r(z) \cdot z - h_D(z)) \\ &\leq \frac{1}{\lambda} \cdot [h_D(z) + (1 - \lambda) \cdot (h_D^r(\bar{s}) \cdot \bar{s} - h_D(\bar{s}))] = \frac{1}{\lambda} \cdot [h_D(z) - C \cdot \varepsilon], \end{aligned}$$

where we have used that $z \mapsto h_D^r(z) \cdot z - h_D(z)$ is decreasing for $z \geq 0$. For $z \notin \text{dom } h_D$ the statement is trivial. \square

Remark 3.7 (Approximation of h_D). If $h_D(\cdot)$ does not satisfy Assumption 3.3, then with the same argument we can find some $h_{D,\varepsilon}$ possessing the same properties from Lemma 3.2 as h_D such that $h_{D,\varepsilon}(\cdot)$ satisfies Assumption 3.3 and $(1 + \varepsilon/\underline{s}) \cdot B_D \subset B_{D,\varepsilon} \subset B_D$ for $\underline{s} = \min h_D^{-1}(\{0\})$. Likewise, if neither $h_D(\cdot)$ nor h_D satisfy Assumption 3.3, then we can construct $h_{D,\varepsilon}$ possessing the same properties from Lemma 3.2 as h_D as well as Assumption 3.3 for $h_{D,\varepsilon}$ and $h_{D,\varepsilon}(\cdot)$ and $(1 - C\varepsilon) \cdot B_D \subset B_{D,\varepsilon} \subset B_D$ for some fixed $C > 0$.

We can now use the flows F_1 and F_1^{inv} to reconstruct for suitable $(\alpha, \beta) \in B_{S,\text{pre}}$ a corresponding candidate function ϕ in (3.1) with $\phi(0) = -\alpha$, $\phi(1) = \beta$ (Lem. 3.9). This allows a more explicit description of the set $B_{S,\text{pre}}$ via F_1 (Prop. 3.10), which we will later need to analyse the transition from $B_{S,\text{pre}}$ to $B_S = \overline{B_{S,\text{pre}}} + (-\infty, 0]^2$ in Proposition 3.1. Then, in Lemma 3.11 we establish that for $(\alpha, \beta) \in C^1(\Omega) \cap \text{Lip}(\Omega)$ with $(\alpha(x), \beta(x)) \in B_{S,\text{pre}}$ for all $x \in \Omega$, applying for each $x \in \Omega$ the interpolation of Lemma 3.9 yields a dynamic dual feasible candidate ϕ for (2.6) with $\phi(0, \cdot) = -\alpha$, $\phi(1, \cdot) = \beta$. The main ingredient of this interpolation are the two auxiliary functions examined in the next lemma, which are careful combinations of the flow functions. With these preliminaries in place we can then provide the proof of Proposition 3.1.

Lemma 3.8. *We introduce the two auxiliary functions*

$$g : [0, 1] \times \text{dom } h_D \ni (t, z) \mapsto \begin{cases} F_t(z) - t \cdot F_1(z) & \text{if } z > 0, \\ 0 & \text{if } z = 0, \\ (1-t) \cdot z & \text{if } z < 0, \end{cases}$$

$$g^{\text{inv}} : [0, 1] \times \text{dom } h_D \ni (t, z) \mapsto \begin{cases} t \cdot z & \text{if } z > 0, \\ 0 & \text{if } z = 0, \\ F_{1-t}^{\text{inv}}(z) - (1-t) \cdot F_1^{\text{inv}}(z) & \text{if } z < 0. \end{cases}$$

These have the following properties:

- (i) g and g^{inv} are continuous and differentiable in t .
- (iI) $g(0, z) = z$, $g(1, z) = 0$, $g^{\text{inv}}(0, z) = 0$, $g^{\text{inv}}(1, z) = z$.
- (iii) Under Assumption 3.3 for h_D and $h_D(\cdot)$ (see Rem. 3.7) g and g^{inv} are differentiable in z .
- (iv) Under the same assumptions, $\partial_z g(t, z) \in [0, 1-t]$ and $\partial_z g^{\text{inv}}(t, z) \in [0, t]$.

Proof. We prove the statements for g . All arguments for g^{inv} work in full analogy by using Lemma 3.5.

- (i) By virtue of Lemma 3.4(iii) and (vi) we have $\text{dom } F_t \supset \text{dom } h_D \cap [0, \infty)$. Therefore, $\text{dom } g = [0, 1] \times \text{dom } h_D$. Since $F_t(0) = 0$ and F_t is continuous on its domain, g is continuous in $z = 0$ and on all of $\text{dom } h_D$ for all $t \in [0, 1]$. Finally, differentiability of g in t for $z \leq 0$ is trivial, for $z > 0$ it is provided by Lemma 3.4(i).
- (ii) This follows immediately from the definition, using the convention $F_0(z) = z$ on $\text{dom } h_D$ (Lem. 3.4).
- (iii) With Assumption 3.3, due to Lemma 3.4(viii) and (x) g is differentiable in z for $z \neq 0$, and the one-sided derivatives coincide in $z = 0$.
- (iv) For $z \leq 0$ the result is clear. The derivative $\partial_z F_s(z)$ for $z \geq 0$ is given by Lemma 3.4(x). For $z \in (0, \bar{s}]$ (see Assumption 3.3) we have $\partial_z F_s(z) = 1$ so that the result is also established. Now fix some $z > \bar{s}$ (thus $h_D(z) < 0$). Then $\partial_z g(t, z) = k(F_t(z)) - t \cdot k(F_1(z))$ with $k(y) = \frac{h_D(y)}{h_D(z)}$. For $y \geq 0$, since h_D is negative, concave, and decreasing for positive arguments, k is positive, convex, and increasing. The map $t \mapsto F_t(z)$ is convex and decreasing by Lemma 3.4(i) and (vii). Therefore, $k(F_t(z)) \geq k(F_1(z)) \geq t \cdot k(F_1(z))$ and thus $\partial_z g(t, z) \geq 0$. In addition, the composition $t \mapsto k(F_t(z))$ is convex and thus $k(F_t(z)) \leq (1-t) \cdot k(F_0(z)) + t \cdot k(F_1(z))$ so that $\partial_z g(t, z) \leq (1-t) \cdot k(z) = (1-t)$.

□

Lemma 3.9. *For $(\alpha, \beta) \in \mathbb{R}^2$ with $(-\alpha, \beta) \in (\text{dom } h_D)^2$ and $\beta \leq F_1(-\alpha)$ the function*

$$\phi_{\alpha, \beta} : [0, 1] \rightarrow \mathbb{R}, \quad t \mapsto g(t, -\alpha) + g^{\text{inv}}(t, \beta) \tag{3.6}$$

satisfies $\alpha = -\phi_{\alpha, \beta}(0)$, $\beta = \phi_{\alpha, \beta}(1)$, $\phi_{\alpha, \beta} \in C^1([0, 1])$, and $(\partial_t \phi_{\alpha, \beta}(t), \phi_{\alpha, \beta}(t)) \in B_D$ for $t \in [0, 1]$.

Proof. The first three properties follow directly from Lemma 3.8. It remains to show $\partial_t \phi_{\alpha, \beta}(t) \leq h_D(\phi_{\alpha, \beta}(t))$ for all $t \in [0, 1]$. If $\alpha \leq 0 \leq \beta$, then by Lemma 3.4(i)

$$\partial_t \phi_{\alpha, \beta}(t) = h_D(F_t(-\alpha)) + \beta - F_1(-\alpha) \leq h_D(F_t(-\alpha)) \leq h_D(F_t(-\alpha) + t(\beta - F_1(-\alpha))) = h_D(\phi_{\alpha, \beta}(t)),$$

where we used that h_D is monotonously decreasing for positive arguments. The result follows analogously for $\beta \leq 0 \leq \alpha$ using Lemma 3.5. Due to $F_t(-\alpha) \leq -\alpha$ (Lem. 3.4(iii)) one cannot have $\alpha, \beta > 0$ so that it remains to consider the case $\alpha, \beta < 0$. Note first that $\phi_\alpha : t \mapsto g(t, -\alpha) = F_t(-\alpha) - t F_1(-\alpha)$ and $\phi_\beta : t \mapsto g^{\text{inv}}(t, \beta) = F_{1-t}^{\text{inv}}(\beta) - (1-t) F_1^{\text{inv}}(\beta)$ both satisfy $\partial_t \phi \leq h_D(\phi)$ by the same argument as above. Furthermore one has $\phi_\beta(t) \leq 0 \leq \phi_\alpha(t)$ by Lemma 3.8(iv) and $g(t, 0) = g^{\text{inv}}(t, 0) = 0$ and thus $\phi_{\alpha, \beta}(t) \in [\phi_\beta(t), \phi_\alpha(t)]$. Now due to

its concavity, h_D attains its minimum on the interval $[\phi_\beta(t), \phi_\alpha(t)]$ at the interval boundary so that with the negativity of h_D we obtain

$$\begin{aligned} \partial_t \phi_{\alpha, \beta}(t) &\leq h_D(\phi_\beta(t)) + h_D(\phi_\alpha(t)) \leq \min\{h_D(\phi_\beta(t)), h_D(\phi_\alpha(t))\} \\ &= \min\{h_D(z) \mid z \in [\phi_\beta(t), \phi_\alpha(t)]\} \leq h_D(\phi_{\alpha, \beta}(t)). \end{aligned}$$

□

The previous result allows a more explicit description of the set $B_{S, \text{pre}}$ from (3.1).

Proposition 3.10. *The set $B_{S, \text{pre}}$ can be characterized by*

$$B_{S, \text{pre}} = \{(\alpha, \beta) \in \mathbb{R}^2 \mid (-\alpha, \beta) \in (\text{dom } h_D)^2, \beta \leq F_1(-\alpha)\}, \quad (3.7)$$

$$\overline{B_{S, \text{pre}}} = \{(\alpha, \beta) \in \mathbb{R}^2 \mid (-\alpha, \beta) \in (\overline{\text{dom } h_D})^2, \beta \leq \overline{F_1}(-\alpha)\}, \quad (3.8)$$

where $\overline{F_1}$ is the concave upper semi-continuous hull of F_1 which is given by

$$\overline{F_1}(z) = \begin{cases} F_1(z) & \text{if } z \in \text{int dom } F_1, \\ \sup F_1(\text{dom } F_1) & \text{if } z = \sup \text{dom } F_1, \\ \inf F_1(\text{dom } F_1) & \text{if } z = \inf \text{dom } F_1. \end{cases} \quad (3.9)$$

Note that $\inf F_1(\text{dom } F_1) = \inf \text{dom } h_D$.

Proof. (3.1) \subset (3.7): For $(\alpha, \beta) \in (3.1)$ by definition $(-\alpha, \beta) \in (\text{dom } h_D)^2$ and there is a feasible candidate ϕ in (3.3) for $F_1(-\alpha)$ with $\phi(0) = -\alpha$ and $\phi(1) = \beta$, thus $F_1(-\alpha) \geq \beta$.

(3.7) \subset (3.1): For $(\alpha, \beta) \in (3.7)$ the function $\phi_{\alpha, \beta}$ constructed in Lemma 3.9 establishes that $(\alpha, \beta) \in (3.1)$.

Since F_1 is concave it is continuous on its domain. Hence, $\overline{F_1}$ only differs at the extreme points of the domain, and the values at those points follow from the monotonicity of F_1 . □

For $(\alpha, \beta) \in C^1(\Omega) \cap \text{Lip}(\Omega)$ with $(\alpha(x), \beta(x)) \in B_{S, \text{pre}}$ for all $x \in \Omega$, the interpolation of Lemma 3.9 at each point $x \in \Omega$ yields a function $\phi \in C^1([0, 1] \times \Omega)$ such that $(\partial_t \phi(t, x), \phi(t, x)) \in B_D$ for all (t, x) . In the following Lemma we show that this pointwise interpolation also satisfies $\|\nabla \phi(t, x)\| \leq 1$ so that ϕ is feasible for the dynamic dual problem. This is a crucial part of proving $W_S(\rho_0, \rho_1) \leq W_D(\rho_0, \rho_1)$ in Proposition 3.1.

Lemma 3.11 (Construction of feasible dynamic dual candidates). *Assume h_D and $\tilde{h}_D = h_D(-\cdot)$ satisfy Assumption 3.3. Let $\alpha, \beta \in C^1(\Omega) \cap \text{Lip}(\Omega)$ with $(\alpha(x), \beta(x)) \in B_{S, \text{pre}}$ for all $x \in \Omega$. For given $(a, b) \in B_{S, \text{pre}}$ let $\phi_{a, b}$ be the corresponding function (3.6). Then $\phi : [0, 1] \times \Omega \rightarrow \mathbb{R}$,*

$$\phi(t, x) = \phi_{\alpha(x), \beta(x)}(t) = g(t, -\alpha(x)) + g^{\text{inv}}(t, \beta(x)),$$

satisfies $\phi \in C^1([0, 1] \times \Omega)$, $\phi(0, \cdot) = -\alpha$ and $\phi(1, \cdot) = \beta$, $\|\nabla \phi(x, t)\| \leq 1$ and $(\partial_t \phi(t, x), \phi(t, x)) \in B_D$ for all $(t, x) \in [0, 1] \times \Omega$.

Proof. By virtue of (3.7) $(\alpha(x), \beta(x))$ satisfy the requirements of Lemma 3.9 for all $x \in \Omega$ so ϕ and $\partial_t \phi$ are well-defined with $(\partial_t \phi(t, x), \phi(t, x)) \in B_D$ at each point (t, x) . With Assumption 3.3 for h_D and $h_D(-\cdot)$ the function ϕ is differentiable in space with $\|\nabla \phi(t, x)\| \leq 1$ (Lem. 3.8(iii) and (iv)). □

Proof of Proposition 3.1. $W_D(\rho_0, \rho_1) \leq W_S(\rho_0, \rho_1)$: Let $\phi \in C^1([0, 1] \times \Omega)$ with $D_D(\phi) > -\infty$ (if no such ϕ exists, the inequality is trivial). Set $\alpha = -\phi(0, \cdot)$, $\beta = \phi(1, \cdot)$. Clearly, $\alpha, \beta \in \text{Lip}(\Omega)$ and $(\alpha(x), \beta(x)) \in B_S$ for all $x \in \Omega$. Hence, $D_S(\alpha, \beta) = D_D(\phi)$.

$W_D(\rho_0, \rho_1) \geq W_S(\rho_0, \rho_1)$: First we show that B_S from (3.2) can be replaced by $\overline{B_{S,\text{pre}}}$ and subsequently $B_{S,\text{pre}}$ in (2.10) without changing the value of (2.11).

Let $\alpha_{\min} = -\sup \text{dom } F_1 = -\sup \text{dom } h_D$ and $\beta_{\min} = \inf \text{dom } F_1^{\text{inv}} = \inf \text{dom } h_D$. Further, let (α, β) be static dual feasible with $D_D(\alpha, \beta) > -\infty$ for the set (3.2). Using the characterization (3.8) of $\overline{B_{S,\text{pre}}}$, we deduce that for every $x \in \Omega$ the condition $(\alpha(x), \beta(x)) \in B_S$ (as defined in (3.2)) implies that there are some $(u, v) \in [0, \infty)^2$ such that $(-\alpha(x) - u, \beta(x) + v) \in \overline{\text{dom } h_D}$ and

$$\beta(x) \leq \beta(x) + v \leq \overline{F_1}(-\alpha(x) - u).$$

Since $-\alpha(x) - u \leq \min\{-\alpha(x), -\alpha_{\min}\}$ and $\overline{F_1}$ is monotonically increasing on $\overline{\text{dom } h_D}$ this implies $\beta(x) \leq \overline{F_1}(-\alpha(x))$ if $-\alpha(x) \in \text{dom } h_D$ and $\beta(x) \leq \overline{F_1}(-\alpha_{\min})$ otherwise. Now consider

$$\hat{\alpha}(x) = \max\{\alpha_{\min}, \alpha(x)\}, \quad \hat{\beta}(x) = \max\{\beta_{\min}, \beta(x)\}.$$

Then $\hat{\alpha}$ and $\hat{\beta}$ still lie in $\text{Lip}(\Omega)$, they majorize α and β , and $(\hat{\alpha}(x), \hat{\beta}(x)) \in \overline{B_{S,\text{pre}}}$ for all $x \in \Omega$. Thus replacing B_S by $\overline{B_{S,\text{pre}}}$ in (2.10) does not change the value of (2.11). Further, note that $\lambda \cdot \overline{B_{S,\text{pre}}} \subset B_{S,\text{pre}}$ for all $\lambda \in (0, 1)$ (since F_1 is concave and $F_1(0) = 0$, see Lem. 3.4). Let $(\lambda_n)_{n=0}^\infty$ be a sequence in $(0, 1)$ with $\lim_{n \rightarrow \infty} \lambda_n = 1$. Then for any $(\hat{\alpha}, \hat{\beta}) \in \text{Lip}(\Omega)^2$ with $(\hat{\alpha}(x), \hat{\beta}(x)) \in \overline{B_{S,\text{pre}}}$ for all $x \in \Omega$, the sequence $(\lambda_n \cdot (\hat{\alpha}, \hat{\beta}))_{n=0}^\infty$ in $\text{Lip}(\Omega)^2$ lies pointwise in $B_{S,\text{pre}}$. Since Ω is compact the sequence converges uniformly to $(\hat{\alpha}, \hat{\beta})$ and so the integral of $\lambda_n \cdot (\hat{\alpha}, \hat{\beta})$ with respect to ρ_0 and ρ_1 converges to the one of $(\hat{\alpha}, \hat{\beta})$. Hence, using $B_{S,\text{pre}}$ instead of B_S yields the same supremum.

Next we show that in (2.11) we may even restrict to $(\alpha, \beta) \in C^1(\Omega)^2$. Indeed, given $\alpha, \beta \in \text{Lip}(\Omega)$ with $(\alpha(x), \beta(x)) \in B_{S,\text{pre}}$ for all $x \in \Omega$, let

$$\alpha_\delta = (1 - \delta)\alpha - \delta M, \quad \beta_\delta = (1 - \delta)\beta - \delta M$$

for $M = \min(|\alpha_{\min}|, |\beta_{\min}|)/2$, $\delta \in (0, 1)$, then also $\alpha_\delta, \beta_\delta \in \text{Lip}(\Omega)$. By convexity of $B_{S,\text{pre}}$ and since $[-2M, 0]^2 \subset B_{S,\text{pre}}$ one finds $(\alpha_\delta(x), \beta_\delta(x)) + [-\delta M, \delta M]^2 = (1 - \delta) \cdot (\alpha(x), \beta(x)) + \delta \cdot [-2M, 0]^2 \subset B_{S,\text{pre}}$ for all $x \in \Omega$. By the density of $C^1(\Omega) \cap \text{Lip}(\Omega)$ in $\text{Lip}(\Omega)$ with respect to uniform convergence due to Proposition 7.5 there exist $\hat{\alpha}_\delta, \hat{\beta}_\delta \in C^1(\Omega) \cap \text{Lip}(\Omega)$ with $|\hat{\alpha}_\delta - \alpha_\delta|, |\hat{\beta}_\delta - \beta_\delta| \leq \delta M$ and thus $(\hat{\alpha}_\delta(x), \hat{\beta}_\delta(x)) \in B_{S,\text{pre}}$ for all $x \in \Omega$. Furthermore, as $\delta \rightarrow 0$ we have $(\hat{\alpha}_\delta, \hat{\beta}_\delta) \rightarrow (\alpha, \beta)$ uniformly so that $D_S(\hat{\alpha}_\delta, \hat{\beta}_\delta) \rightarrow D_S(\alpha, \beta)$. Hence the value of (2.11) does not change if we restrict additionally to $(\alpha, \beta) \in C^1(\Omega)$.

Now let $(\alpha, \beta) \in [C^1(\Omega) \cap \text{Lip}(\Omega)]^2$ with $(\alpha(x), \beta(x)) \in B_{S,\text{pre}}$ for all $x \in \Omega$. If h_D and $h_D(-\cdot)$ satisfy Assumption 3.3 then we can directly use Lemma 3.11 to construct a $\phi \in C^1([0, 1] \times \Omega)$ with $D_D(\phi) = D_S(\alpha, \beta)$ and thus establish equivalence. Otherwise we will now construct a sequence $(\phi_j)_{j=1}^\infty$ with $\phi_j \in C^1([0, 1] \times \Omega)$ such that $\lim_{j \rightarrow \infty} D_D(\phi_j) = D_S(\alpha, \beta)$. Let $\varepsilon_j > 0$ be a sequence converging to zero as $j \rightarrow \infty$ and set $\lambda_j = 1 - C\varepsilon_j$ for C from Remark 3.7. Let $F_{s,j}$ (and $F_{s,j}^{\text{inv}}$) be the flow (and its inverse) of the smoothed set B_{D,ε_j} from Remark 3.7 and let $B_{S,\text{pre},\varepsilon_j}$ be the corresponding set induced via (3.1) or (3.7). Set $(\alpha_j, \beta_j) = \lambda_j \cdot (\alpha, \beta) \in [C^1(\Omega) \cap \text{Lip}(\Omega)]^2$, then $(\alpha_j(x), \beta_j(x)) \in B_S$ for all $x \in \Omega$. Hence, since $\lambda_j \rightarrow 1$ and Ω is compact, $(\alpha_j, \beta_j) \rightarrow (\alpha, \beta)$ uniformly and thus $D_S(\alpha_j, \beta_j) \rightarrow D_S(\alpha, \beta)$. As $\lambda_j \cdot B_D \subset B_{D,\varepsilon_j}$ one has $\lambda_j \cdot B_{S,\text{pre}} \subset B_{S,\text{pre},\varepsilon_j}$ and so $(\alpha_j(x), \beta_j(x)) \in B_{S,\text{pre},\varepsilon_j}$ for all $x \in \Omega$. Therefore, the function ϕ_j , constructed from $F_{s,j}$ and $F_{s,j}^{\text{inv}}$ via Lemma 3.11 for $B_{S,\text{pre},\varepsilon_j}$, satisfies $D_D(\phi_j) = D_S(\alpha_j, \beta_j)$. \square

Proposition 3.12. *The function h_S corresponding to B_S as given by (3.2) is given by*

$$h_S(z) = \begin{cases} F_1(z) & \text{if } z \in \text{int dom } F_1, \\ \sup F_1(\text{dom } F_1) & \text{if } z \geq \sup \text{dom } F_1, \\ \inf F_1(\text{dom } F_1) & \text{if } z = \inf \text{dom } F_1, \\ -\infty & \text{if } z < \inf \text{dom } F_1, \end{cases}$$

where F_1 is the flow of B_D at time $t = 1$, as defined in Lemma 3.4.

Proof. This follows from (3.2) and Proposition 3.10. The properties required for h_S in Proposition 2.5 follow from Lemma 3.4: h_S is increasing and concave by virtue of (iv) and (v), h_S is continuous on $\text{int dom } F_1$ by concavity of F_1 and upper semi-continuous on \mathbb{R} by construction due to monotonicity of F_1 , and $h_S(z) \leq z$ follows from (iii) and $h_S(0) = 0$, $h'_S(0) = 1$ from (viii). \square

3.2. Dynamic primal optimizers from static primal optimizers

The equivalence of the dynamic and static models has been established in Proposition 3.1 *via* equivalence of their dual formulations. In this section we clarify the primal interpretation. Proposition 3.13 provides the relation between the dynamic and the static mass change penalties c_D and c_S from the primal formulations Definitions 2.2 and 2.4. Based on this, Proposition 3.15 shows how to construct optimizers of the primal dynamic problem (2.4) from optimizers of the primal static problem (2.9).

Proposition 3.13 (Relation between dynamic and static mass change penalties). *For a dynamic unbalanced transport problem (2.4) with mass change penalty c_D , the mass change penalty c_S of the equivalent static problem (2.9) is given by*

$$c_S(m_0, m_1) = \min \left\{ \int_0^1 c_D \left(\frac{d\rho}{d\mu}, \frac{d\zeta}{d\mu} \right) d\mu \mid (\rho, \zeta) \in \mathcal{CES}(m_0, m_1) \right\}, \quad (3.10)$$

where μ is any measure in $\mathcal{M}_+([0, 1])$ with $\rho, \zeta \ll \mu$ and

$$\mathcal{CES}(m_0, m_1) = \left\{ (\rho, \zeta) \in \mathcal{M}([0, 1])^2 \mid \int_0^1 (\partial_t \phi) d\rho + \int_0^1 \phi d\zeta = m_1 \cdot \phi(1) - m_0 \cdot \phi(0) \text{ for all } \phi \in C^1([0, 1]) \right\}.$$

If (3.10) is finite, it admits a minimizer.

The set $\mathcal{CES}(m_0, m_1)$ encodes a weak “single point continuity equation” with no transport and only growth, in analogy to Definition 2.1. Thus c_S represents the minimum cost of a time trajectory $(\rho, \zeta) \in \mathcal{CES}(m_0, m_1)$ along which mass change is penalized by c_D .

