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MATERIAL OPTIMIZATION FOR NONLINEARLY ELASTIC PLANAR BEAMS

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Abstract. We consider the problem of an optimal distribution of soft and hard material for nonlinearly elastic planar beams. We prove that under constant vertical load the optimal distribution involves no microstructure and is ordered, and we provide numerical simulations confirming and extending this observation.

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1. Introduction

In this article we study shape optimization for nonlinearly elastic planar beams. We begin by considering the nonlinear bending theory for elastic plates derived in [3]. It assigns to a deformation $u : S \to \mathbb{R}^3$ of a given reference configuration $S \subset \mathbb{R}^2$ the elastic energy (augmented by a potential energy term)

$$\int_S |\mathit{\Pi}|^2 - \int_S \widetilde{f} \cdot u. \tag{1.1}$$

Here $\mathit{\Pi}$ is the second fundamental form of the immersion u and $\widetilde{f} : S \to \mathbb{R}^3$ is an external force. The key constraint on the deformation u is that it must be an isometric immersion. Assume that S is a rectangle $(0,1) \times (-1,1)$ and prescribe clamped boundary conditions with $u(0, x_2) = (0, x_2, 0)$ and $\partial_{x_1} u(0, x_2) = (1, 0, 0)$ for $x_2 \in (-1, 1)$, and assume a constant load in vertical direction, and that the deformation u is of the form $u(x) = (\gamma_1(x_1), x_2, \gamma_2(x_1))$, for some curve $\gamma : (0,1) \to \mathbb{R}^2$. Such an essentially one-dimensional deformation is equivalent to a planar beam.

Allowing for the use of two different materials (soft and hard), a nonlinear planar beam subject to a load f is modelled by the following one-dimensional version of (1.1): for $\gamma \in W^{2,2}((0,1), \mathbb{R}^2)$ with $|\gamma'| = 1$ (this is the one-dimensional equivalent of the isometry constraint) and with $\gamma(0) = 0$ and $\gamma'(0) = (1, 0)$ and $f : [0,1] \to \mathbb{R}^2$, consider the energy functional

$$\mathcal{W}(\gamma, \chi) = \frac{1}{2} \int_0^1 A(t) \kappa^2(t) \, \mathrm{d}t - \int_0^1 f(t) \cdot \gamma(t) \, \mathrm{d}t. \tag{1.2}$$

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Here, κ denotes the curvature of γ and $A(t) = (1-\chi(t))a + \chi(t)b$, where $0 < a < b$ model the two material parameters and $\chi : (0,1) \to \{0,1\}$ describes the distribution of these two materials. Thus, χ and $1-\chi$ are the characteristic functions of the hard phase and the soft phase, respectively.
The compliance of a given material distribution χ is $\chi \mapsto \int_0^1 f \cdot \gamma + c_l \cdot \int_0^1 \chi$. Here c_l is a positive parameter, so that the second term penalizes the use of the harder phase b. We seek to find the optimal design χ, which is the one minimizing the cost functional

$$\chi \mapsto \int_0^1 f \cdot \gamma_\chi + c_l \int_0^1 \chi \tag{1.3}$$

among all $\chi : (0,1) \to \{0,1\}$, where γ_χ is the unique minimizer of $\gamma \mapsto \mathcal{W}(\gamma, \chi)$ for given χ and for f modelling constant load in the negative x_2-direction.
A typical question is whether this minimum is attained, *i.e.*, whether the optimal design is "classical" in the sense that no microstructure occurs. Ideally, one would then like to obtain more precise information about the optimal design. Our main analytical result answers these questions (see Thm. 5.8):

The optimal design under force $f = (0,-1)$ is classical. More precisely, there exists $t^* \in (0,1)$ such that $A = b$ on $(0,t^*)$ and $A = a$ on $(t^*,1)$.

Our numerical simulations confirm this result. We also consider numerically more general clamped boundary conditions allowing $\gamma'(0) \neq e_1$. We numerically compute stationary points for fixed material properties, and we study the optimal design under these more general conditions.

This article is organised as follows. In Section 2 we provide the precise setting. In particular, we rewrite the problem in terms of a new variable K (the phase of γ). This allows us to trade the nonlinear constraint $|\gamma'| = 1$ for a nonlinear equilibrium equation.
In Section 3 we study properties of critical points of $\mathcal{W}$ for generic material distributions, under various extra hypotheses on the force f. For force $f = (0,-1)$, to which most subsequent results are restricted, we prove uniqueness of minimisers.
In Section 4 we introduce the nonlinear compliance to be minimised by the optimal design, and we derive the relaxed optimal design problem by adapting to our nonlinear setting an approach used in linearised elasticity (*cf.* [1]).
In Section 5.1 we derive the equilibrium equation satisfied by material distributions minimising the compliance. It is formulated in terms of the adjoint variable of the phase K. In Section 5.2 we derive some properties about the adjoint variable. Combining these results with properties about K itself, in Section 5.3 we obtain our main result.
Finally, in Sections 6 and 7 we provide the numerical simulations mentioned earlier.

2. Setting

Throughout this article, I denotes the interval $(0,1)$, and $a < b$ are positive real numbers. As stated above, the isometry constraint imposed upon the deformation $\gamma : I \to \mathbb{R}^2$ of the reference configuration I is $|\gamma'| = 1$. For such γ, its curvature is $\kappa = \gamma'' \cdot n$, where $n = (\gamma')^\perp$ is the normal. We are interested in deformations which are clamped at the left edge, *i.e.*, $\gamma(0) = 0$ and $\gamma'(0) = e_1$; here and in what follows, (e_1, e_2) denotes the standard basis in $\mathbb{R}^2$.
For a given function $A \in L^\infty(I; [a,b])$ and given external force $f \in L^2(I; \mathbb{R}^2)$, the total energy (elastic plus potential energy) stored in the deformed configuration $\gamma \in W^{2,2}(I; \mathbb{R}^2)$ is given by (1.2). The typical ansatz for A is $A = \chi b + (1-\chi)a$ for $\chi : I \to \{0,1\}$. However, we will allow more general A as well, because we will eventually work with a relaxed problem.
In order to incorporate the constraint $|\gamma'| = 1$, we introduce the phase K by setting $K(t) = \int_0^t \kappa(s) \mathrm{d}s$ and

identify $\mathbb{C}$ and $\mathbb{R}^2$. Hence

$$\gamma(t) = \int_0^t \mathrm{e}^{iK(s)}\,\mathrm{d}s\,, \tag{2.1}$$

because $\gamma(0) = 0$. Inserting this and integrating by parts, we see that

$$\int_I f(t)\cdot\gamma(t)\mathrm{d}t = \int_I \left(\int_t^1 f(s)\,\mathrm{d}s\right)\cdot \mathrm{e}^{iK(t)}\mathrm{d}t. \tag{2.2}$$

Using (2.1) and (2.2), we see that the right-hand side of (1.2) agrees with $\mathcal{F}(K)$, where $\mathcal{F}$ is defined as follows: for all $A \in L^\infty(I)$ and $f \in L^2(I)$ we define $\mathcal{F} : W^{1,2}(I) \to \mathbb{R}$ by

$$\mathcal{F}(K) = \frac{1}{2}\int_I A(t)(K'(t))^2\ \mathrm{d}t - \int_I \left(\int_t^1 f(s)\ \mathrm{d}s\right)\cdot \mathrm{e}^{iK(t)}\ \mathrm{d}t. \tag{2.3}$$

Note that in terms of K, the clamped boundary condition is equivalent to $K(0) = 0$; the condition $\gamma(0) = 0$ is automatically taken into account by (2.1). Other boundary conditions can also be included into our scheme, *e.g.* following the ideas described in [5].

In view of the boundary conditions, the natural space on which the functional $\mathcal{F}$ given by (2.3) is defined, is the space

$$X = \{K \in W^{1,2}(I) : K(0) = 0\}.$$

From now on we regard $\mathcal{F}$ as a functional on X. Using the direct method in the calculus of variations one easily verifies that there exists a minimizer of $\mathcal{F}$ on X.

