EXISTENCE FOR DISLOCATION-FREE FINITE PLASTICITY

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Abstract. This note addresses finite plasticity under the constraint that plastic deformations are compatible. In this case, the total elastoplastic deformation of the medium is decomposed as $y = y_e \circ y_p$, where the plastic deformation y_p is defined on the fixed reference configuration and the elastic deformation y_e is a mapping from the varying intermediate configuration $y_p(\Omega)$. Correspondingly, the energy of the medium features both Lagrangian (plastic, loads) and not Lagrangian contributions (elastic).

We present a variational formulation of the static elastoplastic problem in this setting and show that a solution is attained in a suitable class of admissible deformations. Possible extensions of the result, especially in the direction of quasistatic evolutions, are also discussed.

1. Introduction

The elastoplastic response of a solid is classically described in terms of its deformation $y : \Omega \to \mathbb{R}^3$ from a reference configuration $\Omega \subset \mathbb{R}^3$ [47]. This deformation results from a combination of reversible elastic and irreversible plastic effects. As the elastic behavior is essentially independent from prior plastic distorsion, the two effects are usually separated [32]. In the finite-deformation regime such separation can be represented by the *multiplicative decomposition* $\mathrm{D}y = F_e F_p$ [36, 40] where the elastic strain F_e describes the elastic response of the medium and the plastic strain F_p encodes plastic distorsion. Although many aspects of finite plasticity are still debated [55] and other decompositions have been set forth [12, 18, 42, 57], this multiplicative decomposition has turned to be the reference in finite plasticity and has recently obtained a new justification based on purely kinematic micromechanical considerations [60, 61].

A tenet of finite plasticity is that the tensor field F_p need not satisfy compatibility conditions, in the sense that it does not necessarily arise as the gradient of a continuous deformation. In case of a simply connected reference configuration such incompatibility can be expressed as $\mathrm{curl}\, F_p \neq 0$, see (2.2). In this case the plastic strain can accommodate the description of *dislocations*, which are crystal defects playing a relevant role in metal plasticity [43]. The tensor $\mathrm{curl}\, F_p$ represents the macroscopic density of geometrically necessary dislocations [9].

We focus here on *dislocation-free* plasticity instead, namely we assume that the plastic strain F_p indeed satisfies compatibility conditions. This is for instance the case of slips along slip-planes, one of the basic plastic mechanisms in crystal plasticity. More precisely, the superposition of infinitely many plastic slips may give rise

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to a continuous plastic strain [60] satisfying compatibility conditions. Such a superposition can be interpreted as a composition of differently oriented plastic shears, each reminiscent of the shearing of a deck of cards.

Dislocation dynamics is strongly size dependent [29, 71] and very small dislocation-free bodies may plasticize without nucleating dislocations. Such compatible deformations can be experimentally realized in ductile metals [34, 46] and are believed to have potential technological impact [67]. In fact, the understanding of plasticity at small scales is attracting increasing attention, for dimensions of applications progressively scale down [66].

The aim of this note is to present a mathematical analysis of finite plasticity under the constraint that F_{p} satisfies compatibility conditions. In this case, F_{e} satisfies compatibility conditions as well, see Section 2.1, and one can interpret F_{e} and F_{p} as gradients of an elastic deformation y_{e} and a plastic deformation y_{p}, respectively. The deformation y is hence expressed via the composition

$$y = y_{\mathrm{e}} \circ y_{\mathrm{p}} \tag{1.1}$$

and the multiplicative decomposition $\mathrm{D}y = F_{\mathrm{e}}F_{\mathrm{p}} = \mathrm{D}y_{\mathrm{e}} \circ y_{\mathrm{p}} \, \mathrm{D}y_{\mathrm{p}}$ follows from (1.1) by the chain rule. Note that the plastic deformation y_{p} maps the *reference* configuration $\Omega \subset \mathbb{R}^d$ to the so-called *intermediate* configuration $y_{\mathrm{p}}(\Omega)$ and the elastic deformation y_{e} maps $y_{\mathrm{p}}(\Omega)$ into the *actual* configuration $y(\Omega)$. This makes the analysis delicate, for the intermediate configuration is a priori unknown.

Our goal it to provide a sound variational frame for the composition assumption (1.1) and to prove an existence result. Elastoplastic evolution is driven by a combination of energy storage and dissipation mechanisms. In addition to elastic and plastic deformations, the state of the material is described by the scalar isotropic hardening parameter z, as well. The *static elastoplastic problem* consists in finding the actual elastoplastic state $(y_{\mathrm{e}}, y_{\mathrm{p}}, z)$ by minimizing

$$\int_{y_{\mathrm{p}}(\Omega)} W_{\mathrm{e}}(\mathrm{D}y_{\mathrm{e}}) \, \mathrm{d}\xi + \int_{\Omega} W_{\mathrm{h}}(z) \, \mathrm{d}x + \int_{\Omega} D(\mathrm{D}y_{\mathrm{p}}\mathrm{D}y_{\mathrm{p}0}^{-1}, z - z_0) \, \mathrm{d}x - \int_{\Omega} f \cdot y_{\mathrm{e}} \circ y_{\mathrm{p}} \, \mathrm{d}x$$

on admissible states. This functional results from the sum of the elastic energy, the hardening potential, the plastic dissipation from the previous plastic state $(y_{\mathrm{p}0}, z_0)$, and a loading, respectively. The elastic energy is defined on the intermediate configuration whereas all other terms are Lagrangian, namely defined on the reference configuration.

Theorem 2.1 states that such a minimization problem admits a solution whenever W_{e} and $P \mapsto \min_z(W_{\mathrm{h}}(z) + D(P, z - z_0))$ are polyconvex and control $|F_{\mathrm{e}}|^3$ and $|P|^3$, respectively. With respect to formulations based on the multiplicative decomposition $\mathrm{D}y = F_{\mathrm{e}}F_{\mathrm{p}}$, assumption (1.1) shows the distinctive advantage of not having to deal with the product $\mathrm{D}y F_{\mathrm{p}}^{-1}$ in the elastic energy. This allows for an existence theory under weaker assumptions with respect to those classically considered in the multiplicative setting. Note that existence results in the context of finite plasticity are just a few. The static problem has been firstly addressed in [50], where nonetheless strong restrictions on the choice of the driving functionals are considered. By augmenting the energy by an integral term of the form $|\mathrm{curl}\, F_{\mathrm{p}} \, F_{\mathrm{p}}^{\top}|$, Mielke and Müller are able to treat more general energies in [51]. The present setting corresponds to that of [51] under the additional constraint $\mathrm{curl}\, F_{\mathrm{p}} = 0$ and weaker coercivity assumptions, see Section 2.5.

Variational methods have proved useful in various evolutionary contexts in mechanics [7, 19, 30, 31]. Among these, dissipative, inelastic evolution of ductile crystals in presence of microstructures has been variationally formulated in terms of an *incremental* minimization problem in Ortiz and Repetto [58], and subsequently investigated in [1, 8, 13, 24, 35, 38, 56, 59] among others. The existence result of Theorem 2.1 can be used in order to implement an incremental minimization scheme, see Section 4.7. Under the effect of time-dependent forcing one finds a time-discrete variational solution $t^i \mapsto (y_{\mathrm{e}}^i, y_{\mathrm{p}}^i, z^i)$ by iteratively minimizing the above functional where the previous plastic state $(y_{\mathrm{p}}^{i-i}, z^{i-1})$ takes the role of $(y_{\mathrm{p}0}, z_0)$.

The quasistatic evolution problem has been proved to admit an *energetic solution* [52] by Mainik and Mielke [45] in case the energy controls the full gradient $|\mathrm{D}F_{\mathrm{p}}|^2$. A similar result for a symmetric formulation in terms of $F_{\mathrm{p}}^{\top} F_{\mathrm{p}}$ is in [27, 28].

The composition assumption (1.1) asks for some specification on the class of admissible plastic deformations. The intermediate configuration $y_{\mathrm{p}}(\Omega)$ need to be properly defined (see Sects. 2.2 and 4.2) and we assume the plastic deformation to conserve volumes by requiring that $\det \mathrm{D}y_{\mathrm{p}} = 1$. As the analysis hinges on the possibility of changing variables in the integrals between the reference and the intermediate configuration via y_{p}, some essential injectivity of y_{p} is instrumental and will be ensured by imposing the classical Ciarlet-Nečas condition [11].

Before moving on let us remark that the idea of considering deformations on a varying domain is not new and has to be traced back at least to Fonseca and Parry [21] and Dacorogna and Fonseca [15]. Moving from a model for slightly defective crystals [16, 17], an existence theory is there presented in terms of the variables (y, y_{p}). More recently, deformations on varying domains have been considered in the frame of the modeling of *nematic elastomers* by Barchiesi and DeSimone [4] and Barchiesi *et al.* [5]. Also *magnetoelasticity* features a mixture of Lagrangian (elastic) and Eulerian (magnetic) energy terms. This again calls for variational problems in varying domain. The Reader is refereed to Rybka and Luskin [65] and Barchiesi *et al.* [5] for results in the static case and Roubíček and Tomassetti [64] and [39] for dissipative dynamics.

2. Existence

This section collects preliminaries as well as the statement of our main existence result, Theorem 2.1. The functional setting for deformations is presented in Section 2.2 and energy and dissipation densities are introduced in Sections 2.3–2.4. The statement of Theorem 2.1 is in Section 2.5 and a differential formulation of the problem is given in Section 2.6.

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be a nonempty, bounded, simply connected, and open reference configuration with the Lipschitz boundary $\partial\Omega$ and let $\Gamma_{\mathrm{Dir}}, \Gamma_{\mathrm{tr}} \subset \partial\Omega$ be disjoint and open with respect to the topology of $\partial\Omega$ with $\overline{\Gamma}_{\mathrm{Dir}} \cup \overline{\Gamma}_{\mathrm{tr}} = \partial\Omega$ (closures in $\partial\Omega$). Moreover, let Γ_{Dir} have positive surface measure. We use a general notation to treat the two cases $d = 2$ and $d = 3$ simultaneously. The less physical case $d > 3$ can be treated as well, up to minor notational changes. Spaces of vector valued functions, *e.g.* $L^p(\Omega, \mathbb{R}^d)$, are indicated by exponents, *e.g.* $L^p(\Omega)^d$. We denote by $|E|$ the d-dimensional Lebesgue measure of the measurable set $E \subset \mathbb{R}^d$ and let $\chi_E : \mathbb{R}^d \to \{0; 1\}$ be its characteristic function.

2.1. Tensors and compatibility conditions

We will use the matrix groups

$$
\begin{aligned}
\mathrm{SL}(d) &:= \{M \in \mathbb{R}^{d \times d} \,:\, \det M = 1\}, \\
\mathrm{SO}(d) &:= \{M \in \mathrm{SL}(d) \,:\, M^{-1} = M^\top\}, \\
\mathrm{GL}(d) &:= \{M \in \mathbb{R}^{d \times d} \,:\, \det M \neq 0\}, \\
\mathrm{GL}_+(d) &:= \{M \in \mathbb{R}^{d \times d} \,:\, \det M > 0\}.
\end{aligned}
$$

The superscript $\top$ denotes transposition, I is the identity 2-tensor, $\operatorname{tr} M = M_{ii}$ (summation over repeated indices) is the trace, $|M|^2 = \operatorname{tr}(M^\top M) = M_{ij}M_{ij}$ defines the norm. For all $M \in \mathrm{GL}(d)$ recall that the cofactor matrix is defined as $\operatorname{cof} M = (\det M)M^{-\top}$. Given the differentiable tensor-valued function $M : \Omega \to \mathbb{R}^{d \times d}$ one defines $\operatorname{curl} M$ as

$$(d = 2) \quad \operatorname{curl} M \in \mathbb{R}^2, \quad (\operatorname{curl} M)_i = \epsilon_{3kj}M_{ij,k} \tag{2.1}$$

$$(d = 3) \quad \operatorname{curl} M \in \mathbb{R}^{3 \times 3}, \quad (\operatorname{curl} M)_{ij} = \epsilon_{j\ell k}M_{ik,\ell} \tag{2.2}$$

where ϵ_{ijk} is the Levi-Civita symbol.