Proof. By Proposition 2.5 we have $\iota_{B_S} = c_S^*$ and thus

$$\begin{aligned} c_S(m_0, m_1) &= \sup \{ m_0 \cdot \alpha + m_1 \cdot \beta - \iota_{B_S}(\alpha, \beta) \} \\ &= \sup \{ m_1 \cdot \phi(1) - m_0 \cdot \phi(0) \mid \phi \in C^1([0, 1]), (\partial_t \phi(t), \phi(t)) \in B_D \text{ for all } t \in [0, 1] \}, \end{aligned}$$

where we have used the relation (3.2) between the sets B_S and B_D from Proposition 3.1. The right-hand side is essentially a variant of the dynamic unbalanced transport problem (2.6) without transport, so what follows is a variant of the proof of Proposition 2.3. Abbreviating

$$G(\phi) = m_1 \cdot \phi(1) - m_0 \cdot \phi(0), \quad F(\partial_t \phi, \phi) = \int_0^1 \iota_{B_D}(\partial_t \phi(t), \phi(t)) dt, \quad A\phi = (\partial_t \phi, \phi),$$

we just repeat the arguments from the proof of Proposition 2.3 to obtain

$$\begin{aligned} c_S(m_0, m_1) &= \sup \{ -F(A\phi) - G(-\phi) \mid \phi \in C^1([0, 1]) \} \\ &= \min \{ F^*(\rho, \zeta) + G^*(A^*(\rho, \zeta)) \mid (\rho, \zeta) \in \mathcal{M}([0, 1])^2 \}, \end{aligned}$$

where $G^*(A^*(\rho, \zeta)) = \iota_{\mathcal{CES}(m_0, m_1)}(\rho, \zeta)$ and $F^*(\rho, \zeta) = \int_0^1 c_D(\frac{d\rho}{d\mu}, \frac{d\zeta}{d\mu}) d\mu$ due to $\iota_{BD} = c_D^*$. \square

In order to construct an optimal triple (ρ, ω, ζ) for the dynamic problem (2.4) from an optimal pair (π_0, π_1) for the static problem (2.9), we first introduce two basic building blocks, a pure mass flow between two points and a pure mass change at a single point.

Definition 3.14 (Singular mass flow and change). For given $x, y \in \Omega$ denote a shortest path between them by $p_{x,y} : [0, 1] \rightarrow \Omega$ with $p_{x,y}(0) = x$ and $p_{x,y}(1) = y$. By the *singular mass flow from x to y* we denote the measure $\omega_{x,y} \in \mathcal{M}(\Omega)^n$ defined via

$$\int_{\Omega} \psi \cdot d\omega_{x,y} = \int_{[0,1]} \psi(p_{x,y}(t)) \cdot \dot{p}_{x,y}(t) dt$$

for all $\psi \in C(\Omega; \mathbb{R}^n)$. By the *singular mass change from $m_0 \geq 0$ to $m_1 \geq 0$* we denote a pair $(\rho[m_0, m_1], \zeta[m_0, m_1]) \in \mathcal{M}([0, 1])^2$ which minimizes (3.10).

Proposition 3.15 (Construction of dynamic optimizers from static optimizers). *Let (π_0, π_1) be a pair of optimal couplings in (2.9), abbreviate $\rho'_0 = P_Y \pi_0$, $\rho'_1 = P_X \pi_1$, and pick $\gamma \in \mathcal{M}_+(\Omega)$ such that $\rho'_0, \rho'_1 \ll \gamma$. For $x \in \Omega$ let $(\rho[\frac{d\rho'_0}{d\gamma}(x), \frac{d\rho'_1}{d\gamma}(x)], \zeta[\frac{d\rho'_0}{d\gamma}(x), \frac{d\rho'_1}{d\gamma}(x)]) \in \mathcal{M}([0, 1])^2$ be the singular mass change from Definition 3.14 between $m_0 = \frac{d\rho'_0}{d\gamma}(x)$ and $m_1 = \frac{d\rho'_1}{d\gamma}(x)$. An optimal triple (ρ, ω, ζ) in (2.4) is obtained by*

$$\begin{aligned} \int_{[0,1] \times \Omega} \phi d\rho &= \int_{\Omega} \int_0^1 \phi(t, x) d\rho[\frac{d\rho'_0}{d\gamma}(x), \frac{d\rho'_1}{d\gamma}(x)](t) d\gamma(x), \\ \int_{[0,1] \times \Omega} \psi \cdot d\omega &= \int_{\Omega} \psi(0, \cdot) \cdot d\omega_0 + \int_{\Omega} \psi(1, \cdot) \cdot d\omega_1 \\ &\quad \text{with } \int_{\Omega} \psi(i, \cdot) \cdot d\omega_i = \int_{\Omega \times \Omega} \int_{\Omega} \psi(i, \cdot) \cdot d\omega_{x,y} d\pi_i(x, y), \quad i = 0, 1, \\ \int_{[0,1] \times \Omega} \phi d\zeta &= \int_{\Omega} \int_0^1 \phi(t, x) d\zeta[\frac{d\rho'_0}{d\gamma}(x), \frac{d\rho'_1}{d\gamma}(x)](t) d\gamma(x) \end{aligned}$$

for all $\phi \in C([0, 1] \times \Omega)$ and $\psi \in C([0, 1] \times \Omega; \mathbb{R}^n)$.

Remark 3.16 (Non-uniqueness of minimizers). Note that the dynamic minimizers (ρ, ω, ζ) constructed above are not necessarily unique (the static minimizers (π_0, π_1) need not be unique either). For instance, if no mass changes occur, the transport can also be performed over an extended interval in time. Vice versa, mass changes may sometimes also occur instantaneously. In the particular case $c_S(m_0, m_1) = |m_0 - m_1|$ for $\min\{m_0, m_1\} \geq 0$, which corresponds to $c_D(\rho, \zeta) = |\zeta|$ for $\rho \geq 0$, mass changes and transport may both happen during arbitrary subintervals or even time points of $[0, 1]$, leading to a large degenerate set of minimizers. In Section 3.3 we give sufficient conditions for c_S such that all dynamic optimizers are of the above form.

Proof. We first show $(\rho, \omega, \zeta) \in \mathcal{CE}(\rho_0, \rho_1)$. Indeed, first note that for $\varphi \in C^1(\Omega)$ we have $\int_{\Omega} \nabla \varphi \cdot d\omega_{x,y} = \int_{[0,1]} \nabla \varphi(p_{x,y}(t)) \cdot \dot{p}_{x,y}(t) dt = \varphi(y) - \varphi(x)$. Therefore, for $\phi \in C^1([0, 1] \times \Omega)$ we have

$$\begin{aligned} \int_{[0,1] \times \Omega} \nabla \phi \cdot d\omega &= \int_{\Omega \times \Omega} \int_{\Omega} \nabla \phi(0, \cdot) \cdot d\omega_{x,y} d\pi_0(x, y) + \int_{\Omega \times \Omega} \int_{\Omega} \nabla \phi(1, \cdot) \cdot d\omega_{x,y} d\pi_1(x, y) \\ &= \int_{\Omega \times \Omega} \phi(0, y) - \phi(0, x) d\pi_0(x, y) + \int_{\Omega \times \Omega} \phi(1, y) - \phi(1, x) d\pi_1(x, y) \\ &= \int_{\Omega} \phi(0, \cdot) d(\rho'_0 - \rho_0) - \int_{\Omega} \phi(1, \cdot) d(\rho'_1 - \rho_1). \end{aligned}$$

In addition, due to $(\rho[m_0, m_1], \zeta[m_0, m_1]) \in \mathcal{CES}(m_0, m_1)$ for $(m_0, m_1) \in [0, \infty)^2$ we obtain

$$\begin{aligned} &\int_{[0,1] \times \Omega} (\partial_t \phi) d\rho + \int_{[0,1] \times \Omega} \phi d\zeta \\ &= \int_{\Omega} \left[\int_0^1 \partial_t \phi(t, x) d\rho \left[\frac{d\rho'_0}{d\gamma}(x), \frac{d\rho'_1}{d\gamma}(x) \right](t) + \int_0^1 \phi(t, x) d\zeta \left[\frac{d\rho'_0}{d\gamma}(x), \frac{d\rho'_1}{d\gamma}(x) \right](t) \right] d\gamma(x) \\ &= \int_{\Omega} \left[\phi(1, x) \cdot \frac{d\rho'_1}{d\gamma}(x) - \phi(0, x) \cdot \frac{d\rho'_0}{d\gamma}(x) \right] d\gamma(x) = \int_{\Omega} \phi(1, \cdot) d\rho'_1 - \int_{\Omega} \phi(0, \cdot) d\rho'_0 \end{aligned}$$

which implies $(\rho, \omega, \zeta) \in \mathcal{CE}(\rho_0, \rho_1)$.

Next we calculate the cost $P_D(\rho, \omega, \zeta)$ from (2.3). Let us abbreviate

$$\bar{C} = \{ \psi \in C([0, 1] \times \Omega; \mathbb{R}^n) \mid \|\psi(t, x)\| \leq 1 \text{ for all } (t, x) \in [0, 1] \times \Omega \},$$

and let us denote the total variation norm of ω by $\|\omega\|_{\mathcal{M}([0,1] \times \Omega)^n} = \sup_{\psi \in \bar{C}} \int_{[0,1] \times \Omega} \psi \cdot d\omega$, then we have

$$\begin{aligned} \int_{[0,1] \times \Omega} \left\| \frac{d\omega}{d\mu} \right\| d\mu &= \|\omega\|_{\mathcal{M}([0,1] \times \Omega)^n} = \sup_{\psi \in \bar{C}} \sum_{i=0}^1 \int_{\Omega \times \Omega} \int_0^1 \psi(i, p_{x,y}(t)) \cdot \dot{p}_{x,y}(t) dt d\pi_i(x, y) \\ &\leq \sum_{i=0}^1 \int_{\Omega \times \Omega} \int_0^1 \|\dot{p}_{x,y}(t)\| dt d\pi_i(x, y) = \int_{\Omega \times \Omega} d(x, y) d\pi_0(x, y) + \int_{\Omega \times \Omega} d(x, y) d\pi_1(x, y). \end{aligned}$$

For the growth term, by optimality of $(\rho[\frac{d\rho'_0}{d\gamma}(x), \frac{d\rho'_1}{d\gamma}(x)], \zeta[\frac{d\rho'_0}{d\gamma}(x), \frac{d\rho'_1}{d\gamma}(x)])$ in (3.10), we get

$$\begin{aligned} \int_{[0,1] \times \Omega} c_D \left(\frac{d\rho}{d\mu}, \frac{d\zeta}{d\mu} \right) d\mu &= \int_{\Omega} \left[\int_0^1 c_D \left(\frac{d\rho[\frac{d\rho'_0}{d\gamma}(x), \frac{d\rho'_1}{d\gamma}(x)]}{d\mu_x}, \frac{d\zeta[\frac{d\rho'_0}{d\gamma}(x), \frac{d\rho'_1}{d\gamma}(x)]}{d\mu_x} \right) d\mu_x \right] d\gamma(x) \\ &= \int_{\Omega} c_S \left(\frac{d\rho'_0}{d\gamma}(x), \frac{d\rho'_1}{d\gamma}(x) \right) d\gamma(x) = C_S(\rho'_0, \rho'_1). \end{aligned}$$

Here, $\{\mu_x\}_{x \in \Omega}$ with $\mu_x \in \mathcal{M}_+([0, 1])$ is a family of measures such that $\rho[\frac{d\rho'_0}{d\gamma}(x), \frac{d\rho'_1}{d\gamma}(x)], \zeta[\frac{d\rho'_0}{d\gamma}(x), \frac{d\rho'_1}{d\gamma}(x)] \ll \mu_x$, and μ is chosen as $\int_{[0,1] \times \Omega} \phi d\mu = \int_{\Omega} \int_0^1 \phi(t, x) d\mu_x(t) d\gamma(x)$ for all $\phi \in C([0, 1] \times \Omega)$. Therefore, $P_D(\rho, \omega, \zeta) \leq P_S(\pi_0, \pi_1) = W_S(\rho_0, \rho_1)$, but $W_S(\rho_0, \rho_1) = W_D(\rho_0, \rho_1) \leq P_D(\rho, \omega, \zeta)$ by Proposition 3.1. Hence the triple (ρ, ω, ζ) minimizes (2.4). \square

Remark 3.17 (Beckmann's problem). The W_1 distance on convex subsets $\Omega \subset \mathbb{R}^n$ can be rewritten as a minimization problem of a cost for mass flows, also known as Beckmann's problem. The measures ω_i constructed

in Proposition 3.15 are the canonical cost minimizing flows corresponding to the optimal transport plans π_i , and one has $W_1(\rho_i, \rho'_i) = \|\omega_i\|_{\mathcal{M}(\Omega)^n}$ (for more details see [20], Thm. 4.6).

Remark 3.18 (Interpretation of dynamic optimizers). The interpretation of the dynamic minimizer (ρ, ω, ζ) constructed in Proposition 3.15 is as follows. At time $t = 0$ mass is instantaneously transported from ρ_0 to ρ'_0 , as encoded in π_0 or rather ω_0 . Then, throughout the interval $[0, 1]$ the mass distribution ρ'_0 is gradually transformed into ρ'_1 via simultaneous mass change throughout Ω . Finally, at $t = 1$ mass is instantaneously transported from ρ'_1 to ρ_1 , as encoded in π_1 and ω_1 . The temporal concentration of the transport into instantaneous shifts is possible since the transport part of the cost, $(\rho, \omega, \zeta) \mapsto \|\omega\|_{\mathcal{M}([0,1] \times \Omega)^n}$, is 1-homogeneous in ω . This is different, for instance, from the Benamou–Brenier formula [4] which is a dynamic formulation of the W_2 distance, or from the Wasserstein–Fisher–Rao distance [8].

3.3. A necessary condition for dynamic primal optimizers

In the previous section we have constructed primal dynamic optimizers from static optimizers. In this section we show that under suitable assumptions on the dynamic mass change cost c_D any dynamic optimizer is of this form.

Proposition 3.19 (Instantaneous mass transport). *Assume $h_D(z) < 0$ for $z \neq 0$ (h_D determines the set B_D via Lemma 3.2, and B_D characterizes c_D via Proposition 2.3). If $(\rho, \omega, \zeta) \in \mathcal{CE}(\rho_0, \rho_1)$ is a dynamic primal optimizer of $W_D(\rho_0, \rho_1)$ in (2.4), then $|\omega|((0, 1) \times \Omega) = 0$, that is, mass transport only happens instantaneously at time 0 and 1.*

Remark 3.20 (Interpretation of assumption). Assumption $h_D(z) < 0$ for $z \neq 0$ is equivalent to $\zeta \rightarrow c_D(1, \zeta)$ being differentiable with derivative 0 at $\zeta = 0$. A prototypical counterexample is $c_D(\rho, \zeta) = |\zeta|$ for $\rho \geq 0$, see Remark 3.16.

Proof. With Proposition 2.3 and Lemma 3.2 we obtain

$$c_D(1, \zeta) = \iota_{B_D}^*(1, \zeta) = \sup\{h_D(\phi) + \zeta \cdot \phi \mid \phi \in \text{dom}(h_D)\} = (-h_D)^*(\zeta).$$

If $h_D(z) < 0$ for all $z \neq 0$, then by concavity and upper semicontinuity of h_D we have

$$z = 0 \quad \Leftrightarrow \quad 0 \in \partial(-h_D)(z) \quad \Leftrightarrow \quad z \in \partial[c_D(1, \cdot)](0)$$

so that $\partial[c_D(1, \cdot)] = \{0\}$. A similar argument works for the reverse implication. \square

Corollary 3.21 (Structure of dynamic primal optimizers). *Under the above assumption on h_D , any dynamic primal optimizer can be constructed from a static primal optimizer as in Proposition 3.15.*

Sketch of proof. By Proposition 3.19 a dynamic primal optimal ω must have the form $\delta_0 \otimes \omega_0 + \delta_1 \otimes \omega_1$ for some $\omega_0, \omega_1 \in \mathcal{M}(\Omega)^n$. Using the continuity equation (2.1) one can show that these have weak divergences $\sigma_0, \sigma_1 \in \mathcal{M}(\Omega)$, that is, $\int_{\Omega} \nabla \psi \, d\omega_i = -\int_{\Omega} \psi \, d\sigma_i$ for $i = 0, 1$ and all $\psi \in C^1(\Omega)$, and that $(\rho, 0, \zeta) \in \mathcal{CE}(\rho'_0 = \rho_0 - \sigma_0, \rho'_1 = \rho_1 + \sigma_1)$. Using arguments similar to those of the following section it is easy to show that ρ'_0 and ρ'_1 are nonnegative (if ρ'_0 were negative at some point, ζ must have a compensating singular positive component at $t = 0$ which implies that mass is first created and then transported by ω_0 , which cannot be optimal). By optimality, ω_0 and ω_1 must then be optimal flows for $W_1(\rho_i, \rho'_i)$ in Beckmann’s formulation, corresponding to optimal couplings π_i (cf. Rem. 3.17). Similarly, (ρ, ζ) must be pointwise optimizers for (3.10) in Proposition 3.13. \square

Proposition 3.19 is proven via a primal-dual argument. On a formal level the primal-dual optimality conditions for the dynamic problems require that $\|\nabla \phi(t, x)\| = 1$ on the support of ω and that the gradient of ϕ is aligned with the orientation of ω . Essentially, we show that $\|\nabla \phi(t, x)\| < 1$ for $t \in (0, 1)$ and sufficiently “many” $x \in \Omega$ for “optimal” ϕ . The remaining $x \in \Omega$ are shown to be harmless via the continuity equation. However, in general

no dynamic dual optimizers exist in $C^1([0, 1] \times \Omega)$, and thus the argument has to be made *via* a maximizing sequence, for which a more explicit bound on the gradient is required. This bound is established *via* some auxiliary lemmas in Proposition 3.22. The duality argument is then made rigorous at the end of the section.

Proposition 3.22 (Lipschitz bound for constructed dynamic dual candidates). *The feasible dynamic dual candidates constructed in Lemma 3.11 satisfy*

$$\|\nabla\phi(t, x)\| \leq 1 - t(1 - t) \cdot [K(-\alpha(x)) + K(-\beta(x))]$$

for some function K with $K(z) = 0$ for $z \leq 0$ and K strictly positive and increasing for $z > 0$.

The proof is distributed over several auxiliary lemmas.

Lemma 3.23. *Let $f : [0, 1] \rightarrow \mathbb{R}$ be convex and decreasing with concave derivative $f' : [0, 1] \rightarrow \mathbb{R}$. Then*

$$f(t) \leq g(t) \quad \text{for } g(t) = (1 - t)f(0) + tf(1) - t(1 - t)\hat{K}$$

with $\hat{K} = f'(1) - f(1) + f(0) \geq 0$.

Proof. The convexity of f implies $f(0) \geq f(1) - f'(1)$ and thus $\hat{K} \geq 0$. Moreover, the concavity of f' implies $f(1) - f(0) = \int_0^1 f'(t)dt \geq \int_0^1 [(1 - t)f'(0) + tf'(1)]dt = \frac{1}{2}(f'(0) + f'(1))$ so that $2(f(1) - f(0)) - f'(1) \geq f'(0)$. Therefore, $g'(0) = f(1) - f(0) - \hat{K} \geq f'(0)$ and $g'(1) = f'(1)$. Now by concavity of f' and thus also of $f' - g'$ there must be some $t_0 \in [0, 1]$ such that

$$f'(s) - g'(s) \leq 0 \text{ on } s \in [0, t_0] \quad \text{and} \quad f'(s) - g'(s) \geq 0 \text{ on } s \in [t_0, 1].$$

Together with $g(0) = f(0)$ and $g(1) = f(1)$ this implies $g(s) \geq f(s)$ for $s \in [0, 1]$. □

Lemma 3.24. *If $h_D(z) < 0$ for $z > 0$ then there is an increasing function $K : \text{dom}(h_D) \cap [0, \infty) \rightarrow [0, \infty)$, $K(z) > 0$ for $z > 0$ such that for $z \in \text{dom}(h_D) \cap [0, \infty)$ one has*

$$F_t(z) \leq (1 - t)z + tF_1(z) - t(1 - t)K(z).$$

Proof. Let $z > 0$. The map $t \mapsto F_t(z)$ is convex and decreasing by Lemma 3.4(i) and (vii). Furthermore, the map $t \mapsto \partial_t F_t(z) = h_D(F_t(z))$ is a composition of a concave decreasing function with a convex function and thus concave. Therefore, we can apply Lemma 3.23 to obtain an upper bound on $F_t(z)$, $t \in [0, 1]$, with coefficient

$$\hat{K}(z) = \partial_t F_1(z) - F_1(z) + F_0(z) = h_D(F_1(z)) - F_1(z) + z = z \left[1 + \frac{F_1(z)}{z} \left(\frac{h_D(F_1(z))}{F_1(z)} - 1 \right) \right].$$

Let us abbreviate $c(z) = \frac{|h_D(F_1(z))|}{F_1(z)} \geq 0$ and note that by the properties of h_D from Lemma 3.2 we have $h_D(z) \leq -c(z)z$ for all $z \geq F_1(z)$. Thus, by Lemma 3.4(i) and $F_t(z) \geq F_1(z)$ for $t \in [0, 1]$, we have $\partial_t F_t(z) = h_D(F_t(z)) \leq -c(z)F_t(z)$. Using $F_0(z) = z$, Grönwall's inequality now implies $F_t(z) \leq z \exp(-c(z)t)$ and thus $\frac{F_1(z)}{z} \leq \exp(-c(z))$. Summarizing, we obtain

$$\hat{K}(z) \geq K(z) \quad \text{for } K(z) = z[1 + \exp(-c(z))(-c(z) - 1)]$$

so that, for $t \in [0, 1]$,

$$F_t(z) \leq (1 - t)F_0(z) + tF_1(z) - t(1 - t)\hat{K}(z) \leq (1 - t)z + tF_1(z) - t(1 - t)K(z).$$

Note that $c(0) = 0$, $c(z) > 0$ for $z > 0$, and that c is increasing by concavity of h_D and monotonicity of F_1 . It is then straightforward to check that K is increasing and strictly positive for $z > 0$. \square

Lemma 3.25. *Let $f : [0, \infty) \rightarrow \mathbb{R}$ be convex and increasing with $f(0) = 0$. Let $x > y \geq 0$, $t \in [0, 1]$, and $\delta \geq 0$ such that $(1-t)x + ty - \delta \geq y$. Then*

$$f((1-t)x + ty - \delta) \leq (1-t)f(x) + tf(y) - \delta \frac{f(x)}{x}.$$

Proof. Since $z = (1-t)x + ty - \delta \in [y, x]$, we can bound $f(z)$ from above by the secant between $(y, f(y))$ and $(x, f(x))$. This can be written as

$$f(z) \leq (1-t)f(x) + tf(y) - \delta \frac{f(x)-f(y)}{x-y}.$$

Since f is convex and increasing, the slope $\frac{f(x)-f(y)}{x-y}$ of this secant is not smaller than the secant slope $\frac{f(x)}{x}$ between $(0, f(0))$ and $(x, f(x))$. \square

Proof of Proposition 3.22. For $z > 0$ we find for g as defined in Lemma 3.8.

$$\partial_z g(t, z) = \frac{h_D(F_t(z))}{h_D(z)} - t \frac{h_D(F_1(z))}{h_D(z)},$$

where we used the formula for the derivative of the flow from Lemma 3.4(x). Let K be the function appearing in Lemma 3.24. Then for $t \in [0, 1]$ we get

$$F_1(z) \leq F_t(z) \leq (1-t)z + tF_1(z) - \delta$$

where $\delta = t(1-t)K(z)$. The function $x \mapsto \frac{h_D(x)}{h_D(z)}$ is convex and increasing and 0 for $x = 0$. Therefore, by Lemma 3.25 we obtain

$$\frac{h_D(F_t(z))}{h_D(z)} \leq \frac{h_D((1-t)z + tF_1(z) - \delta)}{h_D(z)} \leq (1-t) + t \frac{h_D(F_1(z))}{h_D(z)} - t(1-t)K(z)$$

so that $\partial_z g(t, z) \leq (1-t) - t(1-t)K(z)$ for $z > 0$. By extending $K(z) = 0$ for $z \leq 0$ this bound is true for all $z \in \text{dom } h_D$ and also holds in absolute value due to Lemma 3.8(iv), $|\partial_z g(t, z)| \leq (1-t) - t(1-t)K(z)$. Note that Assumption 3.3 is satisfied due to $\bar{s} = 0$. In complete analogy we can show existence of an increasing function \tilde{K} with $\tilde{K}(z) > 0$ for $z > 0$ such that $|\partial_z g^{\text{inv}}(t, z)| \leq t - t(1-t)\tilde{K}(-z)$ for $t \in [0, 1]$. From now on denote by K the pointwise minimum of the two functions, which is still increasing in z with $K(z) > 0$ for $z > 0$. The result then follows from $\phi(t, x) = g(t, -\alpha(x)) + g^{\text{inv}}(t, \beta(x))$ and $\alpha, \beta \in \text{Lip}(\Omega)$. \square

Proof of Proposition 3.19. Let ϕ be feasible for the dynamic dual problem (2.6), set $\alpha = -\phi(0, \cdot)$, $\beta = \phi(1, \cdot)$ and finally $\hat{\phi}(t, x) = g(t, -\alpha(x)) + g^{\text{inv}}(t, \beta(x))$, as constructed in Lemma 3.11. Then $\hat{\phi}$ has the same score as ϕ . Therefore, without loss of generality, we may in the following restrict the dynamic dual problem to candidates of the form constructed in Lemma 3.11.