3. The state equation

In view of (2.3), for a given vector field f we introduce $F(t) = \int_t^1 f(s)\,\mathrm{d}s$. Observe that $F(1) = 0$ and $F' = -f$. With this notation, performing variations of $\mathcal{F}$ within the space X, we see that critical points satisfy

$$\int_I AK'(t)\ \phi'(t)\mathrm{d}t - \int_I F(t)\cdot i\,\mathrm{e}^{iK(t)}\ \phi(t)\mathrm{d}t = 0 \text{ for all } \phi \in C_0^\infty((0,\infty)). \tag{3.1}$$

By density, this is equivalent to the assertion that $K \in X$ satisfies the equilibrium equation

$$(AK')' = -i\,\mathrm{e}^{iK}\cdot F \text{ in } X', \tag{3.2}$$

where X' denotes the topological dual of X. This and formula (3.1) are weak formulations of $(AK')' = -i\,\mathrm{e}^{iK}\cdot F$ subject to the boundary conditions $K(0) = 0$ and $K'(1) = 0$. The condition $K'(1) = 0$ arises as the natural boundary condition. More precisely, we have the following lemma.

Lemma 3.1. *Let $A \in L^\infty(I;[a,b])$, $F \in W^{1,2}(I)$, and assume that $K \in W^{1,2}(I)$ is a distributional solution of*

$$(AK')' = -F\cdot ie^{iK} \text{ in } I. \tag{3.3}$$

Then $k := AK'$ is $C^1(\overline{I})$, K is Lipschitz, and (3.1) is equivalent to the classical formulation

$$k' = -ie^{iK}\cdot F \text{ on } I \text{ and } k(1) = 0. \tag{3.4}$$

Moreover, for any $t_0 \in [0,1]$ and any pair of prescribed values of $K(t_0)$ and of $k(t_0)$, there exists at most one distributional solution $K \in W^{1,2}(I)$ of (3.3).

Proof. The right-hand side of equation (3.3) is continuous up to the boundary by the hypotheses, so clearly k is C^1 up to the boundary. In particular, k is bounded, so $K' = A^{-1}k$ implies that K is Lipschitz.
To prove the asserted equivalence simply note that for $\phi \in C_0^\infty((0,\infty))$ the left-hand side of (3.1) equals

$$\int_I k(t)\phi'(t)\mathrm{d}t - \int_I F(t)\cdot ie^{iK(t)}\phi(t)\mathrm{d}t = k(1)\phi(1) - \int_I \Big(F(t)\cdot ie^{iK(t)} + k'(t)\Big)\,\phi(t)\,\mathrm{d}t.$$

This is zero for all $\phi \in C_0^\infty((0,\infty))$ if and only if (3.4) is satisfied.
To prove uniqueness, we combine (3.3) with the definition of k:

$$\begin{pmatrix} K \\ k \end{pmatrix}' = \begin{pmatrix} A^{-1}k \\ -F\cdot ie^{iK} \end{pmatrix}, \tag{3.5}$$

which is an ODE system of the form $u' = G(t,u)$ with uniformly Lipschitz G. Hence its solutions are uniquely determined by their value at a single point. □

3.1. The state equation for particular forces

In this section we derive some properties of K under suitable additional hypotheses on the force f. We are mainly interested in the case when $f = -e_2$, but for later reference some results are stated under somewhat weaker hypotheses.

Lemma 3.2. *Let $A \in L^\infty(I;[a,b])$, let $f_0 \in \mathbb{S}^1$ and let $F \in W^{1,2}(I;\mathbb{R}^2)$ be parallel to f_0 with $F\cdot f_0 > 0$ everywhere on I. Let $K \in W^{1,2}(I)$ solve*

$$(AK')' = -F\cdot ie^{iK} \text{ in the sense of distributions on } I; \tag{3.6}$$

so we make no assumptions on the boundary data of K. Then the following are true:

(i) If $F\cdot ie^{iK} = 0$ on a set of positive length, then K is constant on I, with $\mathrm{e}^{iK} \parallel f_0$.
(ii) If there exists $c \in \mathbb{R}$ such that $\{t \in I : AK'(t) = c\}$ has positive length, then K is constant on I, with $\mathrm{e}^{iK} \parallel f_0$. In particular, c must be 0.

Proof. Keeping in mind the results of Lemma 3.1, we see that $k = AK'$ is in $C^1(\bar{I})$. Set $n = ie^{iK}$. We claim that

$$k = 0 \text{ almost everywhere on the set } \{F\cdot n = 0\}. \tag{3.7}$$

In fact, almost everywhere on this set we have

$$0 = (F\cdot n)' = F'\cdot n - K'F\cdot \mathrm{e}^{iK} = -K'F\cdot \mathrm{e}^{iK},$$

where we have used that $F'\cdot n = 0$ because $F' \parallel F$. However, by hypothesis $0 \neq F = (F\cdot \mathrm{e}^{iK})\mathrm{e}^{iK}$. The latter equality is true because we are on the set where $F\cdot n = 0$ and because, by its definition, n spans the orthogonal complement of e^{iK}. Hence we conclude that indeed $K' = 0$ and thus $k = 0$ on the set in question. This proves (3.7).
To prove (i), note that $k = 0$ almost everywhere on $\{F\cdot ie^{iK} = 0\}$, by (3.7). In particular there exists a point $t_0 \in (0,1)$ with $k(t_0) = 0$ and $F(t_0)\cdot ie^{iK(t_0)} = 0$, too. But then clearly $K \equiv K(t_0)$ and $k \equiv 0$ is a solution of (3.3), because the direction of F is constant. By Lemma 3.1 this is the only solution. Finally, since $F(t_0) \perp ie^{iK(t_0)}$,

we know that $\mathrm{e}^{iK} \parallel f_0$.
To prove (ii), let $c \in \mathbb{R}$ be such that the set $\{k = c\}$ has positive length. As $k' = 0$ almost everywhere on $\{k = c\}$, by (3.6) we then have $F \cdot n = 0$ on a set of positive length. Hence part (i) implies that K is constant with e^{iK} parallel to f_0. □

Corollary 3.3. *Under the hypotheses of Lemma 3.2 and assuming, in addition, that f_0 is not parallel to $\mathrm{e}^{iK(0)}$, the function $k = AK'$ satisfies $k' \neq 0$ almost everywhere. In particular, the set $\{k = c\}$ has length zero for every $c \in \mathbb{R}$.*

Proof. By (3.6) we have $F \cdot ie^{iK} = 0$ almost everywhere on $\{k' = 0\}$. So if $k' = 0$ on a set of positive length, then Lemma 3.2 (i) would imply that K is a constant satisfying $\mathrm{e}^{iK} \parallel f_0$, contradicting the relation between f_0 and $K(0)$. □

3.2. Properties of minimisers

Lemma 3.4. *Let $\beta \in (-\pi, 0)$ and let $f \in L^2(I; \mathbb{R}^2)$ be such that $f \parallel \mathrm{e}^{i\beta}$ and $f \cdot \mathrm{e}^{i\beta} > 0$ almost everywhere on I. Let $A \in L^\infty(I; [a, b])$ and let K be an absolute minimiser of $\mathcal{F}$ among all $K \in X$. Then $K' \leq 0$ almost everywhere on I, and on $[0, 1)$ the function K takes values in $(\beta, 0]$.*

Proof. By hypothesis there exists $\widetilde{f} : I \to (0, \infty)$ such that $f(t) = \widetilde{f}(t)\mathrm{e}^{i\beta}$. Hence

$$\mathcal{F}(K) = \frac{1}{2}\int_I A(K'(t))^2 \,\mathrm{d}t - \int_I \left(\int_t^1 \widetilde{f}\right) \cos(K(t) - \beta) \,\mathrm{d}t. \tag{3.8}$$

We claim that $K > \beta$ on $[0, 1)$. To prove this, set

$$t_0 = \sup\{t \in [0, 1] : K > \beta \text{ on } [0, t)\}.$$

If the claim were false then $t_0 < 1$ and by continuity $K(t_0) = \beta$. Define

$$\widehat{K}(t) = \begin{cases} K(t) & \text{if } t \leq t_0 \\ \beta & \text{if } t > t_0. \end{cases}$$

Then $\cos(\widehat{K} - \beta) = 1$ on $[t_0, 1]$, so (3.8) shows that $\mathcal{F}(\widehat{K}) < \mathcal{F}(K)$ unless $K \equiv K(t_0) \in \beta + 2\pi\mathbb{Z}$ on $[t_0, 1]$. But then $k = AK' = 0$ on $[t_0, 1]$. Hence, Lemma 3.2 (ii) would imply that $K = K(t_0)$ everywhere on I, contradicting the boundary condition $K(0) = 0$. This concludes the proof of the claim.