Assume now that the tensor-valued function $x \in \Omega \mapsto F_{\mathrm{p}}(x) \in \mathrm{GL}(d)$ satisfies compatibility conditions, namely $F_{\mathrm{p}} = \mathrm{D}y_{\mathrm{p}}$ for some $y_{\mathrm{p}} : \Omega \to \mathbb{R}^d$. Moreover, let $\xi \in y_{\mathrm{p}}(\Omega) \mapsto F_{\mathrm{e}}(\xi)$ be given and define $F(x) :=$

$F_\mathrm{e}(y_\mathrm{p}(x))F_\mathrm{p}(x)$. By assuming sufficient smoothness and following [60], Appendix A one can check that

$$(d = 2) \quad \mathrm{curl}\,_x F = (\mathrm{curl}\,_\xi F_\mathrm{e}) \circ y_\mathrm{p} \det F_\mathrm{p}, \tag{2.3}$$

$$(d = 3) \quad \mathrm{curl}\,_x F = (\mathrm{curl}\,_\xi F_\mathrm{e}) \circ y_\mathrm{p} \mathrm{cof}\, F_\mathrm{p}. \tag{2.4}$$

Indeed, for $d = 2$ one has that

$$
\begin{aligned}
(\mathrm{curl}\,_x F)_i &= \epsilon_{3kj}(F_\mathrm{e}(y_\mathrm{p})F_\mathrm{p})_{ij,k} = \epsilon_{3kj}(F_\mathrm{e}(y_\mathrm{p})_{i\ell}(F_\mathrm{p})_{\ell j}),_k \\
&= \epsilon_{3kj}F_\mathrm{e}(y_\mathrm{p})_{i\ell,n}(F_\mathrm{p})_{nk}(F_\mathrm{p})_{\ell j} + \epsilon_{3kj}(F_\mathrm{e}(y_\mathrm{p}))_{i\ell}(F_\mathrm{p})_{\ell j,k} \\
&= \epsilon_{3n\ell}F_\mathrm{e}(y_\mathrm{p})_{i\ell,n} \det F_\mathrm{p} + (F_\mathrm{e}(y_\mathrm{p})\,\mathrm{curl}\,_x F_\mathrm{p})_i \\
&= ((\mathrm{curl}\,_\xi F_\mathrm{e}) \circ y_\mathrm{p})_i \det F_\mathrm{p}.
\end{aligned}
$$

On the other hand, for $d = 3$ we can compute

$$
\begin{aligned}
(\mathrm{curl}\,_x F)_{ij} &= \epsilon_{j\ell k}(F_\mathrm{e}(y_\mathrm{p})F_\mathrm{p})_{ik,\ell} = \epsilon_{j\ell k}(F_\mathrm{e}(y_\mathrm{p})_{in}(F_\mathrm{p})_{nk}),_\ell \\
&= \epsilon_{j\ell k}F_\mathrm{e}(y_\mathrm{p})_{in,q}(F_\mathrm{p})_{q\ell}(F_\mathrm{p})_{nk} + \epsilon_{j\ell k}F_\mathrm{e}(y_\mathrm{p})_{in}(F_\mathrm{p})_{nk,\ell} \\
&= \epsilon_{s\ell k}F_\mathrm{e}(y_\mathrm{p})_{in,q}(F_\mathrm{p}^{-1})_{jm}(F_\mathrm{p})_{ms}(F_\mathrm{p})_{q\ell}(F_\mathrm{p})_{nk} + (F_\mathrm{e}(y_\mathrm{p})\,\mathrm{curl}\,_x F_\mathrm{p})_{ij} \\
&= \det\,(F_\mathrm{p})\epsilon_{mqn}F_\mathrm{e}(y_\mathrm{p})_{in,q}(F_\mathrm{p}^{-\top})_{mj} \\
&= ((\mathrm{curl}\,_\xi F_\mathrm{e}) \circ y_\mathrm{p})_{im}(\mathrm{cof}\, F_\mathrm{p})_{mj} = ((\mathrm{curl}\,_\xi F_\mathrm{e}) \circ y_\mathrm{p} \mathrm{cof}\, F_\mathrm{p})_{ij}.
\end{aligned}
$$

Moving from (2.3) and (2.4), one readily checks that if $F = \mathrm{D}y$, then $\xi \mapsto F_\mathrm{e}(\xi)$ necessarily satisfies compatibility conditions as well. In other words, from the multiplicative decomposition $F = F_\mathrm{e}F_\mathrm{p}$ one obtains decomposition (1.1) by assuming that both F and F_p satisfy compatibility conditions.

2.2. Deformations

We start introducing the set of admissible deformations. Here we basically follow Fonseca and Gangbo [20], see also Barchiesi and DeSimone [4] for an application of this same frame to a variational model for nematic elastomers. Section 4 presents some alternative settings.

The plastic deformation $y_\mathrm{p} : \Omega \to \mathbb{R}^d$ is assumed to belong to $W^{1,d}(\Omega)^d$ and to be such that $\det \mathrm{D}y_\mathrm{p} = 1$ a.e. This implies that y_p is continuous and almost everywhere differentiable [20], Lemmas 2.1 and 2.7. We hence identify y_p with its continuous representative, so that the image $y_\mathrm{p}(\Omega)$ is well defined. The map y_p possesses the so-called Lusin's N-property ($|E| = 0 \Rightarrow |y_\mathrm{p}(E)| = 0$) as well as the corresponding N^{-1}-property ($|E| = 0 \Rightarrow |y_\mathrm{p}^{-1}(E)| = 0$, where y_p^{-1} indicates here the preimage) [26], page 296. In addition, y_p is *locally invertible almost everywhere*: For a.e. $x_0 \in \Omega$ there exists a ball $B \subset \mathbb{R}^d$ centered in $y_\mathrm{p}(x_0)$, an open neighborhood $U \subset \Omega$ of x_0, and a local inverse $y_\mathrm{p}^{-1} : B \to U$ with $y_\mathrm{p}^{-1} \in W^{1,d/(d-1)}(B)^d$ such that $y_\mathrm{p}|_U$ and y_p^{-1} are onto, $y_\mathrm{p}^{-1} \circ y_\mathrm{p} = \mathrm{id}$ a.e. on U, $y_\mathrm{p} \circ y_\mathrm{p}^{-1} = \mathrm{id}$ a.e. on B, and $\mathrm{D}y_\mathrm{p}^{-1} = (\mathrm{D}y_\mathrm{p})^{-1} \circ y_\mathrm{p}^{-1}$ a.e. on B [20], Theorem 3.1, Corollary 3.3. By letting $D = \{x \in \Omega\,:\,y_\mathrm{p}$ is differentiable in $x\}$, the a.e. local-invertibility property entails that $y_\mathrm{p}(D) \subset y_\mathrm{p}(\Omega)^\circ$ where the latter stands for the interior of $y_\mathrm{p}(\Omega)$. Hence, the N-property ensures that

$$|y_\mathrm{p}(\Omega) \setminus y_\mathrm{p}(\Omega)^\circ| \le |y_\mathrm{p}(\Omega) \setminus y_\mathrm{p}(D)| \le |y_\mathrm{p}(\Omega \setminus D)| = 0$$

since $y_\mathrm{p}(\Omega) \setminus y_\mathrm{p}(D) \subset y_\mathrm{p}(\Omega \setminus D)$ and $|\Omega \setminus D| = 0$. In particular, $y_\mathrm{p}(\Omega)$ is measurable and $|y_\mathrm{p}(\Omega)| = |y_\mathrm{p}(\Omega)^\circ|$. Note that $y_\mathrm{p}(\Omega)$ need not be open, as shown in [3], Example 1.

As we aim at composing the elastic deformation y_e to the plastic one y_p, we additionally require the plastic deformation y_p to be a.e. injective, *i.e.*, injective on $\Omega \setminus N$ with $|N| = 0$. This is equivalent [25], Proposition 2.5

to the classical *Ciarlet-Nečas* condition [11]

$$|\Omega| = \int_\Omega \det \mathrm{D}y_\mathrm{p} \, \mathrm{d}x \le |y_\mathrm{p}(\Omega)|.$$

Note that, under such a.e. injectivity assumption, the change of variables formula

$$\int_E m(y_\mathrm{p}(x)) \, \mathrm{d}x = \int_{y_\mathrm{p}(E)} m(\xi) \, \mathrm{d}\xi \tag{2.5}$$

holds for all measurable functions $m : \mathbb{R}^d \to \mathbb{R}$ and all measurable $E \subset \Omega$ [20], Lemma 2.4.

Given the plastic deformation y_p, we assume that the elastic deformation y_e belongs to $W^{1,d}(y_\mathrm{p}(\Omega)^\circ)^d$ and has $\det \mathrm{D}y_\mathrm{e} > 0$ a.e. In particular, y_e is continuous [20], Lemma 2.1 and it is hence identified with its continuous representative. The total deformation $y = y_\mathrm{e} \circ y_\mathrm{p}$ is then continuous as well. Moreover, one can check the chain rule

$$\mathrm{D}y(x) = \mathrm{D}y_\mathrm{e}(y_\mathrm{p}(x)) \, \mathrm{D}y_\mathrm{p}(x) \quad \text{for a.e. } x \in \Omega. \tag{2.6}$$

Note that such chain rule is not trivial out of the smooth setting as some control on the invertibility of y_p is required [54]. We will use again the local invertibility of y_p on B to check for (2.6) locally on $y_\mathrm{p}^{-1}(B) = U$. To this aim, fix the component $i = 1, \dots, d$, let $\varphi \in C_\mathrm{c}^\infty(U)$ be given, and compute

$$\begin{aligned}
\int_U (y_\mathrm{e}(y_\mathrm{p}(x)))_i \frac{\partial \varphi}{\partial x_j}(x) \, \mathrm{d}x &= \int_B (y_\mathrm{e}(\xi))_i \frac{\partial \varphi}{\partial x_j}(y_\mathrm{p}^{-1}(\xi)) \, \mathrm{d}\xi \\
&= -\int_B \frac{\partial (y_\mathrm{e})_i}{\partial \xi_\ell}(\xi) \frac{\partial (y_\mathrm{p})_\ell}{\partial x_j}(y_\mathrm{p}^{-1}(\xi)) \, \varphi(y_\mathrm{p}^{-1}(\xi)) \, \mathrm{d}\xi \\
&= -\int_U (\mathrm{D}y_\mathrm{e}(y_\mathrm{p}(x)) \mathrm{D}y_\mathrm{p}(x))_{ij} \, \varphi(x) \, \mathrm{d}x.
\end{aligned}$$

This proves (2.6) in U. The relation in Ω follows from the fact that a.e. $x \in \Omega$ belongs to one such a local-invertibility neighborhoods.