Let (ρ, ω, ζ) be a minimizer of the dynamic primal problem (2.4) and let $(\phi_k)_{k \in \mathbb{N}}$ be a maximizing sequence of the dynamic dual problem (2.6). Using the notation from the proof of Proposition 2.3 this implies that

$$F^*(\rho, \omega, \zeta) + G^*(A^*(\rho, \omega, \zeta)) + F(A\phi_k) + G(-\phi_k) \searrow 0 \quad \text{as } k \rightarrow \infty.$$

By the Fenchel–Young inequality one has $F^*(\rho, \omega, \zeta) + F(A\phi_k) - \int_{[0,1] \times \Omega} A\phi_k d(\rho, \omega, \zeta) \geq 0$ and $G^*(A^*(\rho, \omega, \zeta)) + G(-\phi_k) + \int_{[0,1] \times \Omega} A\phi_k d(\rho, \omega, \zeta) \geq 0$, therefore the above implies $F^*(\rho, \omega, \zeta) + F(A\phi_k) -$

$\int_{[0,1] \times \Omega} A\phi_k \, d(\rho, \omega, \zeta) \searrow 0$ as $k \rightarrow \infty$. Further, due to the W_1 -structure of the problem, F and F^* separate into an ω -related part and a (ρ, ζ) -related part,

$$\begin{aligned} & F^*(\rho, \omega, \zeta) + F(A\phi_k) - \int_{[0,1] \times \Omega} A\phi_k \, d(\rho, \omega, \zeta) \\ &= \left(|\omega|([0,1] \times \Omega) + \int_{[0,1] \times \Omega} \iota_{B_1(0)}(\nabla\phi_k) \, d(t, x) - \int_{[0,1] \times \Omega} \nabla\phi_k \, d\omega \right) \\ &+ \left(\int_{[0,1] \times \Omega} c_D \left(\frac{d\rho}{d\mu}, \frac{d\zeta}{d\mu} \right) \, d\mu + \int_{[0,1] \times \Omega} \iota_{B_D}(\partial_t\phi_k, \phi_k) \, d(t, x) - \int_{[0,1] \times \Omega} \partial_t\phi_k \, d\rho - \int_{[0,1] \times \Omega} \phi_k \, d\zeta \right), \end{aligned}$$

where μ is any measure with $\mu \gg \rho, \zeta$, $|\omega| \in \mathcal{M}_+(\Omega)$ denotes the total variation measure, and $B_1(0)$ the closed unit ball in \mathbb{R}^n . Both terms in parentheses are nonnegative due to the Fenchel–Young inequality and thus converge to zero from above separately. Since the ϕ_k are admissible, we thus obtain $|\omega|([0,1] \times \Omega) - \int_{[0,1] \times \Omega} \nabla\phi_k \, d\omega \searrow 0$.

Let $v = \frac{d\omega}{d|\omega|}$ be the local orientation of ω and recall that $\|v\| = 1$ $|\omega|$ -almost everywhere. Then $|\omega|([0,1] \times \Omega) - \int_{[0,1] \times \Omega} \nabla\phi_k \, d\omega = \int_{[0,1] \times \Omega} (1 - \langle \nabla\phi_k, v \rangle) \, d|\omega|$, where the integrand is non-negative $|\omega|$ -almost everywhere since $\|\nabla\phi_k\| \leq 1$. For arbitrary $\delta > 0$ let now k be such that $|\omega|([0,1] \times \Omega) - \int_{[0,1] \times \Omega} \nabla\phi_k \, d\omega < \delta$ and set $\alpha = -\phi_k(0, \cdot)$, $\beta = \phi_k(1, \cdot)$. Let $\varepsilon \in (0, \frac{1}{16})$, set $T_\varepsilon = [\sqrt{\varepsilon}, 1 - \sqrt{\varepsilon}]$ and introduce the following subsets of $[0,1] \times \Omega$:

$$\begin{aligned} S_1 &= \{(t, x) \in T_\varepsilon \times \Omega \mid |\alpha(x)| \geq \varepsilon \text{ or } |\beta(x)| \geq \varepsilon\}, \\ S_2 &= \{(t, x) \in (T_\varepsilon \times \Omega) \setminus S_1 \mid -\nabla\alpha(x) \cdot v(t, x) \leq (1 - \varepsilon) \text{ or } \nabla\beta(x) \cdot v(t, x) \leq (1 - \varepsilon)\}, \\ S_3 &= (T_\varepsilon \times \Omega) \setminus (S_1 \cup S_2), \\ \hat{S}_3 &= ([2\sqrt{\varepsilon}, 1 - 2\sqrt{\varepsilon}] \times \Omega) \setminus (S_1 \cup S_2), \\ S_4 &= ([0,1] \times \Omega) \setminus (S_1 \cup S_2 \cup S_3) = ([0,1] \setminus T_\varepsilon) \times \Omega. \end{aligned}$$

We now estimate the mass of $|\omega|$ on the sets S_1 , S_2 , and \hat{S}_3 . Note that $t(1-t) \geq \varepsilon$ for $t \in T_\varepsilon$ so that $\|\nabla\phi_k(t, x)\| \leq 1 - \varepsilon K(\varepsilon)$ for $(t, x) \in S_1$ by Proposition 3.22. Therefore

$$\delta \geq \int_{S_1} (1 - \|\nabla\phi_k\|) \, d|\omega| \geq \varepsilon K(\varepsilon) |\omega|(S_1).$$

Using $\phi_k = g(t, -\alpha(x)) + g^{\text{inv}}(t, \beta(x))$ and Lemma 3.8(iv) one finds $\nabla\phi_k \cdot v \leq 1 - \varepsilon$ on S_2 and so

$$\delta \geq \int_{S_2} (1 - \nabla\phi_k \cdot v) \, d|\omega| \geq \varepsilon |\omega|(S_2).$$

Let now $h_\varepsilon : [0,1] \rightarrow [0,1]$ be a smooth cutoff function with $h_\varepsilon(t) = 0$ for $t \in [0, \sqrt{\varepsilon}] \cup [1 - \sqrt{\varepsilon}, 1]$, $h_\varepsilon(t) = 1$ on $t \in [2\sqrt{\varepsilon}, 1 - 2\sqrt{\varepsilon}]$ and $|h'_\varepsilon(t)| \leq \frac{2}{\sqrt{\varepsilon}}$ for $t \in [0,1]$. Further, let $g_\varepsilon : \mathbb{R} \rightarrow [-2\varepsilon, 2\varepsilon]$ be a smooth increasing function with $g_\varepsilon(z) = z$ for $z \in [-\varepsilon, \varepsilon]$ and $g'_\varepsilon(z) \in [0,1]$. Set $\psi_\varepsilon(t, x) = h_\varepsilon(t) \cdot g_\varepsilon(-\alpha(x))$. We observe

$$|\psi_\varepsilon(t, x)| \leq 2\varepsilon, \quad \|\nabla\psi_\varepsilon(t, x)\| \leq 1, \quad |\partial_t\psi_\varepsilon(t, x)| \leq 4\sqrt{\varepsilon}$$

for $(t, x) \in [0,1] \times \Omega$ and

$$\nabla\psi_\varepsilon(t, x) \cdot v(t, x) \geq 0 \quad \text{for } (t, x) \in S_3, \quad -\nabla\alpha(x) \cdot v(t, x) > (1 - \varepsilon) \quad \text{for } (t, x) \in \hat{S}_3,$$

$$\nabla\psi_\varepsilon(t, x) = -\nabla\alpha(x) \quad \text{for } (t, x) \in \hat{S}_3, \quad \psi_\varepsilon(t, x) = 0 \quad \text{for } (t, x) \in S_4.$$

We therefore obtain

$$\int_{S_3} \nabla\psi_\varepsilon \, d\omega = \int_{S_3} \nabla\psi_\varepsilon \cdot v \, d|\omega| \geq \int_{\hat{S}_3} -\nabla\alpha \cdot v \, d|\omega| \geq \int_{\hat{S}_3} (1 - \varepsilon) \, d|\omega| = (1 - \varepsilon)|\omega|(\hat{S}_3).$$

Choosing the test function ψ_ε in the continuity equation (2.1) we also find

$$\int_{S_1 \cup S_2 \cup S_3} \nabla\psi_\varepsilon \, d\omega = \int_{[0,1] \times \Omega} \nabla\psi_\varepsilon \, d\omega = - \int_{[0,1] \times \Omega} \partial_t \psi_\varepsilon \, d\rho - \int_{[0,1] \times \Omega} \psi_\varepsilon \, d\zeta.$$

Together with the above bounds on $|\omega|(S_1)$ and $|\omega|(S_2)$ this implies

$$\begin{aligned} (1 - \varepsilon)|\omega|(\hat{S}_3) &\leq \int_{S_3} \nabla\psi_\varepsilon \, d\omega = - \int_{[0,1] \times \Omega} \partial_t \psi_\varepsilon \, d\rho - \int_{[0,1] \times \Omega} \psi_\varepsilon \, d\zeta - \int_{S_1 \cup S_2} \nabla\psi_\varepsilon \, d\omega \\ &\leq 4\sqrt{\varepsilon}|\rho|([0, 1] \times \Omega) + 2\varepsilon|\zeta|([0, 1] \times \Omega) + |\omega|(S_1 \cup S_2) \\ &\leq 4\sqrt{\varepsilon}|\rho|([0, 1] \times \Omega) + 2\varepsilon|\zeta|([0, 1] \times \Omega) + \frac{\delta}{\varepsilon K(\varepsilon)} + \frac{\delta}{\varepsilon} \end{aligned}$$

and thus

$$|\omega|(\hat{S}_3) \leq \frac{1}{1-\varepsilon} \left(4\sqrt{\varepsilon}|\rho|([0, 1] \times \Omega) + 2\varepsilon|\zeta|([0, 1] \times \Omega) + \frac{\delta}{\varepsilon K(\varepsilon)} + \frac{\delta}{\varepsilon} \right).$$

Now we send δ and ε jointly to 0 in a way such that $\frac{\delta}{\varepsilon K(\varepsilon)} + \frac{\delta}{\varepsilon} \rightarrow 0$. Note that then $|\omega|([2\sqrt{\varepsilon}, 1 - 2\sqrt{\varepsilon}] \times \Omega) \leq |\omega|(S_1 \cup S_2 \cup \hat{S}_3) \rightarrow 0$, which implies $|\omega|([\tau, 1 - \tau] \times \Omega) = 0$ for all $\tau > 0$. By inner regularity of Radon measures this implies $\omega((0, 1) \times \Omega) = 0$. \square

3.4. Characterization of static and dynamic primal optimal solutions

In Proposition 3.26, Corollary 3.28, and Proposition 3.29 we will characterize minimizers π_0 and π_1 of the static primal formulation (2.9) in more detail. By the two previous sections, this immediately implies a characterization also of optimizers for the dynamic primal formulation (2.4). Our characterization concerns the possible support of the couplings π_0 and π_1 .

Proposition 3.26 (Necessary optimality condition I for static primal). *Let (π_0, π_1) be a feasible candidate in (2.9). Denote $\rho'_0 = P_Y \pi_0$, $\rho'_1 = P_X \pi_1$, choose some $\gamma \in \mathcal{M}_+(\Omega)$ with $\rho_0, \rho_1, \rho'_0, \rho'_1 \ll \gamma$, and set*

$$\Omega_+ = \left\{ x \in \Omega \mid \frac{d\rho'_0}{d\gamma}(x) < \frac{d\rho'_1}{d\gamma}(x) \right\}, \quad \Omega_- = \left\{ x \in \Omega \mid \frac{d\rho'_0}{d\gamma}(x) > \frac{d\rho'_1}{d\gamma}(x) \right\}, \quad \Omega_0 = \Omega \setminus (\Omega_+ \cup \Omega_-).$$

Clearly, Ω_+ and Ω_- are well-defined up to γ -negligible (and thus ρ'_0, ρ'_1 -negligible) sets. If (π_0, π_1) are minimizers of (2.9), then

$$\int_{\Omega \times \Omega_-} d(x, y) \, d\pi_0(x, y) = 0 \quad \text{and} \quad \int_{\Omega_+ \times \Omega} d(x, y) \, d\pi_1(x, y) = 0. \quad (3.11)$$

Ω_+ and Ω_- are the areas where mass is grown or shrunk in the intermediate C_S -term in (2.8). The above proposition states that in an optimal arrangement or during the optimal dynamics mass is not transported after it has been increased and not transported prior to being decreased.

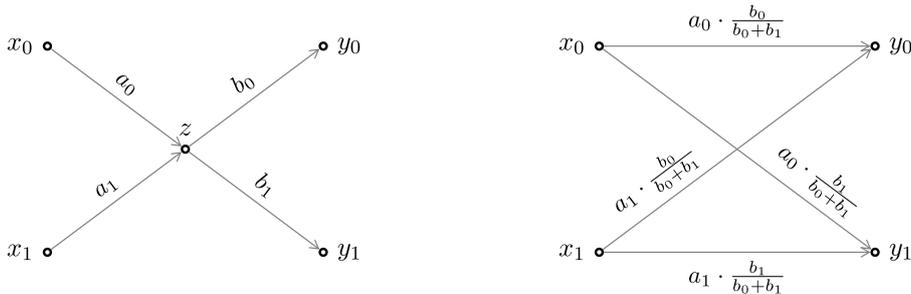


FIGURE 2. Illustration of Proposition 3.26. For pairwise disjoint $x_0, x_1, z, y_0, y_1 \in \Omega$ consider initial and final measure $\rho_0 = a_0 \cdot \delta_{x_0} + a_1 \cdot \delta_{x_1}$ and $\rho_1 = b_0 \cdot \delta_{y_0} + b_1 \cdot \delta_{y_1}$ with $a_0 + a_1 < b_0 + b_1$ (in particular, there is zero mass located at z, y_0, y_1 initially). *Left:* Masses a_i are transported from x_i to intermediate point z by the coupling π_0 . The mass in z is then increased from $a_0 + a_1$ to $b_0 + b_1$ with the C_S term. Finally, masses b_i are transported to y_i by the coupling π_1 . (See (2.8) for the various terms.) *Right:* The cost of the left configuration can always be improved by transporting mass directly from x_i to y_j in suitable fractions *via* the coupling π_0 , then increasing mass at final points y_j appropriately, and performing no more transport *via* π_1 . (See Example 3.27 for details.)

Example 3.27. The intuition behind Proposition 3.26 is illustrated in Figure 2. For the left configuration the combined cost (2.8) for transport and growth is $\sum_{i=0}^1 a_i \cdot d(x_i, z) + c_S(a_0 + a_1, b_0 + b_1) + \sum_{i=0}^1 b_i \cdot d(z, y_i)$. For the configuration on the right it is $\sum_{i,j=0}^1 a_i \frac{b_j}{b_0+b_1} \cdot d(x_i, y_j) + \sum_{j=0}^1 c_S(b_j \frac{a_0+a_1}{b_0+b_1}, b_j)$. By using $a_0 + a_1 < b_0 + b_1$, $z \neq y_i$ and the triangle inequality one finds for the transport costs

$$\begin{aligned} \sum_{i,j=0}^1 a_i \frac{b_j}{b_0+b_1} \cdot d(x_i, y_j) &\leq \sum_{i,j=0}^1 a_i \frac{b_j}{b_0+b_1} \cdot [d(x_i, z) + d(z, y_j)] \\ &= \sum_{i=0}^1 a_i \cdot d(x_i, z) + \frac{a_0 + a_1}{b_0 + b_1} \sum_{j=0}^1 b_j \cdot d(z, y_j) < \sum_{i=0}^1 a_i \cdot d(x_i, z) + \sum_{j=0}^1 b_j \cdot d(z, y_j). \end{aligned}$$

Moreover, by 1-homogeneity of c_S one obtains

$$\sum_{j=0}^1 c_S(b_j \frac{a_0+a_1}{b_0+b_1}, b_j) = c_S(a_0 + a_1, b_0 + b_1).$$

Thus, the second configuration has a strictly lower cost. Analogously, if $a_0 + a_1$ were strictly smaller than $b_0 + b_1$ and $x_i \neq z$, the left configuration can be strictly improved by letting the first coupling π_0 perform no transport at all, then decreasing masses at x_i and finally transporting the decreased masses to the points y_j in appropriate fractions *via* π_1 .

Proof of Proposition 3.26. The strategy of the proof is as follows: for any feasible candidate (π_0, π_1) that does not satisfy condition (3.11) we will construct a new candidate $(\hat{\pi}_0, \hat{\pi}_1)$ with a strictly better cost. To this end, Example 3.27 is generalized to arbitrary configurations.

(Step i) First we construct the improved candidate. Let the family $\{\pi_{0,z}\}_{z \in \Omega}$ be the disintegration of π_0 with respect to its second marginal $P_Y \pi_0 = \rho'_0$ (see for instance [2], Thm. 5.3.1). This means $\pi_{0,z} \in \mathcal{P}(\Omega)$ for

ρ'_0 -a.e. $z \in \Omega$ and, for any measurable $\phi : \Omega \rightarrow [0, \infty]$,

$$\int_{\Omega \times \Omega} \phi \, d\pi_0 = \int_{\Omega} \left[\int_{\Omega} \phi(x, z) \, d\pi_{0,z}(x) \right] d\rho'_0(z).$$

Intuitively, $\pi_{0,z}$ describes the distribution of mass before transport that arrives at z according to the transport plan π_0 . It corresponds to the normalized coefficients $a_i/(a_0 + a_1)$ in Example 3.27. Similarly, let $\{\pi_{1,z}\}_{z \in \Omega}$ be the disintegration of π_1 according to its first marginal $P_X \pi_1 = \rho'_1$ such that

$$\int_{\Omega \times \Omega} \phi \, d\pi_1 = \int_{\Omega} \left[\int_{\Omega} \phi(z, y) \, d\pi_{1,z}(y) \right] d\rho'_1(z).$$

We now modify π_0 and π_1 as follows: All mass that is transported by π_0 into Ω_- (and then after a mass decrease is further transported by π_1) will no longer be moved by $\hat{\pi}_0$ and will instead be transported by $\hat{\pi}_1$ directly from its original to its final position. Likewise, all mass that is transported by π_0 into Ω_+ (and then after a mass increase is further transported by π_1) will no longer be moved by $\hat{\pi}_1$ but will instead be transported by $\hat{\pi}_0$ directly from its original to its final position. In detail, we set $\hat{\pi}_i = \hat{\pi}_{i,+} + \hat{\pi}_{i,-} + \hat{\pi}_{i,=}$ for $i \in \{0, 1\}$, where we define the measures $\hat{\pi}_{i,+}, \hat{\pi}_{i,-}, \hat{\pi}_{i,=} \in \mathcal{M}_+(\Omega \times \Omega)$ for $i \in \{0, 1\}$ by how they act on measurable functions $\phi : \Omega^2 \rightarrow [0, \infty]$, using the above disintegrations,

$$\begin{aligned} \int_{\Omega \times \Omega} \phi \, d\hat{\pi}_{0,+} &= \int_{\Omega_+} \left[\int_{\Omega \times \Omega} \phi(x, y) \, d\pi_{0,z}(x) \, d\pi_{1,z}(y) \right] d\rho'_0(z), \\ \int_{\Omega \times \Omega} \phi \, d\hat{\pi}_{1,+} &= \int_{\Omega_+ \times \Omega} \phi(y, y) \, d\pi_1(z, y), \\ \int_{\Omega \times \Omega} \phi \, d\hat{\pi}_{0,-} &= \int_{\Omega \times \Omega_-} \phi(x, x) \, d\pi_0(x, z), \\ \int_{\Omega \times \Omega} \phi \, d\hat{\pi}_{1,-} &= \int_{\Omega_-} \left[\int_{\Omega \times \Omega} \phi(x, y) \, d\pi_{0,z}(x) \, d\pi_{1,z}(y) \right] d\rho'_1(z), \\ \int_{\Omega \times \Omega} \phi \, d\hat{\pi}_{0,=} &= \int_{\Omega \times \Omega=} \phi \, d\pi_0, \\ \int_{\Omega \times \Omega} \phi \, d\hat{\pi}_{1,=} &= \int_{\Omega_+ \times \Omega} \phi \, d\pi_1. \end{aligned}$$

The measure $\hat{\pi}_{0,+}$ corresponds to the coefficients $a_i \frac{b_j}{b_0 + b_1}$ in Example 3.27. $\hat{\pi}_{1,+}$ is supported on the diagonal which indicates that these particles do not move in the second transport term. $\hat{\pi}_{i,-}$ are the appropriate counterparts when mass is decreased and $\hat{\pi}_{i,=}$ are the corresponding parts of the original couplings in areas where the amount of mass is not changed. One can readily verify $P_X \hat{\pi}_0 = P_X \pi_0 = \rho_0$ and $P_Y \hat{\pi}_1 = P_Y \pi_1 = \rho_1$.

(Step ii) Now we assume that (π_0, π_1) do not satisfy (3.11) and then show that the transport costs implied by $(\hat{\pi}_0, \hat{\pi}_1)$ are strictly smaller than those of (π_0, π_1) . By definition we find

$$\begin{aligned} \int_{\Omega \times \Omega} d(x, y) \, d\hat{\pi}_{0,=}(x, y) &= \int_{\Omega \times \Omega=} d(x, y) \, d\pi_0(x, y), & \int_{\Omega \times \Omega} d(x, y) \, d\hat{\pi}_{0,-}(x, y) &= 0, \\ \int_{\Omega \times \Omega} d(x, y) \, d\hat{\pi}_{1,=}(x, y) &= \int_{\Omega_+ \times \Omega} d(x, y) \, d\pi_1(x, y), & \int_{\Omega \times \Omega} d(x, y) \, d\hat{\pi}_{1,+}(x, y) &= 0. \end{aligned}$$

Now assume $\int_{\Omega_+ \times \Omega} d(x, y) d\pi_1(x, y) > 0$. Then

$$\begin{aligned} \int_{\Omega \times \Omega} d d\hat{\pi}_{0,+} &= \int_{\Omega_+} \left[\int_{\Omega \times \Omega} d(x, y) d\pi_{0,z}(x) d\pi_{1,z}(y) \right] d\rho'_0(z) \\ &\leq \int_{\Omega_+} \left[\int_{\Omega} d(x, z) d\pi_{0,z}(x) + \int_{\Omega} d(z, y) d\pi_{1,z}(y) \right] d\rho'_0(z) \\ &< \int_{\Omega \times \Omega_+} d d\pi_0 + \int_{\Omega_+ \times \Omega} d d\pi_1. \end{aligned}$$

The first inequality is obtained using $d(x, y) \leq d(x, z) + d(z, y)$ for any $z \in \Omega$. The strictness of the second inequality follows from $\frac{d\rho'_0}{d\gamma}(z) < \frac{d\rho'_1}{d\gamma}(z)$ for $z \in \Omega_+$ and the assumption $\int_{\Omega_+ \times \Omega} d(x, y) d\pi_1(x, y) > 0$. Analogously one shows that $\int_{\Omega \times \Omega_-} d(x, y) d\pi_0(x, y) > 0$ implies $\int_{\Omega \times \Omega} d d\hat{\pi}_{1,-} < \int_{\Omega \times \Omega_-} d d\pi_0 + \int_{\Omega_- \times \Omega} d d\pi_1$. Consequently, by assumption

$$\int_{\Omega \times \Omega} d d\hat{\pi}_0 + \int_{\Omega \times \Omega} d d\hat{\pi}_1 < \int_{\Omega \times \Omega} d d\pi_0 + \int_{\Omega \times \Omega} d d\pi_1.$$

(Step iii) Let $\hat{\rho}'_0 = P_Y \hat{\pi}_0$, $\hat{\rho}'_1 = P_X \hat{\pi}_1$. We now show $C_S(\hat{\rho}'_0, \hat{\rho}'_1) \leq C_S(\rho'_0, \rho'_1)$, which together with the previous step implies that $(\hat{\pi}_0, \hat{\pi}_1)$ has a strictly smaller cost. Let

$$\hat{\rho}'_{0,\chi} = P_Y \hat{\pi}_{0,\chi} \quad \text{and} \quad \hat{\rho}'_{1,\chi} = P_X \hat{\pi}_{1,\chi} \quad \text{for } \chi \in \{+, -, =\},$$

such that

$$\hat{\rho}'_i = \sum_{\chi \in \{+, -, =\}} \hat{\rho}'_{i,\chi} \quad \text{for } i \in \{0, 1\}.$$

Since $\{\Omega_+, \Omega_-, \Omega_=\}$ is a partition of Ω , one has

$$C_S(\rho'_0, \rho'_1) = C_S(\rho'_0 \llcorner_{\Omega_+}, \rho'_1 \llcorner_{\Omega_+}) + C_S(\rho'_0 \llcorner_{\Omega_-}, \rho'_1 \llcorner_{\Omega_-}) + C_S(\rho'_0 \llcorner_{\Omega_=}, \rho'_1 \llcorner_{\Omega_=}),$$

and by joint subadditivity of C_S in its two arguments

$$C_S(\hat{\rho}'_0, \hat{\rho}'_1) \leq \sum_{\chi \in \{+, -, =\}} C_S(\hat{\rho}'_{0,\chi}, \hat{\rho}'_{1,\chi}).$$

One finds $\hat{\rho}'_{i,=} = \rho'_i \llcorner_{\Omega_=}$, $i \in \{0, 1\}$ and thus $C_S(\rho'_0 \llcorner_{\Omega_=}, \rho'_1 \llcorner_{\Omega_=}) = C_S(\hat{\rho}'_{0,=}, \hat{\rho}'_{1,=})$. In the following let $\pi_\gamma \in \mathcal{M}_+(\Omega \times \Omega)$ be defined by

$$\int_{\Omega \times \Omega} \phi d\pi_\gamma = \int_{\Omega} \left[\int_{\Omega} \phi(z, y) d\pi_{1,z}(y) \right] d\gamma(z),$$

where we combine the disintegration $\{\pi_{1,z}\}_{z \in \Omega}$ with a different marginal γ (as chosen in the statement of the result). Let $\hat{\gamma} = P_Y \pi_\gamma$ (note that by construction $\gamma = P_X \pi_\gamma$) and let $\{\pi_{\gamma,y}\}_{y \in \Omega}$ be the disintegration of π_γ with

respect to the second marginal, that is

$$\int_{\Omega \times \Omega} \phi \, d\pi_\gamma = \int_{\Omega} \left[\int_{\Omega} \phi(z, y) \, d\pi_{\gamma, y}(z) \right] d\hat{\gamma}(y).$$

Now consider

$$\begin{aligned} C_S(\rho'_0 \llcorner_{\Omega_+}, \rho'_1 \llcorner_{\Omega_+}) &= \int_{\Omega_+} c_S \left(\frac{d\rho'_0}{d\gamma}(z), \frac{d\rho'_1}{d\gamma}(z) \right) d\gamma(z) = \int_{\Omega_+} \left[\int_{\Omega} c_S(\dots) \, d\pi_{1, z}(y) \right] d\gamma(z) \\ &= \int_{\Omega_+ \times \Omega} c_S(\dots) \, d\pi_\gamma(z, y) = \int_{\Omega} \left[\int_{\Omega_+} c(\dots) \, d\pi_{\gamma, y}(z) \right] d\hat{\gamma}(y) \\ &\geq \int_{\Omega} c_S \left(\int_{\Omega_+} \frac{d\rho'_0}{d\gamma}(z) \, d\pi_{\gamma, y}(z), \int_{\Omega_+} \frac{d\rho'_1}{d\gamma}(z) \, d\pi_{\gamma, y}(z) \right) d\hat{\gamma}(y) \\ &= C_S(\sigma_0, \sigma_1), \end{aligned}$$

where the inequality is due to Jensen's inequality and σ_0 and $\sigma_1 \in \mathcal{M}_+(\Omega)$ are defined for $i \in \{0, 1\}$ by

$$\int_{\Omega} \phi \, d\sigma_i = \int_{\Omega} \left[\int_{\Omega_+} \frac{d\rho'_i}{d\gamma}(z) \, d\pi_{\gamma, y}(z) \right] \phi(y) \, d\hat{\gamma}(y).$$

One finds

$$\begin{aligned} \int_{\Omega} \phi \, d\sigma_0 &= \int_{\Omega_+} \left[\int_{\Omega} \phi(y) \frac{d\rho'_0}{d\gamma}(z) \, d\pi_{1, z}(y) \right] d\gamma(z) = \int_{\Omega_+} \left[\int_{\Omega} \phi(y) \, d\pi_{1, z}(y) \right] d\rho'_0(z) \\ &= \int_{\Omega} \phi(y) \, d(P_Y \hat{\pi}_{0, +})(y) \end{aligned}$$

and therefore $\sigma_0 = \hat{\rho}'_{0, +}$. Similarly one finds $\sigma_1 = \hat{\rho}'_{1, +}$ and thus $C_S(\rho'_0 \llcorner_{\Omega_+}, \rho'_1 \llcorner_{\Omega_+}) \geq C_S(\hat{\rho}'_{0, +}, \hat{\rho}'_{1, +})$. With analogous arguments one shows $C_S(\rho'_0 \llcorner_{\Omega_-}, \rho'_1 \llcorner_{\Omega_-}) \geq C_S(\hat{\rho}'_{0, -}, \hat{\rho}'_{1, -})$ and thus $C_S(\rho'_0, \rho'_1) \geq C_S(\hat{\rho}'_0, \hat{\rho}'_1)$. \square

As a further consequence we have that, in an optimal arrangement or during the optimal dynamics, not only is mass never transported after it has been increased or before it is decreased, but also no material is transported out of a region of mass increase before this increase happens, and no material is transported into a region of mass decrease after this decrease happened.