In order to show that $K' \leq 0$ almost everywhere, let t_1 be the minimum over all t such that $-\int_0^t |K'| = \beta$; if no such t exists then set $t_1 = 1$. Observe that $t_1 > 0$. Define

$$\widetilde{K}(t) = \begin{cases} -\int_0^t |K'| & \text{for } t \in [0, t_1] \\ \beta & \text{for } t > t_1. \end{cases}$$

In view of (3.8) we have

$$\mathcal{F}(\widetilde{K}) - \mathcal{F}(K) = -\frac{1}{2}\int_{t_1}^1 A(K'(t))^2 \mathrm{d}t + \int_I \left(\int_t^1 \widetilde{f}\right) \Big(\cos(K(t) - \beta) - \cos(\widetilde{K}(t) - \beta)\Big) \,\mathrm{d}t.$$

The first term is nonpositive. We claim that the second term is negative (which would contradict the minimising property of K) unless $K' \leq 0$ almost everywhere. To prove this, it is enough to show that

$$\cos(\widetilde{K} - \beta) \geq \cos(K - \beta) \text{ on } I, \tag{3.9}$$

and that this inequality is strict on a set of positive length unless $K' \leq 0$ almost everywhere on I.
Since $\widetilde{K} = \beta$ on $[t_1, 1]$, inequality (3.9) is clearly satisfied on this interval.
On $[0, t_1]$ we have $|K| \leq |\beta|$. Hence (3.9) is satisfied here provided that

$$|\widetilde{K} - \beta| \leq |K - \beta| \tag{3.10}$$

and

$$|\widetilde{K} - \beta| < |K - (\beta + 2\pi)|. \tag{3.11}$$

But, by construction, $\widetilde{K} \leq K$, with strict inequality on a set of positive length unless $K' \leq 0$ almost everywhere. So since K, $\widetilde{K} \geq \beta$, inequality (3.10) follows.
Next notice that $\beta + 2\pi \geq \pi > K$ because $|K| \leq |\beta|$ on $[0, t_1]$. Hence

$$|\widetilde{K} - \beta|' = \widetilde{K}' \leq -K' = |K - (\beta + 2\pi)|'.$$

Hence the strict inequality (3.11) follows from the fact that it is satisfied at $t = 0$ because $\beta > -\pi$. □

Proposition 3.5. *Let $A \in L^\infty(I; [a, b])$, let $\beta \in [-\frac{\pi}{2}, 0)$ and let $f = \mathrm{e}^{i\beta}$. Then the following are true:*

(i) There exists precisely one global minimizer of $\mathcal{F}$ within X, namely the (unique) function $K \in X$ satisfying

$$K(t) \in (\beta, 0] \text{ for all } t \in [0, 1) \tag{3.12}$$

and solving

$$(AK')' = (1 - t) \sin(K - \beta) \text{ in } X'. \tag{3.13}$$

(ii) If $K \in X$ satisfies (3.13) and K takes values in $[\beta, \beta + \pi]$, then K satisfies (3.12).

Proof. The first statement in fact is a direct consequence of Lemma 3.4 and convexity of the energy density. We include the details for the reader's convenience. The function

$$W(t, z, p) = \frac{1}{2} A(t) p^2 - (1 - t) \cos(z - \beta). \tag{3.14}$$

satisfies

$$W\left(t, \frac{z + \widetilde{z}}{2}, \frac{p + \widetilde{p}}{2}\right) \leq \frac{1}{2} W(t, z, p) + \frac{1}{2} W(t, \widetilde{z}, \widetilde{p}) \tag{3.15}$$

whenever $t \in I$ and $p, \widetilde{p} \in \mathbb{R}$ and $z, \widetilde{z} \in [\beta - \frac{\pi}{2}, \beta + \frac{\pi}{2}]$. The inequality in (3.15) is strict unless $(z, p) = (\widetilde{z}, \widetilde{p})$. These facts follow from the concavity of the cosine function on $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

If $K_1, K_2 \in X$ are minimizers of $\mathcal{F}$ within X, then K_1 and K_2 satisfy (3.12) by Lemma 3.4. Set $\widehat{K} = \frac{1}{2}(K_1 + K_2)$. Since $\beta \in [-\frac{\pi}{2}, 0)$, the inclusion (3.12) implies that K_1 and K_2 take values in $[\beta, \beta + \frac{\pi}{2}]$. So (3.15) applies. Hence

$$\min_X \mathcal{F} \le \mathcal{F}(\widehat{K}) = \int_I W\left(t, \frac{K_1(t)+K_2(t)}{2}, \frac{K_1'(t)+K_2'(t)}{2}\right) \mathrm{d}t$$
$$\le \int_I \left(\frac{1}{2}W(t, K_1(t), K_1'(t)) + \frac{1}{2}W(t, K_2(t), K_2'(t))\right) \mathrm{d}t \le \min_X \mathcal{F}.$$

Hence we have equality throughout. Again by (3.15) this implies

$$W\left(\cdot, \frac{K_1+K_2}{2}, \frac{K_1'+K_2'}{2}\right) = \frac{1}{2}W(\cdot, K_1, K_1') + \frac{1}{2}W(\cdot, K_2, K_2') \text{ a.e. on } I, \tag{3.16}$$

so $K_1 = K_2$ almost everywhere. Hence the minimiser is unique.

Now let $K \in X$ satisfy (3.13) and (3.12) (the minimiser of $\mathcal{F}$ in X satisfies these conditions due to Lem. 3.4). By Lemma 3.4 it is enough to show that K is minimizing among functions satisfying (3.12). By convexity of $(z, p) \mapsto W(t, z, p)$ we have

$$W(t, \widetilde{z}, \widetilde{p}) \ge W(t, z, p) + (\partial_z W)(t, z, p)(\widetilde{z} - z) + (\partial_p W)(t, z, p)(\widetilde{p} - p)$$

whenever p, $\widetilde{p} \in \mathbb{R}$ and z, $\widetilde{z} \in [\beta - \frac{\pi}{2}, \beta + \frac{\pi}{2}]$. If $\widetilde{K} \in X$ satisfies (3.12), then we may insert $(z, p) = (K, K')$ and $(\widetilde{z}, \widetilde{p}) = (\widetilde{K}, \widetilde{K}')$. Then we integrate and use the equation satisfied by K to find that indeed $\mathcal{F}(\widetilde{K}) \ge \mathcal{F}(K)$.

