POINTWISE FEEDBACK STABILIZATION OF AN EULER-BERNOULLI BEAM IN OBSERVATIONS WITH TIME DELAY[☆]

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Abstract. This paper considers a one-dimensional Euler-Bernoulli beam equation where two collocated actuators/sensors are presented at the internal point with pointwise feedback shear force and angle velocity at the arbitrary position ξ in the bounded domain (0,1). The boundary $x = 0$ is simply supported and at the other boundary $x = 1$ there is a shear hinge end. Both of the observation signals are subjected to a given time delay τ (>0). Well-posedness of the open-loop system is shown to illustrate availability of the observer. An observer is then designed to estimate the state at the time interval when the observation is available, while a predictor is designed to predict the state at the time interval when the observation is not available. Pointwise output feedback controllers are introduced to guarantee the closed-loop system to be exponentially stable for the smooth initial values when $\xi \in (0, 1)$ is a rational number satisfying $\xi \neq 2l/(2m - 1)$ for any integers l, m. Simulation results demonstrate that the proposed feedback design effectively stabilizes the performance of the pointwise control system with time delay.

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1. Introduction

Effect of time delay exists in lots of engineering systems, aircraft control systems, laser models, biology etc. [3, 11], for example, sampled data control [24]. Unfortunately, time delay is, in many cases, a source of instability. It often has been involved in challenging areas of communication or information technologies [23]. An example of the one-dimensional wave equation in [13, 14, 15] illustrates that even small time delay in the boundary output feedback may make the originally stable system unstable, also the elastic control system is destabilized when time delay occurs for an Euler-Bernoulli beam [12, 13, 14, 15, 29]. Therefore, stability analysis and robust control of time-delay systems are of theoretical and practical importance [17]. However, stabilization of distributed parameter control systems with time delay in observation or control represents a difficult mathematical challenge [16]. Recently we solved stabilization of a one-dimensional wave equation, an

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Euler-Bernoulli beam equation, and a Schrödinger equation where observation or control is suffered from given time delay by the separation principle [20, 21, 22]. It seems that the feedback law based on the observer and predictor is efficient to stabilize the hyperbolic distributed parameter system with time delay in observation.

It is a common phenomenon that the actuators have been jointed with each other which is called the pointwise control system. Over the past two decades much attention has been devoted to pointwise stabilization of flexible structures modeling infinite dimensional linear systems [32, 33, 34]. A string equation with a pointwise interior actuator is considered and the uniform decay rate is obtained by [1] and [26]. The general model of serially connected Euler-Bernoulli beams with joint linear feedback control was proposed in 1980s [8]. Later on many efforts have been made to study the pointwise feedback control stabilization for two connected beam equations, see for example [1, 2, 4, 5, 6, 7, 9, 10, 19, 22, 25] and the references therein.

From [2] it is found that exponential stability of the pointwise control Euler-Bernoulli beam equation is independent of the position of the actuators in which one controls the velocity of the beam at the arbitrary point ξ $(0 < \xi < 1)$ and the other controls the shear force at the same point. It seems that exponential stability of the pointwise control Euler-Bernoulli beam equation is robust to the position of the actuators. However, we will show that for the Euler-Bernoulli beam equation where two actuators are presented: one controls the angle velocity of the beam at the arbitrary point ξ $(0 < \xi < 1)$ and the other controls the shear force at the same point, exponential stability depends on the location of the joints for the boundary conditions different from that in [2].