Corollary 3.28 (Necessary optimality condition II for static primal). *If (π_0, π_1) are minimizers of (2.9), then*

$$\pi_0 \llcorner (\Omega_+ \times (\Omega \setminus \Omega_+)) = 0, \quad \pi_1 \llcorner ((\Omega \setminus \Omega_-) \times \Omega_-) = 0.$$

Proof. We show $\pi_0 \llcorner (\Omega_+ \times (\Omega \setminus \Omega_+)) = 0$, the other statement follows analogously. We need to show $\pi_0 \llcorner (\Omega_+ \times \Omega_-) = 0$ since $\pi_0 \llcorner (\Omega_+ \times \Omega_-) = 0$ by the previous proposition. Assume $\pi_0 \llcorner (\Omega_+ \times \Omega_-) \neq 0$, then one can modify π_0 and π_1 to $\hat{\pi}_0$ and $\hat{\pi}_1$ such that their cost does not increase. Indeed, consider all the mass being transported by π_0 from Ω_+ to Ω_- and then further by π_1 from Ω_- to Ω . The modification of the couplings is to transport this mass not at all during the transport associated with $\hat{\pi}_0$ and to transport it instead *via* $\hat{\pi}_1$ directly from Ω_+ to the final position in Ω without the intermediate deposition in Ω_- . In detail, using the disintegration notation

from the previous proof, we choose $\hat{\pi}_0$ and $\hat{\pi}_1$ according to

$$\begin{aligned} \int_{\Omega \times \Omega} \phi \, d\hat{\pi}_0 &= \int_{(\Omega \times \Omega) \setminus (\Omega_+ \times \Omega_=)} \phi \, d\pi_0 + \int_{\Omega_+ \times \Omega_=} \phi(x, x) \, d\pi_0(x, y), \\ \int_{\Omega \times \Omega} \phi \, d\hat{\pi}_1 &= \int_{\Omega \times \Omega} \phi \, d\pi_1 - \int_{\Omega_=} \int_{\Omega} \phi(z, y) \, d\pi_{1,z}(y) \, dP_Y(\pi_0 \llcorner (\Omega_+ \times \Omega_=))(z) \\ &\quad + \int_{\Omega_+ \times \Omega_=} \int_{\Omega} \phi(x, y) \, d\pi_{1,z}(y) \, d\pi_0(x, z) \end{aligned}$$

for all measurable $\phi : \Omega^2 \rightarrow [0, \infty]$. It is readily checked that $\hat{\pi}_0, \hat{\pi}_1 \in \mathcal{M}_+(\Omega \times \Omega)$ as well as $P_X \hat{\pi}_0 = P_X \pi_0 = \rho_0$ and $P_Y \hat{\pi}_1 = P_Y \pi_1 = \rho_1$ so that $\hat{\pi}_0$ and $\hat{\pi}_1$ are admissible. Likewise it is straightforward to check $\Delta \rho \stackrel{\text{def}}{=} \hat{\rho}'_1 - \hat{\rho}'_0 = P_X \hat{\pi}_1 - P_Y \hat{\pi}_0 = P_X \pi_1 - P_Y \pi_0 = \rho'_1 - \rho'_0$ so that the same mass change happens as before. In particular, the domains $\Omega_+, \Omega_-, \Omega_=$ did not change, and $\hat{\rho}'_0 \llcorner \Omega_+ \geq \rho'_0 \llcorner \Omega_+$ as well as $\hat{\rho}'_0 \llcorner \Omega_- = \rho'_0 \llcorner \Omega_-$. Therefore,

$$C_S(\hat{\rho}'_0, \hat{\rho}'_1) = C_S(\hat{\rho}'_0 \llcorner (\Omega \setminus \Omega_=), \hat{\rho}'_0 \llcorner (\Omega \setminus \Omega_=) + \Delta \rho) \leq C_S(\rho'_0 \llcorner (\Omega \setminus \Omega_=), \rho'_0 \llcorner (\Omega \setminus \Omega_=) + \Delta \rho) = C_S(\rho'_0, \rho'_1)$$

since $m \mapsto c_S(m, m + \Delta m)$ is nonincreasing. (Indeed, using the 1-homogeneity and convexity of c_S one has $c_S(m + \delta, m + \delta + \Delta m) = \frac{m+\delta}{m} c_S(m, m + \frac{m}{m+\delta} \Delta m) \leq \frac{m+\delta}{m} \left((1 - \frac{m}{m+\delta}) \cdot c_S(m, m) + \frac{m}{m+\delta} c_S(m, m + \Delta m) \right) = c_S(m, m + \Delta m)$. In words, the mass increase in Ω_+ is identical to before, but may have a cheaper cost since it starts from an already higher mass.) Finally, the transported mass and the transport distance per particle (and thus the total transport cost) is no larger than before,

$$\begin{aligned} &\int_{\Omega \times \Omega} d \, d\hat{\pi}_0 + \int_{\Omega \times \Omega} d \, d\hat{\pi}_1 \\ &= \int_{(\Omega \times \Omega) \setminus (\Omega_+ \times \Omega_=)} d \, d\pi_0 + \int_{\Omega \times \Omega} d \, d\pi_1 + \int_{\Omega_+ \times \Omega_=} \int_{\Omega} d(x, y) \, d\pi_{1,z}(y) \, d\pi_0(x, z) \\ &\quad - \int_{\Omega_=} \int_{\Omega} d(z, y) \, d\pi_{1,z}(y) \, dP_Y(\pi_0 \llcorner (\Omega_+ \times \Omega_=))(z) \\ &\leq \int_{(\Omega \times \Omega) \setminus (\Omega_+ \times \Omega_=)} d \, d\pi_0 + \int_{\Omega \times \Omega} d \, d\pi_1 + \int_{\Omega_+ \times \Omega_=} \int_{\Omega} d(x, z) + d(z, y) \, d\pi_{1,z}(y) \, d\pi_0(x, z) \\ &\quad - \int_{\Omega_=} \int_{\Omega} d(z, y) \, d\pi_{1,z}(y) \, dP_Y(\pi_0 \llcorner (\Omega_+ \times \Omega_=))(z) \\ &= \int_{\Omega \times \Omega} d \, d\pi_0 + \int_{\Omega \times \Omega} d \, d\pi_1. \end{aligned} \tag{3.12}$$

Summarizing, the modification does not increase the cost so that $(\hat{\pi}_0, \hat{\pi}_1)$ are optimal as well. However, using the optimality of (π_0, π_1) and $(\hat{\pi}_0, \hat{\pi}_1)$ and the previous proposition,

$$0 = \int_{\Omega_+ \times \Omega} d \, d\hat{\pi}_1 = \int_{\Omega_+ \times \Omega_=} \int_{\Omega} d(x, y) \, d\pi_{1,z}(y) \, d\pi_0(x, z)$$

so that the inequality in (3.12) is actually strict and thus (π_0, π_1) cannot be optimal. \square

The characterization from Proposition 3.26 and Corollary 3.28 can be refined even further, in particular for special cases.

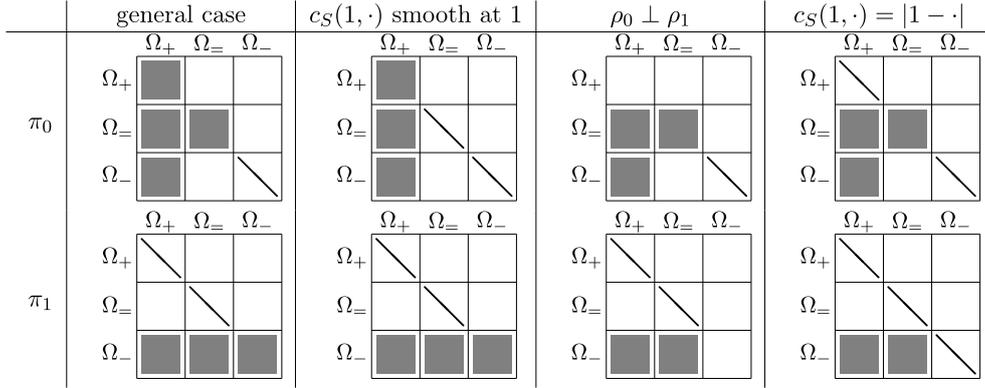


FIGURE 3. Illustration of Proposition 3.29: there are optimal couplings π_0, π_1 that take the shown form for the different particular cases. Each square field corresponds to the direct product of the regions indicated by the row and column. An empty field corresponds to π being zero, a grey field indicates that π may be nonzero, and a field with a diagonal line corresponds to diagonal support of π (that is, mass is not moved in that region). If $c_S(1, \cdot)$ is differentiable at 1 and at the same time $\rho_0 \perp \rho_1$, then in addition $\pi_0 \llcorner (\Omega_- \times \Omega_-) = \pi_1 \llcorner (\Omega_- \times \Omega_-) = 0$.

Proposition 3.29 (Characterization of optimal unbalanced mass transport). *For all (ρ_0, ρ_1) and among all possible optimizers of (2.9), there is always a particular one (π_0, π_1) such that its support is as shown in Figure 3 left, that is, in addition to Proposition 3.26 and Corollary 3.28 we have*

$$\pi_0(\Omega_- \times \Omega_-) = \pi_1(\Omega_- \times \Omega_+) = 0 \quad \text{and} \quad \int_{\Omega \setminus \Omega_- \times \Omega} d(x, y) d\pi_1(x, y) = 0.$$

Furthermore, $\rho'_0 + \rho'_1 \ll \rho_0 + \rho_1$. If in addition

1. $c_S(1, \cdot)$ is differentiable at 1 or
2. $\rho_0 \perp \rho_1$ or
3. $c_S(1, \cdot) = |1 - \cdot|$,

then the supports of the optimal (π_0, π_1) can be chosen as shown in Figure 3 in the second, third, and fourth column, respectively. If case (1) and (2) hold simultaneously, then in addition $\pi_0 \llcorner (\Omega_- \times \Omega_-) = \pi_1 \llcorner (\Omega_- \times \Omega_-) = 0$.

Remark 3.30. Of course, the cases from Figure 3 can be combined. For instance, if $c_S(1, \cdot)$ is differentiable at 1 and $\rho_0 \perp \rho_1$, then the transport can be interpreted as follows (see Fig. 4): From points contributing to ρ_0 there is first mass transported to ρ_1 (blue), then the mass is shrunk in the ρ_0 -region (orange) and grown in the ρ_1 -region, and finally the rest is transported to ρ_1 (yellow). Those steps are essentially a manifestation of the nonconvexity of the so-called path cost and its convexification being a combination of transport with subsequent growth and shrinkage with subsequent transport, see Section 4.2.

Proof of Proposition 3.29. Note first that by Proposition 2.5, if the cost is finite then minimizing pairs (π_0, π_1) exist.

General case. By Proposition 3.26 and Corollary 3.28 it only remains to show that optimal (π_0, π_1) can be found such that π_1 is diagonal on $\Omega_- \times \Omega_-$ (that is, $\int_{\Omega_- \times \Omega_-} \phi(x, y) d\pi_1(x, y) = \int_{\Omega_- \times \Omega_-} \phi(x, x) d\pi_1(x, y)$ for all test functions ϕ) and that $\pi_0 \llcorner (\Omega_- \times \Omega_-) = 0$ as well as $\pi_1 \llcorner (\Omega_- \times \Omega_+) = 0$ (of which we shall only show the first, the second follows analogously). To this end we just modify any optimal couplings (π_0, π_1) without increasing their cost. In particular, consider the mass transported from Ω to Ω_- by π_0 and then transported

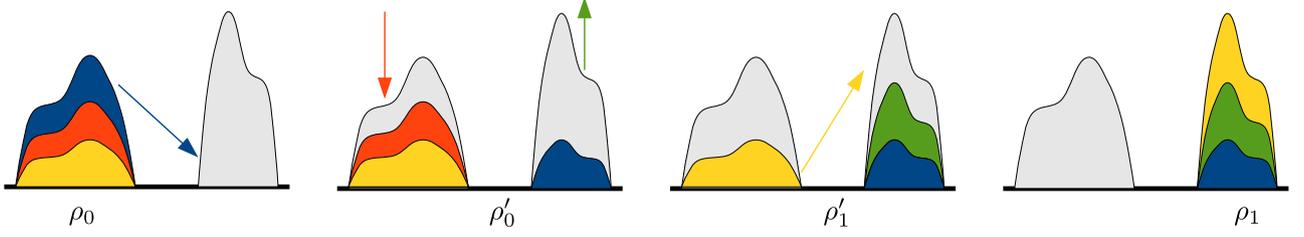


FIGURE 4. Illustration of the optimal dynamic mass transport as characterized by Proposition 3.29, consisting of an initial transport (*blue*), subsequent mass shrinkage (*orange*) and growth (*green*), and a final transport (*yellow*). See Remark 3.30 for more details.

further within Ω_- by π_1 . We modify π_0 and π_1 such that the transport of this mass from its initial distribution inside ρ_0 to its final distribution inside ρ_1 happens already in π_0 and thus π_1 no longer transports mass within Ω_- (corresponding formulas for the modified π_0 and π_1 are straightforward to obtain as in the two previous proofs). Obviously, the mass change cost is identical after the modification, and the transport cost has not increased so that we may assume π_1 to be diagonal on $\Omega_- \times \Omega_-$. Next assume $\pi_0 \llcorner (\Omega_- \times \Omega_-) \neq 0$ and modify π_0 and π_1 as follows. Consider the mass transported from Ω_- to Ω_- by π_0 and then transported further by π_1 . We modify π_0 and π_1 such that π_0 no longer transports mass from Ω_- to Ω_- , but instead the transport of this mass happens in π_1 (again, corresponding formulas for the modified π_0 and π_1 are straightforward to obtain as in the previous proofs). Obviously, the transport cost has not increased by the modification. Furthermore, the amount of mass decrease on Ω_- is identical to before, but may have a cheaper cost since it starts from a higher initial mass and $c_S(m, m - \Delta m)$ is nonincreasing in m . Finally, the amount and cost of mass increase on Ω_+ stays unchanged.

The property $\rho'_0 + \rho'_1 \ll \rho_0 + \rho_1$ can now be inferred as follows: By definition of Ω_+ we have $\rho'_0 \llcorner \Omega_+ \leq \rho'_1 \llcorner \Omega_+$ and therefore $\rho'_0 \llcorner \Omega_+ \ll \rho'_1 \llcorner \Omega_+$, which is absolutely continuous with respect to $\rho_1 \llcorner \Omega_+$ since π_1 is diagonal on $\Omega_+ \times \Omega_+$. Likewise, $\rho'_1 \llcorner \Omega_- \ll \rho'_0 \llcorner \Omega_- \ll \rho_0 \llcorner \Omega_-$ (mass decreases on Ω_- and π_0 is diagonal on $\Omega_- \times \Omega_-$). Finally, $\rho'_1 \llcorner \Omega_- \ll \rho_1 \llcorner \Omega_-$ by the structure of π_1 . If we now use Lebesgue's decomposition theorem to decompose $\rho'_0 \llcorner \Omega_- = \nu_0 + \nu_1$ with $\nu_0 \ll \rho_0 + \rho_1$ and $\nu_1(\Omega \setminus \Omega') = 0$ for some $\Omega' \subset \Omega_-$ with $(\rho_0 + \rho_1)(\Omega') = 0$, then $\nu_1 \ll \rho'_0 \llcorner \Omega' = \rho'_1 \llcorner \Omega' \ll \rho_1 \llcorner \Omega' = 0$ (note that $\rho'_0 \llcorner \Omega_- = \rho'_1 \llcorner \Omega_-$ was not obvious a priori). Thus, $\rho'_0, \rho'_1 \ll \rho_0 + \rho_1$.

Case $c_S(1, \cdot)$ differentiable at 1. For a contradiction assume $\pi_0 \llcorner (\Omega_- \times \Omega_-)$ is not diagonal. Define $\rho_0^\varepsilon = P_X(\pi_0 \llcorner (\Omega_- \times \Omega_-))$ as well as the diagonal coupling $\pi_0^\varepsilon(A \times B) = \rho_0^\varepsilon(A \cap B)$ for all Borel sets $A, B \subset \mathbb{R}^n$. Next define the coupling π_0^ε by setting $\pi_0^\varepsilon = \pi_0$ on $(\Omega \times \Omega) \setminus (\Omega_- \times \Omega_-)$ and $\pi_0^\varepsilon = (1 - \varepsilon)\pi_0 + \varepsilon\pi_0^\varepsilon$ on $\Omega_- \times \Omega_-$ for some $\varepsilon \in [0, 1]$. Note that the pair $(\pi_0^\varepsilon, \pi_1)$ is an admissible candidate in (2.9). Denote its cost by W^ε and note $W_S(\rho_0, \rho_1) = W^0$ due to the assumption of (π_0, π_1) being optimal. Then at $\varepsilon = 0$ we have

$$\begin{aligned} \frac{dW^\varepsilon}{d\varepsilon} &= -\varepsilon \int_{\Omega_- \times \Omega_-} d(x, y) d\pi_0(x, y) + \frac{d}{d\varepsilon} C_S(P_Y \pi_0^\varepsilon, P_X \pi_1) \\ &= -\varepsilon \int_{\Omega_- \times \Omega_-} d(x, y) d\pi_0(x, y) + \int_{\Omega_-} \frac{d}{d\varepsilon} c_S\left(\frac{dP_Y \pi_0^\varepsilon}{d\rho_1}, 1\right) d\rho_1 \\ &= -\varepsilon \int_{\Omega_- \times \Omega_-} d(x, y) d\pi_0(x, y) < 0 \end{aligned}$$

since $\frac{d}{d\varepsilon} c_S\left(\frac{dP_Y \pi_0^\varepsilon}{d\rho_1}, 1\right)|_{\varepsilon=0} = 0$. This contradicts the optimality of (π_0, π_1) .

Case $\rho_0 \perp \rho_1$. Using the general structure of π_0 and π_1 , for $E \subset \Omega_+$ measurable we have

$$\rho_1 \llcorner E = P_Y(\pi_1 \llcorner (\Omega \times E)) \gg P_Y(\pi_1 \llcorner (E \times E)) = P_X(\pi_1 \llcorner (E \times E)) = \rho'_1 \llcorner E \gg \rho'_0 \llcorner E.$$

Thus, $\rho_1 \llcorner E = 0$ would imply $\rho'_1 \llcorner E = \rho'_0 \llcorner E = 0$ so that, due to $E \subset \Omega_+$, E can only be a γ -nullset (where γ is the reference measure used in Proposition 3.26 to define Ω_+ , Ω_- and $\Omega_=\$). Now assume $\rho_0 \llcorner \Omega_+ \neq 0$, then due to the singularity with ρ_1 there is a measurable $E \subset \Omega_+$ with $\rho_0 \llcorner E \neq 0$ and $\rho_1 \llcorner E = 0$. By the above, E must be a γ -nullset, which contradicts $\rho_0 \llcorner E \neq 0$. Therefore, $\rho_0 \llcorner \Omega_+ = 0$ and thus also $\pi_0 \llcorner (\Omega_+ \times \Omega) = 0$.

Likewise one shows that $\rho_0 \llcorner E = 0$ with $E \subset \Omega_-$ can only happen for γ -nullsets E and that $\pi_1 \llcorner (\Omega \times \Omega_-) = 0$.

If in addition $c_S(1, \cdot)$ is differentiable at 1, then from the previous case we know

$$\rho_0 \llcorner \Omega_ = \gg P_X(\pi_0 \llcorner (\Omega_ = \times \Omega_ =)) = P_Y(\pi_0 \llcorner (\Omega_ = \times \Omega_ =)) = \rho'_0 \llcorner \Omega_ =,$$

$$\rho_1 \llcorner \Omega_ = \gg P_Y(\pi_1 \llcorner (\Omega_ = \times \Omega_ =)) = P_X(\pi_1 \llcorner (\Omega_ = \times \Omega_ =)) = \rho'_1 \llcorner \Omega_ =.$$

Hence, since ρ_0 and ρ_1 are singular with respect to each other, so are $\rho'_0 \llcorner \Omega_ =$ and $\rho'_1 \llcorner \Omega_ =$. However, due to $\rho'_0 \llcorner \Omega_ = = \rho'_1 \llcorner \Omega_ =$ this is only possible if both are zero, which in turn implies $\pi_0 \llcorner (\Omega \times \Omega_ =) = \pi_1 \llcorner (\Omega_ = \times \Omega) = 0$.

Case $c_S(1, \cdot) = |1 - \cdot|$. Note that by Proposition 2.5 the corresponding function h_S is given by $h_S(z) = \min(z, 1) - \nu_{[-1, \infty)}(z)$ so that the dual formulation (2.11) can be transformed into

$$W_S(\rho_0, \rho_1) = \sup\{\int_{\Omega} \alpha d(\rho_0 - \rho_1) \mid \alpha \in \text{Lip}(\Omega), |\alpha| \leq 1\}.$$

This implies that the cost actually only depends on $\rho_0 - \rho_1$. Therefore, defining

$$\rho = \min\left(\frac{d\rho_0}{d(\rho_0 + \rho_1)}, \frac{d\rho_1}{d(\rho_0 + \rho_1)}\right) (\rho_0 + \rho_1), \quad \tilde{\rho}_0 = \rho_0 - \rho \in \mathcal{M}_+(\Omega), \quad \tilde{\rho}_1 = \rho_1 - \rho \in \mathcal{M}_+(\Omega),$$

we see $W_S(\tilde{\rho}_0, \tilde{\rho}_1) = W_S(\rho_0, \rho_1)$. Now given optimal couplings $(\tilde{\pi}_0, \tilde{\pi}_1)$ for $W_S(\tilde{\rho}_0, \tilde{\rho}_1)$ we simply set $\pi_0 = \tilde{\pi}_0 + \pi$, $\pi_1 = \tilde{\pi}_1 + \pi$, where π is the diagonal coupling defined via $\int_{\Omega \times \Omega} \phi d\pi = \int_{\Omega} \phi(x, x) d\rho(x)$ for all measurable $\phi : \Omega^2 \rightarrow [0, \infty]$. It is straightforward to check that (π_0, π_1) is a feasible candidate in (2.9) and that it has the same cost as $(\tilde{\pi}_0, \tilde{\pi}_1)$. Therefore, (π_0, π_1) must be optimal. Note also that Ω_{\pm} are the same for (π_0, π_1) as for $(\tilde{\pi}_0, \tilde{\pi}_1)$. Now by the previous case, $\tilde{\pi}_0 \llcorner (\Omega_+ \times \Omega_+) = \tilde{\pi}_1 \llcorner (\Omega_- \times \Omega_-) = 0$ so that $\pi_0 \llcorner (\Omega_+ \times \Omega_+) = \pi \llcorner (\Omega_+ \times \Omega_+)$ and $\pi_1 \llcorner (\Omega_- \times \Omega_-) = \pi \llcorner (\Omega_- \times \Omega_-)$ are diagonal. \square

The final result of this section shows that the maximal transport distance may be bounded depending on the model. It implies for instance that, if the supports of ρ_0 and ρ_1 have distance larger than some threshold (which for some models is infinite), then no transport happens at all. Since the transport component of the total cost depends on the transport distance while the mass change cost is independent of it and only depends on how much mass is created or removed, such a threshold effect is not unexpected.