To prove part (ii), note that by Lemma 3.1 and (3.13) the function $k = AK'$ satisfies

$$k(t) = -\int_t^1 (1-s)\sin(K(s) - \beta)\ \mathrm{d}s,$$

which is nonpositive because K takes values in $[\beta, \beta + \pi]$. Hence K is nonincreasing. In particular, $K \le K(0) = 0$ on I. Moreover, if $K(t_0) = \beta$ for some $t_0 < 1$, then $K = \beta$ on $[t_0, 1]$, because K is nonincreasing and by hypothesis $K \ge \beta$. This would contradict *e.g.* Lemma 3.2 (ii). ∎

4. Relaxation by the homogenization method

For $\theta \in [0, 1]$ define

$$A(\theta) = \left(\frac{1-\theta}{a} + \frac{\theta}{b}\right)^{-1}. \tag{4.1}$$

If $\theta = \chi$ only takes values in $\{0, 1\}$, then

$$A(\chi) = (1-\chi)a + \chi b.$$

The coefficient (4.1) will arise naturally for the usual reason: if $\chi_n \in L^\infty(I; \{0, 1\})$ converges weakly-$*$ in $L^\infty(I)$ to θ, then

$$((1-\chi_n)a + \chi_n b)^{-1} = (A(\chi_n))^{-1} = \frac{1-\chi_n}{a} + \frac{\chi_n}{b} \overset{*}{\rightharpoonup} (A(\theta))^{-1} \tag{4.2}$$

in $L^\infty(I)$. We define the compliance $J : X \times L^\infty(I;[0,1]) \to \mathbb{R}$ as follows:

$$J(K,\theta) = \int_I F(t)\cdot \mathrm{e}^{iK(t)}\mathrm{d}t + c_l\int_I \theta(t)\,\mathrm{d}t.$$

The constant c_l is strictly positive, so the second term penalises the use of the hard material.
The optimal design χ should minimise $J(K,\chi)$, under the constraint that K be a solution to (3.2) with $A = A(\chi)$, among all $\chi \in L^\infty(I;\{0,1\})$. Following the work [1] in the context of linearised elasticity, we begin by deriving the corresponding relaxed problem and obtain the following result:

Proposition 4.1. *Let $\theta_n \in L^\infty(I;[0,1])$ and let $\theta \in L^\infty(I;[0,1])$ be such that $\theta_n \overset{*}{\rightharpoonup} \theta$ weakly-$*$ in $L^\infty(I)$ as $n\to\infty$. Let $F \in W^{1,2}(I;\mathbb{R}^2)$ and let $K_n \in X$ solve*

$$(A(\theta_n)K')' = -i\,\mathrm{e}^{iK}\cdot F \text{ in } X'.$$

Then, after passing to a subsequence, K_n converges weakly in $W^{1,2}(I)$ to a solution $K \in X$ of

$$(A(\theta)K')' = -i\,\mathrm{e}^{iK}\cdot F \text{ in } X'. \tag{4.3}$$

Proposition 4.1 is a consequence of the fact that under its hypotheses we have $(A(\theta_n))^{-1} \overset{*}{\rightharpoonup} (A(\theta))^{-1}$ weakly-$*$ in L^∞, and of the following lemma.

Lemma 4.2. *Let $F\in W^{1,2}(I;\mathbb{R}^2)$, let $A_n \in L^\infty(I;[a,b])$ and let $K_n \in X$ be a solution of (3.2) with $A = A_n$, and suppose that there is $B \in L^\infty(I)$ such that $A_n^{-1} \overset{*}{\rightharpoonup} B$ weakly-$*$ in $L^\infty(I)$. Then there exists $K\in X$ such that, after passing to subsequences, $K_n \rightharpoonup K$ in $W^{1,2}(I)$. Moreover, K solves (3.2) with $A = B^{-1}$.*

Proof. The state equation (3.2) implies an *a priori* estimate for K_n: in fact, testing (3.2) with K_n we have

$$a\int_I (K_n')^2\,\mathrm{d}t \le \int_I A_n(K_n')^2\mathrm{d}t = \int_I K_n\; ie^{iK_n}\cdot F\,\mathrm{d}t \le \|F\|_{L^2}\|K_n\|_{L^2}.$$

Since $K_n(0) = 0$ we have $\|K_n\|_{L^2} \le \|K_n'\|_{L^2}$, so the above estimate implies

$$\|K_n\|_{L^2} \le \frac{1}{a}\|F\|_{L^2}. \tag{4.4}$$

But then using the above chain of estimates again,

$$\|K_n'\|_{L^2} \le \frac{1}{a}\|F\|_{L^2}.$$

Hence, after taking subsequences, there is $K \in X$ such that $K_n \rightharpoonup K$ in $W^{1,2}(I)$.
Since $(A_nK_n')(1) = 0$ by Lemma 3.1, we can write (3.2) as

$$K_n' = A_n^{-1}\int_t^1 F\cdot ie^{iK_n}\,\mathrm{d}t.$$

Since $K_n \to K$ uniformly, we have

$$\int_t^1 F\cdot ie^{iK_n}\mathrm{d}t \to \int_t^1 F\cdot ie^{iK}\,\mathrm{d}t$$

uniformly on I. Since $A_n^{-1} \overset{*}{\rightharpoonup} B$ in L^∞, we deduce that K satisfies

$$K'(t) = B \int_t^1 F \cdot ie^{iK}\,\mathrm{d}t.$$

This is equivalent to (3.2) with $A = B^{-1}$. □

Proposition 4.1 can be viewed as a homogenization result for the equilibrium equation of the nonlinear bending energy functional (1.2). Since the nonlinearity is of lower order, its proof closely resembles arguments used in the context of linearised elasticity, *cf.* [1].
Related (general) homogenization results for nonlinearly elastic rods can be found in [6], where the homogenization process is carried out on a variational level (not on the equilibrium equation). The starting point in [6] is the genuinely three-dimensional nonlinear elasticity functional for a rod of finite positive thickness, and the homogenization limit is combined with the zero thickness limit.

5. Optimal design

Throughout this chapter we assume that $f = -e_2$. So

$$F(t) = \int_t^1 f = (t-1)e_2.$$

Motivated by Lemma 3.4 we introduce the convex subset

$$\widetilde{X} = \left\{ K \in X : K(t) \in \left(-\frac{\pi}{2}, 0\right] \text{ for all } t \in [0,1) \right\}.$$

Observe that $K(1) = -\frac{\pi}{2}$ is not excluded.
Proposition 3.5 shows that for every $\theta \in L^\infty(I;[0,1])$ there exists a unique solution $K \in \widetilde{X}$ of (4.3). Abusing notation we will henceforth denote this solution $K \in \widetilde{X}$ by $K(\theta)$. We define $\widehat{J} : L^\infty(I;[0,1]) \to \mathbb{R}$ by

$$\widehat{J}(\theta) = J\left(K(\theta), \theta\right).$$

Since now $F(t) = (t-1)e_2$ this amounts to

$$\widehat{J}(\theta) = \int_I (t-1)\sin K(\theta)(t)\mathrm{d}t + c_l \int_I \theta(t)\,\mathrm{d}t.$$

Proposition 5.1. *The infimum*

$$\inf_{\theta \in L^\infty(I;[0,1])} \widehat{J}(\theta) \tag{5.1}$$

is attained and agrees with

$$\inf_{\chi \in L^\infty(I;\{0,1\})} \widehat{J}(\chi). \tag{5.2}$$

Proof. In order to see that (5.1) is attained, let $\theta_n \in L^\infty(I;[0,1])$ be such that $\widehat{J}(\theta_n)$ converges to (5.1). After taking subsequences (not relabelled), we may assume that $\theta_n \overset{*}{\rightharpoonup} \theta$ in $L^\infty(I)$. Hence by Proposition 4.1 we know that after taking another subsequence, $K_n = K(\theta_n)$ converge weakly in $W^{1,2}(I)$ to a solution $K \in X$ of (4.3) (with $F = (t-1)e_2$). We know that K takes values in $[-\frac{\pi}{2}, 0]$, because the same is true of K_n and because this interval is a convex and closed set. Hence $K \in \widetilde{X}$ by Proposition 3.5 (ii). Hence $K = K(\theta)$ by Proposition 3.5 (i). Hence $\widehat{J}(\theta_n) \to \widehat{J}(\theta)$.

In order to prove that (5.2) does not exceed (5.1) (the other estimate is trivial), let θ minimise $\widehat{J}$ among all functions in $L^\infty(I; [0,1])$. Let $\chi_n \in L^\infty(I; \{0,1\})$ be such that $\chi_n \overset{*}{\rightharpoonup} \theta$ weakly-$*$ in $L^\infty(I)$. Then as before we see that $K(\chi_n)$ subconverge to $K(\theta)$ weakly in $W^{1,2}$, and therefore $\widehat{J}(\chi_n) \to \widehat{J}(\theta)$. $\square$

The next natural question is whether microstructure actually occurs, *i.e.*, whether the minimum in (5.2) is attained or not. Following the abstract approach in [4] we introduce the operator $G : X \times L^\infty(I) \to X'$ by setting

$$G(K, \theta) = (A(\theta)K')' - (1-t)\cos K.$$

The optimal design is a function $\theta : I \to [0,1]$ minimising $\widehat{J}(\theta) = J(K(\theta), \theta)$, *i.e.*, minimising $J(K, \theta)$ subject to the constraint that K be the (unique) solution in $\widetilde{X}$ of the state equation $G(K, \theta) = 0$ in X', *i.e.*,

$$(A(\theta)K')' = (1-t)\cos K \text{ in } X'. \tag{5.3}$$

As mentioned earlier, by Proposition 3.5, this K (denoted by $K(\theta)$) is the unique absolute minimiser of the functional $\mathcal{F}$ (with $A = A(\theta)$ and $f = -e_2$).