As both $\mathrm{D}y_\mathrm{e}$ and $\mathrm{D}y_\mathrm{p}$ belong to L^d, relation (2.6) entails that $\mathrm{D}y$ is in $L^{d/2}$, namely

$$\int_\Omega |\mathrm{D}y(x)|^{d/2} \mathrm{d}x \le \left(\int_{y_\mathrm{p}(\Omega)} |\mathrm{D}y_\mathrm{e}(\xi)|^d \mathrm{d}\xi \right)^{1/2} \left(\int_\Omega |\mathrm{D}y_\mathrm{p}(x)|^d \mathrm{d}x \right)^{1/2}.$$

By assuming the body to be clamped along Γ_Dir, namely $y = y_\mathrm{e} \circ y_\mathrm{p} = \mathrm{id}$ on Γ_Dir, the Poincaré inequality and the latter integral control ensure that $y \in W^{1,d/2}(\Omega)^d$. More precisely, there exists a constant $c_\mathrm{P} > 0$ such that

$$\|y\|_{W^{1,d/2}(\Omega)^d} \le c_\mathrm{P} \|\mathrm{D}y\|_{L^{d/2}(\Omega)^{d \times d}} \le c_\mathrm{P} \|\mathrm{D}y_\mathrm{e}\|_{L^d(y_\mathrm{p}(\Omega)^\circ)^{d \times d}} \|\mathrm{D}y_\mathrm{p}\|_{L^d(\Omega)^{d \times d}}. \tag{2.7}$$

Note that a nonhomogeneous Dirichlet boundary condition on y can be considered as well, see Section 4.6.

2.3. Energy

We assume the elastic energy density $W_{\mathrm{e}} : \mathrm{GL}(d) \to [0, \infty]$ to be finite just on $\mathrm{GL}_+(d)$ and polyconvex [2], namely

$$W_{\mathrm{e}}(F_{\mathrm{e}}) = \widehat{W}_{\mathrm{e}}(F_{\mathrm{e}}, \mathrm{cof}\, F_{\mathrm{e}}, \det F_{\mathrm{e}}), \tag{2.8}$$

for $F_{\mathrm{e}} \in \mathbb{R}^{d \times d}$, where $\widehat{W}_{\mathrm{e}} : \mathbb{R}^{d \times d} \times \mathbb{R}^{d \times d} \times (0, \infty) \to [0, \infty)$ is convex and $\widehat{W}_{\mathrm{e}} = \infty$ iff $\det F_{\mathrm{e}} \leq 0$. Note that the notation corresponds to the case $d = 3$, for definiteness. If $d = 2$ no dependence on $\mathrm{cof}\, F_{\mathrm{e}}$ is actually needed and for $d > 3$ additional dependences on other minors can be considered as well. We will ask some coercivity of W_{e} in the form

$$W_{\mathrm{e}}(F_{\mathrm{e}}) \geq c|F_{\mathrm{e}}|^d + c(\det F_{\mathrm{e}})^r - \frac{1}{c} \tag{2.9}$$

for all $F_{\mathrm{e}} \in \mathrm{GL}_+(d)$ and some $c > 0$ and $r > 1$. Additionally, we prescribe the isotropic hardening potential $W_{\mathrm{h}} : (0, \infty) \to (0, \infty)$ to be monotone increasing. The stored energy of the medium is obtained by integration as

$$E(y_{\mathrm{p}}, y_{\mathrm{e}}, z) = \int_{y_{\mathrm{p}}(\Omega)} W_{\mathrm{e}}(\mathrm{D}y_{\mathrm{e}}(\xi)) \, \mathrm{d}\xi + \int_{\Omega} W_{\mathrm{h}}(z(x)) \, \mathrm{d}x.$$

Note that E is indifferent to superimposed translations of the intermediate configuration: By letting $T : y_{\mathrm{p}} \mapsto y_{\mathrm{p}} + v$ with $v \in \mathbb{R}^d$ we have that

$$E(y_{\mathrm{e}} \circ T^{-1}, T \circ y_{\mathrm{p}}, z) = E(y_{\mathrm{e}}, y_{\mathrm{p}}, z).$$

In order to settle such an indetermination, we assume without loss of generality that y_{p} has zero mean, namely

$$\int_{\Omega} y_{\mathrm{p}}(x) \, \mathrm{d}x = 0.$$

Before moving on, let us remark that assumptions (2.8) and (2.9) are compatible with *frame indifference* [70]. In particular, one could additionally ask for

$$W_{\mathrm{e}}(QF_{\mathrm{e}}) = W_{\mathrm{e}}(F_{\mathrm{e}}) \quad \forall Q \in \mathrm{SO}(d), \, Fe \in \mathrm{GL}_+(d)$$

by simply requiring $\widehat{W}_{\mathrm{e}}(QF_{\mathrm{e}}, Q\mathrm{cof}\, F_{\mathrm{e}}, \det F_{\mathrm{e}}) = \widehat{W}_{\mathrm{e}}(F_{\mathrm{e}}, \mathrm{cof}\, F_{\mathrm{e}}, \det F_{\mathrm{e}})$. Examples of a frame-indifferent elastic energy densities W_{e} complying with (2.8) and (2.9) can be found in the class of *Ogden* materials [10],Section 4.9. These are given by

$$W_{\mathrm{e}}(F_{\mathrm{e}}) = \sum_{i=1}^n a_i \, \mathrm{tr}(F_{\mathrm{e}}^\top F_{\mathrm{e}})^{\alpha_i/2} + \sum_{j=1}^m b_j \, \mathrm{tr}(\mathrm{cof}\, F_{\mathrm{e}}^\top F_{\mathrm{e}})^{\beta_j/2} + \Gamma(\det F_{\mathrm{e}})$$

for $F_{\mathrm{e}} \in \mathrm{GL}_+(d)$ and $W_{\mathrm{e}} = \infty$ otherwise, where $n, m \geq 1$, $a_i, b_j \geq 0$, $\alpha_i, \beta_j \geq 1$, $\Gamma : (0, \infty) \to (0, \infty)$ is convex, grows suitably at ∞, and $\lim_{\delta \to 0+} \Gamma(\delta) = \infty$. All Ogden elastic-energy densities are clearly polyconvex. By letting $\alpha_j \geq d$ for some $j = 1, \ldots, n$ and $\liminf_{\delta \to \infty} \Gamma(\delta) \delta^{-r} > 0$ the coercivity (2.9) follows.

2.4. Dissipation

Following Mielke [49, 50], we define the *dissipation distance* $\Delta : (\mathrm{SL}(d) \times (0, \infty))^2 \to [0, \infty]$ as

$$\Delta((F_{\mathrm{p}0}, z_0), (F_{\mathrm{p}1}, z_1)) = \inf \left\{ \int_0^1 R(P(t), z(t), \dot{P}(t)) \, \mathrm{d}t \; : \; (P, z) \in C^1([0, 1])^{d \times d + 1}, \right.$$

$$\left. \det P(t) = 1, \; z(t) > 0 \text{ for all } t \in [0, 1], \; P(i) = F_{\mathrm{p}i}, \; z(i) = z_i \text{ for } i = 0, 1 \right\}$$

where $R : T(\mathrm{SL}(d)\mathbb{R}) \to [0, \infty]$ (here T denotes the tangent bundle) is chosen as

$$R(P, \zeta, \dot{P}) = \begin{cases} |(\dot{P}P^{-1})_{\mathrm{sym}}| & \text{if } |(\dot{P}P^{-1})_{\mathrm{sym}}| \leq \zeta \\ \infty & \text{else} \end{cases}$$

and $(\dot{P}P^{-1})_{\mathrm{sym}} = ((\dot{P}P^{-1}) + (\dot{P}P^{-1})^\top)/2$ is the symmetric part of $\dot{P}P^{-1}$. This choice is inspired by the classical von Mises theory with isotropic hardening [32, 44] and is among the very few allowing an explicit expression for Δ. Namely, we have [48], Corollary 6.2.

$$\Delta((F_{\mathrm{p}0}, z_0), (F_{\mathrm{p}1}, z_1)) = D(F_{\mathrm{p}1}F_{\mathrm{p}0}^{-1}, z_1 - z_0), \tag{2.10}$$

$$D(P, \zeta) = \begin{cases} |\log(P^\top P)^{1/2}| & \text{if } |\log(P^\top P)^{1/2}| \leq \zeta \\ \infty & \text{else.} \end{cases} \tag{2.11}$$

Note that Δ satisfies *plastic invariance*

$$\Delta((F_{\mathrm{p}0}, z_0), (F_{\mathrm{p}1}, z_1)) = \Delta((I, z_0), (F_{\mathrm{p}1}F_{\mathrm{p}0}^{-1}, z_1))$$

as well as the *triangle inequality*

$$\Delta((F_{\mathrm{p}0}, z_0), (F_{\mathrm{p}1}, z_1)) \leq \Delta((F_{\mathrm{p}0}, z_0), (F_{\mathrm{p}2}, z_2)) + \Delta((F_{\mathrm{p}2}, z_2), (F_{\mathrm{p}1}, z_1))$$

for all $F_{\mathrm{p}0}, F_{\mathrm{p}1}, F_{\mathrm{p}2} \in \mathrm{SL}(d)$ and $z_0, z_1, z_2 > 0$.

For all $z_0 > 0$ we follow Mielke and Müller [51], and define

$$H(P, z_0) = \min_{z > 0} \left(W_{\mathrm{h}}(z) + D(P, z - z_0) \right)$$

$$= W_{\mathrm{h}}\left(z_0 + |\log(P^\top P)^{1/2}|\right) + |\log(P^\top P)^{1/2}|. \tag{2.12}$$

In [51], Section 4 necessary and sufficient conditions on W_{h} are discussed ensuring the polyconvexity and the coercivity of $P \mapsto H(P, z_0)$. If W_{h} is exponential, namely

$$W_{\mathrm{h}}(z) = \gamma \exp(\kappa z) - z - \frac{1}{\gamma} \tag{2.13}$$

with γ and $\kappa \geq (d(d-1))^{1/2}$ the map $P \mapsto H(P, z_0)$ is coercive, namely

$$H(P, z_0) \geq c|P|^d - \frac{1}{c}$$

for some $c > 0$. Polyconvexity for all $z_0 > 0$ holds for $d = 2$ if $\kappa \geq 1/2$ [51], Proposition 4.1 and is conjectured to hold for $\kappa \geq 1$ if $d = 3$ [51], Remark 4.2.

2.5. Main result

We are now ready to state our existence result.

Theorem 2.1 (Existence). *Let $z_0 \in L^d(\Omega)$ and $y_{p0} \in W^{1,\infty}(\Omega)^d$ with $\det \mathrm{D}y_{p0} = 1$ a.e. in Ω be given. Assume $W_e : \mathrm{GL}(d) \to [0, \infty)$ to be polyconvex, $W_h : (0, \infty) \to (0, \infty)$ to be increasing, and define Δ as in (2.10) and (2.11). Define H as in (2.12) and assume that $P \in \mathrm{SL}(d) \mapsto H(P, z_0)$ is polyconvex. Moreover, let the coercivity*

$$W_e(F_e) + H(P, z_0) \geq c|F_e|^d + c(\det F_e)^r + c|P|^d - \frac{1}{c} \tag{2.14}$$

hold for all $F_e \in \mathrm{GL}_+(d)$, $z_0 > 0$, and $P \in \mathrm{SL}(d)$ and some $c > 0$ and $r > 1$ and $W_e(F_e) = \infty$ iff $\det F_e \leq 0$. Finally, let the body force $f \in L^{d/(d-1)}(\Omega)^d$ and the traction $g \in L^{(d-1)/(d-2)}(\Gamma_{tr})^d$ be given. Then, the functional

$$\begin{aligned} F(y_e, y_p, z) = &\int_{y_p(\Omega)} W_e(\mathrm{D}y_e(\xi)) \, \mathrm{d}\xi + \int_{\Omega} W_h(z(x)) \, \mathrm{d}x \\ &+ \int_{\Omega} \Delta((\mathrm{D}y_{p0}(x), z_0(x)), (\mathrm{D}y_p(x), z(x))) \, \mathrm{d}x \\ &- \int_{\Omega} f(x) \cdot y_e(y_p(x)) \, \mathrm{d}x - \int_{\Gamma_{tr}} g(x) \cdot y_e(y_p(x)) \, \mathrm{d}S \end{aligned}$$

admits a minimizer on the set of admissible deformations

$$\begin{aligned} A = \Big\{ &(y_e, y_p, z) \, : \, y_p \in W^{1,d}(\Omega)^d, \ \int_{\Omega} y_p(x) \, \mathrm{d}x = 0, \ \det \mathrm{D}y_p = 1 \text{ a.e. in } \Omega, \ |\Omega| \leq |y_p(\Omega)|, \\ &y_e \in W^{1,d}(y_p(\Omega)^\circ)^d, \ \det \mathrm{D}y_e > 0 \text{ a.e. in } \Omega, \ y_e \circ y_p = \mathrm{id} \text{ a.e. on } \Gamma_{\mathrm{Dir}}, \ z \in L^d(\Omega) \Big\}. \end{aligned}$$

Note that the assumptions on f and g in the statement imply that the forcing term

$$\int_{\Omega} f(x) \cdot y_e(y_p(x)) \, \mathrm{d}x + \int_{\Gamma_{tr}} g(x) \cdot y_e(y_p(x)) \, \mathrm{d}S$$

is well defined for all $y_e \circ y_p \in W^{1,d/2}(\Omega)^d \subset L^d(\Omega)^d$ and weakly continuous with respect to the weak topology of $W^{1,3/2}(\Omega)^3$ for $d = 3$ and the weak-star topology of $BV(\Omega)^2$ for $d = 2$, see also [41], Theorem 15.16. The forcing term above features Lagrangian contributions only. Let us however mention that Eulerian forcings as, for instance, nonconstant gravity, could be easily included in the picture.