Proposition 3.31 (Maximal transport distance). *Any optimizer (π_0, π_1) of (2.9) satisfies*

$$\pi_0 \llcorner \{(x, y) \in \Omega \times \Omega \mid d(x, y) > L_0\} = \pi_1 \llcorner \{(x, y) \in \Omega \times \Omega \mid d(x, y) > L_1\} = 0$$

for $L_0 = \lim_{a \rightarrow \infty} \partial_1 c_S(a, 1) - \lim_{a \rightarrow 0} \partial_1 c_S(a, 1) \in (0, \infty]$, $L_1 = \lim_{a \rightarrow \infty} \partial_2 c_S(1, a) - \lim_{a \rightarrow 0} \partial_2 c_S(1, a) \in (0, \infty]$, where ∂_i denotes the i th (left or right) partial derivative.

Proof. We only prove $\pi_1 \llcorner \{(x, y) \in \Omega \times \Omega \mid d(x, y) > L_1\} = 0$, the other statement follows analogously. Let $\Delta \geq 0$ arbitrary and abbreviate

$$L^r = \lim_{a \rightarrow \infty} \partial_2 c_S(1, a), \quad L^l = - \lim_{a \rightarrow 0} \partial_2 c_S(1, a).$$

By convexity of $c_S(1, \cdot)$ we have for any $b \geq 0$

$$c_S(1, b + \Delta) \leq c_S(1, b) + \partial_2 c_S(1, b + \Delta) \Delta \leq c_S(1, b) + L^r \Delta$$

and thus by the one-homogeneity $c_S(a, b + \Delta) \leq c_S(a, b) + L^r \Delta$ for all $a \geq 0$. Analogously one obtains $c_S(a, b - \Delta) \leq c_S(a, b) + L^l \Delta$ for $a \geq 0, b \geq \Delta$. As a consequence, for any nonnegative measures $\rho', \tilde{\rho}', \hat{\rho}'$ with $\rho' - \tilde{\rho}' + \hat{\rho}'$ nonnegative we have

$$C_S(\rho'_0, \rho' - \tilde{\rho}' + \hat{\rho}') \leq C_S(\rho'_0, \rho') + L^l \tilde{\rho}'(\Omega) + L^r \hat{\rho}'.$$

Now assume (π_0, π_1) optimizes (2.9) with $\pi_1^{\text{far}} \equiv \pi_1 \llcorner \{(x, y) \in \Omega \times \Omega \mid d(x, y) > L_1\} \neq 0$. We will construct a competitor with strictly lower cost. Define π_1^{cls} and π_1^{dgl} via

$$\pi_1^{\text{cls}} = \pi_1 - \pi_1^{\text{far}} \quad \text{and} \quad \int_{\Omega \times \Omega} \phi \, d\pi_1^{\text{dgl}} = \int_{\Omega \times \Omega} \phi(y, y) \, d\pi_1^{\text{far}}(x, y)$$

for all continuous $\phi : \Omega \times \Omega \rightarrow \mathbb{R}$, that is, π_1^{far} describes the transport of mass that moves farther than L_1 and π_1^{dgl} is the coupling corresponding to not moving that mass at all. Our competitor is $\tilde{\pi}_1 = \pi_1^{\text{cls}} + \pi_1^{\text{dgl}} = \pi_1 - \pi_1^{\text{far}} + \pi_1^{\text{dgl}}$, and we estimate

$$\begin{aligned} P_S(\pi_0, \tilde{\pi}_1) &= \int_{\Omega \times \Omega} d \, d\pi_0 + C_S(P_Y \pi_0, P_X \pi_1 - P_X \pi_1^{\text{far}} + P_X \pi_1^{\text{dgl}}) + \int_{\Omega \times \Omega} d \, d(\pi_1 - \pi_1^{\text{far}} + \pi_1^{\text{dgl}}) \\ &< \int_{\Omega \times \Omega} d \, d\pi_0 + C_S(P_Y \pi_0, P_X \pi_1) + L^l \pi_1^{\text{far}}(\Omega \times \Omega) + L^r \pi_1^{\text{dgl}}(\Omega \times \Omega) + \int_{\Omega \times \Omega} d \, d\pi_1 - L_1 \pi_1^{\text{far}}(\Omega \times \Omega) \\ &= P_S(\pi_0, \pi_1) + (L^l + L^r - L_1) \pi_1^{\text{far}}(\Omega \times \Omega) = P_S(\pi_0, \pi_1), \end{aligned}$$

where we used $(P_X \pi_1^{\text{far}})(\Omega) = \pi_1^{\text{far}}(\Omega \times \Omega) = (P_X \pi_1^{\text{dgl}})(\Omega) = \pi_1^{\text{dgl}}(\Omega \times \Omega)$. This contradicts the optimality of (π_0, π_1) and thus concludes the proof. \square

4. OVERVIEW OF AND RELATION BETWEEN STATIC AND DYNAMIC FORMULATIONS

In this section we continue and extend the discussion from the introduction on different model formulations and their relations. In particular, we provide a semi-coupling formulation of the static W_1 -type problem, and we identify, which static W_1 -type problems have an equivalent dynamic formulation.

4.1. Semi-coupling formulation of unbalanced transport

In this section we establish the relation between the formulations for transport problems from Section 2 with the semi-coupling formulation introduced in [7]. A main result of [7], Theorem 4.3, is the relation between dynamic problems and the semi-coupling formulation. Using the results of Section 3 we give an alternative proof for this statement in the special case of the W_1 -type setting (Prop. 4.4).

Definition 4.1 (Primal semi-coupling formulation [7], Def. 3.3). For $\rho_0, \rho_1 \in \mathcal{M}_+(\Omega)$ the corresponding set of *semi-couplings* is given by

$$\Gamma(\rho_0, \rho_1) = \{(\gamma_0, \gamma_1) \in (\mathcal{M}_+(\Omega \times \Omega))^2 \mid P_X \gamma_0 = \rho_0, P_Y \gamma_1 = \rho_1\}. \quad (4.1)$$

Let further $c_{\text{sc}} : (\Omega \times \mathbb{R})^2 \rightarrow \mathbb{R} \cup \{\infty\}$ be a lower semi-continuous function such that for all fixed $(x_0, x_1) \in \Omega \times \Omega$ the function $(m_0, m_1) \mapsto c_{\text{sc}}(x_0, m_0, x_1, m_1)$ is convex and 1-homogeneous with $c_{\text{sc}}(x_0, m_0, x_1, m_1) = \infty$

if $m_0 < 0$ or $m_1 < 0$. The *primal semi-coupling problem* is given by

$$W_{SC}(\rho_0, \rho_1) = \inf \left\{ \int_{\Omega^2} c_{sc} \left(x_0, \frac{d\gamma_0}{d\gamma}(x_0, x_1), x_1, \frac{d\gamma_1}{d\gamma}(x_0, x_1) \right) d\gamma(x_0, x_1) \mid (\gamma_0, \gamma_1) \in \Gamma(\rho_0, \rho_1) \right\}, \quad (4.2)$$

where $\gamma \in \mathcal{M}_+(\Omega \times \Omega)$ is such that $\gamma_0, \gamma_1 \ll \gamma$.

The semi-coupling cost $c_{sc}(x_0, m_0, x_1, m_1)$ may be interpreted as the cost for starting out at x_0 with mass m_0 destined for x_1 , but only arriving at x_1 with changed mass m_1 . Again, there is a corresponding dual problem.

Proposition 4.2 (Dual semi-coupling formulation [7], Prop. 3.11). *For a function c_{sc} as in Definition 4.1, let*

$$B_{sc}(x_0, x_1) = \{(\alpha, \beta) \in \mathbb{R}^2 \mid [c_{sc}(x_0, \cdot, x_1, \cdot)]^*(\alpha, \beta) < \infty\}.$$

Then

$$W_{SC}(\rho_0, \rho_1) = \sup \left\{ \int_{\Omega} \alpha d\rho_0 + \int_{\Omega} \beta d\rho_1 \mid (\alpha, \beta) \in C(\Omega)^2, \right. \\ \left. (\alpha(x_0), \beta(x_1)) \in B_{sc}(x_0, x_1) \text{ for all } (x_0, x_1) \in \Omega \times \Omega \right\}. \quad (4.3)$$

A main result of [7] was to show that for suitable dynamic problems, including those of the form of Definition 2.2, there is an equivalent semi-coupling formulation.

Proposition 4.3 (Equivalent semi-coupling formulation for dynamic problem [7], Thm. 4.3). *For a dynamic primal problem as in Definition 2.2, the associated path cost $c_p : (\Omega \times \mathbb{R})^2 \mapsto [0, \infty]$ is given by*

$$c_p(x_0, m_0, x_1, m_1) = \inf \left\{ \int_0^1 m(t) \cdot \|\partial_t x(t)\| + c_D(m(t), \partial_t m(t)) dt \mid \right. \\ \left. (x, m) \in C^1([0, 1], \Omega \times \mathbb{R}), (x(i), m(i)) = (x_i, m_i) \text{ for } i \in \{0, 1\} \right\}.$$

Assume that there exists some $C < \infty$ such that $c_D(\rho, 2\zeta) \leq C \cdot c_D(\rho, \zeta)$ for all $(\rho, \zeta) \in \mathbb{R}^2$, and let c_{sc} be the convex envelope of c_p with respect to the arguments (m_0, m_1) . Then $W_D = W_{SC}$.

A similar result can be established by finding an equivalent semi-coupling formulation for static W_1 -type problems and then using the equivalence between dynamic and static W_1 -type problems (Prop. 3.1, see also Prop. 3.13). This strategy has the advantage that one avoids computing the path cost c_p and obtains a more explicit form of the semi-coupling cost function c_{sc} , involving only the static mass change cost c_S from (3.10).

Proposition 4.4. *For a given static W_1 -type problem according to Definition 2.4 and Proposition 2.5, characterized by a function c_S or equivalently a set B_S , let*

$$c_{pre}(x_0, m_0, x_1, m_1) = c_S(m_0, m_1) + \min\{m_0, m_1\} \cdot d(x_0, x_1)$$

and denote by $c_{pre}^{**}(x_0, m_0, x_1, m_1)$ its convex envelope with respect to the arguments (m_0, m_1) . Then for the choice $c_{sc} = c_{pre}^{**}$ or equivalently

$$B_{sc}(x_0, x_1) = [B_S + \{(0, d(x_0, x_1))\}] \cap [B_S + \{(d(x_0, x_1), 0)\}]$$

one has $W_S = W_{SC}$.

The proof is split into the two following lemmas. First, equivalence of the dual formulations is established.

Lemma 4.5. *For $B_{\text{sc}}(x_0, x_1)$ as given in Proposition 4.4, the dual static W_1 -type problem from Proposition 2.5 is equivalent to the dual semi-coupling problem from Proposition 4.2.*

Proof. Throughout this proof we write $\Delta x = d(x_0, x_1)$. Note that the set $B_{\text{sc}}(x_0, x_1)$ can be written as

$$B_{\text{sc}}(x_0, x_1) = \{(a, b) \in \mathbb{R}^2 \mid [b \leq h_S(-a + \Delta x)] \text{ and } [b - \Delta x \leq h_S(-a)]\}. \quad (4.4)$$

We show first that every feasible candidate of (2.11) is also feasible for (4.3). Let $(\alpha, \beta) \in \text{Lip}(\Omega)^2$, $(\alpha(x), \beta(x)) \in B_S$ for all $x \in \Omega$. Then for all $(x_0, x_1) \in \Omega \times \Omega$ we have

$$\beta(x_1) \leq h_S(-\alpha(x_1)) \leq h_S(-\alpha(x_0) + \Delta x), \quad \beta(x_1) - \Delta x \leq \beta(x_0) \leq h_S(-\alpha(x_0)).$$

Hence, $(\alpha(x_0), \beta(x_1)) \in B_{\text{sc}}(x_0, x_1)$ and thus $W_{SC} \geq W_S$.

Conversely, for every feasible candidate (α, β) of (4.3) we construct a feasible candidate with potentially better score for (2.11). Since α and β might not be in $\text{Lip}(\Omega)$ we may have to alter them. We first modify β for fixed α and then vice versa. Recall that h_S is concave and $h'_S(0) = 1$. Hence, for some $a \geq \Delta x$ any super-gradient ζ of h_S at $-a + \Delta x$ satisfies $\zeta \geq 1$ and consequently

$$h_S(-a + \Delta x) \geq h_S(-a) + \Delta x \cdot \zeta \geq h_S(-a) + \Delta x.$$

We introduce the auxiliary function

$$\hat{h}_S(z) = \begin{cases} h_S(z) & \text{if } z \geq 0, \\ z & \text{if } z \leq 0. \end{cases}$$

Note that \hat{h}_S is Lipschitz and $\hat{h}_S(z) \geq h_S(z)$ so that

$$\begin{aligned} B_{\text{sc}}(x_0, x_1) &= \{(a, b) \in \mathbb{R}^2 \mid b \leq \min\{h_S(-a + \Delta x), h_S(-a) + \Delta x\}\} \\ &= \{(a, b) \in \mathbb{R}^2 \mid b \leq \min\{\hat{h}_S(-a + \Delta x), h_S(-a) + \Delta x\}\}. \end{aligned}$$

Now set

$$\beta_1(x_1) = \inf\{h_S(-\alpha(x_0)) + \Delta x \mid x_0 \in \Omega\}, \quad \beta_2(x_1) = \inf\{\hat{h}_S(-\alpha(x_0) + \Delta x) \mid x_0 \in \Omega\}.$$

Using that \hat{h}_S is Lipschitz, it is easy to show that β_1 and β_2 are Lipschitz, and consequently so is $\tilde{\beta} : x \mapsto \min\{\beta_1(x), \beta_2(x)\}$. Moreover, $\tilde{\beta} \geq \beta$, so $(\alpha, \tilde{\beta})$ is still feasible for (4.3) with a score not lower than the one for (α, β) . Now, one performs the analogous modification for α , yielding a pair $(\tilde{\alpha}, \tilde{\beta})$ which does not decrease the score. Then $(\tilde{\alpha}, \tilde{\beta}) \in \text{Lip}(\Omega)^2$, and from the constraint $(\tilde{\alpha}(x), \tilde{\beta}(x)) \in B_{\text{sc}}(x, x)$ for $x \in \Omega$ we find that $(\tilde{\alpha}, \tilde{\beta})$ is feasible for (2.11). Hence, $W_{SC} \leq W_S$ and finally $W_{SC} = W_S$. \square

The next lemma establishes the corresponding primal semi-coupling formulation.

Lemma 4.6. *For B_{sc} and c_{sc} as given in Proposition 4.4 we have $\iota_{B_{\text{sc}}(x_0, x_1)} = c_{\text{sc}}(x_0, \cdot, x_1, \cdot)^*$, and c_{sc} is a valid cost function in the sense of Definition 4.1.*

Proof. Again, we write $\Delta x = d(x_0, x_1)$. For fixed $(x_0, x_1) \in \Omega \times \Omega$ the map $\mathbb{R}^2 \ni (m_0, m_1) \mapsto c_{\text{pre}}(x_0, m_0, x_1, m_1)$ is 1-homogeneous and lower semi-continuous. Therefore, by Carathéodory's theorem ([19], Cor. 17.1.6) its convex

envelope can be written as

$$c_{\text{pre}}^{**}(x_0, m_0, x_1, m_1) = \min\{c_S(a_0, a_1) + \Delta x \cdot \min\{a_0, a_1\} + c_S(b_0, b_1) + \Delta x \cdot \min\{b_0, b_1\} \mid a_0, a_1, b_0, b_1 \in \mathbb{R}, a_0 + b_0 = m_0, a_1 + b_1 = m_1\}. \quad (4.5)$$

Let us show that this is equivalent to

$$\min\{c_S(a_0, a_1) + a_0 \Delta x + c_S(b_0, b_1) + b_1 \Delta x \mid a_0, a_1, b_0, b_1 \in \mathbb{R}, a_0 + b_0 = m_0, a_1 + b_1 = m_1\}. \quad (4.6)$$

It trivially holds (4.5) \leq (4.6), so now we consider the opposite inequality. For a feasible candidate (a_0, a_1, b_0, b_1) of (4.5) we distinguish the following cases.

- $[a_0 \leq a_1]$ and $[b_1 \leq b_0]$: The candidate is also feasible for (4.6) with the same score.
- $[a_0 > a_1]$ and $[b_1 > b_0]$: The candidate $(\tilde{a}_0, \tilde{a}_1, \tilde{b}_0, \tilde{b}_1) = (b_0, b_1, a_0, a_1)$ has the same score in (4.5) and satisfies $\tilde{a}_0 \leq \tilde{a}_1, \tilde{b}_1 \leq \tilde{b}_0$. Thus it also has the same score in (4.6).
- $[a_0 > a_1]$ and $[b_1 \leq b_0]$: The candidate $(\tilde{a}_0, \tilde{a}_1, \tilde{b}_0, \tilde{b}_1) = (0, 0, a_0 + b_0, a_1 + b_1)$ has at least an equally good score in (4.5) (using the subadditivity of c_S). Due to $\tilde{a}_0 \leq \tilde{a}_1, \tilde{b}_1 \leq \tilde{b}_0$ it also has the same score in (4.6). The case $[a_0 \leq a_1]$ and $[b_1 > b_0]$ works analogously.

Thus (4.5) = (4.6). Note that (4.6) is the infimal convolution of $(m_0, m_1) \mapsto c_S(m_0, m_1) + \Delta x \cdot m_0$ and $(m_0, m_1) \mapsto c_S(m_0, m_1) + \Delta x \cdot m_1$.

Now abbreviate $B_S(a, b) = B_S + \{(a, b)\}$ for $(a, b) \in \mathbb{R}^2$. Then one finds (cf. Prop. 2.5) $\iota_{B_S(a, b)}^*(m_0, m_1) = c_S(m_0, m_1) + m_0 \cdot a + m_1 \cdot b$ so that $c_{\text{pre}}^{**}(x_0, \cdot, x_1, \cdot) = \iota_{B_S(\Delta x, 0)}^* \square \iota_{B_S(0, \Delta x)}^*$, where \square denotes the infimal convolution. Therefore,

$$c_{\text{sc}}^*(x_0, \cdot, x_1, \cdot) = c_{\text{pre}}^{***}(x_0, \cdot, x_1, \cdot) = \left(\iota_{B_S(\Delta x, 0)}^* \square \iota_{B_S(0, \Delta x)}^* \right)^* = \iota_{B_S(\Delta x, 0)} + \iota_{B_S(0, \Delta x)} = \iota_{B_{\text{sc}}(x_0, x_1)}.$$

For fixed (x_0, x_1) , convexity, lower semi-continuity, and 1-homogeneity of c_{sc} are properties of the Fenchel–Legendre conjugate, while $c_{\text{sc}}(x, m_0, y, m_1) = \infty$ for $\min\{m_0, m_1\} < 0$ follows from $(-\infty, 0]^2 \subset B_{\text{sc}}(x, y)$. For bounded, non-negative sequences $\{(m_{0,k}, m_{1,k})\}_k$, the family of functions $(x_0, x_1) \mapsto c_{\text{sc}}(x_0, m_{0,k}, x_1, m_{1,k})$ is uniformly Lipschitz continuous. Therefore, the function c_{sc} is jointly lower semi-continuous in all four arguments. \square

Remark 4.7 (Dirac cost interpretation). Proposition 4.3 suggests the interpretation of $c_{\text{sc}}(x_0, m_0, x_1, m_1)$ as the dynamic cost for unbalanced transport from a Dirac mass m_0 at x_0 to a Dirac mass m_1 at x_1 . The explicit characterization of $c_{\text{sc}} = c_{\text{pre}}^{**}$ via (4.6) now reveals the same transportation structure as obtained in Proposition 3.29 and Figure 4: first some mass a_0 is transported from x_0 to x_1 , then the remaining mass b_0 at x_0 is changed to b_1 while the mass a_0 at x_1 changes to a_1 , and finally the remaining b_1 is transported from x_0 to x_1 . Example 5.1 will give a more detailed analysis of transport between two Dirac masses.

4.2. Relation between different formulations of unbalanced transport

We now turn to a more detailed discussion of Figure 1 and the relation between the various dynamic and static formulations of transport problems. In the following we write (A) \rightarrow (B) for

“A problem of class (A) induces a corresponding problem of class (B)”.

Throughout this article we consider six different formulations, three pairs of primal and dual problems. We use the shorthands (Dyn) for dynamic formulations, (SC) for the semi-coupling formulations, and (W1T) for the W_1 -type formulations; suffixes (P) and (D) indicate the primal or dual form.

In [7] a family of dynamic problems is defined, and corresponding dual formulations are given, $(\text{Dyn-P}) \leftrightarrow (\text{Dyn-D})$. The Wasserstein-1-type dynamic problems defined in Section 2.1 are a subset of this family. Moreover, [7] introduces the primal and dual semi-coupling formulations and establishes their equivalence $(\text{SC-P}) \leftrightarrow (\text{SC-D})$ (cf. Sect. 4.1). One of the main results in [7] then is the equivalence $(\text{Dyn}) \rightarrow (\text{SC})$, which was proven using the primal formulations, involving the integration of the flow of the (smoothed) momentum field ω .

In [21] families of primal and dual static Wasserstein-1-type problems are proposed, and their equivalence is shown, $(\text{W1T-P}) \leftrightarrow (\text{W1T-D})$. The static Wasserstein-1-type problems defined in Section 2.2 are a subset of these families.

The primal and dual Kantorovich formulation of standard optimal transport with fixed marginals can be interpreted as special cases of (SC-P) and (SC-D) . Similarly, the celebrated Benamou–Brenier formulation of the Wasserstein-2 distance [4] is a special instance of (Dyn) . For this setting $(\text{Dyn}) \rightarrow (\text{SC})$ is shown in [5] by establishing the cyclic inequality $(\text{SC-P}) \geq (\text{Dyn-P}) = (\text{Dyn-D}) \geq (\text{SC-D}) = (\text{SC-P})$. For the crucial step $(\text{Dyn-D}) \geq (\text{SC-D})$ the Hopf–Lax solution for the Hamilton–Jacobi equation corresponding to (Dyn-D) is used, hence the link in Figure 1 is marked as $(\text{Dyn-D}) \rightarrow (\text{SC-D})$.

Analogously, for the Wasserstein–Fisher–Rao/Hellinger–Kantorovich distance one can give formulations in terms of (SC) and (Dyn) . For this setting the link $(\text{Dyn-D}) \rightarrow (\text{SC-D})$ is established in [12] by a generalized Hopf–Lax formula. Note that the primal formulation paired with (SC-D) in [12] is a soft-marginal formulation and differs from the semi-coupling formulation of [7], but is equivalent ([7], Cor. 5.9). In [12] the formulation of (Dyn-D) is given by Theorem 8.13 and (SC-D) by Theorem 6.3, equation (6.14). Different forms of the Hopf–Lax solution are given in equation (8.41) and Theorem 8.12.

Finally, the main result of Section 3.1 is the relation $(\text{Dyn-D}) \rightarrow (\text{W1T-D})$, Proposition 3.1. The equivalence proof involved constructing a feasible candidate for the dynamic dual problem from feasible candidates of the static dual problem. In the next Section we will study when it is possible to derive a dynamic formulation from a static problem, $(\text{W1T-D}) \rightarrow (\text{Dyn-D})$.

4.3. Which static models have an equivalent dynamic formulation?

We already know that any dynamic model of type (2.4) can equivalently be expressed as a static model of type (2.9) (Prop. 3.1). In this section we answer the question under what conditions the reverse statement is true, that is, which static models have an equivalent dynamic formulation. We will see that the class of static models is richer than the dynamic class: there are static models that cannot be expressed in a dynamic form (see Rem. 5.7 later). At first glance this is surprising, since unlike the static formulation the dynamic one models the full time evolution of the measures and thus might be expected to have more design freedom. In detail, the main result of this section is the following.

Proposition 4.8 (Characterization of static models admitting an equivalent dynamic one). *Consider the functions h_S and $\bar{h}_S = -h_S^{-1}(\cdot)$ from Proposition 2.5, which by Proposition 2.5 uniquely specify a static model. Abbreviate*

$$\begin{aligned} \underline{d} &= -\min\{z \in \mathbb{R} \mid \bar{h}_S(z) = \sup \bar{h}_S\} \in [-\infty, 0), \\ \bar{d} &= \min\{z \in \mathbb{R} \mid h_S(z) = \sup h_S\} \in (0, \infty], \\ \underline{s} &= -\max\{z \in \mathbb{R} \mid \bar{h}_S(z) = z\} \in [-\infty, 0), \\ \bar{s} &= \max\{z \in \mathbb{R} \mid h_S(z) = z\} \in (0, \infty] \\ \underline{m} &= \lim_{\varepsilon \searrow 0} (\bar{h}_S(\underline{s} + \varepsilon) - \underline{s})/\varepsilon \in [0, 1], \\ \bar{m} &= \lim_{\varepsilon \searrow 0} (h_S(\bar{s} + \varepsilon) - \bar{s})/\varepsilon \in [0, 1]. \end{aligned}$$

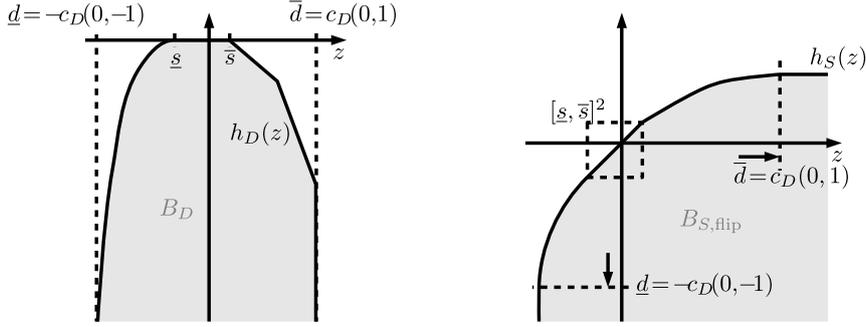


FIGURE 5. Sketch of the functions h_D and h_S from Lemma 3.2 and Proposition 2.5. $B_{S,\text{flip}}$ denotes the set B_S flipped along the vertical axis.