In what follows, we will denote by D the Fréchet derivative of a functional depending on one variable. For functionals depending on more than one variable, we denote by D_i the partial Fréchet derivative with respect to the i-th variable.

Lemma 5.2. *For $\varepsilon > 0$ small enough (depending on a and b), the map $K : L^\infty(I; (-\varepsilon, 1+\varepsilon)) \to W^{1,2}(I)$ taking θ into $K(\theta)$ is continuously Fréchet differentiable.*

Proof. It is easy to verify that $G : X \times L^\infty(I; (-\varepsilon, 1+\varepsilon)) \to X'$ is continuously Fréchet differentiable. Its partial Fréchet derivative $D_1 G(K, \theta)$ with respect to K is the operator taking $\eta \in X$ into

$$D_1 G(K, \theta)(\eta) = (A(\theta)\eta')' + (1-t)(\sin K)\eta. \tag{5.4}$$

For $K \in \widetilde{X}$ the linear operator $D_1 G(K, \theta) : X \to X'$ is easily seen to be bijective, because $\sin K$ is nonpositive for $K \in \widetilde{X}$. Hence the claim follows from the implicit function theorem. $\square$

Clearly $D_1 J(K, \theta) = F \cdot ie^{iK} = -(1-t)\cos K$ and $D_2 J(K, \theta) = c_l$. Next we compute the dual operator of $D_2 G(K, \theta)$, which we will need later on. The partial derivative $D_2 G(K, \theta) : L^\infty(I; (-\varepsilon, 1+\varepsilon)) \to X'$ is the linear map given by

$$D_2 G(K, \theta)(\eta) = \left(\dot{A}(\theta)\ \eta\ K'\right)'.$$

Here

$$\dot{A}(\theta) = \left(\frac{1}{a} - \frac{1}{b}\right) A^2(\theta), \tag{5.5}$$

where as always $A(\theta)$ is as in (4.1). Using this, we see that the dual operator to $D_2 G(K, \theta)$ is

$$D_2 G(K, \theta)^* : X \to (L^\infty(I; (-\varepsilon, 1+\varepsilon)))'$$

given by

$$D_2G(K,\theta)^* = \left(\left(\frac{1}{a}-\frac{1}{b}\right)A^2(\theta)K'\right)'.$$

5.1. Equilibrium equation for the optimal design

We will now derive the equilibrium equation satisfied by $\widehat{J}$-minimising θ. Denoting the Fréchet derivative of K with respect to θ by DK, we compute (using Lemma 5.2)

$$D\widehat{J}(\theta)(\eta) = D_1J(K(\theta),\theta)\,(DK(\theta)\eta) + D_2J(K(\theta),\theta)\eta$$

for all $\eta \in L^\infty(I)$, that is,

$$D\widehat{J}(\theta) = (DK(\theta))^*\,(D_1J(K(\theta),\theta)) + D_2J(K(\theta),\theta).$$

In order to compute the first term on the right-hand side, we differentiate the state equation $D_1G(K(\theta),\theta) = 0$ with respect to θ and take adjoints to see that

$$(DK(\theta))^* = -D_2G\left(K(\theta),\theta\right)^*\left((D_1G(K(\theta),\theta))^{-1}\right)^*.$$

Therefore, denoting by $P \in X$ the unique solution in X of

$$(D_1G(K(\theta),\theta))^*\,(P) = -D_1J(K(\theta),\theta) \text{ in } X', \tag{5.6}$$

we have

$$D\widehat{J}(\theta) = D_2G(K(\theta),\theta)^*P + D_2J(K(\theta),\theta). \tag{5.7}$$

By the computations above equation (5.7) becomes

$$D\widehat{J}(\theta) = -\left(\frac{1}{a}-\frac{1}{b}\right)kp + c_l. \tag{5.8}$$

Here and in what follows we write $p = A(\theta)P'$ and $k = A(\theta)K(\theta)'$. The adjoint equation (5.6) becomes

$$(A(\theta)P')' = p' = (1-t)\left(\cos K(\theta) - P\sin K(\theta)\right). \tag{5.9}$$

(In particular, $p \in C^1(\bar{I})$.) The equilibrium equation satisfied by designs θ minimising $\widehat{J}$ asserts that

$$D\widehat{J}(\theta)(\eta) \geq 0$$

for all $\eta \in L^\infty(I)$ satisfying $\eta \geq 0$ almost everywhere on the set $\{\theta = 0\}$ and satisfying $\eta \leq 0$ almost everywhere on the set $\{\theta = 1\}$.
This leads to the following pointwise conditions:

$$\left(\frac{1}{a}-\frac{1}{b}\right)kp - c_l \begin{cases} \leq 0 & \text{on } \{\theta = 0\} \\ \geq 0 & \text{on } \{\theta = 1\} \\ = 0 & \text{on } \{\theta \in (0,1)\}. \end{cases}$$

Since $\left(\frac{1}{a}-\frac{1}{b}\right)>0$, with

$$\lambda = \left(\frac{1}{a}-\frac{1}{b}\right)^{-1} c_l \tag{5.10}$$

this can be written as follows:

$$kp \begin{cases} \leq \lambda & \text{on } \{\theta = 0\} \\ \geq \lambda & \text{on } \{\theta = 1\} \\ = \lambda & \text{on } \{\theta \in (0,1)\}. \end{cases} \tag{5.11}$$

5.2. Properties of the adjoint variable

For brevity, for given θ we will from now on write K instead of $K(\theta)$ and A instead of $A(\theta)$, and $P \in X$ will always denote the solution of (5.6), and $k = AK'$ and $p = AP'$.
Since the right-hand side of (5.9) is continuous, we see that $p \in C^1([0,1])$. Recall from (5.3) that K satisfies

$$(AK')' = (1-t)\cos K \text{ and } K(0) = 0,\ K'(1) = 0,$$

and K is decreasing and on $[0,1)$ takes values in $(-\frac{\pi}{2}, 0]$.
In order to study the behaviour of P, we introduce $\rho : I \to \mathbb{R}$ by

$$\rho(t) = \cot K(t) = \frac{\cos K(t)}{\sin K(t)},$$

so clearly $\rho < 0$ on $(0,1)$, and $\rho(t) \to -\infty$ as $t \downarrow 0$. Moreover,

$$\rho' = -\frac{K'}{\sin^2 K} \text{ and } A\rho' = -\frac{k}{\sin^2 K}. \tag{5.12}$$

By Lemma 3.1 we see that $A\rho'$ is continuous, positive and strictly decreasing on $(0,1)$. The relevance of ρ is that p' is a positive multiple of $Q := P - \rho$, namely

$$p' = -(1-t)\cdot \sin K \cdot Q.$$

In particular, $p' = 0$ if and only if $Q = 0$, and the sign of p' equals that of Q.
We introduce

$$q := AQ' = p - A\rho' = p + \frac{k}{\sin^2 K}$$

and we compute

$$q' = -(1-t)Q\sin K + \left(\frac{k}{\sin^2 K}\right)'. \tag{5.13}$$

Lemma 5.3. *We have $q(1) = p'(1) = p(1) = P(0) = 0$, as well as $p(0) < 0$ and $p'(0) = 1$.*