The coercivity assumption (2.14) is weaker than the corresponding one from [51]. In particular, in [51] it is assumed that the density of F controls $|F_e|^a + |P|^b$ for $1/a + 1/b < 1/d$, which does not necessarily hold true here. As a consequence, a larger class of materials can be considered under (2.14). Moreover, requiring plastic deformations to be in $W^{1,d}(\Omega)^d$ corresponds to the minimal integrability frame, which still guarantees the integrability of plastic increments. More precisely, by letting y_{p0} and y_{p1} be two plastic deformations in $W^{1,d}(\Omega)^d$ with $\det \mathrm{D}y_{p0} = 1$, one has that $\mathrm{cof}\,\mathrm{D}y_{p0}^\top \in L^{d/(d-1)}(\Omega)^{d \times d}$ and the plastic increment $\mathrm{D}y_{p1}\mathrm{D}y_{p0}^{-1} = \mathrm{D}y_{p1}\,\mathrm{cof}\,\mathrm{D}y_{p0}^\top$ belongs to $L^1(\Omega)^{d \times d}$.

Theorem 2.1 is proved in Section 3. The rest of this section investigates the possibility of defining an incremental solution to the elastoplastic evolution problem via successive minimization, in case of time-varying external actions $f(t)$ and $g(t)$. Section 4 collects a number of remarks on possible variants and generalizations of the existence result.

2.6. Differential formulation

The Euler-Lagrange equations for the minimization problem in Theorem 2.1 can be formally computed by taking variations with respect to $y = y_\mathrm{e} \circ y_\mathrm{p}$, y_p, and z. Suitably smooth minimizers of F in A solve the nonlinear system

$$\mathrm{div}\, \mathrm{D}W_\mathrm{e}(\mathrm{D}y_\mathrm{e} \circ y_\mathrm{p})\mathrm{D}y_\mathrm{p}^{-\top} + f = 0 \quad \text{in } \Omega, \tag{2.15}$$

$$\partial_1 D(\mathrm{D}y_\mathrm{p}\mathrm{D}y_{\mathrm{p}0}^{-1}, z-z_0)\mathrm{D}y_{\mathrm{p}0}^{-\top} - (\mathrm{D}y_\mathrm{e} \circ y_\mathrm{p})^\top \mathrm{D}W_\mathrm{e}(\mathrm{D}y_\mathrm{e} \circ y_\mathrm{p})\mathrm{D}y_\mathrm{p}^{-\top} \ni 0 \quad \text{in } \Omega, \tag{2.16}$$

$$\partial_2 D(\mathrm{D}y_\mathrm{p}\mathrm{D}y_{\mathrm{p}0}^{-1}, z-z_0) + W_\mathrm{h}'(z) \ni 0 \quad \text{in } \Omega. \tag{2.17}$$

Relation (2.15) is the quasistatic equilibrium system. The Dirichlet condition $y_\mathrm{e} \circ y_\mathrm{p} = \mathrm{id}$ on Γ_Dir is complemented by Neumann condition

$$\mathrm{D}W_\mathrm{e}(\mathrm{D}y_\mathrm{e} \circ y_\mathrm{p})\mathrm{D}y_\mathrm{p}^{-\top} n = g \quad \text{on } \Gamma_\mathrm{tr}$$

where $n \in \mathbb{R}^d$ is the outward normal to Γ_tr. The inclusion (2.16) corresponds to the variation of F with respect to y_p, which is assumed to have zero mean. In particular, ∂_1 denotes partial (sub)differentiation with respect to the first variable in D and (2.16) is a balance between the dissipative force $\partial_1 D(\mathrm{D}y_\mathrm{p}\mathrm{D}y_{\mathrm{p}0}^{-1}, z-z_0)\mathrm{D}y_{\mathrm{p}0}^{-\top}$ and the conservative force $\partial_{\mathrm{D}y_\mathrm{p}}W_\mathrm{e}(\mathrm{D}y_\mathrm{e} \circ y_\mathrm{p}) = -(\mathrm{D}_\xi y_\mathrm{e} \circ y_\mathrm{p})^\top \mathrm{D}W_\mathrm{e}(\mathrm{D}y_\mathrm{e} \circ y_\mathrm{p})\mathrm{D}y_\mathrm{p}^{-\top}$. Eventually, relation (2.17) is the variation of F with respect to the hardening parameter z and corresponds to a balance of dissipative and conservative actions as well.

2.7. Role of the intermediate configuration

As mentioned in the Introduction, the coupling between elastic and plastic variables in the functional F is realized *via* the specification of the intermediate configuration $y_\mathrm{p}(\Omega)$. Assume $y_{\mathrm{p}0}$ to be an open map with $\det \mathrm{D}y_{\mathrm{p}0} = 1$ a.e. Then, the set $\Omega^1 := y_{\mathrm{p}0}(\Omega)$ is open and one can rewrite the problem solely on Ω^1 by changing variables in the dissipation term as

$$\int_\Omega D(\mathrm{D}y_\mathrm{p}(x)(\mathrm{D}y_{\mathrm{p}0}(x))^{-1}, z(x)-z_0(x))\,\mathrm{d}x = \int_{\Omega^1} D(\mathrm{D}\tilde{y}_\mathrm{p}(\xi^1), \tilde{z}(\xi^1)-\tilde{z}_0(\xi^1))\,\mathrm{d}\xi^1$$

where $y_\mathrm{p} = \tilde{y}_\mathrm{p} \circ y_{\mathrm{p}0}$, $z = \tilde{z} \circ y_{\mathrm{p}0}$, $z_0 = \tilde{z}_0 \circ y_{\mathrm{p}0}$, and $\xi^i \mapsto \tilde{y}_\mathrm{p}(\xi^i) : \Omega^1 \to y_\mathrm{p}(\Omega)$. One can hence reformulate the incremental elastoplastic problem (with $g = 0$) as that of minimizing the functional

$$\tilde{F}(y_\mathrm{e}, \tilde{y}_\mathrm{p}, \tilde{z}) = \int_{\tilde{y}_\mathrm{p}(\Omega^1)} W_\mathrm{e}(\mathrm{D}y_\mathrm{e}(\xi))\,\mathrm{d}\xi + \int_{\Omega^1} W_\mathrm{h}(\tilde{z}(\xi^1))\,\mathrm{d}\xi^1$$
$$+ \int_{\Omega^1} D(\mathrm{D}\tilde{y}_\mathrm{p}(\xi^1), \tilde{z}_0(\xi^1))\,\mathrm{d}\xi^1 - \int_{\Omega^1} \tilde{f}(\xi) \cdot y_\mathrm{e}(\tilde{y}_\mathrm{p}(\xi))\,\mathrm{d}\xi$$

where $\tilde{f} = f \circ y_{\mathrm{p}0}^{-1} : \Omega^1 \to \mathbb{R}^3$. The influence of the previous plastic history is recorded into the set Ω^1, as well as in the modified initial isotropic hardening parameter $\tilde{z}_0$ and in the modified load $\tilde{f}$. In particular, the problem is independent of the reference configuration Ω.

3. PROOF OF THEOREM 2.1

The argument follows the classical path of the Direct Method [14] and is divided into five steps. Within this section, C denotes a positive constant depending on data but not on k and possibly varying from line to line.

Step 1: Minimization in z. As F depends only locally on z, one directly minimizes it out as

$$\min_{z \in L^d(\Omega)} F(y_{\mathrm{e}}, y_{\mathrm{p}}, z) = G(y_{\mathrm{e}}, y_{\mathrm{p}})$$

$$:= \int_{y_{\mathrm{p}}(\Omega)^\circ} W_{\mathrm{e}}(\mathrm{D}y_{\mathrm{e}}(\xi)) \, \mathrm{d}\xi + \int_\Omega H(\mathrm{D}y_{\mathrm{p}}(x)\mathrm{D}y_{\mathrm{p}0}^{-1}(x), z_0(x)) \, \mathrm{d}x$$

$$- \int_\Omega f(x) \cdot y_{\mathrm{e}}(y_{\mathrm{p}}(x)) \, \mathrm{d}x - \int_{\Gamma_{\mathrm{tr}}} g(x) \cdot y_{\mathrm{e}}(y_{\mathrm{p}}(x)) \, \mathrm{d}S.$$

Given z_0 and $\mathrm{D}y_{\mathrm{p}}$, this minimization corresponds to the position

$$z(x) = z_0(x) + |\log(\mathrm{D}y_{\mathrm{p}}(x)^\top \mathrm{D}y_{\mathrm{p}}(x))^{1/2}|.$$

As soon as $\mathrm{D}y_{\mathrm{p}}$ will be checked to be bounded in $L^d(\Omega)^{d \times d}$ in Step 2 below, the L^d bound on z will follow from the sublinearity of the logarithm.

The assertion of the Theorem can hence be equivalently reformulated as that of finding a minimizer for G on $B := \{(y_{\mathrm{e}}, y_{\mathrm{p}}) : (y_{\mathrm{e}}, y_{\mathrm{p}}, z_0) \in A\}$.