Denoting the left-sided derivative of a function g by g' and the j -fold composition of g with itself by g^j , the function

$$q[h_S](z) = \begin{cases} -\infty & \text{if } z < \underline{d}, \\ \underline{c} \lim_{j \rightarrow \infty} \frac{\overline{h}_S^j(-z) - \overline{h}_S^{j-1}(-z)}{(\overline{h}_S^j)'(-z)} & \text{if } z \in (\underline{d}, 0), \\ \overline{c} \lim_{j \rightarrow \infty} \frac{h_S^j(z) - h_S^{j-1}(z)}{(h_S^j)'(z)} & \text{if } z \in [0, \overline{d}), \\ -\infty & \text{if } z > \overline{d}, \end{cases} \quad \underline{c} = \begin{cases} 1 & \text{if } \underline{m} = 1, \\ \frac{\log \underline{m}}{1 - 1/\underline{m}} & \text{else,} \end{cases}$$

$$\overline{c} = \begin{cases} 1 & \text{if } \overline{m} = 1, \\ \frac{\log \overline{m}}{1 - 1/\overline{m}} & \text{else,} \end{cases}$$

is well-defined (at \underline{d} and \overline{d} we simply extend $q[h_S]$ upper semi-continuously), and the following holds.

- (1). Existence. The static model (2.9) has an equivalent dynamic formulation (2.4) (that is, $W_S(\rho_0, \rho_1) = W_D(\rho_0, \rho_1)$ for all measures ρ_0, ρ_1) if and only if $q[h_S]$ is concave.
- (2). Necessary condition. A necessary condition for the above is that h_S is locally Lipschitz differentiable on $(\overline{s}, \overline{d})$ and \overline{h}_S is so on $(\underline{d}, \underline{s})$.
- (3). Sufficient condition. A sufficient condition for the above is that $\frac{1}{\overline{h}_S}$ and $\frac{1}{h_S}$ are convex on $[0, \infty)$.
- (4). Uniqueness and reconstruction formula. If the static model (2.9) has an equivalent dynamic formulation (2.4), then necessarily $h_D = q[h_S]$. (Recall that any dynamic model is uniquely specified by the function h_D from Lemma 3.2. In that case, $\underline{d} = -c_D(0, -1)$, $\overline{d} = c_D(0, 1)$.)

For a better intuition, in Figure 5 we illustrate two corresponding functions h_D and h_S , specifying a dynamic model and its equivalent static formulation, respectively. As an example, Proposition 4.8(2) implies that the static model with the piecewise linear $h_S(z) = \min(z, 1 + \frac{z}{2}, 2 + \frac{z}{3})$ does not admit a dynamic formulation, while by Proposition 4.8(3) the static model with $h_S(z) = \frac{z}{1+z}$ does. Note that Proposition 4.8(3) is only sufficient: for instance, it is readily checked that the function h_S belonging to a piecewise linear h_D typically has multiple linear segments of different slope so that $\frac{1}{h_S}$ or $\frac{1}{\overline{h}_S}$ cannot be convex.

Before proving the proposition, let us provide some motivation for the definition of $q[h_S]$, which will be made rigorous later. First observe that h_S as induced by h_D is given by Proposition 3.12 where F_t is the solution to (3.4) and by Lemma 3.4(vii) one has $\sup \text{dom } F_1 = \sup \text{dom } h_D$. Thus we can read off

$$\overline{d} = \sup \text{dom } h_D = c_D(0, 1) \quad \text{as well as} \quad h_S^j(z) = F_j(z) \quad \text{for all } z \in \text{dom } h_D. \quad (4.7)$$

If the static model with h_S indeed comes from a dynamic model with (some as yet unknown) h_D , then for small z we can approximate $h_D(z)$ as follows. For $\varepsilon \in \mathbb{R}$ of small magnitude, with (3.10) we estimate

$$c_S(1, 1 + \varepsilon) = \inf_{(\rho, \zeta) \in \mathcal{CE}\mathcal{S}(1, 1 + \varepsilon)} \int_0^1 c_D(\rho, \zeta) dt \approx c_D(1, \varepsilon).$$

Now by Propositions 2.3 and 2.5 and Lemma 3.2 we have

$$\begin{aligned} c_D(1, \varepsilon) &= \sup\{\alpha + \varepsilon \cdot \beta \mid \alpha, \beta \in \mathbb{R}, \alpha \leq h_D(\beta)\}, \\ c_S(1, 1 + \varepsilon) &= \sup\{\alpha + (1 + \varepsilon) \cdot \beta \mid \alpha, \beta \in \mathbb{R}, \alpha \leq \bar{h}_S(-\beta)\} \\ &= \sup\{\tilde{\alpha} + \varepsilon \cdot \tilde{\beta} \mid \tilde{\alpha}, \tilde{\beta} \in \mathbb{R}, \tilde{\alpha} \leq \bar{h}_S(-\tilde{\beta}) + \tilde{\beta}\}, \end{aligned} \quad (4.8)$$

where in the last line we substituted $(\tilde{\alpha}, \tilde{\beta}) = (\alpha + \beta, \beta)$. Since for small ε we have $c_S(1, 1 + \varepsilon) \approx c_D(1, \varepsilon)$, for small β we expect

$$h_D(\beta) \approx \bar{h}_S(-\beta) + \beta = -h_S^{-1}(\beta) + \beta. \quad (4.9)$$

For large z on the other hand (analogously for large negative z) we can now obtain an estimate of $h_D(z)$ by exploiting that h_S is the flow of h_D . The flow can be differentiated, yielding $(h_S^j)'(z) = F_j'(z) = \frac{h_D(h_S^j(z))}{h_D(z)}$, so that finally we arrive at

$$h_D(z) = \begin{cases} -\infty & \text{if } z > \bar{d}, \\ \frac{h_D(h_S^j(z))}{(h_S^j)'(z)} & \text{if } z \in [0, \bar{d}] \end{cases} \approx \begin{cases} -\infty & \text{if } z > \bar{d}, \\ \frac{h_S^j(z) - h_S^{j-1}(z)}{(h_S^j)'(z)} & \text{if } z \in [0, \bar{d}], \end{cases}$$

where we have used our approximation of $h_D(z)$ for small z and that $h_S^j(z)$ is small for j large enough. This is exactly the expression used in the definition of $q[h_S]$.

Remark 4.9 (Separate treatment on positive and negative real line). Analogously to (4.7) one obtains

$$\underline{d} = -c_D(0, -1) = \inf \text{dom } h_D \quad \text{as well as} \quad \bar{h}_S^j(z) = F_j^{\text{inv}}(z) \text{ for all } z \in \text{dom } h_D.$$

For statements about $h_D(z)$ with positive z we will therefore always work with h_S and F_j , while for negative z we shall use \bar{h}_S as well as F_j^{inv} .

We shall now begin proving Proposition 4.8 in multiple steps, not necessarily in the order of its different statements.

Proof of Proposition 4.8(2). Assume that the static model with h_S has an equivalent dynamic formulation with h_D . By (4.7) we have $h_S(z) = F_1(z)$ which is locally Lipschitz differentiable on (\bar{s}, \bar{d}) by Lemma 3.4(x). The statement for \bar{h}_S follows analogously with Remark 4.9. \square

The following lemma rigorously establishes the heuristic relation (4.9) “for small arguments”, as required for the proof of Proposition 4.8.

Lemma 4.10 (Approximation of h_D for small arguments). *Let the static model with h_S have an equivalent dynamic formulation with h_D . Then*

$$\lim_{z \nearrow \bar{s}} \frac{h_D(z)}{-\bar{h}_S^{-1}(-z) - z} = \lim_{z \nearrow \bar{s}} \frac{h_D(z)}{h_S(z) - z} = \underline{c}, \quad \lim_{z \searrow \bar{s}} \frac{h_D(z)}{z - \bar{h}_S^{-1}(z)} = \bar{c}.$$

Proof. We only prove the second limit, the first one following analogously *via* Remark 4.9. First consider the case that h_S is differentiable in \bar{s} , that is, $\bar{c} = \bar{m} = 1$ and $h'_S(z) \rightarrow \bar{m} = 1$ as $z \rightarrow \bar{s}$. Using (3.10), for all $\varepsilon > 0$ we have

$$\begin{aligned} c_S(1, 1 + \varepsilon) &= \min \left\{ \int_0^1 c_D\left(\frac{d\rho}{d\mu}, \frac{d\xi}{d\mu}\right) d\mu \mid (\rho, \xi) \in \mathcal{CES}(1, 1 + \varepsilon) \right\} \\ &\leq \int_0^1 c_D(1 + \varepsilon t, \varepsilon) dt \leq \int_0^1 c_D(1, \varepsilon) dt = c_D(1, \varepsilon), \end{aligned}$$

where we exploited $1 + t\varepsilon \geq 1$ and the convexity of c_D *via*

$$c_D(\rho, \varepsilon) = \rho c_D(1, \frac{\varepsilon}{\rho}) \leq \rho \left[\frac{\varepsilon/\rho}{\varepsilon} c_D(1, \varepsilon) + \frac{\varepsilon - \varepsilon/\rho}{\varepsilon} c_D(1, 0) \right] = c_D(1, \varepsilon)$$

for $\rho \geq 1$. Note also that (see (4.8))

$$\begin{aligned} c_D(1, \varepsilon) &= \sup \{ \alpha + \varepsilon\beta \mid \alpha \leq h_D(\beta) \} = (-h_D)^*(\varepsilon), \\ c_S(1, 1 + \varepsilon) &= \sup \{ \alpha + (1 + \varepsilon)\beta \mid \alpha \leq -h_S^{-1}(\beta) \} = (h_S^{-1} - \text{id})^*(\varepsilon). \end{aligned}$$

Therefore, $c_S(1, 1 + \varepsilon) \leq c_D(1, \varepsilon)$ for $\varepsilon > 0$ implies $h_D(z) \geq z - h_S^{-1}(z)$ for $z \geq 0$ and thus

$$\limsup_{z \searrow \bar{s}} \frac{h_D(z)}{z - h_S^{-1}(z)} \leq 1.$$

Furthermore, by (4.7), for $z > 0$ sufficiently close to \bar{s} we have

$$\begin{aligned} h_S(z) &= F_1(z) = \sup \{ \phi(1) \mid \partial_t \phi(t) \leq h_D(\phi(t)) \text{ for all } t \in [0, 1], \phi(0) = z \} \\ &\geq \sup \left\{ \phi(1) \mid \phi(t) = \phi(0) \exp\left(\frac{h_D(\phi(0))}{\phi(0)} t\right), \phi(0) = z \right\} = z \exp \frac{h_D(z)}{z}, \end{aligned}$$

where we used the concavity of h_D . This implies $h_D(z) \leq z \log \frac{h_S(z)}{z}$. Since h_S is differentiable in \bar{s} and thus $h'_S(h_S^{-1}(z)) \rightarrow 1$ as $z \rightarrow \bar{s}$, using $h'_S(h_S^{-1}(z)) \cdot (h_S^{-1}(z) - z) \leq z - h_S(z)$ due to the concavity of h_S as well as $\lim_{z \rightarrow \bar{s}} h_S(z) = \bar{s}$ we thus obtain

$$\begin{aligned} \liminf_{z \searrow \bar{s}} \frac{h_D(z)}{z - h_S^{-1}(z)} &\geq \liminf_{z \searrow \bar{s}} \frac{z \log \frac{h_S(z)}{z}}{z - h_S^{-1}(z)} \\ &\geq \liminf_{z \searrow \bar{s}} \frac{z \log \frac{h_S(z)}{z}}{h_S(z) - z} h'_S(h_S^{-1}(z)) = \liminf_{z \searrow \bar{s}} \frac{-\log \frac{h_S(z)}{z}}{1 - \frac{h_S(z)}{z}} h'_S(h_S^{-1}(z)) = 1. \end{aligned}$$

The case $\bar{m} < 1$ is treated more explicitly. Let $\tilde{m} = \lim_{\varepsilon \searrow 0} h_D(\bar{s} + \varepsilon)/\varepsilon$ denote the right derivative of h_D in \bar{s} so that

$$h_D(z) = \tilde{m}(z - \bar{s}) - o(z - \bar{s}) \quad \text{for } z \geq \bar{s},$$

where $o(z - \bar{s})$ denotes a nonnegative term decreasing to zero faster than $z - \bar{s}$. The flow of h_D can readily be approximated: $h_D(z) \leq \tilde{m}(z - \bar{s})$ implies $F_1(z) \leq ze^{\tilde{m}}$, and the same calculation as above yields $F_1(z) \geq z \exp \frac{h_D(z)}{z} = ze^{\tilde{m} - o(1)}$, where $o(1)$ is a nonnegative term decreasing to zero as $z \rightarrow \bar{s}$. Now (4.7) implies that

the right derivative of h_S in \bar{s} is given by $\bar{m} = F_1'(\bar{s}) = e^{\bar{m}}$. Furthermore, by the inverse function theorem, the right derivative of h_S^{-1} in \bar{s} is given by $\frac{1}{\bar{m}}$. Thus,

$$\lim_{z \searrow \bar{s}} \frac{h_D(z)}{z - h_S^{-1}(z)} = \lim_{z \searrow \bar{s}} \frac{\tilde{m}(z - \bar{s}) - o(z - \bar{s})}{z - \bar{s} - \frac{1}{\bar{m}}(z - \bar{s}) - o(z - \bar{s})} = \lim_{z \searrow \bar{s}} \frac{\log \bar{m}(z - \bar{s}) - o(z - \bar{s})}{(1 - \frac{1}{\bar{m}})(z - \bar{s}) - o(z - \bar{s})} = \bar{c}.$$

□

Proof of Proposition 4.8(4). We first show that the limits in the definition of $q[h_S]$ are well-defined and finite. In fact, defining

$$q_j(z) = \frac{h_S^j(z) - h_S^{j-1}(z)}{(h_S^j)'(z)},$$

for all $z \in [0, \bar{d})$ we can show the monotonicity property

$$-\infty < q_1(z) \leq q_{j-1}(z) \leq q_j(z) \leq 0 \quad \text{for all } j > 1. \quad (4.10)$$

Indeed, we have $q_1(z) > -\infty$ since $h_S'(z) > 0$ (the strict positivity follows by the concavity of h_S and the definition of \bar{d}). Likewise, $q_j(z) \leq 0$ follows from $h_S^j \leq h_S^{j-1}$ (which is true since $h_S(z) \leq z$, see Prop. 2.5) and $(h_S^j)' > 0$. Finally, the monotonicity of the sequence q_j is obtained as follows: The composition of a concave, monotonically increasing function with a concave function is again concave, thus h_S^j is concave and increasing for all j . Therefore,

$$\begin{aligned} h_S(h_S^{j-2}(z)) &\leq h_S(h_S^{j-1}(z)) + h_S'(h_S^{j-1}(z))(h_S^{j-2}(z) - h_S^{j-1}(z)) \\ \text{so that } q_j(z) &= \frac{h_S(h_S^{j-1}(z)) - h_S(h_S^{j-2}(z))}{h_S'(h_S^{j-1}(z))(h_S^{j-1})'(z)} \geq \frac{h_S^{j-1}(z) - h_S^{j-2}(z)}{(h_S^{j-1})'(z)} = q_{j-1}(z). \end{aligned}$$

This monotonicity property then implies the existence of the finite limit $\lim_{j \rightarrow \infty} q_j(z)$ (the case of negative z is treated analogously).

Note further that $q_j(0) = q_j'(0) = 0$ and q_j is nonincreasing on $(0, \infty)$. Indeed, the latter follows from

$$q_j'(z) = \frac{(h_S^j)'(z) - (h_S^{j-1})'(z)}{(h_S^j)'(z)} - q_j(z) \frac{(h_S^j)''(z)}{(h_S^j)'(z)} = 1 - \frac{1}{h_S'(h_S^{j-1}(z))} - q_j(z) \frac{(h_S^j)''(z)}{(h_S^j)'(z)} \leq 0,$$

for $z \in [0, \bar{d})$, where $(h_S^j)''$ is well-defined as a weak derivative due to Proposition 4.8(2). Analogously one can show that q_j is nondecreasing on $(-\infty, 0)$ (Rem. 4.9). By the monotone convergence $\bar{c} \cdot q_j \nearrow q[h_S]$ the limit $q[h_S]$ shares the same properties.

Now we prove $h_D(z) = q[h_S](z)$ for positive z (the case of negative z follows analogously *via* Rem. 4.9). By (4.7), $h_D(z) = q[h_S](z) = -\infty$ for $z > \bar{d}$. On the other hand, for $z \in [0, \bar{s})$ we have $h_S(z) = z$ so that $q[h_S](z) = 0$ as well as $h_D(z) = 0$ by (4.7). Finally, for $z \in (\bar{s}, \bar{d})$ Lemma 3.4(x) together with (4.7) implies

$$(h_S^j)'(z) = F_j'(z) = \frac{h_D(F_j(z))}{h_D(z)} = \frac{h_D(h_S^j(z))}{h_D(z)}$$

so that application of Lemma 4.10 yields

$$\frac{h_D(z)}{q[h_S](z)} = \lim_{j \rightarrow \infty} \frac{h_D(z) (h_S^j)'(z)}{\bar{c} \cdot (h_S^j(z) - h_S^{j-1}(z))} = \lim_{j \rightarrow \infty} \frac{h_D(h_S^j(z))}{\bar{c} \cdot (h_S^j(z) - h_S^{j-1}(z))} = \lim_{\tilde{z} \searrow \bar{s}} \frac{h_D(\tilde{z})}{\bar{c} \cdot (\tilde{z} - h_S^{-1}(\tilde{z}))} = 1.$$

□

Proof of Proposition 4.8(3). We prove by induction that q_j is concave on $[0, \infty)$ for all j , which automatically implies the desired concavity of the limit $q[h_S]$ on $[0, \infty)$ (the proof for $(-\infty, 0]$ is analogous by Rem. 4.9). First note that $q_0(z) = z - h_S^{-1}(z)$ is indeed concave. Furthermore, due to $q_{j+1} = \frac{q_j \circ h_S}{h_S'}$ we have

$$q'_{j+1}(z) = q'_j(h_S(z)) - q_j(h_S(z)) \frac{h_S''(z)}{(h_S'(z))^2}.$$

We need to show that q'_{j+1} is non-increasing. Now q'_j and thus also $q'_j \circ h_S$ is non-increasing on $[0, \infty)$. Likewise, $|q_j|$ and $|q_j \circ h_S|$ are non-decreasing, so it remains to show that $|h_S''(z)|/(h_S'(z))^2 = -h_S''(z)/(h_S'(z))^2 = (1/h_S'(z))'$ is non-decreasing. However, this is equivalent to the convexity of $\frac{1}{h_S}$. □

Proof of Proposition 4.8(1). If the static model has an equivalent dynamic formulation, then by Proposition 4.8(4) we must have $h_D = q[h_S]$. Since h_D is concave by Lemma 3.2, $q[h_S]$ is so as well.

It remains to show that concavity of $q[h_S]$ implies the existence of an equivalent dynamic formulation. We have already shown $q[h_S] \leq 0$ with $q[h_S](0) = q[h_S]'(0) = 0$. Together with the concavity this means that $q[h_S]$ specifies an admissible dynamic model of type (2.4) via

$$c_D(m_0, m_1) = \sup\{m_0\alpha + m_1\beta \mid \alpha \leq q[h_S](\beta)\}.$$

Denote by \tilde{h}_S the function specifying the corresponding static problem. We need to show $h_S = \tilde{h}_S$ (which we do on the positive real line; the negative case follows analogously with Rem. 4.9). However, we have $h_S(z) = z = \tilde{h}_S(z)$ for all $z \in [0, \bar{s}]$ and both h_S and \tilde{h}_S constant on $[\bar{d}, \infty)$. Furthermore, for $z \in (\bar{s}, \bar{d})$, by the monotone convergence theorem (recall (4.10)) and a change of variables,

$$\int_z^{h_S(z)} \frac{1}{q[h_S](x)} dx = \frac{1}{\bar{c}} \lim_{j \rightarrow \infty} \int_z^{h_S(z)} \frac{1}{q_j(x)} dx = \frac{1}{\bar{c}} \lim_{j \rightarrow \infty} \int_{h_S^j(z)}^{h_S^{j+1}(z)} \frac{1}{y - h_S^{-1}(y)} dy.$$

Now first assume that h_S is differentiable in \bar{s} and thus $\bar{m} = \bar{c} = 1$ as well as $h_S'(h_S^{j-1}(z)) \rightarrow 1$ as $j \rightarrow \infty$. Note that for a negative increasing function f and $a \leq b$ one has $(a - b)f(b) \leq \int_b^a f(y) dy \leq (a - b)f(a)$. Choosing $a = h_S^{j+1}(z)$, $b = h_S^j(z)$, and $f(y) = \frac{1}{y - h_S^{-1}(y)}$ (which is indeed negative and increasing on (\bar{s}, \bar{d})) and using the concavity $h_S(h_S^{j-1}(z)) \geq h_S(h_S^j(z)) + h_S'(h_S^{j-1}(z)) \cdot (h_S^{j-1}(z) - h_S^j(z))$, we have

$$h_S'(h_S^{j-1}(z)) \leq \frac{h_S^j(z) - h_S^{j+1}(z)}{h_S^{j-1}(z) - h_S^j(z)} \leq \int_{h_S^j(z)}^{h_S^{j+1}(z)} \frac{1}{y - h_S^{-1}(y)} dy \leq \frac{h_S^j(z) - h_S^{j+1}(z)}{h_S^j(z) - h_S^{j+1}(z)} = 1$$

so that $\int_z^{h_S(z)} \frac{1}{q[h_S](x)} dx = 1$. In the case that $\bar{m} < 1$, by the inverse function theorem the right derivative of h_S^{-1} in \bar{s} is $\frac{1}{\bar{m}}$. Further, for $z \in (\bar{s}, \bar{d})$ we have $h_S^j(z) \searrow \bar{s}$ as $j \rightarrow \infty$ so that (using little o-notation)

$$\frac{1}{\bar{c}} \lim_{j \rightarrow \infty} \int_{h_S^j(z)}^{h_S^{j+1}(z)} \frac{1}{y - h_S^{-1}(y)} dy = \frac{1}{\bar{c}} \lim_{z \searrow \bar{s}} \int_z^{h_S(z)} \frac{1}{y - h_S^{-1}(y)} dy$$

$$\begin{aligned}
&= \frac{1}{\tilde{c}} \lim_{z \searrow \bar{s}} \int_z^{h_S(z)} \frac{1}{y - \bar{s} - \frac{1}{\bar{m}}(y - \bar{s}) + o(z - \bar{s})} dy \\
&= \frac{1}{\tilde{c}} \lim_{z \searrow \bar{s}} \int_z^{h_S(z)} \frac{1}{(1 - \frac{1}{\bar{m}})(y - \bar{s})} dy \\
&= \frac{1}{\tilde{c}} \lim_{z \searrow \bar{s}} \frac{1}{1 - \frac{1}{\bar{m}}} \log \frac{h_S(z) - \bar{s}}{z - \bar{s}} = \frac{1}{\tilde{c}} \frac{1}{1 - \frac{1}{\bar{m}}} \log \bar{m} = 1
\end{aligned}$$

and thus again $\int_z^{h_S(z)} \frac{1}{q[h_S](x)} dx = 1$. Now this implies that h_S is the flow of $q[h_S]$ at $t = 1$ (*cf.* proof of Lem. 3.4(i)). However, \tilde{h}_S is so as well by construction so that \tilde{h}_S and h_S coincide. \square

5. EXAMPLES

This section will provide some examples of possible static and dynamic models, some of which are well-known. We shall also calculate in detail the optimal solution to some simple unbalanced mass transport problems, which will give some additional insight into the solution structure beyond Section 3.4.

5.1. Unbalanced transport between two Dirac masses

Here we consider the simple problem of unbalanced transport between two measures ρ_0 and ρ_1 , where both consist of just two Dirac masses at the same positions. Since the unbalanced transport problem is invariant under translation and rotation of the coordinate system and since according to Proposition 3.29 transport only happens between the measure supports $\text{spt } \rho_0$ and $\text{spt } \rho_1$, we may without loss of generality assume the Dirac masses to lie on the one-dimensional real line, one at the origin and the other at some distance $L > 0$ (in fact, our example covers the behaviour of unbalanced transport between two Dirac masses at distance L in any metric space). Due to the one-homogeneity of the unbalanced transport cost $W_S(\rho_0, \rho_1)$ we might even restrict one of the measures ρ_0 and ρ_1 to be a probability measure, however, this does not simplify the exposition substantially. We shall consider the static formulation (2.9), since by Proposition 3.15 the dynamic model behaviour can be directly inferred from the static behaviour.