Proof. We have $P(0) = 0$ because $P \in X$, and $p(1) = 0$ (hence $q(1) = 0$ since $k(1) = 0$) because (5.9) is an equation in X' involving natural boundary conditions. From (5.9) and since $K(0) = P(0) = 0$, we have

$$p'(0) = \cos K(0) - P(0)\sin K(0) = 1.$$

Also from (5.9), we see $p'(1) = 0$. Finally, the inequality $p(0) < 0$ follows easily from the boundary conditions $p(1) = 0$ and $P(0) = 0$ and the observation from (5.9) that $p' \geq 0$ on $\{P \geq 0\}$. Indeed, assuming $p(0) > 0$ we obtain a straightforward contradiction to $p(1) = 0$ and assuming $p(0) = 0$ we deduce that $p' \equiv 0$, which implies $K \equiv -\frac{\pi}{2}$ and this contradicts $K(0) = 0$. □

Lemma 5.4. *There is $t_0 \in [0, 1]$ such that $Q > 0$ on $[0, t_0)$ and $Q \leq 0$ on $[t_0, 1]$.*

Proof. As $Q(0) = +\infty$, it is enough to show that $Q' \leq 0$ almost everywhere on $\{Q \geq 0\}$. As A is positive, this is equivalent to the assertion that $q \leq 0$ almost everywhere on $\{Q \geq 0\}$.
By (5.13) we have

$$q' \geq 0 \text{ almost everywhere on } \{Q \geq 0\} \tag{5.14}$$

because $k \cdot \sin^{-2} K$ is an increasing function. So if t_0 is such that $Q(t_0) \geq 0$ and $q(t_0) > 0$, then q is nondecreasing on $(t_0, 1)$, which can be seen as follows: Since $q(t_0) > 0$, by continuity of q the set

$$\{t \in (t_0, 1) : q(t) > 0 \text{ on } (t_0, t)\}$$

is nonempty. Denote by t_1 the supremum over this set. Then Q is increasing on (t_0, t_1) because $Q' = q/A$. Since $Q(t_0) \geq 0$, this implies that $Q \geq 0$ on (t_0, t_1). Hence q is nondecreasing on (t_0, t_1) by (5.14). Hence $q(t_1) > 0$, so by continuity necessarily $t_1 = 1$.
Therefore, one obtains $q(1) > 0$, contradicting Lemma 5.3. □

Proposition 5.5. *There exists $t_0 \in (0, 1]$ such that $p' > 0$ on $[0, t_0)$ and $p' \leq 0$ on $[t_0, 1]$. Moreover, the following is true:*

- *If $t_0 = 1$ then $p < 0$ on $[0, 1)$.*
- *If $t_0 < 1$ then there exists $t_1 \in (0, t_0)$ such that $p < 0$ on $[0, t_1)$ and $p > 0$ on $(t_1, 1)$.*

Proof. The first part follows from Lemma 5.4 and our initial observation that the sign of p' is determined by that of Q.
To prove the second part, first note that if $t_0 = 1$ then $p < 0$ on $[0, 1)$ because $p(1) = 0$ and p is increasing.
If $t_0 < 1$ then $p > 0$ on $(t_0, 1)$. In fact, since p is nonincreasing on this interval and since $p(1) = 0$, if we had $p(t') = 0$ at some $t' \in (t_0, 1)$ then $p = 0$ on $(t', 1)$. By (5.9) this would imply that $P = \cot K$ on this interval. And by $AP' = p = 0$ the function $\cot K$ and therefore K and thus k would be constant on $(t', 1)$. This would contradict Corollary 3.3.
Since p is strictly increasing on $(0, t_0)$ and $p(0) < 0$ by Lemma 5.3, and since $p(t_0) > 0$, by continuity there exists precisely one t_1 as in the statement. □

Corollary 5.6. *There exists $t_2 \in (0, 1]$ such that $kp > 0$ and kp is strictly decreasing on $(0, t_2)$ and $kp \leq 0$ on $[t_2, 1]$. In particular, the set $\{kp = c\}$ has zero length for any $c > 0$.*

Proof. Recall that k is negative and strictly increasing. Let t_0 and t_1 be as in the conclusion of Proposition 5.5. If $t_0 = 1$ then p is negative and strictly increasing $[0, 1)$, so kp is positive and strictly decreasing. In this case, therefore, the claim is satisfied with $t_2 = 1$. Finally, if $t_0 \in (0, 1)$, then the claim is satisfied with $t_2 = t_1$. □

The above proof of Lemma 5.4 is self-contained. For variety, we also include a shorter proof based on the following maximum principle:

Lemma 5.7. *Let $T > 0$, let g, $m : [0,T] \to \mathbb{R}$ be measurable with $m > 0$ and $g \leq 0$ almost everywhere. Let u be locally absolutely continuous and such that mu' is locally absolutely continuous, and such that*

$$(mu')' + gu \leq 0 \text{ almost everywhere on } (0,T)$$

and $u(0)$, $u(T) \geq 0$. Then $u \geq 0$ on $(0,T)$.

A proof of Lemma 5.7 can be found in [9]. In order to apply Lemma 5.7, we extend A, P and K (and thus ρ) evenly to $[0,2]$ by setting

$$B(1+t) = B(1-t) \text{ for } t \in (0,1]$$

for $B = A$, P, K. We introduce the operator $Lu = (Au')' + ((1-t)\sin K)\,u$. So (5.13) becomes

$$LQ = \left(\frac{k}{\sin^2 K}\right)' \text{ on } (0,2). \tag{5.15}$$

As mentioned below (5.12), the quantity $k \cdot \sin^{-2} K$ is strictly increasing on $(0,1)$, hence the right-hand side of (5.15) is positive on $(0,1)$. As A and K are even about 1, the function $k = AK'$ is odd about 1, hence so is $k \cdot \sin^{-2} K$. Therefore the right-hand side of (5.15) is positive on $(0,2)$.

As $P(0) = 0$ and $\rho(0) = -\infty$, either $Q > 0$ on $[0,1)$ or there exists a smallest $t_0 \in (0,1)$ such that $Q(t_0) = 0$. In the latter case, in view of (5.15) and since both P and ρ are even about 1, the function Q satisfies the boundary value problem

$$\begin{aligned} LQ &> 0 \text{ in } (t_0, 2-t_0) \\ Q &= 0 \text{ on } \partial(t_0, 2-t_0). \end{aligned}$$

Applying Lemma 5.7 with $u = -Q$ shows that $Q \leq 0$ on $[t_0, 2-t_0]$; in particular on $[t_0, 1]$. And by definition $Q > 0$ on $[0, t_0)$. Therefore we have recovered Lemma 5.4.

5.3. The optimal design

Since $c_l > 0$ and $0 < a < b$, we have $\lambda > 0$ by its definition in (5.10). Combining (5.11) with Corollary 5.6, we therefore obtain the following result (with $t^* < t_2$):

Theorem 5.8. *The optimal design is classical and ordered. More precisely, if θ is a critical point of $\widehat{J}$ within $L^\infty(I;[0,1])$, then there exists $t^* \in (0,1)$ such that $A(\theta) = b$ almost everywhere on $(0,t^*)$ and $A(\theta) = a$ almost everywhere on $(t^*, 1)$.*

In [2] the *worst* design for nonlinearly elastic membranes was studied, with a nonlinear compliance consisting of the sum of the compliance used here plus the elastic energy. (We refer to [8] for a discussion of various choices of compliances in the context of nonlinear elasticity.)
In our setting, too, this worst design problem is much easier to handle than the optimal design. In fact, there is no need to consider the adjoint variable p: instead of Corollary 5.6 one merely needs the observation that k^2 is not constant on any set of positive length, which follows readily *via* the Leibniz rule from the results in Section 3.2. One can then show that the worst design is also classical and ordered. As expected, the order is reversed with respect to the optimal design: first the soft phase is used and then the hard phase. We leave the details to the interested reader.