Step 2: Bounds. Note that the set B of admissible deformations is nonempty: By letting $\overline{x}$ denote the barycenter of Ω and T be the translation $T(x) = x - \overline{x}$ one has that

$$(\mathrm{id} \circ T^{-1}, T \circ \mathrm{id}) \in B.$$

Let $(y_{\mathrm{e}}^k, y_{\mathrm{p}}^k) \in B$ be an infimizing sequence and assume with no loss of generality that $\sup_k G(y_{\mathrm{e}}^k, y_{\mathrm{p}}^k) < \infty$. Denote the total deformation by $y^k = y_{\mathrm{e}}^k \circ y_{\mathrm{p}}^k$ and observe that, given $f \in L^{d/(d-1)}(\Omega)^d$ and $g \in L^{(d-1)/(d-2)}(\Gamma_{\mathrm{tr}})^d$, the forcing term in G can be controlled as follows

$$\int_\Omega f(x) \cdot y_{\mathrm{e}}^k(y_{\mathrm{p}}^k(x)) \, \mathrm{d}x + \int_{\Gamma_{\mathrm{tr}}} g(x) \cdot y_{\mathrm{e}}^k(y_{\mathrm{p}}^k(x)) \, \mathrm{d}S$$

$$\leq \|f\|_{L^{d/(d-1)}(\Omega)^d} \|y^k\|_{L^d(\Omega)^d} + \|g\|_{L^{(d-1)/(d-2)}(\Gamma_{\mathrm{tr}})^d} \|\gamma y^k\|_{L^{d-1}(\partial\Omega)^d}$$

$$\leq C \left(\|f\|_{L^{d/(d-1)}(\Omega)^d} + \|g\|_{L^{(d-1)/(d-2)}(\Gamma_{\mathrm{tr}})^d} \right) \|y^k\|_{W^{1,d/2}(\Omega)^d} \qquad (3.1)$$

where $\gamma : W^{1,d/2}(\Omega)^d \to L^{d-1}(\partial\Omega)^d$ is the (continuous) trace operator [52], Theorem B.4.4. As the tensor norm is submultiplicative we have $|A|/|B| \leq |A\,B^{-1}|$ for all $A \in \mathbb{R}^{d \times d}$ and $B \in \mathrm{GL}(d)$. The coercivity (2.14) ensures that

$$c\|\mathrm{D}y_{\mathrm{e}}^k\|_{L^d(y_{\mathrm{p}}^k(\Omega)^\circ)^{d \times d}}^d + c\|\det \mathrm{D}y_{\mathrm{e}}^k\|_{L^r(y_{\mathrm{p}}^k(\Omega)^\circ)}^r + c\|\mathrm{D}y_{\mathrm{p}}^k\|_{L^d(\Omega)^{d \times d}}^d / \|\mathrm{D}y_{\mathrm{p}0}\|_{L^\infty(\Omega)^{d \times d}}^d$$

$$\leq G(y_{\mathrm{p}}^k, y_{\mathrm{e}}^k) + \int_\Omega f(x) \cdot y^k(x) \, \mathrm{d}x + \int_{\Gamma_{\mathrm{tr}}} g(x) \cdot y^k(x) \, \mathrm{d}S + \frac{|\Omega|}{c}$$

$$\overset{(3.1)}{\leq} G(y_{\mathrm{p}}^k, y_{\mathrm{e}}^k) + C \left(\|f\|_{L^{d/(d-1)}(\Omega)^d} + \|g\|_{L^{(d-1)/(d-2)}(\Gamma_{\mathrm{tr}})^d} \right) \|y^k\|_{W^{1,d/2}(\Omega)^d} + C$$

$$\overset{(2.7)}{\leq} G(y_{\mathrm{p}}^k, y_{\mathrm{e}}^k) + C\|\mathrm{D}y_{\mathrm{e}}^k\|_{L^d(y_{\mathrm{p}}^k(\Omega)^\circ)^{d \times d}} \|\mathrm{D}y_{\mathrm{p}}^k\|_{L^d(\Omega)^{d \times d}} + C$$

$$\leq G(y_{\mathrm{p}}^k, y_{\mathrm{e}}^k) + \frac{c}{2}\|\mathrm{D}y_{\mathrm{e}}^k\|_{L^d(y_{\mathrm{p}}^k(\Omega)^\circ)^{d \times d}}^d + \frac{c}{2}\|\mathrm{D}y_{\mathrm{p}}^k\|_{L^d(\Omega)^{d \times d}}^d + C.$$

In particular, as we have that

$$\int_{\Omega} |\mathrm{D}y_{\mathrm{e}}^k(y_{\mathrm{p}}^k(x))|^d \, \mathrm{d}x = \int_{y_{\mathrm{p}}^k(\Omega)^\circ} |\mathrm{D}y_{\mathrm{e}}^k(\xi)|^d \, \mathrm{d}\xi,$$

one concludes that

$$\|\mathrm{D}y_{\mathrm{e}}^k\|_{L^d(y_{\mathrm{p}}^k(\Omega)^\circ)^{d\times d}}^d + \|\mathrm{D}y_{\mathrm{e}}^k \circ y_{\mathrm{p}}^k\|_{L^d(\Omega)^{d\times d}} + \|\det \mathrm{D}y_{\mathrm{e}}^k\|_{L^r(y_{\mathrm{p}}^k(\Omega)^\circ)}^r + \|\mathrm{D}y_{\mathrm{p}}^k\|_{L^d(\Omega)^{d\times d}}^d \leq C. \tag{3.2}$$

As y_{p}^k has zero mean on Ω, the Poincaré-Wirtinger inequality entail that y_{p}^k are bounded in $W^{1,d}(\Omega)^d$. On the other hand, as $y^k = \mathrm{id}$ a.e. on Γ_{Dir} one has that y^k are bounded in $W^{1,d/2}(\Omega)^d$ by the Poincaré inequality and (2.6). The change of variables formula (2.5) ensures that

$$\int_{y_{\mathrm{p}}^k(\Omega)^\circ} |y_{\mathrm{e}}^k(\xi)| \, \mathrm{d}\xi = \int_{\Omega} |y^k(x)| \, \mathrm{d}x$$

which, again in combination with bound (3.2), entails that y_{e}^k is bounded in $W^{1,d}(y_{\mathrm{p}}^k(\Omega)^\circ)^d$ as well. Namely, we have checked that

$$\|y_{\mathrm{e}}^k\|_{W^{1,d}(y_{\mathrm{p}}^k(\Omega)^\circ)^d} + \|y_{\mathrm{p}}^k\|_{W^{1,d}(\Omega)^d} + \|y^k\|_{W^{1,d/2}(\Omega)^d} \leq C. \tag{3.3}$$

Step 3: Lower semicontinuity of the dissipative term. By extracting a not relabeled subsequence, we have that $y_{\mathrm{p}}^k \to y_{\mathrm{p}}$ weakly in $W^{1,d}(\Omega)^d$ and uniformly on all $\overline{\Omega}'$ where $\Omega' \subset\subset \Omega$ open [20], Lemma 2.1. In particular, one has that y_{p} has zero mean, $\mathrm{cof}\,\mathrm{D}y_{\mathrm{p}}^k \to \mathrm{cof}\,\mathrm{D}y_{\mathrm{p}}$ weakly in $L^{d/(d-1)}(\Omega)^{d\times d}$, and $1 = \det \mathrm{D}y_{\mathrm{p}}^k \to \det \mathrm{D}y_{\mathrm{p}}$ weakly star in the sense of measures, so that $\det \mathrm{D}y_{\mathrm{p}} = 1$ a.e. [14], Theorem 8.20, see also [53, 62]. As $\mathrm{cof}\,(F_{\mathrm{p}}\mathrm{D}y_{\mathrm{p}0}^{-1}) = \mathrm{cof}\,F_{\mathrm{p}}\mathrm{cof}\,\mathrm{D}y_{\mathrm{p}0}^{-1}$, from the polyconvexity of $P \mapsto H(P, z_0)$ we deduce that $F_{\mathrm{p}} \mapsto H(F_{\mathrm{p}}\mathrm{D}y_{\mathrm{p}0}^{-1}, z_0)$ is polyconvex as well. We can hence pass to the lower limit in the dissipative term and find

$$\int_{\Omega} H(\mathrm{D}y_{\mathrm{p}}(x)\mathrm{D}y_{\mathrm{p}0}^{-1}(x), z_0(x)) \, \mathrm{d}x \leq \liminf_{k\to\infty} \int_{\Omega} H(\mathrm{D}y_{\mathrm{p}}^k(x)\mathrm{D}y_{\mathrm{p}0}^{-1}(x), z_0(x)) \, \mathrm{d}x.$$

Moreover, the Ciarlet-Nečas condition $|\Omega| \leq |y_{\mathrm{p}}(\Omega)|$ holds. Indeed, for all $\Omega' \subset\subset \Omega$ one can argue as in [11] and exploit the uniform convergence of $y_{\mathrm{p}}^k \to y_{\mathrm{p}}$ in $\overline{\Omega}'$ in order to check that $|\Omega'| \leq |y_{\mathrm{p}}(\Omega')|$. Letting $\Omega_n' \subset\subset \Omega$ be a sequence with $|\Omega_n'| \to |\Omega|$ we hence conclude that $|\Omega| = \lim_{n\to\infty} |\Omega_n'| \leq \limsup_{n\to\infty} |y_{\mathrm{p}}(\Omega_n')| \leq |y_{\mathrm{p}}(\Omega)|$.

Step 4: Convergence of the elastic deformations. Extend now y_{e}^k, $\mathrm{D}y_{\mathrm{e}}^k$, $\mathrm{cof}\,\mathrm{D}y_{\mathrm{e}}^k$, and $\det \mathrm{D}y_{\mathrm{e}}^k$ by zero outside $y_{\mathrm{p}}^k(\Omega)^\circ$, without changing notation. The bounds (3.2) and (3.3) entail that

$$\|y_{\mathrm{e}}^k\|_{L^d(\mathbb{R}^d)^d} + \|\mathrm{D}y_{\mathrm{e}}^k\|_{L^d(\mathbb{R}^d)^{d\times d}} + \|\mathrm{cof}\,\mathrm{D}y_{\mathrm{e}}^k\|_{L^{d/(d-1)}(\mathbb{R}^d)^{d\times d}} + \|\det \mathrm{D}y_{\mathrm{e}}^k\|_{L^r(\mathbb{R}^d)} \leq C.$$

By extracting without relabelling we find that $y_{\mathrm{e}}^k \to u$ weakly in $L^d(\mathbb{R}^d)^d$, $\mathrm{D}y_{\mathrm{e}}^k \to G$ weakly in $L^d(\mathbb{R}^d)^{d\times d}$, $\mathrm{cof}\,\mathrm{D}y_{\mathrm{e}}^k \to V$ weakly in $L^{d/(d-1)}(\mathbb{R}^d)^{d\times d}$, and $\det \mathrm{D}y_{\mathrm{e}}^k \to v$ weakly in $L^r(\mathbb{R}^d)$.

We now follow the argument of Barchiesi and DeSimone [4] (see also [5], Prop. 7.1) and check that, by defining $y_{\mathrm{e}} = u|_{y_{\mathrm{p}}(\Omega)^\circ}$, one has $y_{\mathrm{e}} \in W^{1,d}(y_{\mathrm{p}}(\Omega)^\circ)^d$ and $G = \mathrm{D}y_{\mathrm{e}}$ a.e. in $y_{\mathrm{p}}(\Omega)^\circ$. Owing to [20], Lemma 4.3, for all $\Omega' \subset\subset \Omega$, there exists a not relabeled subsequence y_{p}^k such that for a.e. $x_0 \in \Omega'$ one can find open neighborhoods $U, U^k \subset \Omega'$ of x_0, a ball B centered in $z_0 = y_{\mathrm{p}}(x_0)$, and local inverses $y_{\mathrm{p}}^{-1} : B \to U$ and $(y_{\mathrm{p}}^k)^{-1} : B \to U^k$

such that $y_{\mathrm{p}}(U) = y_{\mathrm{p}}^k(U^k) = B \subset y_{\mathrm{p}}(\Omega)^\circ$. The bound (3.3) ensures that y_{e}^k is bounded in $W^{1,d}(B)^d$. Hence, $Dy_{\mathrm{e}}^k \to Dy_{\mathrm{e}}$ weakly in $L^d(B)^{d \times d}$ and $G = Dy_{\mathrm{e}}$ a.e. in B.