The following beam equation with both two boundary end fixed, and two force stabilizers at the arbitrary intermediate joint $0 < \xi < 1$ has been considered:

$$
\begin{cases}
w_{tt}(x,t) + w_{xxxx}(x,t) = 0, \quad 0 < x < \xi, \ \xi < x < 1, \ t > 0, \\
w(0,t) = w_x(1,t) = w_{xx}(0,t) = w_{xxx}(1,t) = 0, \quad t \geq 0, \\
w(\xi^-,t) = w(\xi^+,t), \quad t \geq 0, \\
w_{xx}(\xi^-,t) = w_{xx}(\xi^+,t), \quad t \geq 0, \\
w_{xt}(\xi^-,t) - w_{xt}(\xi^+,t) = u_1(t), \quad t \geq 0, \\
w_{xxx}(\xi^+,t) - w_{xxx}(\xi^-,t) = u_2(t), \quad t \geq 0, \\
y_1(t) = w_{xx}(\xi, t-\tau), \quad t > \tau, \\
y_2(t) = w_t(\xi, t-\tau), \ t > \tau, \\
w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), \quad 0 \leq x \leq 1,
\end{cases}
\tag{1.1}
$$

with two sensors: one measures the angle velocity of the beam at an arbitrary point $\xi \in (0,1)$ and the other measures the curvature of the beam at the same point. $u_1(t)$ and $u_2(t)$ are control or input, $y_1(t)$ and $y_2(t)$ are observation or output which are suffered from given time delay τ (>0). For both of the boundaries, one boundary $x = 0$ is simply supported and at the other boundary $x = 1$ there is a shear hinge end. The jointed condition satisfies that when a physical variable is discontinuous its conjugate variable must be continuous. Besides, the physical meaning of the jointed condition is related to the fact that there is only internal horizontal dampers with a damping coefficient C_{dH} while there is none of internal vertical dampers, moreover, the external dampers with a damping coefficient C_d is perpendicular to the horizontal line [9]. Our aim is to design a feedback law to make the closed-loop system exponentially stable for certain sensor location ξ.

This paper firstly shows that the open-loop system is well-posed in the sense of Salamon-Weiss for the infinite-dimensional systems. Then the observer is proposed at the time interval when the observation signal is available, while the predictor is designed at the time interval when the observation signal is not available. A control law is constructed to give the estimated state feedback. It is shown that the closed-loop system is exponentially stable when the displacement of the actuator $\xi \in (0,1)$ is a rational number satisfying $\xi \neq 2l/(2m-1)$ for any integers l, m. In this sense, this paper is of great significance to deal with pointwise stabilization of distributed parameter systems with time delay.

This paper is arranged below. Section 2 will show well-posedness of the open-loop system. The following section is devoted to designing observer and predictor systems based on which output feedback is introduced. The closed-loop system is illustrated to be exponentially stable for smooth initial values in Section 4. Numerical simulation results are given in Section 5.

2. Well-posedness of the open-loop system

The system (1.1) above is considered in the state Hilbert space

$$\mathcal{H} = H_e^2(0,1) \times L^2(0,1), \quad H_e^2(0,1) = \{f \in H^2(0,1) \mid f(0) = f'(1) = 0\}, \tag{2.1}$$

with the inner product induced norm:

$$\left\| (f,g)^T \right\|_{\mathcal{H}}^2 = \int_0^1 \left[f''(x)^2 + g(x)^2 \right] \mathrm{d}x. \tag{2.2}$$

In this section we will demonstrate well-posedness of the open-loop system which may guarantee availability of the observer system shown at following part. After introducing the variable

$$z_1(x,t) = w_{xx}(\xi, t - \tau x), \quad z_2(x,t) = w_t(\xi, t - \tau x), \quad 0 \le x \le 1, \ t \ge 0,$$

the system (1.1) becomes

$$\begin{cases}
w_{tt}(x,t) + w_{xxxx}(x,t) = 0, \quad 0 < x < \xi, \ \xi < x < 1, \ t > 0, \\
w(0,t) = w_x(1,t) = w_{xx}(0,t) = w_{xxx}(1,t) = 0, \quad t \ge 0, \\
w(\xi^-,t) = w(\xi^+,t), \quad t \ge 0, \\
w_{xx}(\xi^-,t) = w_{xx}(\xi^+,t), \quad t \ge 0, \\
w_{xt}(\xi^-,t) - w_{xt}(\xi^+,t) = u_1(t), \quad t \ge 0, \\
w_{xxx}(\xi^+,t) - w_{xxx}(\xi^-,t) = u_2(t), \quad t \ge 0, \\
\tau z_{it}(x,t) + z_{ix}(x,t) = 0, \quad 0 < x < 1, \ t > 0, \ i = 1,2, \\
z_1(0,t) = w_{xx}(\xi,t), \quad t \ge 0, \\
z_2(0,t) = w_t(\xi,t), \quad t \ge 0, \\
w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), \quad 0 \le x \le 1, \\
z_1(x,0) = z_{10}(x), \quad z_2(x,0) = z_{20}(x), \quad 0 < x < 1, \\
y_1(t) = z_1(1,t), \quad y_2(t) = z_2(1,t), \quad t > \tau,
\end{cases} \tag{2.3}$$