Example 5.1 (Unbalanced transport between two Dirac masses). For simplicity we shall assume c_S to be differentiable (the exact same calculation can also be performed in the non-differentiable case if one simply replaces derivatives by subderivatives; however, the rigorous justification of the calculation then involves a few more technical arguments that we avoid here for an easier exposition). We set

$$\rho_0 = m_0^0 \delta_0 + m_0^L \delta_L, \quad \rho_1 = m_1^0 \delta_0 + m_1^L \delta_L \quad \text{with } m_1^0 < m_0^0, \quad m_1^L > m_0^L,$$

that is, mass is removed at the origin and added at L . We seek the optimal transport couplings π_0 and π_1 in (2.9). To this end, we may parameterize them as

$$\begin{aligned}
\pi_0 &= a \delta_{(0,L)} + (m_0^0 - a) \delta_{(0,0)} + m_0^L \delta_{(L,L)}, \quad \pi_1 = b \delta_{(0,L)} + m_1^0 \delta_{(0,0)} + (m_1^L - b) \delta_{(L,L)} \\
&\quad \text{with } (a, b) \in S = [0, m_0^0] \times [0, m_1^L],
\end{aligned} \tag{5.1}$$

that is, π_0 transports mass a and π_1 transports mass b from the origin to L . That π_0 and π_1 must have the above form follows from Proposition 3.29, which implies $\text{spt } \pi_0, \text{spt } \pi_1 \subset \text{spt}(\rho_0 + \rho_1)^2 = \{0, L\}^2$ and $\pi_0(\{(L, 0)\}) = \pi_1(\{(L, 0)\}) = 0$. Indeed, suppose $\pi_0(\{(L, 0)\}) > 0$ (and thus $\pi_0(\{(0, L)\}) = 0$ by optimality of π_0), then by Proposition 3.29

1. either $\{0, L\} = \Omega_+$ so that no mass change happens at all (thus both π_0 and π_1 must transport from 0 to L , leading to a contradiction),

2. or $\{0\} \subset \Omega_+$ so that necessarily $\pi_1(\{(0, L)\}) > 0$ in order to achieve a net mass removal at the origin. This again leads to a contradiction since by Proposition 3.26 π_1 may not transport any mass from Ω_+ .

Analogously one obtains $\pi_1(\{(L, 0)\}) = 0$. The cost (2.8) associated with π_0, π_1 can be calculated as

$$\begin{aligned} P(a, b) &= L(a + b) + c_S(m_0^0 - a, m_1^0 + b) + c_S(m_0^L + a, m_1^L - b) \\ &= L(a + b) + (m_0^0 - a)c_S(1, \alpha) + (m_0^L + a)c_S(1, \beta) \end{aligned} \quad (5.2)$$

for the relative mass changes

$$\alpha = \frac{m_1^0 + b}{m_0^0 - a} \geq 0 \quad \text{and} \quad \beta = \frac{m_1^L - b}{m_0^L + a} \geq 0. \quad (5.3)$$

Since P is convex, its optimum is well-defined. To find the optimal $(a, b) \in S$ there are thus only two possible cases.

1. *Case $0 \neq \nabla P(a, b)$ for all $(a, b) \in S$.* In that case the optimum satisfies $a = 0$ or $b = 0$. Indeed, the optimum must lie in ∂S , and if $b = m_1^L$ then $a = 0$ is optimal (analogously one shows that $a = m_0^0$ implies $b = 0$): using the notation of Proposition 3.26 we have $\rho'_1(\{L\}) = 0$ and thus necessarily $L \notin \Omega_+$. If $L \in \Omega_-$, then by Corollary 3.28 the structure of π_1 implies $0 \in \Omega_-$ as well and thus $a = 0$ since by Proposition 3.26 π_0 is diagonal on $\Omega_- \times \Omega_-$. If on the other hand $L \in \Omega_+$, then $\rho'_0(\{L\}) = \rho'_1(\{L\}) = 0$ and thus again $a = 0$.
2. *Case $0 = \nabla P(a, b)$ for some $(a, b) \in S$.* In that case the optimum (a, b) has to satisfy

$$\begin{aligned} 0 &= \frac{\partial P}{\partial a} = L - c_S(1, \alpha) + \alpha c_{S,2}(1, \alpha) + c_S(1, \beta) - \beta c_{S,2}(1, \beta), \\ 0 &= \frac{\partial P}{\partial b} = L + c_{S,2}(1, \alpha) - c_S(1, \beta), \end{aligned}$$

where subscript $,2$ denotes the derivative with respect to the second argument. These equations can be transformed into

$$L = c_{S,2}(1, \beta) - c_{S,2}(1, \alpha) = T[\beta]' - T[\alpha]', \quad (5.4)$$

$$\begin{aligned} 0 &= \frac{\partial P}{\partial a} - \frac{\partial P}{\partial b} = [c_S(1, \beta) - (\beta - 1)c_{S,2}(1, \beta)] - [c_S(1, \alpha) + (1 - \alpha)c_{S,2}(1, \alpha)] \\ &= T[\beta](1) - T[\alpha](1), \end{aligned} \quad (5.5)$$

where $T[\gamma]$ denotes the tangent to $c_S(1, \cdot)$ at γ . Note that (5.4) implies $\beta > \alpha$ and (5.5) thus implies $T[\alpha]' < 0$ and $T[\beta]' > 0$ or equivalently

$$\alpha \leq 1 \quad \text{and} \quad \beta \geq 1.$$

Thus the minimizer (a, b) can be constructed geometrically as follows (compare Fig. 6): For $s < 0$ let

$$\begin{aligned} T_s^l : [0, 1] &\rightarrow \mathbb{R}, & x &\mapsto \sup\{s + t(x - 1) \mid t < 0, s + t(y - 1) \leq c_S(1, y) \forall y \in [0, 1]\}, \\ T_s^r : [1, \infty) &\rightarrow \mathbb{R}, & x &\mapsto \sup\{s + t(x - 1) \mid t > 0, s + t(y - 1) \leq c_S(1, y) \forall y \in [1, \infty)\} \end{aligned}$$

denote the left and the right tangent line to c_S running through the point $(1, s)$, and set

$$L(s) = (T_s^r)' - (T_s^l)' > 0.$$

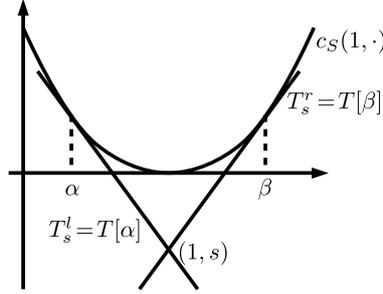


FIGURE 6. Sketch for the construction of the optimal relative mass changes α and β from Example 5.1.

Obviously, $L(s)$ is continuous and strictly decreasing in s . Note that s parameterizes the solutions to (5.5) and that by (5.4) there must be one s with

$$L(s) = L. \quad (5.6)$$

The optimal relative mass decrease α and mass increase β can now be identified *via* the condition

$$T_s^l = T[\alpha], \quad T_s^r = T[\beta] \quad (5.7)$$

(note that there may be a closed interval of α s or β s satisfying this condition). From these we can solve for (a, b) as

$$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\beta - \alpha} \begin{pmatrix} -1 & -1 \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} \alpha m_0^0 - m_1^0 \\ \beta m_0^L - m_1^L \end{pmatrix}. \quad (5.8)$$

Obviously, in addition to the obvious constraints

$$\alpha \geq m_1^0/m_0^0 \text{ and } \beta \leq m_1^L/m_0^L, \quad (5.9)$$

the condition $(a, b) \in S$ imposes four constraints on (α, β) ,

$$\beta \leq \frac{\rho_1(\mathbb{R}) - \alpha m_0^0}{m_0^L}, \quad \beta \geq \frac{\alpha m_1^L}{\alpha \rho_0(\mathbb{R}) - m_1^0}, \quad \beta \geq \frac{\rho_1(\mathbb{R})}{\rho_0(\mathbb{R})}, \quad \alpha \leq \frac{\rho_1(\mathbb{R})}{\rho_0(\mathbb{R})}. \quad (5.10)$$

We learn several interesting facts from the above example and particularly optimality conditions (5.6)–(5.8) for s , (α, β) , and (a, b) :

1. A minimizer with $a, b > 0$ has to satisfy (5.6)–(5.8). Consequently there is a minimum distance $L_{\min} = \lim_{s \nearrow 0} L(s) \in [0, \infty)$ between both Dirac masses, only depending on c_S , below which mass is either only transported before or after the mass change (that is, $a = 0$ or $b = 0$). L_{\min} is the difference between left and right derivative of $c_S(1, \cdot)$ at 1, so $L_{\min} > 0$ only if $c_S(1, \cdot)$ is not differentiable at 1.
2. For the same reason and consistent with Proposition 3.31 there is a maximum distance $L_{\max} = \lim_{s \rightarrow -\infty} L(s) \in (0, \infty]$ between both Dirac masses, only depending on c_S , beyond which there is no mass transport before or after the mass change ($a = 0$ or $b = 0$).
3. If there is transport before and after the mass change, then by optimality conditions (5.4) and (5.5) the relative mass change α and β at each Dirac mass only depends on c_S and the distance between both Dirac masses.

4. If the distance between both Dirac masses satisfies $L \in [L_{\min}, L_{\max}]$ and α, β are the corresponding relative mass changes, then mass transport happens before as well as after the mass change (that is, $a, b > 0$) unless α, β violate (5.9) and (5.10).
5. For any c_S we can easily engineer ρ_0 and ρ_1 , each consisting of two Diracs at the same locations, such that transport optimally happens only before, only after, or before as well as after mass change. Indeed, choose arbitrary $a, b \geq 0$ and $s < 0$ and let α, β be the relative mass changes corresponding to s via (5.7). Then set $L = L(s)$ and choose any $m_0^0, m_1^0, m_0^L, m_1^L$ satisfying $\alpha = \frac{m_1^0 + b}{m_0^0 - a}$ and $\beta = \frac{m_1^L - b}{m_0^L + a}$. Optimality conditions (5.6)–(5.8) are now satisfied by construction.
6. Even in the simple setting with just two Dirac masses nonuniqueness of minimizers can occur in various ways. First, if $c_S(1, \cdot)$ is not strictly convex, then the tangents $T[\gamma]$ to $c_S(1, \cdot)$ for multiple different values of γ coincide so that (5.4) and (5.5) may have multiple solutions. Second, if $c_S(1, \cdot)$ has nonzero left or right derivative in 1 and L is small enough, (5.4) and (5.5) are solved by $\alpha = \beta = 1$ so that (5.3) has infinitely many solutions (a, b) as long as $\rho_0(\Omega) = \rho_1(\Omega)$. Third, there may exist optimal (π_0, π_1) which are not of the form (5.1). Remark 5.2 examines when uniqueness can be expected.

Remark 5.2 (Uniqueness for unbalanced transport between two Dirac masses). If $c_S(1, \cdot)$ is strictly convex and differentiable at 1, then the minimizer (π_0, π_1) of (2.9) for Example 5.1 is unique. Indeed, we show below that any minimizer (π_0, π_1) satisfies $\text{spt } \pi_0, \text{spt } \pi_1 \subset \{0, L\}^2$, and then following the same argument as given in Example 5.1, any minimizer must have the form (5.1). Furthermore, for strictly convex $c_S(1, \cdot)$, (5.4) and (5.5) can have at most one solution (α, β) , which necessarily satisfies $\alpha, \beta \neq 1$ (due to the differentiability of $c_S(1, \cdot)$ in 1) and thus results in at most one solution (a, b) . Thus, there is at most one minimizer (a, b) in the interior of S . Likewise, there can be at most one minimizer on each line segment of ∂S due to the strict convexity of the cost (5.2) if either a or b are held fixed. Together with the convexity of the cost (5.2) these two statements imply that there is a unique minimizer (a, b) and thus a unique minimizer (π_0, π_1) .

It remains to show $\text{spt } \pi_0, \text{spt } \pi_1 \subset \{0, L\}^2$ or equivalently $\rho'_0(\Omega \setminus \{0, L\}) = \rho'_1(\Omega \setminus \{0, L\}) = 0$. For a contradiction, assume the converse and note that by Proposition 3.26 we have $\Omega \setminus \{0, L\} \subset \Omega_-$ so that $\rho'_0 \llcorner (\Omega \setminus \{0, L\}) = \rho'_1 \llcorner (\Omega \setminus \{0, L\})$. The coupling π_0 must have transported this mass to $\Omega \setminus \{0, L\}$ from $\{0, L\}$ (without loss of generality assume that part of it comes from 0). Now note $\pi_0((\Omega_+ \cup \Omega_-) \times \Omega_-) = 0$ (due to Props. 3.26 and 3.29, whose proof in the case of strictly convex $c_S(1, \cdot)$ actually implies $\pi_0(\Omega_- \times \Omega_-) = \pi_1(\Omega_- \times \Omega_+) = 0$ for all minimizers (π_0, π_1) and not just a particular one) so that we must have $0 \in \Omega_-$. The optimality of (π_0, π_1) now implies that π_1 does not transport mass back from $\Omega_- \setminus \{0\}$ to 0. Thus, π_1 must transport the mass $\rho'_0 \llcorner (\Omega \setminus \{0, L\}) = \rho'_1 \llcorner (\Omega \setminus \{0, L\})$ from $\Omega \setminus \{0, L\} = \Omega_- \setminus \{0, L\}$ to L . The fact $\pi_1(\Omega_- \times (\Omega_+ \cup \Omega_-)) = 0$ (again due to Props. 3.26 and 3.29) now implies $L \in \Omega_-$ and thus $\Omega_- = \Omega$. Hence there is no mass change at all, however, this contradicts the differentiability of $c_S(1, \cdot)$ at 1 with vanishing derivative, since one can readily reduce the cost of pure transport by introducing a sufficiently small amount of mass change.

As a further illustration of unbalanced transport between two Dirac masses we specify the previous example to the special case in which ρ_0 and ρ_1 each only consist of a single Dirac mass and in which mass change is penalized by the so-called squared Hellinger distance

$$c_S(m_0, m_1) = (\sqrt{m_0} - \sqrt{m_1})^2.$$

Example 5.3 (Unbalanced transport between Dirac masses with squared Hellinger metric). Here we pick

$$\rho_0 = m_0^0 \delta_0, \quad \rho_1 = m_1^L \delta_L$$

as well as the squared Hellinger cost $c_S(m_0, m_1) = (\sqrt{m_0} - \sqrt{m_1})^2$. As in the previous example we can parameterize π_0 and π_1 by $(a, b) \in S = [0, m_0^0] \times [0, m_1^L]$. This can be further restricted to $S = [0, \min(m_0^0, m_1^L)]$. Indeed, if $m_1^L < a \leq m_0^0$, then $L \in \Omega_-$ which must be suboptimal by virtue of Proposition 3.26 and one argues

analogously for b . The optimality conditions (5.4) and (5.5) turn into

$$L = \frac{1}{\sqrt{\alpha}} - \frac{1}{\sqrt{\beta}}, \quad 0 = \sqrt{\alpha} - \sqrt{\beta} + \frac{1}{\sqrt{\alpha}} - \frac{1}{\sqrt{\beta}},$$

which can be solved to yield

$$\alpha = \frac{1}{S(L)} \quad \text{and} \quad \beta = S(L) \quad \text{for} \quad S(L) = \left(\frac{L}{2} + \sqrt{\frac{L^2}{4} + 1} \right)^2 > 1$$

and thus

$$a = \frac{S(L)m_1^L - m_0^0}{S(L)^2 - 1}, \quad b = \frac{S(L)m_0^0 - m_1^L}{S(L)^2 - 1}.$$

This is admissible (that is, $(a, b) \in S$) as long as

$$\frac{m_1^L}{m_0^0} \geq \frac{1}{S(L)} \quad \text{and} \quad \frac{m_1^L}{m_0^0} \leq S(L).$$

Otherwise, from the previous example we know $a = 0$ or $b = 0$ with cost

$$P(0, b) = Lb - 2\sqrt{bm_0^0} + m_0^0 + m_1^L,$$

$$P(a, 0) = La - 2\sqrt{am_1^L} + m_0^0 + m_1^L,$$

from which we obtain $b = \min(m_0^0, m_1^L, \frac{m_0^0}{L^2})$ or $a = \min(m_0^0, m_1^L, \frac{m_1^L}{L^2})$, respectively, by optimization. Comparing the respective costs, we obtain

$$a = \begin{cases} 0 & \text{if } \frac{m_1^L}{m_0^0} \leq \frac{1}{S(L)}, \\ \frac{S(L)m_1^L - m_0^0}{S(L)^2 - 1} & \text{if } \frac{1}{S(L)} < \frac{m_1^L}{m_0^0} < S(L), \\ m_0^0 & \text{if } S(L) \leq \frac{m_1^L}{m_0^0}, \end{cases} \quad b = \begin{cases} m_1^L & \text{if } \frac{m_1^L}{m_0^0} \leq \frac{1}{S(L)}, \\ \frac{S(L)m_0^0 - m_1^L}{S(L)^2 - 1} & \text{if } \frac{1}{S(L)} < \frac{m_1^L}{m_0^0} < S(L), \\ 0 & \text{if } S(L) \leq \frac{m_1^L}{m_0^0}, \end{cases}$$

which is illustrated *via* the phase diagram in Figure 7.

As can be seen from Figure 7, $L_{\min} = 0$ and $L_{\max} = \infty$ (it is straightforward to check $T_s^{l/r} = T[1 - \frac{s}{2} \mp \sqrt{s^2/4 - s}]$ with $L(s) = (T_s^r)' - (T_s^l)' = \sqrt{s^2 - 4s} \rightarrow \infty$ as $s \rightarrow -\infty$). Likewise, $L_0 = L_1 = \infty$ for the maximal transport distances from Proposition 3.31. Thus there will always occur some mass transport, no matter how far the Dirac masses are apart. This behaviour is fundamentally different from the corresponding unbalanced transport with the Wasserstein-2 transport metric (the Wasserstein–Fisher–Rao or Hellinger–Kantorovich distance [7, 12]) in which mass transport is known to cease for distances larger than π .

5.2. Examples of different static and corresponding dynamic models

In the literature there are several examples of static mass change penalties, a few of which are described in [21] and summarized in Table 1. By 4.8(3), all of these models admit a corresponding dynamic formulation. For some of them we calculate the corresponding dynamic functions h_D and c_D below (they are also provided in

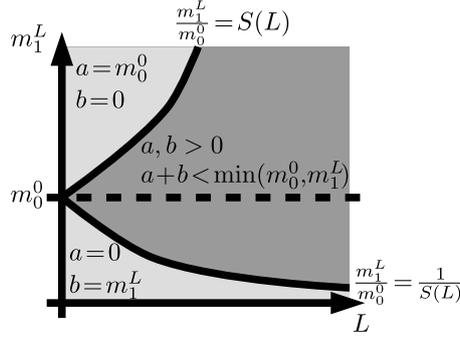


FIGURE 7. Phase diagram for the unbalanced transport of one Dirac mass onto another one from Example 5.3. In the bottom parameter region, the first Dirac mass m_0^0 is first shrunk to size m_1^L and then completely moved to the new position; in the top parameter region it is first completely moved and then grown to m_1^L . In the intermediate region, only part a of the mass is moved initially, then the first Dirac mass is shrunk while the second one is simultaneously grown, and finally the rest b of the mass is moved.

Tab. 1 for reference), for others (for which a closed formula is not straightforward to obtain) we just numerically approximate h_D and c_D using Proposition 4.8(4). The numerical approximations are displayed in Table 2.

Example 5.4 (Dynamic formulation for piecewise linear metric C_S^{pw1}). The mass change penalty of C_S^{pw1} from Table 1 has four linear segments: $c_S(1, \cdot)$ is zero at 1, and it increases to the right first with slope $-\underline{s}$ and then with slope $-\underline{d}$ while it increases to the left first with slope \bar{s} and then with slope \bar{d} . The resulting h_S is piecewise linear as well and satisfies

$$\bar{h}_S(z) = -\underline{s} + \frac{1}{a}(z + \underline{s}) \quad \text{for } z \in [-\underline{s}, -\underline{d}], \quad h_S(z) = \bar{s} + b(z - \bar{s}) \quad \text{for } z \in [\bar{s}, \bar{d}].$$

Note that $\bar{h}_S^j(z) = \frac{1}{a^j}z - (1 - \frac{1}{a^j})\underline{s}$ for $z \in [\underline{d}, \underline{s}]$ and $h_S^j(z) = b^j z + (1 - b^j)\bar{s}$ for $z \in [\bar{s}, \bar{d}]$. Proposition 4.8(4) thus implies

$$h_D(z) = \begin{cases} -\infty & \text{if } z \notin [\underline{d}, \bar{d}], \\ (\underline{s} - z) \log \frac{1}{a} & \text{if } z \in [\underline{d}, \underline{s}], \\ 0 & \text{if } z \in [\underline{s}, \bar{s}], \\ (z - \bar{s}) \log b & \text{if } z \in [\bar{s}, \bar{d}] \end{cases}$$

so that c_D can readily be computed *via* $c_D(\rho, \zeta) = \sup\{\rho h_D(z) + \zeta z \mid z \in \mathbb{R}\}$. Note that the mass change penalties $C_S^{\underline{d}}$ and C_S^{TV} can be seen as special cases with $-\underline{d} = -\underline{s} = \bar{s} = \bar{d} = \infty$ and $-\underline{d} = -\underline{s} = \bar{s} = \bar{d} = 1$, respectively.

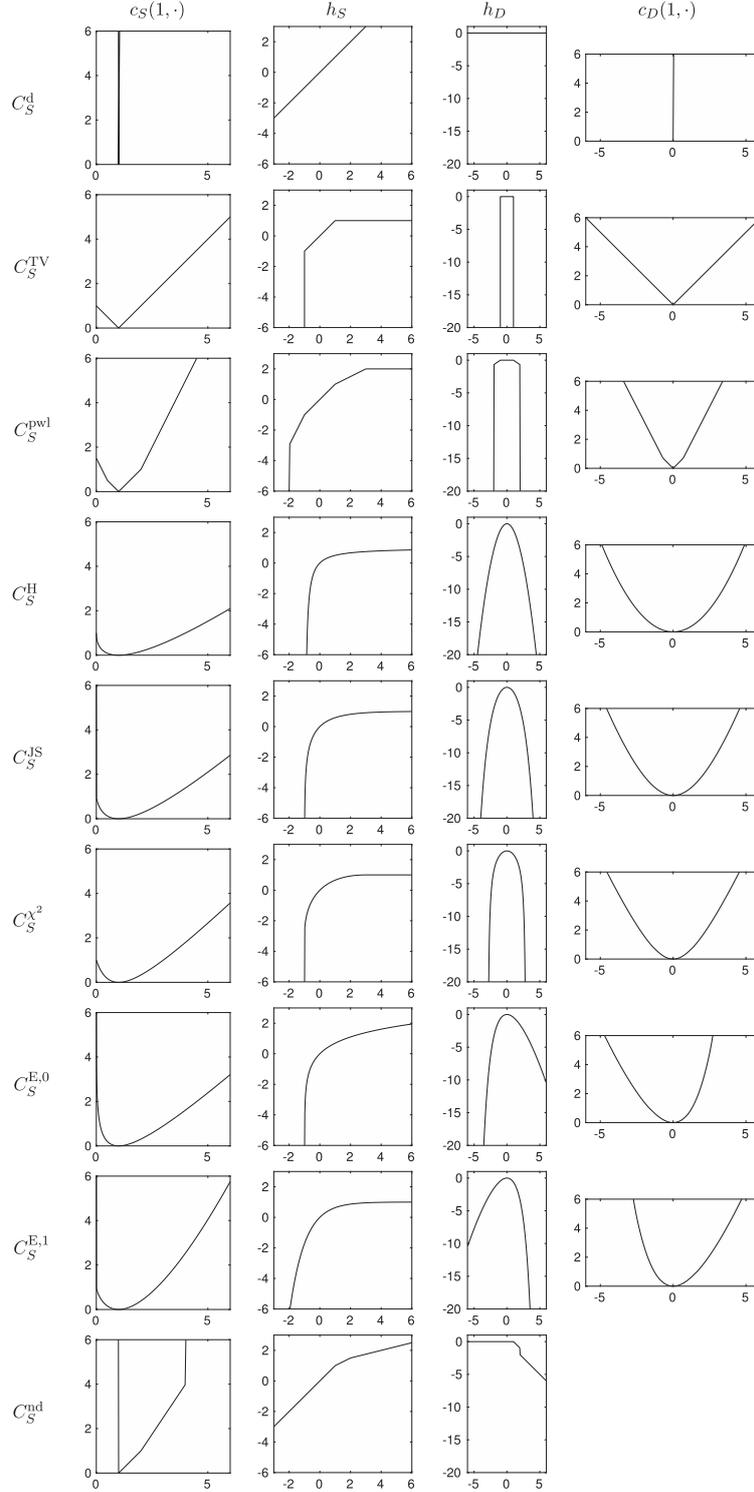
Example 5.5 (Dynamic formulation for Hellinger metric C_S^{H}). The squared Hellinger metric $c_S(m_0, m_1) = (\sqrt{m_0} - \sqrt{m_1})^2$ readily results in $h_S(z) = \bar{h}_S(z) = \frac{z}{1+z}$ for $z \in (-1, \infty)$. Note $h_S^j(z) = \bar{h}_S^j(z) = \frac{z}{1+jz}$, as can easily be shown by induction. Proposition 4.8(4) thus implies

$$h_D(z) = \lim_{j \rightarrow \infty} \frac{h_S^j(z) - h_S^{j-1}(z)}{(h_S^j)'(z)} = \lim_{j \rightarrow \infty} \frac{z/(1+jz) - z/(1+(j-1)z)}{1/(1+jz)^2} = -z^2$$

TABLE 1. Formulas of different static and dynamic model functions.