By the N-property, we have hence proved that, for all $\Omega' \subset\subset \Omega$ there exists a negligible set $N \subset \mathbb{R}^d$ such that for all $z_0 \in y_{\mathrm{p}}(\Omega') \cap y_{\mathrm{p}}(\Omega)^\circ \setminus N$ there exists a ball $B(z_0) \subset y_{\mathrm{p}}(\Omega)^\circ$ such that $G = Dy_{\mathrm{e}}$ a.e. in $B(z_0)$. Let now a sequence $\Omega'_n \subset\subset \Omega$ invade Ω and denote the corresponding negligible sets by N_n. Then, $G = Dy_{\mathrm{e}}$ holds a.e. in a neighborhood of each $z_0 \in \cup_n (y_{\mathrm{p}}(\Omega'_n) \cap y_{\mathrm{p}}(\Omega)^\circ \setminus N_n)$, thus a.e. in $y_{\mathrm{p}}(\Omega)^\circ$ as $y_{\mathrm{p}}(\Omega)^\circ \setminus \cup_n (y_{\mathrm{p}}(\Omega'_n) \cap y_{\mathrm{p}}(\Omega)^\circ \setminus N_n)$ is negligible. We hence conclude that $V = \operatorname{cof} Dy_{\mathrm{e}}$ and $\det Dy_{\mathrm{e}} = v$ a.e. in $y_{\mathrm{p}}(\Omega)^\circ$ as well.

Step 5: Lower semicontinuity of the elastic energy. By possibly extracting again without relabelling, bound (3.3) yields that $y^k \to y$ weakly in $W^{1,d/2}(\Omega)^d$ for $d \geq 3$ and weakly star in $BV(\Omega)^2$ for $d = 2$. Let again U, U^k, B, and the local inverses $(y_{\mathrm{p}}^k)^{-1}$ be as in Step 4 and fix $\varphi \in C_c^\infty(U)^d$. Since $\chi_{U^k} \to \chi_U$ a.e. in $\mathbb{R}^d$ [20], Remark 4.4.2 one has

$$\lim_{k\to\infty} \int_{U^k} y_{\mathrm{e}}^k(y_{\mathrm{p}}^k(x)) \cdot \varphi(x)\,\mathrm{d}x = \lim_{k\to\infty} \int_\Omega \chi_{U^k}(x)\, y_{\mathrm{e}}^k(y_{\mathrm{p}}^k(x)) \cdot \varphi(x)\,\mathrm{d}x$$
$$= \int_\Omega \chi_U(x)\, y(x) \cdot \varphi(x)\,\mathrm{d}x = \int_U y(x) \cdot \varphi(x)\,\mathrm{d}x. \tag{3.4}$$

On the other hand, as $\varphi \circ (y_{\mathrm{p}}^k)^{-1} \to \varphi \circ y_{\mathrm{p}}^{-1}$ weakly in $W^{1,1+\delta}(B)^d$ [20], Lemma 4.5. (ii) with $0 < \delta < d/(d-1)$ and strongly in $L^q(B)^d$ for all $q < \infty$ we have that

$$\lim_{k\to\infty} \int_{U^k} y_{\mathrm{e}}^k(y_{\mathrm{p}}^k(x)) \cdot \varphi(x)\,\mathrm{d}x = \lim_{k\to\infty} \int_B y_{\mathrm{e}}^k(\xi) \cdot \varphi((y_{\mathrm{p}}^k)^{-1}(\xi))\,\mathrm{d}\xi$$
$$\int_B y_{\mathrm{e}}(\xi) \cdot \varphi(y_{\mathrm{p}}^{-1}(\xi))\,\mathrm{d}\xi = \int_U y_{\mathrm{e}}(y_{\mathrm{p}}(x)) \cdot \varphi(x)\,\mathrm{d}x. \tag{3.5}$$

Relations (3.4) and (3.5) entail that $y = y_{\mathrm{e}} \circ y_{\mathrm{p}}$ a.e. in U, hence in Ω. Analogous arguments ensure that $Dy_{\mathrm{e}}^k \circ y_{\mathrm{p}}^k \to Dy_{\mathrm{e}} \circ y_{\mathrm{p}}$ weakly in $L^d(\Omega)^{d\times d}$, $\operatorname{cof} Dy_{\mathrm{e}}^k \circ y_{\mathrm{p}}^k \to \operatorname{cof} Dy_{\mathrm{e}} \circ y_{\mathrm{p}}$ weakly in $L^{d/(d-1)}(\Omega)^{d\times d}$, and $\det Dy_{\mathrm{e}}^k \circ y_{\mathrm{p}}^k \to \det Dy_{\mathrm{e}} \circ y_{\mathrm{p}}$ weakly in $L^r(\Omega)$. One can hence pass to the lower limit in the elastic part of the energy by polyconvexity (2.8) as

$$\int_{y_{\mathrm{p}}(\Omega)} W_{\mathrm{e}}(Dy_{\mathrm{e}}(\xi))\,\mathrm{d}\xi = \int_\Omega W_{\mathrm{e}}(Dy_{\mathrm{e}}(y_{\mathrm{p}}(x)))\,\mathrm{d}x$$
$$= \int_\Omega \widehat{W}_{\mathrm{e}}(Dy_{\mathrm{e}}(y_{\mathrm{p}}(x)), \operatorname{cof} Dy_{\mathrm{e}}(y_{\mathrm{p}}(x)), \det Dy_{\mathrm{e}}(y_{\mathrm{p}}(x)))\,\mathrm{d}x$$
$$\leq \liminf_{k\to\infty} \int_\Omega \widehat{W}_{\mathrm{e}}(Dy_{\mathrm{e}}^k(y_{\mathrm{p}}^k(x)), \operatorname{cof} Dy_{\mathrm{e}}^k(y_{\mathrm{p}}^k(x)), \det Dy_{\mathrm{e}}^k(y_{\mathrm{p}}^k(x)))\,\mathrm{d}x$$
$$= \liminf_{k\to\infty} \int_\Omega W_{\mathrm{e}}(Dy_{\mathrm{e}}^k(y_{\mathrm{p}}^k(x)))\,\mathrm{d}x = \liminf_{k\to\infty} \int_{y_{\mathrm{p}}^k(\Omega)} W_{\mathrm{e}}(Dy_{\mathrm{e}}^k(\xi))\,\mathrm{d}\xi.$$

In particular, $\det Dy_{\mathrm{e}} > 0$ a.e. in $y_{\mathrm{p}}(\Omega)^\circ$.

Eventually, as $y^k \to y = y_{\mathrm{e}} \circ y_{\mathrm{p}}$ weakly in $W^{1,d/2}(\Omega)^d$ for $d \geq 3$ and weakly star in $BV(\Omega)^2$ for $d = 2$, we can pass to the limit in the forcing terms, which are weakly continuous. For $d = 2$, we use here the continuity of the trace operator $\gamma : BV(\Omega)^2 \to L^1(\partial\Omega)^2$ [41], Theorem 15.16. We have hence checked that $G(y_{\mathrm{p}}, y_{\mathrm{e}}) \leq \liminf_{k\to\infty} G(y_{\mathrm{p}}^k, y_{\mathrm{e}}^k) = \inf_B G$ and the minimality of $(y_{\mathrm{p}}, y_{\mathrm{e}})$ follows.

4. MODIFICATIONS AND EXTENSIONS

As it is often the case in continuum mechanics, the existence result of Theorem 2.1 admits a number of possible variants and extensions. We devote this last section to comment on some of these.

4.1. Higher integrability of y_{p}

The existence theory is somewhat simplified by asking the plastic deformation y_{p} to belong to $W^{1,p}(\Omega)^d$ for $p > d$. This is precisely the functional setting in [51]. In this case, y_{p} is continuous up to boundary. Although $y_{\mathrm{p}}(\Omega)$ is not necessarily open, one can prove that $\overline{y_{\mathrm{p}}(\Omega)} = y_{\mathrm{p}}(\overline{\Omega})$ and, by the N-property, $|y_{\mathrm{p}}(\overline{\Omega}) \setminus y_{\mathrm{p}}(\Omega)| = 0$. One can conclude that the deformed boundary $\partial y_{\mathrm{p}}(\Omega) = \overline{y_{\mathrm{p}}(\Omega)} \setminus y_{\mathrm{p}}(\Omega)^{\circ}$ is negligible, namely $|\partial y_{\mathrm{p}}(\Omega)| = |y_{\mathrm{p}}(\overline{\Omega}) \setminus y_{\mathrm{p}}(\Omega)| + |y_{\mathrm{p}}(\Omega) \setminus y_{\mathrm{p}}(\Omega)^{\circ}| = 0$.

In this more regular context, the coercivity requirement (2.14) could be modified as

$$W_{\mathrm{e}}(F_{\mathrm{e}}) + H(P, z_0) \geq c|F_{\mathrm{e}}|^q + c|\mathrm{cof}\, F_{\mathrm{e}}|^{d/(d-1)} + c(\det F_{\mathrm{e}})^r + c|P|^p - \frac{1}{c}$$

for all $F_{\mathrm{p}} \in \mathrm{SL}(d)$ and $F_{\mathrm{e}} \in \mathrm{GL}(d)$, $q > p' \wedge (d-1)$, $1/p' + 1/p = 1$, $r > 1$, and some $c > 0$. Indeed, the weak continuity of the minors of $\mathrm{D}y_{\mathrm{e}}^k$ along sequences which are bounded in energy follows from [22], Theorem 2.

4.2. Lower integrability of y_{p}

One can reformulate the variational principle in the less regular case of $y_{\mathrm{p}} \in W^{1,p}(\Omega)^d$ for $d - 1 < p < d$ by resorting to the cavitation-free frame presented by Barchiesi *et al.* [5]. This would have the advantage of requiring a weaker coercivity assumption on the energy densities with respect to (2.14), by still retaining the compactifying term on the determinants.

This weaker formulation of the problem needs to cope with the fact that the set $y_{\mathrm{p}}(\Omega)$ is not well defined, as y_{p} has no continuous representative. Indeed, one would be asked to use the *topological image* $\mathrm{im}\,(y_{\mathrm{p}}, \Omega)$ instead. This is defined as

$$\mathrm{im}\,(y_{\mathrm{p}}, \Omega) := \bigcup_{U \in \mathcal{U}_{y_{\mathrm{p}}}} \mathrm{im}\,(y_{\mathrm{p}}, U)$$

where $\mathcal{U}_{y_{\mathrm{p}}}$ is a y_{p}-dependent class of open subsets $U \subset\subset \Omega$ with C^2 boundary, see [5], Definitions 2.17 and 5.11, and $\mathrm{im}\,(y_{\mathrm{p}}, U)$ is the set of those points $z \in \mathbb{R}^d \setminus y_{\mathrm{p}}(\partial U)$ such that the degree $\deg(y_{\mathrm{p}}, U, z) \neq 0$, see [68]. Within this weaker functional frame, a properly reformulated Ciarlet-Nečas condition still yields a.e. injectivity and is closed with respect to weak convergence [69]. The lower semicontinuity can be addressed by suitably combining the arguments of [5], Proposition 7.3 and 7.8.

Recall however that, by letting $y_{\mathrm{p}}, y_{\mathrm{p}0} \in W^{1,p}(\Omega)^d$ for $p < d$ with $\det \mathrm{D}y_{\mathrm{p}} = \det \mathrm{D}y_{\mathrm{p}0} = 1$ a.e. one has that the term $\mathrm{D}y_{\mathrm{p}} \mathrm{D}y_{\mathrm{p}0}^{-1}$ in the dissipation may be not integrable. This does not pair well with the linear growth of H with respect to $\mathrm{D}y_{\mathrm{p}} \mathrm{D}y_{\mathrm{p}0}^{-1}$, recall (2.13). In order to restore such integrability, one would strengthen the coercivity requirements by asking the energy density W_{p} to additionally control $|\mathrm{cof}\, F_{\mathrm{p}}|^{p/(p-1)}$.