where z_{10} and z_{20} are the initial values of the new variables z_1 and z_2.

With the state variable $(w(\cdot,t), w_t(\cdot,t), z_1(\cdot,t), z_2(\cdot,t))^T$, we consider the system (2.3) in the energy state space $\mathbb{H} = \mathcal{H} \times L^2(0,1) \times L^2(0,1)$. The norm of $(w(\cdot,t), w_t(\cdot,t), z_1(\cdot,t), z_2(\cdot,t))^T$ in $\mathbb{H}$ is defined by the energy as follows:

$$\begin{aligned}
E_1(t) &= \frac{1}{2} \left\| (w(\cdot,t), w_t(\cdot,t), z_1(\cdot,t), z_2(\cdot,t))^T \right\|_{\mathbb{H}}^2 \\
&= \frac{1}{2} \int_0^1 \left[w_{xx}^2(x,t) + w_t^2(x,t) \right] \mathrm{d}x + \frac{1}{2} \int_0^1 \left[z_1^2(x,t) + z_2^2(x,t) \right] \mathrm{d}x.
\end{aligned}$$

The input space and the output space are the same $U = Y = \mathbb{C}^2$.

Theorem 2.1. *The system (2.3) is well-posed: For each $u_1 \in L^2_{loc}(0, \infty)$, $u_2 \in L^2_{loc}(0, \infty)$ and initial datum $(w_0, w_1, z_{10}, z_{20})^T \in \mathbb{H}$, there exists a unique solution $(w(\cdot, t), w_t(\cdot, t), z_1(\cdot, t), z_2(\cdot, t))^T \in C(0, \infty; \mathbb{H})$ of (2.3) and for each $T > 0$ there exists a constant $C_T > 0$ such that*

$$\| (w(\cdot, T), w_t(\cdot, T), z_1(\cdot, T), z_2(\cdot, T))^T \|^2_{\mathbb{H}} + \int_0^T \left(|y_1(t)|^2 + |y_2(t)|^2 \right) \mathrm{d}t$$

$$\leq C_T \left[\| (w_0, w_1, z_{10}, z_{20})^T \|^2_{\mathbb{H}} + \int_0^T \left(|u_1(t)|^2 + |u_2(t)|^2 \right) \mathrm{d}t \right]. \tag{2.4}$$

Proof. It is known that the following system

$$\begin{cases} w_{tt}(x, t) + w_{xxxx}(x, t) = 0, \quad 0 < x < \xi, \ \xi < x < 1, \ t > 0, \\ w(0, t) = w_x(1, t) = w_{xx}(0, t) = w_{xxx}(1, t) = 0, \quad t \geq 0, \\ w(\xi^-, t) = w(\xi^+, t), \quad t \geq 0, \\ w_{xx}(\xi^-, t) = w_{xx}(\xi^+, t), \quad t \geq 0, \\ w_{xt}(\xi^-, t) - w_{xt}(\xi^+, t) = u_1(t), \quad t \geq 0, \\ w_{xxx}(\xi^+, t) - w_{xxx}(\xi^-, t) = u_2(t), \quad t \geq 0, \\ y_{w1}(t) = w_{xx}(\xi, t), \quad y_{w2}(t) = w_t(\xi, t), \quad t \geq 0, \end{cases} \tag{2.5}$$