6. Numerical discretization of the state equation

In this section we consider a force $f = -\delta\ e_2$ for $\delta \in \mathbb{R}$, and we allow inhomogeneous clamped boundary conditions $K(0) = K_0$ (*i.e.* $\gamma'(0) = \mathrm{e}^{iK_0}$). The corresponding curve γ is given by

$$\gamma(t) = \int_0^t \mathrm{e}^{i(K(s)+K_0)}\ \mathrm{d}s\,,$$

where $K \in X$. The associated stored energy is given by

$$E(K) = \int_0^1 \frac{1}{2}A(K')^2 + \delta(1-t)\sin(K(t)+K_0)\ \mathrm{d}t\,.$$

We use Newton's method to find local minimizers of the stored energy. It requires to compute the first and second derivatives of the stored energy:

$$DE(K)(\phi) = \int_0^1 AK'\phi' + \delta(1-t)\cos(K(t)+K_0)\phi\ \mathrm{d}t\,,$$

$$D^2E(K)(\phi)(\psi) = \int_0^1 A\phi'\psi' - \delta(1-t)\sin(K(t)+K_0)\phi\psi\ \mathrm{d}t\,,$$

where $\phi, \psi \in X$.

Observe that considering $f = -\delta e_2$ and initial data $K(0) = K_0$ is equivalent to choosing $f = \delta \mathrm{e}^{i\beta}$ for suitable β, with initial data $K(0) = 0$. Translated into the former setting, Proposition 3.5 guarantees the uniqueness of minimisers provided $K_0 \in [-\frac{\pi}{2}, 0]$.

In this section, however, we are mainly interested in the case $K_0 \in [0, \frac{\pi}{2})$. Existence of a minimizer is still guaranteed by the direct method in the calculus of variations. But for $K_0 > 0$ one no longer expects uniqueness. We will solve the state equation in the phase variable K with the intention to explore experimentally some characteristic solutions of the state equation.

For the numerical implementation we consider piecewise affine and continuous finite elements. More precisely, we consider an equidistant grid with N nodes $x_n = \frac{n}{N-1}$ for $n = 0, \dots, N-1$ and associated $N-1$ cells $C_n = (x_{n-1}, x_n)$ for $n = 1, \dots, N-1$. The corresponding grid width is given by $h = \frac{1}{N-1}$. Then we approximate K in the space V_h of functions, which are continuous and piecewise affine on the above cells. Here and in what follows, we identify finite element functions and the corresponding coordinate vectors in the hat basis. We denote the nodal basis functions of V_h by ξ_h^n for $n = 0, \dots, N-1$. For the numerical integration, we choose a Gaussian quadrature with Q quadrature points per element, where we use $Q = 5$ in the implementation and obtain the approximation

$$\int_0^1 g(t)\mathrm{d}t \approx \sum_{C_n} |C_n| \sum_{q=0}^{Q-1} w_q^n g(x_q^n) \tag{6.1}$$

with w_q^n denoting the weight at the quadrature point x_q^n and $|C_n| = h$ the length of cell C_n. Applying this quadrature to the stored energy and its derivatives, we get a discrete stored energy E_h on V_h and associated derivatives DE_h, and D^2E_h.

Testing the first derivative with the basis functions, we obtain a vector $R[K] := (R[K]_j)_{j=0,\dots,N-1}$ with $R[K]_j = DE_h(K_h)(\xi_h^j)$. Analogously, testing the second derivative, we are led to a matrix $M[K] = (M[K]_{ij})_{i,j=0,\dots,N-1}$ with

$$M[K]_{ij} = D^2E_h(K_h)(\xi_h^j)(\xi_h^i)\,.$$

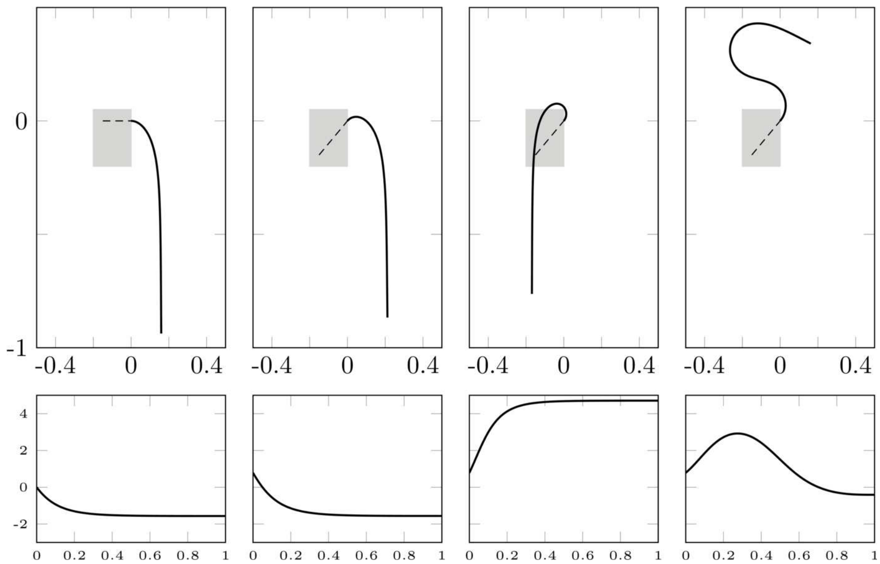

FIGURE 1. Different solutions of the state equation (*top row*) with corresponding phase variable K (*bottom row*) are shown (*from left to right*): simple configurations with $K_0 = 0$ and $K_0 = \frac{\pi}{4}$, a twisted beam with $K_0 = \frac{\pi}{4}$, and an S-shaped configuration with $K_0 = \frac{\pi}{4}$. Here, we have chosen $\delta = 100$, $A = 1$.

Because of the clamped boundary conditions we modify the first row and column of $M[K]$ by setting $M[K]_{0,0} = 0$ and $M[K]_{0,j} = 0 = M[K]_{i,0}$ for $i, j = 1, \dots, N-1$, and we set $R[K]_0 = 0$. Finally, Newton's method for minimization of the stored energy computes a sequence $(K_h^i)_{i=1,\dots}$ with

$$M[K_h^i](K_h^{i+1} - K_h^i) = R[K_h^i]$$

for given initial data K_h^0. To cope with the nonlinearity, we use a multilevel scheme, first solving the problem on a coarse grid, prolongate the obtained result onto a finer grid, and proceed iteratively. Here, we take into account a dyadic sequence $N = 2^l + 1$ with $l = L_c, \dots, L_f$, where we usually use $L_c = 3$ and $L_f \in 9, 10, 11$.
For a homogeneous material $A \equiv 1$ we experimentally observe essentially three types of stationary points (see Fig. 1). First, there is of course a simple configuration where the curve is just turning downwards. In fact, this appears to be an approximation of the global minimizer of the energy functional E discussed in the first part of this article. Secondly, we get a twisted curve, which can be interpreted physically as turning the free end of the beam to the other side. These two configurations are relatively stable under a change of material, *i.e.*, taking some simple (resp. twisted) beam as initialization for a different material, the computed discrete solution in our experiments always turned out to be a simple (resp. twisted) beam again. However, there is also a highly unstable configuration in between, where the beam neither decides to fall towards the left side nor towards the right side.

7. COMPUTING OPTIMAL DESIGNS

Our numerical scheme to compute the optimal design is based on a phase field approach. Following [8], we consider a phase field function $v : [0,1] \to \mathbb{R}$ which takes values either approximately 1 for hard material with elasticity constant b and approximately -1 for soft material with elasticity constant a. Thus, the material coefficient A is assumed to be a function of v and at each point $t \in [0,1]$

$$A(v) = b\chi(v) + a(1 - \chi(v)),$$

where we approximate the characteristic function χ by

$$\chi(v) = \frac{1}{4}(v+1)^2 \,.$$

To ensure the phase-field function to be smooth and essentially to take values $v \in \{-1, 1\}$, we use the 1D version of the perimeter functional proposed by Modica and Mortola [7]

$$\mathrm{Per}^{\epsilon}(v) = \frac{1}{2}\int_0^1 \epsilon(v')^2 + \frac{1}{\epsilon}\frac{9}{16}(v^2-1)^2 \;\mathrm{d}t$$

as regularizer, where ϵ describes the width of the diffuse interface. Further, the definition of χ allows us to approximate the length covered by hard material by

$$\mathrm{Len}(v) = \int_0^1 \chi(v) \;\mathrm{d}t \;.$$

Altogether, this allows us to define in analogy to Section 5 the (augmented) compliance functional as

$$J(K, v) = \int_0^1 -\delta(1-t)\sin(K(t)+K_0) \;\mathrm{d}t + c_l \mathrm{Len}(v) + c_p \mathrm{Per}^{\epsilon}(v)\,, \tag{7.1}$$