4.3. Injectivity of y_{p}

As already mentioned, the Ciarlet-Nečas condition $|\Omega| \leq |y_{\mathrm{p}}(\Omega)|$ is equivalent to a.e. injectivity for functions $y_{\mathrm{p}} \in W^{1,d}(\Omega)^d$ with $\det \mathrm{D}y_{\mathrm{p}} = 1$ a.e. Global injectivity can also be considered by further specifying the assumption frame. In particular, in addition to the Ciarlet-Nečas condition we may impose y_{p} to have $\mathrm{D}y_{\mathrm{p}} \in L^s(\Omega)^{d \times d}$ with $s > d(d-1)$ for $d \geq 3$ and $s = 2$ for $d = 2$. As soon as the latter integrability of the gradient holds, such maps are either constant or both open and discrete [33], Theorem 3.4, where *discrete* means here that the preimage of each point in $y_{\mathrm{p}}(\Omega)$ is a discrete set. As the Ciarlet-Nečas condition is assumed, y_{p} cannot be

constant. Being the map y_{p} open and discrete, one can easily prove that the a.e. injectivity is actually global [37], Theorem 3.3.7. Such higher integrability of the gradient does not directly follow from the coercivity (2.14) which would need to be suitably strengthened.

Let us mention that a global invertibility result requiring the prescription of a Dirichlet condition for y_{p} on the whole boundary $\partial\Omega$ as in [3] is not well suited in this setting. At first, imposing boundary conditions on the internal variable y_{p} is surely questionable from the mechanical viewpoint. Secondly, in case of a constant body force (gravity, for instance), no traction ($g = 0$), and a constant initial isotropic-hardening level z_0, the intermediate configuration $y_{\mathrm{p}}(\Omega)$ is the only coupling between plastic and elastic dynamics. By prescribing it a priori, the elastoplastic problem essentially decouples into two separate equilibrium problems.

4.4. Kinematic hardening

The existence theory can be extended to include the case of combined isotropic-kinematic hardening. This results from including an additional density $F_{\mathrm{p}} \mapsto W_{\mathrm{p}}(F_{\mathrm{p}})$ to the energy. If W_{p} is polyconvex one can readily pass to the lim inf in this term as well. Additionally, this term could provide some coercivity, so that the corresponding assumption on H could be relaxed. Let us however mention that the purely kinematic case $W_{\mathrm{h}} = 0$ cannot be directly treated within the frame of Theorem 2.1, as the function H reduces then to D, which fails to be polyconvex.

4.5. Material symmetry

One could additionally require the invariance of W_{p} with respect to right-multiplications by rotations of the material symmetry group $S \subset \mathrm{SO}(d)$, namely

$$W_{\mathrm{p}}(F_{\mathrm{p}}Q) = W_{\mathrm{p}}(F_{\mathrm{p}}) \quad \forall Q \in S, \, F_{\mathrm{p}} \in \mathrm{SL}(d).$$

On the other hand, invariance of W_{p} with respect to left-multiplication by rotations

$$W_{\mathrm{p}}(QF_{\mathrm{p}}) = W_{\mathrm{p}}(F_{\mathrm{p}}) \quad \forall Q \in \mathrm{SO}(d), \, F_{\mathrm{p}} \in \mathrm{SL}(d)$$

corresponds to frame indifference on the intermediate configuration, namely indifference to plastic rotations [27, 28]. In this case, the energy E is invariant by all rigid-body motions of the intermediate configuration, not just by translations.

4.6. Nonhomogeneous Dirichlet boundary conditions

The Dirichlet boundary condition on y can be made inhomogeneous, namely $y = y_{\mathrm{Dir}}$ a.e. on Γ_{Dir}, by means of a successive composition. Following [23], we let the datum y_{Dir} be extended smoothly to $\mathbb{R}^d$. We will be looking for deformations of the form $y = y_{\mathrm{Dir}} \circ y_{\mathrm{e}} \circ y_{\mathrm{p}}$ by imposing $y_{\mathrm{e}} \circ y_{\mathrm{p}} = \mathrm{id}$ a.e. on Γ_{Dir}. This change effects just the elastic energy and the loadings which now read

$$\int_{y_{\mathrm{p}}(\Omega)} W_{\mathrm{e}}(\mathrm{D}y_{\mathrm{Dir}}(y_{\mathrm{e}}(\xi))\mathrm{D}y_{\mathrm{e}}(\xi)) \, \mathrm{d}\xi - \int_{\Omega} f(x) \cdot y_{\mathrm{Dir}}(y_{\mathrm{e}}(y_{\mathrm{p}}(x))) \, \mathrm{d}x - \int_{\Gamma_{\mathrm{tr}}} g(x) \cdot y_{\mathrm{Dir}}(y_{\mathrm{e}}(y_{\mathrm{p}}(x))) \, \mathrm{d}S.$$

The reach of Theorem 2.1 can be extended to deal with this case. By assuming

$$y_{\mathrm{Dir}} \in C_{\mathrm{c}}^2(\mathbb{R}^d)^d, \ \mathrm{D}y_{\mathrm{Dir}}(x) \in \mathrm{GL}_+(d), \ \mathrm{D}y_{\mathrm{Dir}}^{-1} \in L^{\infty}(\mathbb{R}^d)^{d \times d} \tag{4.1}$$

one can still obtain bounds (3.2) and (3.3), now of course depending on y_{Dir} as well. The identification of (3.4) and (3.5) can still be performed by observing that $y_{\mathrm{Dir}} \circ y_{\mathrm{e}}^k \to y_{\mathrm{Dir}} \circ y_{\mathrm{e}}$ strongly in $L^d(\Omega)^d$. Analogously, one

identifies the limits of $\operatorname{cof} \mathrm{D}(y_{\mathrm{Dir}} \circ y_{\mathrm{e}}^k) \circ y_{\mathrm{p}}^k$ and $\det \mathrm{D}(y_{\mathrm{Dir}} \circ y_{\mathrm{e}}^k) \circ y_{\mathrm{p}}^k$ by simply computing

$$\operatorname{cof} \mathrm{D}(y_{\mathrm{Dir}} \circ y_{\mathrm{e}}^k) = (\operatorname{cof} \mathrm{D}y_{\mathrm{Dir}} \circ y_{\mathrm{e}}^k)\operatorname{cof} \mathrm{D}y_{\mathrm{e}}^k, \quad \det \mathrm{D}(y_{\mathrm{Dir}} \circ y_{\mathrm{e}}^k) = (\det \mathrm{D}y_{\mathrm{Dir}} \circ y_{\mathrm{e}}^k)\det \mathrm{D}y_{\mathrm{e}}^k$$

and observing that the extra terms $\operatorname{cof} \mathrm{D}y_{\mathrm{Dir}} \circ y_{\mathrm{e}}^k$ and $\det \mathrm{D}y_{\mathrm{Dir}} \circ y_{\mathrm{e}}^k$ strongly converge to the corresponding limits because of the smoothness of y_{Dir}. Some time-dependent version of assumptions (4.1) would allow to obtain time-discrete incremental solution as in Section 4.7.

4.7. Incremental problem

The existence result of Theorem 2.1 can be used to find a discrete-in-time elastoplastic evolution starting from the initial state $(y_{\mathrm{e}0}, y_{\mathrm{p}0}, z_0)$ and driven by time-dependent loadings

$$f \in C^1([0, T], L^{d/(d-1)}(\Omega)^d), \quad g \in C^1([0, T], L^{(d-1)/(d-2)}(\Gamma_{\mathrm{tr}})^d). \tag{4.2}$$

Let a time partition $\{0 = t^0 < t^1 < \cdots < t^N = T\}$ be given, define $y_{\mathrm{p}}^0 = y_{\mathrm{p}0}$ and $z^0 = z_0$, and recursively solve for $i = 1, \ldots N$ the minimization problems

$$
\begin{aligned}
(y_{\mathrm{e}}^i, y_{\mathrm{p}}^i, z^i) \in \operatorname*{argmin}_{(y_{\mathrm{e}}, y_{\mathrm{p}}, z) \in A} \Bigg\{ & \int_{y_{\mathrm{p}}(\Omega)} W_{\mathrm{e}}(\mathrm{D}y_{\mathrm{e}}(\xi)) \, \mathrm{d}\xi + \int_{\Omega} W_{\mathrm{h}}(z(x)) \, \mathrm{d}x + \\
& + \int_{\Omega} D(\mathrm{D}y_{\mathrm{p}}(x)(\mathrm{D}y_{\mathrm{p}}^{i-1}(x))^{-1}, z(x) - z^{i-1}(x)) \, \mathrm{d}x \\
& - \int_{\Omega} f(t^i, x) \cdot y_{\mathrm{e}}(y_{\mathrm{p}}(x)) \, \mathrm{d}x - \int_{\Gamma_{\mathrm{tr}}} g(t^i, x) \cdot y_{\mathrm{e}}(y_{\mathrm{p}}(x)) \, \mathrm{d}S \Bigg\}.
\end{aligned}
\tag{4.3}
$$

Such an incremental solution can be proved to exist by induction. For $i = 1$ this is exactly the statement of Theorem 2.1. Assume to have solved the minimization problems up to level $i - 1$, namely to have found a time-discrete, incremental solution $t^k \mapsto (y_{\mathrm{p}}^k, y_{\mathrm{e}}^k, z^k) \in A$ for $k \leq i - 1$. By arguing as in Section 3, one can check that such time-discrete solution fulfills the bound

$$\sum_{k=1}^{i-1} \int_{\Omega} \Delta((\mathrm{D}y_{\mathrm{p}}^{i-1}(x), z^{i-1}(x)), (\mathrm{D}y_{\mathrm{p}}^i(x), z^i(x))) \, \mathrm{d}x \leq C \tag{4.4}$$

where $C > 0$ depends on $y_{\mathrm{p}0}$, f, g, and T, see [52], Proposition 2.1.4.

We shall prove that a solution $(y_{\mathrm{p}}^i, y_{\mathrm{e}}^i, z^i)$ exists. To this aim, Theorem 2.1 cannot be directly applied as y_{p}^{k-1} may fail to be Lipschitz. This in turn makes the functional not necessarily coercive in the L^d norm for $\mathrm{D}y_{\mathrm{p}}$. One can nevertheless restore this coercivity by arguing as in [51], (3.10), namely by using

$$
\begin{aligned}
\int_{\Omega} H(\mathrm{D}y_{\mathrm{p}}(x)(\mathrm{D}y_{\mathrm{p}}^{i-1}(x))^{-1}, z^{i-1}(x)) \, \mathrm{d}x \geq & \int_{\Omega} H(\mathrm{D}y_{\mathrm{p}}(x)(\mathrm{D}y_{\mathrm{p}0}(x))^{-1}, z_0(x)) \, \mathrm{d}x \\
& - \int_{\Omega} \Delta((\mathrm{D}y_{\mathrm{p}0}(x), z_0(x)), (\mathrm{D}y_{\mathrm{p}}^{i-1}(x), z^{i-1}(x))) \, \mathrm{d}x \\
\geq & \int_{\Omega} H(\mathrm{D}y_{\mathrm{p}}(x)(\mathrm{D}y_{\mathrm{p}0}(x))^{-1}, z_0(x)) \, \mathrm{d}x - C
\end{aligned}
$$

where we have used the triangle inequality and estimate (4.4). This allows to control the L^d norm of $\mathrm{D}y_{\mathrm{p}}$ and the argument of Theorem 2.1 can once again be followed in order to ensure the existence of a minimizer $(y_{\mathrm{p}}^i, y_{\mathrm{e}}^i, z^i)$.

Now that the time discrete incremental solution $t^k \mapsto (y_{\mathrm{p}}^k, y_{\mathrm{e}}^k, z^k) \in A$ is available up to $i = N$, we can perform the classical energy estimate in this setting [52], Proposition 2.1.4 and obtain

$$\sup_{k=1,\dots,i-1} \left(\|y_{\mathrm{p}}^k\|_{W^{1,d}(\Omega)^d} + \|y_{\mathrm{e}}^k\|_{W^{1,d}(y_{\mathrm{p}}^k(\Omega))^d} \right)$$
$$+ \sum_{k=1}^{i-1} \int_\Omega \Delta((\mathrm{D}y_{\mathrm{p}}^{i-1}(x), z^{i-1}(x)), (\mathrm{D}y_{\mathrm{p}}^i(x), z^i(x)))\, \mathrm{d}x \le C, \qquad (4.5)$$

see (3.3).