can be written as

$$\sum(\mathcal{A}, \mathcal{B}, \mathcal{C}) : \begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} w(\cdot, t) \\ w_t(\cdot, t) \end{pmatrix} = \mathcal{A} \begin{pmatrix} w(\cdot, t) \\ w_t(\cdot, t) \end{pmatrix} + \mathcal{B} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \\ \begin{pmatrix} y_{w1}(t) \\ y_{w2}(t) \end{pmatrix} = \mathcal{C} \begin{pmatrix} w(\cdot, t) \\ w_t(\cdot, t) \end{pmatrix} = \begin{pmatrix} w_{xx}(\xi, t) \\ w_t(\xi, t) \end{pmatrix}, \end{cases} \tag{2.6}$$

where $\mathcal{A} : D(\mathcal{A})(\subset \mathcal{H}) \longrightarrow \mathcal{H}$ is defined below:

$$\begin{cases} \mathcal{A}(f, g)^T = (g, -f^{(4)})^T, \ \forall (f, g)^T \in D(\mathcal{A}), \\ D(\mathcal{A}) = \{(f, g)^T \in \mathcal{H} \cap (H^4(0, 1) \times H_e^2(0, 1)) \mid f''(0) = f'''(1) = 0\}, \end{cases} \tag{2.7}$$

and simple computation shows that $\mathcal{A}^* = -\mathcal{A}$ that is, $\mathcal{A}$ is a skew-adjoint operator.

At the same time,

$$\begin{pmatrix} y_{w1}(t) \\ y_{w2}(t) \end{pmatrix} = \begin{pmatrix} w_{xx}(\xi, t) \\ w_t(\xi, t) \end{pmatrix}$$

$$= \mathcal{C} \begin{pmatrix} w(x, t) \\ w_t(x, t) \end{pmatrix}, \tag{2.8}$$

where

$$\mathcal{C} = \begin{pmatrix} \langle \cdot, \delta''(x - \xi) \rangle & 0 \\ 0 & \langle \cdot, \delta(x - \xi) \rangle \end{pmatrix}. \tag{2.9}$$

Here $\delta(\cdot)$ denotes the Dirac function, and both $\mathcal{B}$ and $\mathcal{C}$ are unbounded operators. Simple computation gives that $\mathcal{C} = \mathcal{B}^*$.

Since

$$\mathcal{B}^* \mathcal{A}^{*-1}(f, g)^T = \left(\int_1^{\xi} (\xi - \eta) g(\eta) \mathrm{d}\eta - \int_0^1 \eta g(\eta) \mathrm{d}\eta, -f(\xi) \right)^T, \tag{2.10}$$

the operator $\mathcal{B}^* \mathcal{A}^{*-1}$ is bounded.

Simple computation gives the eigenvalues and eigenfunctions of the operator $\mathcal{A}^*$:

$$\{\lambda_n, \overline{\lambda}_n\}_{n \in \mathbb{N}},$$
$$\left\{ (\varphi_n, \psi_n)^T, (\overline{\varphi}_n, \psi_n)^T \right\}_{n \in \mathbb{N}},$$

respectively where

$$\begin{cases} \lambda_n = i(n - 1/2)^2 \pi^2, \quad n \in \mathbb{N}, \\ \varphi_n(x) = -i(n - 1/2)^{-2} \pi^{-2} \sin((n - 1/2)\pi x), \quad \psi_n(x) = \sin((n - 1/2)\pi x), \quad n \in \mathbb{N}. \end{cases} \tag{2.11}$$