with coefficients $c_l, c_p > 0$. Thus, the total cost functional in terms of a phase field function is given by

$$\widehat{J}(v) = J(K(v), v)\,, \tag{7.2}$$

where $K(v)$ is a solution to $D_1E(K, v)(\phi) = 0$ for all test functions $\phi \in X$ and E takes into account the material coefficient $A(v)$. As mentioned in Section 6, the solution $K(v)$ is not necessarily unique. As in [8] there is a set of stationary points and we aim to compute one such solution numerically *via* a minimization of $\widehat{J}$ over all phase fields v. For this purpose we can apply the same abstract approach as in Section 5 and obtain as derivative

$$D\widehat{J}(v)(w) = D_2J(K(v), v)(w) + (D_2D_1E)^*(K(v), v)P \tag{7.3}$$

where P is the adjoint variable solving

$$(D_1D_1E)^*(K(v), v)P = -D_1J(K(v), v)\,. \tag{7.4}$$

This requires the derivatives

$$\begin{aligned}
D_2J(K, v)(w) &= c_l\int_0^1 \frac{1}{2}(v+1)w\mathrm{d}t + c_p\int_0^1 \epsilon v'w' + \frac{9}{8\epsilon}(v^2-1)vw\mathrm{d}t\\
D_1J(K, v)(\phi) &= \int_0^1 -\delta(1-t)\cos(K(t)+K_0)\phi\mathrm{d}t\\
D_1D_2E(K, v)(\phi)(w) &= \int_0^1 \frac{1}{2}(b-a)(v+1)wK'\phi'\mathrm{d}t\\
D_1D_1E(K, v)(\phi)(\psi) &= \int_0^1 A(v)\phi'\psi' - \delta(1-t)\sin(K(t)+K_0)\phi\psi\mathrm{d}t.
\end{aligned}$$

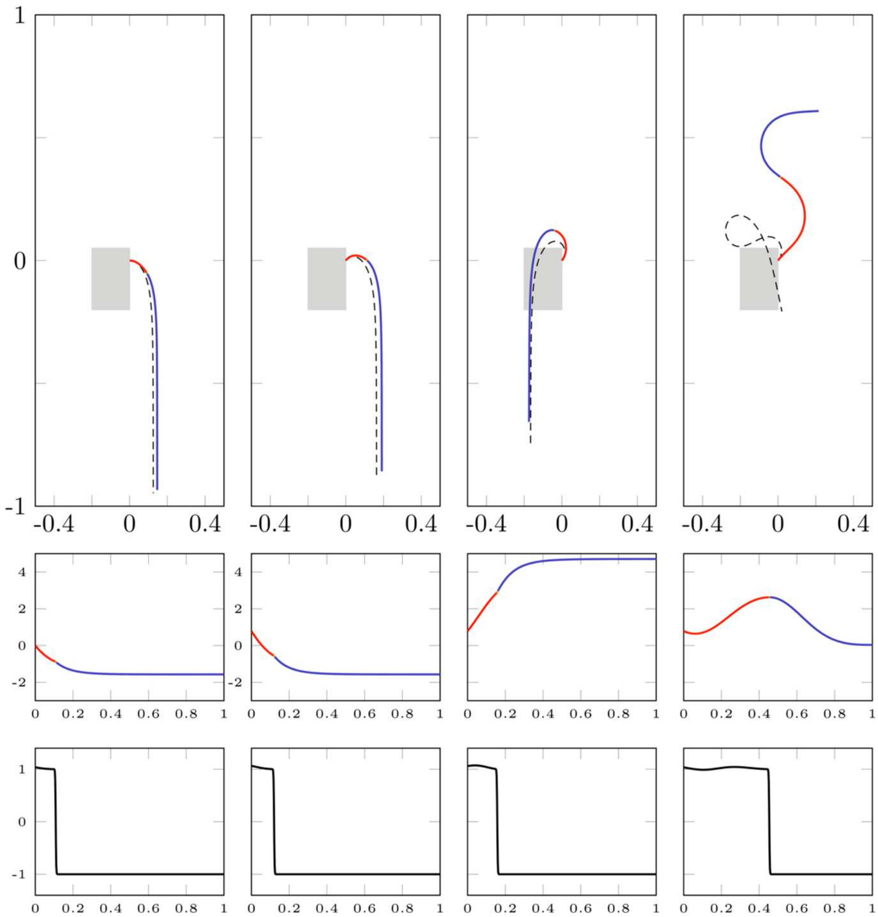

FIGURE 2. Top row: Starting from different initializations with $v \equiv 0$ (*dotted black*) we obtain optimal designs (*from left to right*) for a simple configuration with $K_0 = 0$ and with $K_0 = \frac{\pi}{4}$, as well as a twisted configuration with $K_0 = \frac{\pi}{4}$, and an S-configuration with $K_0 = \frac{\pi}{4}$. In the middle and bottom row we see the corresponding plots of the phase K and phase field v. Here, we have chosen $b = 1$, $a = 0.5$, $\delta = 100$, $c_l = 1$, $c_p = 1$, $N = 513$, and $\epsilon = \frac{1}{N-1}$. The phase field v is colored red for hard material and blue for soft material.

We choose v_h in the finite element space V_h defined in Section 6. Let us emphasize that we have to impose the Dirichlet boundary condition for P, *i.e.* $P_h(0) = 0$. Using the numerical quadrature in (6.1), we obtain discrete operators $\widehat{J}_h$, J_h, Len_h, Per_h^ϵ, $D_1 E_h$, and the corresponding derivatives. With these functionals and operators at hand, we use the Quasi-Newton-Method BFGS to compute minimizers of $\widehat{J}_h$.
For a given phase field function v_h the solution $K_h(v_h)$ is an element of the set of solutions to the state equation. It is computed by the method described in Section 6. Thus, starting with some initial phase, the Newton-Method converges to a state $K_h(v_h)$ which depends upon this initialization.
Our numerical experiments reflect the result from Theorem 5.8 (see Fig. 2). Furthermore, they suggest that a similar result remains true for solutions of the state equation other than the absolute minimizer. In fact, in our numerical simulations for clamped boundary conditions at 0 the optimal design always gathers the hard material on the left in some interval $[0, t^*]$ independent from the initial distribution taken into account. In Figure 2 we only depict one instance of many tests we performed with three different numerically computed local minimizers of the cost functional.

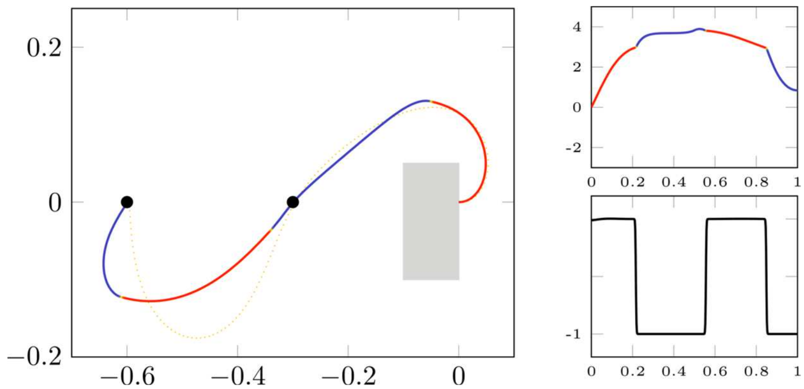

Figure 3. Optimal designs for a beam under the constraint that three fixed beam positions $(0,0)$, $(-0.3,0)$ and $(-0.6,0)$ at times $t=0$, 0.5, 1. Here $b=4.0$, $a=0.5$, $\delta=100$, $c_l=1$, $c_p=1$, $N=513$, and $\epsilon=\frac{1}{N-1}$. Plots and color coding are as in Figure 2.

Finally, we have implemented additional constraints prescribing a set of beam positions on $(0,1]$. In this case, the resulting optimal designs is characterized by separated subintervales with hard material. Also in these tests we never observed the microstructures even for small values of c_p. Figure 3 shows an instance of these computational results with additional point constraints.

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