The incremental problem (4.3) can be reformulated along the lines of Section 2.7 by recursively minimizing the functional $\tilde{F}$ with time-dependent loading (again $g = 0$), namely the functionals

$$(y_{\mathrm{e}}, \tilde{y}_{\mathrm{p}}, \tilde{z}) \mapsto \int_{\tilde{y}_{\mathrm{p}}^i(\Omega^{i-1})} W_{\mathrm{e}}(\mathrm{D}y_{\mathrm{e}}(\xi^i))\, \mathrm{d}\xi^i + \int_{\Omega^{i-1}} W_{\mathrm{h}}(\tilde{z}(\xi^{i-1}))\, \mathrm{d}\xi^{i-1}$$
$$+ \int_{\Omega^{i-1}} D(\mathrm{D}\tilde{y}_{\mathrm{p}}(\xi^{i-1}), \tilde{z}(\xi^{i-1}) - \tilde{z}^{i-1}(\xi^{i-1}))\, \mathrm{d}\xi^{i-1} - \int_{\Omega^{i-1}} \tilde{f}^i(\xi^{i-1}) \cdot y_{\mathrm{e}}(\tilde{y}_{\mathrm{p}}(\xi^{i-1}))\, \mathrm{d}\xi^{i-1}. \quad (4.6)$$

Here, the *incremental* deformation maps $\xi^i \mapsto \tilde{y}_{\mathrm{p}}^i(\xi^i)$ and the *incremental* isotropic hardening parameters $\xi^i \mapsto \tilde{z}^i(\xi^i)$ are defined on $\Omega^i = \tilde{y}_{\mathrm{p}}^{i-1}(\Omega^{i-1})$ for $i = 1, \dots, N$. In particular, the plastic deformations y_{p}^i from (4.3) and those from (4.6) are related through $y_{\mathrm{p}}^i = \tilde{y}_{\mathrm{p}}^i \circ \cdots \circ \tilde{y}_{\mathrm{p}}^0$ so that the total deformation fulfills

$$y^i = y_{\mathrm{e}}^i \circ (\tilde{y}_{\mathrm{p}}^i \circ \cdots \circ \tilde{y}_{\mathrm{p}}^0).$$

At time t^i, the composition of the incremental deformation maps $\tilde{y}_{\mathrm{p}}^{i-1} \circ \cdots \circ \tilde{y}_{\mathrm{p}}^0$ represents the past plastic deformation history.

Before closing this discussion, let us note that the equivalence of formulations (4.3) and (4.6) is presently just formal. To make it rigorous one would need to ensure that the incremental deformation maps $\tilde{y}_{\mathrm{p}}^i$ are open. This is not guaranteed in the setting of Theorem 2.1 but would follow under some slightly stronger assumptions on the coercivity of H, as commented in Section 4.3.

4.8. Quasistatic evolution

Owing to bound (4.5) and the Helly Selection Principle [52], Theorem 2.1.24 one can extract a convergent subsequence of incremental solutions of (4.3) as the time step of the partition $\tau = \max_i(t^i - t^{-1})$ goes to 0. On the other hand, the setting of Theorem 2.1 is too weak to prove that the limit of such convergent subsequence is an *energetic solution* [52], Definition 1.6.1. The latter are trajectories $t \in [0,T] \mapsto (y_{\mathrm{e}}(t), y_{\mathrm{p}}(t), z(t)) \in A$ such that $(y_{\mathrm{e}}(0), y_{\mathrm{p}}(0), z(0)) = (y_{\mathrm{e}0}, y_{\mathrm{p}0}, z_0)$ and, for all $t \in [0,T]$, the conditions hold

$$E(y_{\mathrm{e}}(t), y_{\mathrm{p}}(t), z(t), t) \le E(\hat{y}_{\mathrm{e}}, \hat{y}_{\mathrm{p}}, \hat{z}, t) + \int_\Omega \Delta((\mathrm{D}y_{\mathrm{p}}(t,x), z(t,x)), (\mathrm{D}\hat{y}_{\mathrm{p}}(x), \hat{z}(x)))\, \mathrm{d}x \qquad \forall (\hat{y}_{\mathrm{e}}, \hat{y}_{\mathrm{p}}, \hat{z}) \in A, \quad (4.7)$$

$$E(y_{\mathrm{e}}(t), y_{\mathrm{p}}(t), z(t), t) + \mathrm{Diss}_{[0,t]}(y_{\mathrm{p}}, z) = E(y_{\mathrm{e}0}, y_{\mathrm{p}0}, z_0, 0) - \int_0^t \int_\Omega \partial_s f(s,x) \cdot y_{\mathrm{e}}(s, y_{\mathrm{p}}(s,x))\, \mathrm{d}x\, \mathrm{d}s$$
$$- \int_0^t \int_{\Gamma_{\mathrm{tr}}} \partial_s g(s,x) \cdot y_{\mathrm{e}}(s, y_{\mathrm{p}}(s,x))\, \mathrm{d}S\, \mathrm{d}s \qquad (4.8)$$

where the complementary energy $E : A \times [0, T] \to (0, \infty)$ and the total dissipation on $[0, t]$ are given as

$$E(y_\mathrm{e}, y_\mathrm{p}, z, t) = \int_{y_\mathrm{p}(\Omega)} W_\mathrm{e}(\mathrm{D}y_\mathrm{e}) \, \mathrm{d}\xi + \int_\Omega W_\mathrm{h}(z) \, \mathrm{d}x + \int_\Omega D(\mathrm{D}y_\mathrm{p}\mathrm{D}y_\mathrm{p0}^{-1}, z - z_0) \, \mathrm{d}x$$

$$- \int_\Omega f(t, x) \cdot y_\mathrm{e}(y_\mathrm{p}(x)) \, \mathrm{d}x - \int_{\Gamma_\mathrm{tr}} g(t, x) \cdot y_\mathrm{e}(y_\mathrm{p}(x)) \, \mathrm{d}S,$$

$$\mathrm{Diss}_{[0,t]}(y_\mathrm{p}, z) = \sup \left\{ \sum_{i=1}^{N} \int_\Omega \Delta((\mathrm{D}y_\mathrm{p}(t^{i-1}, x), z(t^{i-1}, x)), (\mathrm{D}y_\mathrm{p}(t^i, x), z(t^i, x))) \, \mathrm{d}x \right\}$$

and the supremum is taken over all partitions $\{0 = t^0 < t^1 < \cdots < t^N = t\}$ of $[0, t]$. Condition (4.7) is usually referred to as *global stability*. It expresses the optimality of the current state $(y_\mathrm{e}(t), y_\mathrm{p}(t), z(t))$ against competitors $(\hat{y}_\mathrm{e}, \hat{y}_\mathrm{p}, \hat{z})$ with respect to the complementary energy, augmented by the dissipation from $(y_\mathrm{p}(t), z(t))$ to $(\hat{y}_\mathrm{p}, \hat{z})$. Relation (4.8) imposes the balance between the actual complementary energy $E(y_\mathrm{e}(t), y_\mathrm{p}(t), z(t), t)$ plus the total dissipation $\mathrm{Diss}_{[0,t]}(y_\mathrm{p}, z)$ up to time t and initial energy $E(y_\mathrm{e0}, y_\mathrm{p0}, z_0, 0)$ plus work of the external actions. It hence corresponds to *energy conservation*.

Exactly as in [51], the obstruction in proving that incremental solutions converge to energetic ones is a lack of compactness which prevents to pass to the limit in the dissipation term, which is not upper semicontinuous with respect to the weak $W^{1,d}$ topology for y_p. The dissipation distance D is however continuous with respect to the strong $W^{1,1}$ topology. This opens the way to proving the existence of energetic solutions in case the energy is augmented by a strongly compactifying term in $W^{1,1}$ for the plastic strain $\mathrm{D}y_\mathrm{p}$. A classical choice [45] is to augment the energy by interfacial terms as

$$\tilde{E}(y_\mathrm{e}, y_\mathrm{p}, z, t) = E(y_\mathrm{e}, y_\mathrm{p}, z, t) + \frac{\kappa}{q} \int_\Omega |\mathrm{D}^2 y_\mathrm{p}(x)|^q \, \mathrm{d}x + \frac{h}{r} \int_\Omega |\mathrm{D}z(x)|^r \, \mathrm{d}x \qquad (4.9)$$

for some $q, r \geq 1$ ($q = 2$ in [45]), where $\kappa, h > 0$ play the role of interfacial energy densities. Such a modification of the energy allows to prove that the limit of incremental solutions is indeed energetic. An application of the general theory of [52], Theorem 2.1.6 entails the following.

Proposition 4.1. *Assume that data f and g fulfill* (4.2) *and that $(y_\mathrm{e0}, y_\mathrm{p0}, z_0)$ fulfill* (4.7) *for $t = 0$. Given a sequence of partitions of $[0, T]$ indexed by their diameter τ, indicate by $(y_\mathrm{e}^\tau, y_\mathrm{p}^\tau, z^\tau)$ a corresponding sequence of piecewise-constant-in-time interpolants of incremental solutions for energy $\tilde{E}$, starting from $(y_\mathrm{e0}, y_\mathrm{p0}, z_0)$. By letting $\tau \to 0$ we can find a (not relabeled) subsequence such that $(y_\mathrm{e}^\tau, y_\mathrm{p}^\tau, z^\tau)$ converges to $(y_\mathrm{e}, y_\mathrm{p}, z)$ pointwise weakly in $(W^{1,d}(\Omega)^d, W^{1,d}(\Omega)^d \times W^{2,q}(\Omega)^d, W^{1,r}(\Omega))$, where $(y_\mathrm{e}, y_\mathrm{p}, z)$ is an energetic solution for energy $\tilde{E}$.*

Nonlocal variants of the interfacial term in (4.9) in the form of Sobolov-Slobodeckij seminorms [63], Section 1.2.3 can be considered as well. Alternatively, by assuming isotropy of the elastic response, the quasistatic elastoplastic problem is reformulated in [28] solely in term of the *plastic metric tensor* $\mathrm{D}y_\mathrm{p}^\top \mathrm{D}y_\mathrm{p}$. Correspondingly, the above mentioned compactness is granted by the gradient term

$$\kappa \int_\Omega |\mathrm{D}(\mathrm{D}y_\mathrm{p}^\top(x)\mathrm{D}y_\mathrm{p}(x))| \, \mathrm{d}x.$$

More precisely, the problem is formulated in the space of plastic metric tensors $\mathrm{D}y_\mathrm{p}^\top \mathrm{D}y_\mathrm{p}$ of bounded variation and the latter intergral corresponds to the total variation of $\mathrm{D}y_\mathrm{p}^\top \mathrm{D}y_\mathrm{p}$. Another option is to resort to the *gradient-polyconvexity* frame by Benešová *et al.* [6] by augmenting the energy by the term

$$\frac{\kappa}{q} \int_\Omega |\mathrm{D}(\mathrm{cof}\, \mathrm{D}y_\mathrm{p}(x))|^q \, \mathrm{d}x.$$

Note that, as we assume $\det \mathrm{D}y_\mathrm{p} = 1$, this gradient term actually corresponds to a control on the gradient of the inverse $(\mathrm{D}y_\mathrm{p})^{-1}$. This ensures the compactness of $(\mathrm{D}y_\mathrm{p})^{-1}$, which in turn entails that $\mathrm{D}y_\mathrm{p} = \mathrm{cof}\,(\mathrm{D}y_\mathrm{p})^{-\top} = (\mathrm{cof}\,(\mathrm{D}y_\mathrm{p})^{-1})^{\top}$ is compact as well [6], Proposition 5.1.

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