For any $(w_0, w_1)^T \in \mathcal{H}$ with

$$(w_0, w_1)^T = \sum_{n=1}^{\infty} A_n (\varphi_n, \psi_n)^T + B_n (\overline{\varphi}_n, \psi_n)^T,$$

we have that

$$e^{\mathcal{A}^* t}(w_0, w_1)^T = \sum_{n=1}^{\infty} A_n e^{\lambda_n t} (\varphi_n, \psi_n)^T + B_n e^{\overline{\lambda}_n t} (\overline{\varphi}_n, \psi_n)^T, \tag{2.12}$$

which together with (2.9) illustrates that

$$\mathcal{B}^* e^{\mathcal{A}^* t}(w_0, w_1)^T = \begin{pmatrix} \sum\limits_{n=1}^{\infty} \left(A_n e^{\lambda_n t} \varphi_n''(\xi) + B_n e^{\overline{\lambda}_n t} \overline{\varphi}_n''(\xi) \right) \\ \sum\limits_{n=1}^{\infty} \left(A_n e^{\lambda_n t} + B_n e^{\overline{\lambda}_n t} \right) \psi_n(\xi) \end{pmatrix}.$$

Thus, Ingham's Inequality of page 59 in the book [28] shows that

$$\int_0^T \left| \mathcal{B}^* e^{\mathcal{A}^* t}(w_0, w_1)^T \right|^2 \mathrm{d}t \leq C_T \sum_{n=1}^{\infty} \left(| A_n |^2 + | B_n |^2 \right), \tag{2.13}$$

for some positive constant $C_T > 0$ with any $T > 0$.

Therefore, $\mathcal{B}^*$ is admissible for $e^{\mathcal{A}^* t}$, moreover, for each $u_1 \in L_{loc}^2(0, \infty)$, $u_2 \in L_{loc}^2(0, \infty)$ and initial datum $(w_0, w_1, z_{10}, z_{20})^T \in \mathbb{H}$, there exists a unique solution $(w(\cdot, t), w_t(\cdot, t), z_1(\cdot, t), z_2(\cdot, t))^T \in C(0, \infty; \mathbb{H})$ of (2.3).

Next we show the transfer function matrix is bounded on some right half complex plane.

A direct computation tells us the transfer function matrix:

$$\frac{1}{\Delta(s)} \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix},$$

where

$$G_{11}(s) = \sin\left(\sqrt{si}\xi\right)\cos\left(\sqrt{si}(1-\xi)\right)\cosh\left(\sqrt{si}\right) + \cos\left(\sqrt{si}\right)\sinh\left(\sqrt{si}\xi\right)\cosh\left(\sqrt{si}(1-\xi)\right),$$
$$G_{12}(s) = -iG_{11}(s),$$
$$G_{21}(s) = -\sin\left(\sqrt{si}\xi\right)\cos\left(\sqrt{si}(1-\xi)\right)\cosh\left(\sqrt{si}\right) + \cos\left(\sqrt{si}\right)\sinh\left(\sqrt{si}\xi\right)\cosh\left(\sqrt{si}(1-\xi)\right),$$
$$G_{22}(s) = iG_{21}(s),$$

and

$$\Delta(s) = 2\sqrt{si}\cos\left(\sqrt{si}\right)\cosh\left(\sqrt{si}\right).$$

Since obviously there exists no root for the equation $\Delta(s)$ on the right-half complex plane, the transfer function matrix is bounded on some right-half complex plane. By the well-posed linear infinite-dimensional system theory [11, 27, 31, 35], that is to say that $\mathcal{C}$ is admissible for $e^{\mathcal{A}t}$, $\mathcal{B}^*$ is admissible for $e^{\mathcal{A}^*t}$ and the transfer function is bounded on some right half complex plane. It was shown that the system (2.5) is well-posed in the sense of D. Salamon: for any $u_1 \in L^2_{loc}(0,\infty)$, $u_2 \in L^2_{loc}(0,\infty)$ and $(w_0, w_1)^T \in \mathcal{H}$, there exists a unique solution $(w(\cdot,t), w_t(\cdot,t))^T \in C(0,\infty;\mathcal{H})$ of (2.5) and for any $T > 0$, there exists a constant $D_T > 0$ such that

$$\|(w(\cdot,T), w_t(\cdot,T))^T\|_{\mathcal{H}}^2 + \int_0^T \left(|y_{w1}(t)|^2 + |y_{w2}(t)|^2\right)\mathrm{d}t \le D_T\left[\|(w_0, w_1)^T\|_{\mathcal{H}}^2 + \int_0^T\left(|u_1(t)|^2 + |u_2(t)|^2\right)\mathrm{d}t\right]. \tag{2.14}$$