	$c_S(m_0, m_1)$ (for $m_0, m_1 \geq 0$)	$h_S(z)$	$h_D(z)$	$c_D(\rho, \zeta)$ (for $\rho \geq 0$)
C_S^d	$\begin{cases} 0 & \text{if } m_0 = m_1 \\ \infty & \text{else} \end{cases}$	z	0	$\begin{cases} 0 & \text{if } \zeta = 0 \\ \infty & \text{else} \end{cases}$
C_S^{TV}	$ m_1 - m_0 $	$\min(z, 1) - \iota_{[-1, \infty)}(z)$	$-\iota_{[-1, 1]}$	$ \zeta $
C_S^{pwl}	$\begin{cases} m_1 b(\bar{d} - \bar{s}) + \bar{s} m_1 - \underline{d} m_0 & \text{if } \frac{m_1}{m_0} \leq b, \\ (m_1 - m_0) \bar{s} & \text{if } b < \frac{m_1}{m_0} \leq 1, \\ (m_1 - m_0) \underline{s} & \text{if } 1 < \frac{m_1}{m_0} \leq a, \\ m_1 a(\underline{d} - \underline{s}) + \underline{s} m_1 - \underline{d} m_0 & \text{if } a < \frac{m_1}{m_0}. \end{cases}$	$\begin{cases} -\infty & \text{if } z < \underline{d}, \\ \underline{s} + a(z - \underline{s}) & \text{if } z \in [\underline{d}, \underline{s}], \\ z & \text{if } z \in [\underline{s}, \bar{s}], \\ \bar{s} + b(z - \bar{s}) & \text{if } z \in [\bar{s}, \bar{d}], \\ \bar{s} + b(\bar{d} - \bar{s}) & \text{else.} \end{cases}$	$\begin{cases} -\infty & \text{if } z \notin [\underline{d}, \bar{d}], \\ (z - \underline{s}) \log a & \text{if } z \in [\underline{d}, \underline{s}], \\ 0 & \text{if } z \in [\underline{s}, \bar{s}], \\ (z - \bar{s}) \log b & \text{if } z \in [\bar{s}, \bar{d}]. \end{cases}$	$\begin{cases} \underline{d}\zeta + \rho \log a(\underline{d} - \underline{s}) & \text{if } \zeta \leq \rho \log \frac{1}{a}, \\ \underline{s}\zeta & \text{if } \rho \log \frac{1}{a} \leq \zeta \leq 0, \\ \bar{s}\zeta & \text{if } 0 \leq \zeta \leq \rho \log \frac{1}{b}, \\ \bar{d}\zeta + \rho \log b(\bar{d} - \bar{s}) & \text{if } \zeta \geq \rho \log \frac{1}{b}. \end{cases}$
C_S^{H}	$(\sqrt{m_1} - \sqrt{m_0})^2$	$\frac{z}{1+z} - \iota_{(-1, \infty)}(z)$	$-z^2$	$\frac{\zeta^2}{4\rho}$
C_S^{JS}	$m_0 \log_2 \frac{2m_0}{m_0+m_1} + m_1 \log_2 \frac{2m_1}{m_0+m_1}$	$\log_2(2 - \frac{1}{z^2}) - \iota_{(-1, \infty)}(z)$	$\frac{(2^{z/2} - 2^{-z/2})^2}{-\log 2}$	$\frac{(\sqrt{g-1}/\sqrt{g})^2}{-\log 2} \rho + \zeta \log_2 g$ with $g = \frac{\zeta}{2\rho} + \sqrt{\frac{\zeta^2}{4\rho^2} + 1}$
$C_S^{\chi^2}$	$\frac{(m_1 - m_0)^2}{m_1 + m_0}$	$\begin{cases} 1 & \text{if } z > 3 \\ 1 - (2 - \sqrt{1+z})^2 & \text{if } z \in [-1, 3] \\ -\infty & \text{else} \end{cases}$		Num. approx. in Table 2
$C_S^{\text{E},0}$	$m_1 - m_0 - m_0 \log \frac{m_1}{m_0}$	$\log(1+z) - \iota_{(-1, \infty)}(z)$		Num. approx. in Table 2
$C_S^{\text{E},1}$	$m_1 \log \frac{m_1}{m_0} - m_1 + m_0$	$1 - \exp(-z)$		Num. approx. in Table 2
$C_S^{\text{E},p}$	$m_0 \frac{1}{p(p-1)} \left(\left(\frac{m_1}{m_0} \right)^p - p \left(\frac{m_1}{m_0} \right) - 1 \right) - 1$	$\frac{1 - (1 + (1-p)z)^{\frac{p-1}{p}}}{p} - \iota_{[-1, \infty)}((1-p)z)$		Num. approx. in Table 2

TABLE 2. Graphs of the functions c_S , h_S , h_D , and c_D for the dissimilarities from Table 1. The last row shows a dissimilarity C_S^{nd} for which $q[h_S]$ is not convex so that there is no corresponding dynamic model.



for $z \geq 0$ and likewise $h_D(z) = -z^2$ for $z \leq 0$. From this we can now readily compute

$$c_D(\rho, \zeta) = \sup\{\rho h_D(z) + \zeta z \mid z \in \mathbb{R}\} = \begin{cases} \infty & \text{if } \rho < 0, \\ \frac{\zeta^2}{4\rho} & \text{else.} \end{cases}$$

Example 5.6 (Dynamic formulation for Jensen–Shannon divergence C_S^{JS}). The Jensen–Shannon divergence $c_S(m_0, m_1) = m_0 \log_2 \frac{2m_0}{m_0+m_1} + m_1 \log_2 \frac{2m_1}{m_0+m_1}$ results in $h_S(z) = \bar{h}_S(z) = \log_2(2 - \frac{1}{2^z})$ on $(-1, \infty)$. Note that $h_S^j(z) = \bar{h}_S^j(z) = \log_2 \frac{(j+1)2^z - j}{j2^z - (j-1)}$. Thus Proposition 4.8(4) implies

$$h_D(z) = \lim_{j \rightarrow \infty} \frac{h_S^j(z) - h_S^{j-1}(z)}{(h_S^j)'(z)} = \frac{(2^{z/2} - 2^{-z/2})^2}{-\log 2}$$

for $z \geq 0$ and likewise $h_D(z) = \frac{(2^{z/2} - 2^{-z/2})^2}{-\log 2}$ for $z \leq 0$, from which we obtain

$$c_D(\rho, \zeta) = \sup\{\rho h_D(z) + \zeta z \mid z \in \mathbb{R}\} = \begin{cases} \infty & \text{if } \rho < 0, \\ \frac{(\sqrt{g}-1/\sqrt{g})^2}{-\log 2} \rho + \zeta \log_2 g & \text{else} \end{cases}$$

with $g = \frac{\zeta}{2\rho} + \sqrt{\frac{\zeta^2}{4\rho^2} + 1}$.

The graphs of the functions c_S , h_S , h_D and c_D for the above examples and a few further ones are provided in Table 2.

Remark 5.7 (Static model without dynamic correspondence). There are static models which are not induced by a corresponding dynamic model. Indeed, the last row of Table 2 shows a static model with

$$c_S(1, z) = \begin{cases} +\infty & \text{if } z \notin [1, 4], \\ \max\{z - 1, \frac{3}{2}z - 2\} & \text{if } z \in [1, 4], \end{cases} \quad h_S(z) = \min\{z, \frac{1}{2}z + \frac{1}{2}, \frac{1}{4}z + 1\}.$$

By Proposition 4.8(2) it does not admit a corresponding dynamic model; and indeed, the calculated $q[h_S]$ shown in Table 2 is nonconvex. Note that the non-existence of a dynamic model in this example is unrelated to the behaviour of $c_S(1, z)$ for $z \leq 1$.

As a final remark, let us mention that the cost (2.9) with C_S^{TV} is also known as the flat norm difference between two zero-dimensional currents or flat chains ρ_0, ρ_1 ([9], Sect. 4.1-2). Hence, the unbalanced transport cost with C_S^{TV} metrizes flat convergence. Since flat convergence of uniformly bounded measures coincides with weak convergence, the following proposition implies that weak convergence is equivalent to convergence of the unbalanced transport cost for all static models with dissimilarities from Table 1 except for C_S^{d} , $C_S^{\text{E},0}$, and $C_S^{\text{E},1}$.

Proposition 5.8 (Topological equivalence). *The cost W_S from (2.9) for the dissimilarities C_S^{TV} , C_S^{pw1} with $(\underline{d}, \underline{s}, a) = (-\bar{d}, -\bar{s}, \frac{1}{b})$, C_S^{H} , C_S^{JS} , or $C_S^{\text{X}^2}$ is a semimetric on the space $\{\rho \in \mathcal{M}_+(\Omega) \mid \rho(\Omega) \leq M\}$ of nonnegative measures with mass uniformly bounded by $M > 1$. Convergence in this semimetric is equivalent among all those dissimilarities.*

Proof. It is straightforward to see that the cost $W_S(\rho_0, \rho_1)$ is positive unless $\rho_0 = \rho_1$, and that $W_S(\rho_0, \rho_1) = W_S(\rho_1, \rho_0)$. Thus, W_S is a semimetric.

Consider two sequences of nonnegative measures ρ_j, σ_j , $j = 1, 2, \dots$. Since all listed dissimilarities can be bounded above by a multiple of C_S^{TV} , convergence $W_S(\rho_j, \sigma_j) \rightarrow 0$ for C_S^{TV} also implies $W_S(\rho_j, \sigma_j) \rightarrow 0$ for any

of the other dissimilarities. On the other hand, all dissimilarities can be bounded below by a multiple of the dissimilarity $C_S(\rho_0, \rho_1) = \int_{\Omega} c_S(\frac{d\rho_0}{d\gamma}, \frac{d\rho_1}{d\gamma}) d\gamma$ for

$$c_S(m_0, m_1) = \begin{cases} \infty & \text{if } m_0 < 0 \text{ or } m_1 < 0, \\ 2(m_0 - m_1) - m_1 & \text{if } \frac{m_1}{m_0} < \frac{1}{2}, \\ \frac{(m_0 - m_1)^2}{m_1} & \text{if } \frac{m_1}{m_0} \in [\frac{1}{2}, 1), \\ \frac{(m_1 - m_0)^2}{m_0} & \text{if } \frac{m_1}{m_0} \in [1, 2), \\ 2(m_1 - m_0) - m_0 & \text{if } \frac{m_1}{m_0} \geq 2. \end{cases}$$

Thus it remains to show that $W_S(\rho_j, \sigma_j) \rightarrow 0$ for C_S implies $W_S(\rho_j, \sigma_j) \rightarrow 0$ also for C_S^{TV} . For $\varepsilon \in (0, 1)$ let N be large enough such that $W_S(\rho_j, \sigma_j) < \varepsilon$ for C_S and all $j > N$. Let $\rho'_j = P_Y \pi_0$ and $\sigma'_j = P_X \pi_1$ for the optimal π_0 and π_1 in (2.9) with C_S and abbreviate $\gamma = \rho'_j + \sigma'_j$ and $\tilde{\Omega} = \{x \in \Omega \mid \frac{d\rho'_j}{d\gamma} / \frac{d\sigma'_j}{d\gamma} \in (\frac{1}{2}, 2)\}$, then

$$\begin{aligned} C_S(\rho'_j, \sigma'_j) &= \int_{\Omega} c_S(\frac{d\rho'_j}{d\gamma}, \frac{d\sigma'_j}{d\gamma}) d\gamma = \int_{\Omega \setminus \tilde{\Omega}} 2|\frac{d\rho'_j}{d\gamma} - \frac{d\sigma'_j}{d\gamma}| - \min(\frac{d\rho'_j}{d\gamma}, \frac{d\sigma'_j}{d\gamma}) d\gamma + \int_{\tilde{\Omega}} \frac{|\frac{d\rho'_j}{d\gamma} - \frac{d\sigma'_j}{d\gamma}|^2}{\min(\frac{d\rho'_j}{d\gamma}, \frac{d\sigma'_j}{d\gamma})} d\gamma \\ &\geq \int_{\Omega \setminus \tilde{\Omega}} |\frac{d\rho'_j}{d\gamma} - \frac{d\sigma'_j}{d\gamma}| d\gamma + \int_{\tilde{\Omega}} |\frac{d\rho'_j}{d\gamma} - \frac{d\sigma'_j}{d\gamma}|^2 d\gamma \geq \int_{\Omega \setminus \tilde{\Omega}} |\frac{d\rho'_j}{d\gamma} - \frac{d\sigma'_j}{d\gamma}| d\gamma + \frac{1}{\gamma(\tilde{\Omega})} \left(\int_{\tilde{\Omega}} |\frac{d\rho'_j}{d\gamma} - \frac{d\sigma'_j}{d\gamma}| d\gamma \right)^2 \\ &\geq \int_{\Omega \setminus \tilde{\Omega}} |\frac{d\rho'_j}{d\gamma} - \frac{d\sigma'_j}{d\gamma}| d\gamma + \frac{1}{2M} \left(\int_{\tilde{\Omega}} |\frac{d\rho'_j}{d\gamma} - \frac{d\sigma'_j}{d\gamma}| d\gamma \right)^2 \geq \frac{1}{4M} \left(\int_{\Omega} |\frac{d\rho'_j}{d\gamma} - \frac{d\sigma'_j}{d\gamma}| d\gamma \right)^2 = \frac{1}{4M} C_S^{\text{TV}}(\rho'_j, \sigma'_j)^2. \end{aligned}$$

Thus, the cost with C_S^{TV} satisfies $W_S(\rho_j, \sigma_j) \leq \varepsilon + \sqrt{4M\varepsilon}$ for all $j > N$. The result follows now from the arbitrariness of ε . \square

The same argument shows that weak convergence is also equivalent to convergence with respect to the cost W_S using the dissimilarities $C_S^{E,p}$ with $p \in (0, 1)$ or a general C_S^{pw1} . However, those costs are not symmetric and thus not a semimetric.

6. CONCLUSION

We considered a dynamic formulation of general unbalanced transport models with 1-homogeneous transport costs as in the Wasserstein-1 distance. We showed its equivalence to a corresponding static formulation, and we showed how the dynamic and static model parameters (the functions c_D and c_S) as well as the dynamic and static optimizers relate to each other. This is particularly relevant from an application viewpoint, since the static models have the advantage of being much lower-dimensional (in particular in their dual formulation in Prop. 2.5) and thus easier to compute.

We also answered the question which static models allow a dynamic formulation, arriving at a necessary and sufficient condition that can for instance be checked numerically. Due to our focus on W_1 -type models, this characterization becomes much simpler than a characterization of what dynamic and static unbalanced transport models correspond to each other in the general case as in Theorem 4.3 of [8]. An interesting, non-obvious result of this analysis is that the class of static models is richer than the class of dynamic models, even though one might have thought that modelling the full time dynamics allows more modelling freedom. This phenomenon is connected to the fact that the optimal dynamics are controlled by an autonomous ordinary differential equation (ode) (3.4) describing the mass change, which imposes some a priori regularity on the resulting mass change. In other settings without our particular W_1 structure, a similar situation should be expected. Allowing the mass change penalty to change over time (thereby turning the above autonomous into a nonautonomous ode) might remove this asymmetry between dynamic and static models and could be further investigated.

The analysis of which static models allow a dynamic formulation turns out to be rather decoupled from the actual transport problem. Thus the question arises whether it might have a broader applicability. Essentially it answers the question what mass change penalties (that is, convex, 1-homogeneous premetrics) are generated by geodesic paths with respect to a convex 1-homogeneous infinitesimal cost in the space of positive measures. A completely different, dynamical systems or inverse problems perspective is the following: If the static model has an equivalent dynamic formulation, then we showed that the static model function h_S is the time-one map of the autonomous scalar ode governed by the dynamic model function h_D . Of course, for any time-one map there are many possible compatible right-hand sides of the ode. Thus we essentially answered the question which monotone maps $h_S : \mathbb{R} \rightarrow \mathbb{R}$ with a single interval of fixed points are the time-one map of an autonomous ode with concave right-hand side h_D , and we provided an explicit formula to recover this h_D .

We furthermore gave a detailed characterization of the structure of dynamic optimizers. This characterization heavily exploits the W_1 -type nature of our problem, and it will be fundamentally different for other dynamic unbalanced transport models, in which typically mass change and transport occur simultaneously. In such cases no clean separation of dynamic processes into transport and mass change can be expected. This clean separation is actually advantageous from the application perspective, since it avoids that even a simple, mere transport task will be accompanied by mass loss throughout the first half of the transport and subsequent mass gain throughout the second half, just to reduce transport costs. A similar situation of clean separation between both processes is expected if not the transport costs, but rather the mass change costs are restricted to being 1-homogeneous, a case which seems worth investigating.

7. A VERSION OF THE STONE–WEIERSTRASS THEOREM

Throughout, let $\Omega \subset \mathbb{R}^n$ be the closure of an open bounded connected set, and let the metric d on Ω be induced by shortest curves inside Ω (those exist, since Ω is geodesically complete). Let $B_\delta(x) \subset \mathbb{R}^n$ denote the closed ball of radius $\delta > 0$ around $x \in \mathbb{R}^n$.

Lemma 7.1 (Approximation property of distance). *For $\delta > 0$ denote the δ -dilation of Ω by $\Omega_\delta = \{x \in \mathbb{R}^n \mid \exists y \in \Omega : \|x - y\| \leq \delta\}$, and denote the metric induced by shortest curves on Ω_δ by d_δ . For any $x, y \in \Omega$ we have $d_\delta(x, y) \rightarrow d(x, y)$ monotonically from below as $\delta \rightarrow 0$.*

Proof. Obviously, $d_\delta(x, y)$ is monotonically increasing as $\delta \rightarrow 0$ and satisfies $d_\delta(x, y) \leq d(x, y)$. Let $\gamma_\delta \subset \Omega_\delta$ be a curve connecting x and y with minimal length, that is, $\mathcal{H}^1(\gamma_\delta) = d_\delta(x, y)$. By the Blaschke Compactness theorem ([1], Thm. 4.4.15) there exists a subsequence γ_{δ_i} converging in the Hausdorff distance to a compact set γ connecting x and y . By Gołab's theorem ([1], Thm. 4.4.17), γ is connected with $\mathcal{H}^1(\gamma) \leq \liminf_{i \rightarrow \infty} \mathcal{H}^1(\gamma_{\delta_i}) = \liminf_{i \rightarrow \infty} d_{\delta_i}(x, y)$. Furthermore, $\gamma \subset \Omega$ so that $d(x, y) \leq \liminf_{i \rightarrow \infty} d_{\delta_i}(x, y)$, concluding the proof. Indeed, if $z \in \gamma \setminus \Omega$, let $D = \text{dist}(z, \Omega)$. Then for all i large enough there is some $z_i \in \gamma_{\delta_i}$ with $\|z - z_i\| \leq \frac{D}{4}$, which contradicts $\gamma_{\delta_i} \subset \Omega_{\delta_i}$ for $\delta_i < \frac{D}{4}$. \square

Lemma 7.2 (Point separation). *For any nonidentical $x, y \in \Omega$ and $a, b \in \mathbb{R}$ with $|a - b| < d(x, y)$ there is a function $f \in C^1(\Omega) \cap \text{Lip}(\Omega)$ with $f(x) = a$ and $f(y) = b$.*

Proof. Without loss of generality $b > a$. For $\delta > 0$ define $g_\delta : \Omega_\delta \rightarrow \mathbb{R}$, $g_\delta(z) = d_\delta(x, z)$, and extend g_δ by zero outside Ω_δ . Then convolve g_δ with a smooth mollifier of support in $B_{\frac{\delta}{2}}(0)$ to obtain $h_\delta \in C^1(\mathbb{R}^n)$. The value of ∇h_δ at each point $z \in \Omega$ is the weighted mean of ∇g_δ over $B_{\frac{\delta}{2}}(z)$, which lies completely inside Ω_δ so that $g_\delta \in \text{Lip}(B_{\frac{\delta}{2}}(z))$ and thus $\|\nabla h_\delta(z)\| \leq 1$ (since g_δ satisfied the same condition). Thus $h_\delta|_\Omega \in C^1(\Omega) \cap \text{Lip}(\Omega)$. Finally, define $f_\delta : \Omega \rightarrow \mathbb{R}$, $f_\delta(z) = a + (h_\delta(z) - h_\delta(x)) \frac{b-a}{h_\delta(y) - h_\delta(x)}$. By definition, $f_\delta(x) = a$ and $f_\delta(y) = b$. Furthermore, as $\delta \rightarrow 0$, $h_\delta \rightarrow g_\delta$ uniformly on Ω and $g_\delta(y) \rightarrow d(x, y)$ by the previous lemma. Thus, $\frac{b-a}{h_\delta(y) - h_\delta(x)} \leq 1$ for small enough δ so that $f_\delta \in C^1(\Omega) \cap \text{Lip}(\Omega)$. \square

Lemma 7.3 (Smoothed Heaviside function). *For any $\varepsilon, \delta > 0$ there exists a function $H_{\varepsilon, \delta} \in C^\infty(\mathbb{R})$ with $|xH'_{\varepsilon, \delta}(x)| \leq \delta$ for all $x \in \mathbb{R}$ and $H_{\varepsilon, \delta}(x) = 0$ for all $x \leq -\varepsilon$ and $H_{\varepsilon, \delta}(x) = 1$ for all $x \geq \varepsilon$.*

Proof. Since we can always mollify $H_{\varepsilon, \delta}$, it suffices to show the existence of a piecewise differentiable $H_{\varepsilon, \delta}$ satisfying the above conditions. Set $\eta = \varepsilon / (\exp(\frac{1}{2\delta}) - 1)$ and define

$$H_{\varepsilon, \delta}(x) = \begin{cases} 0 & \text{if } x \leq -\varepsilon, \\ \frac{1}{2} - \delta \log \frac{|x| + \eta}{\eta} & \text{if } x \in (-\varepsilon, 0], \\ \frac{1}{2} + \delta \log \frac{|x| + \eta}{\eta} & \text{if } x \in (0, \varepsilon], \\ 1 & \text{if } x > \varepsilon. \end{cases}$$

It is straightforward to check that $H_{\varepsilon, \delta}$ is continuous and satisfies the required conditions almost everywhere. \square

Lemma 7.4 (Smooth min and max operation). *For $f, g \in C^1(\Omega) \cap \text{Lip}(\Omega)$ and $\delta > 0$ define*

$$\begin{aligned} f \wedge_\delta g &= g H_{\delta, \delta}(f - g) + f(1 - H_{\delta, \delta}(f - g)), \\ f \vee_\delta g &= f H_{\delta, \delta}(f - g) + g(1 - H_{\delta, \delta}(f - g)). \end{aligned}$$

Then $f \wedge_\delta g, f \vee_\delta g \in C^1(\Omega) \cap (1 + 2\delta) \text{Lip}(\Omega)$, and

$$\begin{aligned} \min(f, g) - \delta &\leq f \wedge_\delta g \leq \min(f, g) + \delta, \\ \max(f, g) - \delta &\leq f \vee_\delta g \leq \max(f, g) + \delta. \end{aligned}$$

Proof. The continuous differentiability follows from the continuous differentiability of f , g , and $H_{\delta, \delta}$. Furthermore,

$$\begin{aligned} \|\nabla(f \wedge_\delta g)\| &= \|\nabla g H_{\delta, \delta}(f - g) + \nabla f(1 - H_{\delta, \delta}(f - g)) + (g - f)H'_{\delta, \delta}(f - g)\nabla(f - g)\| \\ &\leq \|\nabla g H_{\delta, \delta}(f - g) + \nabla f(1 - H_{\delta, \delta}(f - g))\| + |(g - f)H'_{\delta, \delta}(f - g)|(\|\nabla f\| + \|\nabla g\|) \leq 1 + \delta(1 + 1). \end{aligned}$$

Analogously one shows $f \vee_\delta g \in (1 + 2\delta) \text{Lip}(\Omega)$. Finally, if $|f - g| \geq \delta$, then $\min(f, g) = f \wedge_\delta g$ and $\max(f, g) = f \vee_\delta g$, while otherwise we obtain a convex combination of f and g , which implies the rest of the statement. \square

The following is a simple variant of the classical proof by Stone.

Proposition 7.5 (Stone–Weierstraß). *$C^1(\Omega) \cap \text{Lip}(\Omega)$ is dense in $\text{Lip}(\Omega)$ with respect to the supremum norm.*

Proof. Given $g \in \text{Lip}(\Omega)$ and $\varepsilon > 0$ we seek some $f \in C^1(\Omega) \cap \text{Lip}(\Omega)$ with $\|f - g\| \leq \varepsilon$. Actually, since $\lambda g \rightarrow g$ uniformly as $\lambda \rightarrow 1$, it suffices to consider $g \in \lambda \text{Lip}(\Omega)$ for an arbitrary $\lambda \in (0, 1)$.

For given $x \in \Omega$, by Lemma 7.2 there exists for each $y \in \Omega$ a function $f_y \in C^1(\Omega) \cap \text{Lip}(\Omega)$ with $f_y(x) = g(x)$ and $f_y(y) = g(y)$. Introduce the open set $V_y = \{z \in \Omega \mid f_y(z) < g(z) + \frac{\varepsilon}{4}\}$. Since the family V_y for $y \in \Omega$ forms an open cover of Ω (note that $x, y \in V_y$) and Ω is compact, we can extract a finite subcover V_{y_1}, \dots, V_{y_k} . For $\delta > 0$ define

$$\tilde{F}_x = (\dots((f_{y_1} \wedge_\delta f_{y_2}) \wedge_\delta f_{y_3}) \dots \wedge_\delta f_{y_k}).$$

By the previous lemma, $\tilde{F}_x \in C^1(\Omega) \cap (1 + 2\delta)^k \text{Lip}(\Omega)$ with $|\tilde{F}_x - \min(f_{y_1}, \dots, f_{y_k})| \leq k\delta$. Thus, by choosing δ small enough (depending on x) we have $F_x = \frac{1}{(1+2\delta)^k} \tilde{F}_x \in C^1(\Omega) \cap \text{Lip}(\Omega)$ with $F_x \leq g + \frac{\varepsilon}{2}$.

Now introduce the open set $W_x = \{z \in \Omega \mid F_x(z) > g(z) - \frac{\varepsilon}{4}\}$, which contains x . Again the W_x form an open cover of Ω from which we can extract a subcover W_{x_1}, \dots, W_{x_l} , and analogously to before we set

$$\tilde{f} = (\dots((F_{x_1} \vee_\delta F_{x_2}) \vee_\delta F_{x_3}) \dots \vee_\delta F_{x_l}).$$

Again, by choosing δ small enough we have $f = \frac{1}{(1+2\delta)^l} \tilde{f} \in C^1(\Omega) \cap \text{Lip}(\Omega)$ with $g - \varepsilon \leq f \leq g + \varepsilon$. \square

Remark 7.6 (Higher smoothness). One may replace C^1 by C^∞ .

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