On the other hand, for the "z_1" part equation:

$$\begin{cases} \tau z_{1t}(x,t) + z_{1x}(x,t) = 0, & 0 < x < 1,\ t > 0, \\ z_1(0,t) = w_{xx}(\xi,t), & t \ge 0, \\ z_1(x,0) = z_{10}(x), & 0 < x < 1, \end{cases} \tag{2.15}$$

by integrating along the characteristic line, the analytic solution can be obtained:

$$z_1(x,t) = \begin{cases} z_{10}\left(x - \dfrac{t}{\tau}\right), & x \ge \dfrac{t}{\tau}, \\ w_{xx}\left(\xi, t - x\tau\right), & x < \dfrac{t}{\tau}. \end{cases} \tag{2.16}$$

Therefore, we have, for any $T > 0$,

$$\int_0^1 z_1^2(x,T)\mathrm{d}x = \begin{cases} \displaystyle\int_0^{1-T/\tau} z_{10}^2(x)\mathrm{d}x + \dfrac{1}{\tau}\int_0^T w_{xx}^2(\xi,t)\mathrm{d}t, & T \le \tau, \\ \dfrac{1}{\tau}\displaystyle\int_{T-\tau}^T w_{xx}^2(\xi,t)\mathrm{d}t, & T > \tau, \end{cases} \tag{2.17}$$

and

$$\int_0^T |y_1(t)|^2\mathrm{d}t = \begin{cases} \tau\displaystyle\int_{1-T/\tau}^1 z_{10}^2(x)\mathrm{d}x, & T \le \tau, \\ \tau\displaystyle\int_0^1 z_{10}^2(x)\mathrm{d}x + \int_0^{T-\tau} w_{xx}^2(\xi,t)\mathrm{d}t, & T > \tau. \end{cases} \tag{2.18}$$

Similarly, the analytic expression of the "z_2" part equation

$$
\begin{cases}
\tau z_{2t}(x,t) + z_{2x}(x,t) = 0, & 0 < x < 1, \ t > 0, \\
z_2(0,t) = w_t(\xi,t), & t \geq 0, \\
z_2(x,0) = z_{20}(x), & 0 < x < 1,
\end{cases}
\tag{2.19}
$$

can be obtained:

$$
z_2(x,t) = \begin{cases}
z_{20}\left(x - \dfrac{t}{\tau}\right), & x \geq \dfrac{t}{\tau}, \\[2mm]
w_t(\xi, t - x\tau), & x < \dfrac{t}{\tau},
\end{cases}
\tag{2.20}
$$

which results that, for any $T > 0$,

$$
\int_0^1 z_2^2(x,T)\mathrm{d}x = \begin{cases}
\displaystyle\int_0^{1-T/\tau} z_{20}^2(x)\mathrm{d}x + \frac{1}{\tau}\int_0^T w_t^2(\xi,t)\mathrm{d}t, & T \leq \tau, \\[4mm]
\displaystyle\frac{1}{\tau}\int_{T-\tau}^T w_t^2(\xi,t)\mathrm{d}t, & T > \tau,
\end{cases}
\tag{2.21}
$$

and

$$
\int_0^T |y_2(t)|^2\mathrm{d}t = \begin{cases}
\displaystyle\tau\int_{1-T/\tau}^1 z_{20}^2(x)\mathrm{d}x, & T \leq \tau, \\[4mm]
\displaystyle\tau\int_0^1 z_{20}^2(x)\mathrm{d}x + \int_0^{T-\tau} w_t^2(\xi,t)\mathrm{d}t, & T > \tau.
\end{cases}
\tag{2.22}
$$

Collecting (2.14), (2.17), (2.18), (2.21) and (2.22) gives that (2.4). □

3. Observer and predictor design

This section will focus on two steps of constructing the observer and predictor systems in order to estimate the state of (1.1). For the state $\{(w(x,s), w_s(x,s))^T, s \in [0, t-\tau], t > \tau\}$ which satisfies

$$
\begin{cases}
w_{ss}(x,s) + w_{xxxx}(x,s) = 0, & 0 < x < \xi, \ \xi < x < 1, \ s > 0, \\
w(0,s) = w_x(1,s) = w_{xx}(0,s) = w_{xxx}(1,s) = 0, & s \geq 0, \\
w(\xi^-,s) = w(\xi^+,s), & s \geq 0, \\
w_{xx}(\xi^-,s) = w_{xx}(\xi^+,s), & s \geq 0, \\
w_{xs}(\xi^-,s) - w_{xs}(\xi^+,s) = u_1(s), & s \geq 0, \\
w_{xxx}(\xi^+,s) - w_{xxx}(\xi^-,s) = u_2(s), & s \geq 0, \\
y_1(s) = w_{xx}(\xi, s-\tau), \quad y_2(s) = w_s(\xi, s-\tau), & s > \tau, \\
w(x,0) = w_0(x), \quad w_s(x,0) = w_1(x), & 0 \leq x \leq 1,
\end{cases}
\tag{3.1}
$$

the first step is committed to a Luenberger observer system as following, as $k_1 > 0$ and $k_2 > 0$,

$$
\begin{cases}
\widehat{w}_{ss}(x,s) + \widehat{w}_{xxxx}(x,s) = 0, \quad 0 < x < \xi, \ \xi < x < 1, \ 0 < s \leq t - \tau, \ t > \tau, \\
\widehat{w}(0,s) = \widehat{w}_x(1,s) = \widehat{w}_{xx}(0,s) = \widehat{w}_{xxx}(1,s) = 0, \quad 0 \leq s \leq t - \tau, \ t \geq \tau, \\
\widehat{w}(\xi^-,s) = \widehat{w}(\xi^+,s), \quad 0 \leq s \leq t - \tau, \ t \geq \tau, \\
\widehat{w}_{xx}(\xi^-,s) = \widehat{w}_{xx}(\xi^+,s), \quad 0 \leq s \leq t - \tau, \ t \geq \tau, \\
\widehat{w}_{xs}(\xi^-,s) - \widehat{w}_{xs}(\xi^+,s) = u_1(s) - k_1[\widehat{w}_{xx}(\xi,s) - y_1(s+\tau)], \quad 0 \leq s \leq t - \tau, \ t \geq \tau, \\
\widehat{w}_{xxx}(\xi^+,s) - \widehat{w}_{xxx}(\xi^-,s) = u_2(s) - k_2[\widehat{w}_s(\xi,s) - y_2(s+\tau)], \quad 0 \leq s \leq t - \tau, \ t \geq \tau, \\
\widehat{w}(x,0) = \widehat{w}_0(x), \quad \widehat{w}_s(x,0) = \widehat{w}_1(x), \quad 0 \leq x \leq 1,
\end{cases}
\tag{3.2}
$$

where $(\widehat{w}_0, \widehat{w}_1)^T$ is the initial state of observer (arbitrarily assigned).

In order to show that it is available for (3.2) to be the observer of (3.1), we have to show convergence between these two systems. To do this, let

$$
\varepsilon(x,s) = \widehat{w}(x,s) - w(x,s), \quad 0 \leq s \leq t - \tau, t > \tau,
\tag{3.3}
$$

from both of (3.1) and (3.2), $\varepsilon(x,s)$ satisfies

$$
\begin{cases}
\varepsilon_{ss}(x,s) + \varepsilon_{xxxx}(x,s) = 0, \quad 0 < x < \xi, \ \xi < x < 1, \ 0 < s \leq t - \tau, \ t > \tau, \\
\varepsilon(0,s) = \varepsilon_x(1,s) = \varepsilon_{xx}(0,s) = \varepsilon_{xxx}(1,s) = 0, \quad 0 \leq s \leq t - \tau, \ t \geq \tau, \\
\varepsilon(\xi^-,s) = \varepsilon(\xi^+,s), \quad 0 \leq s \leq t - \tau, \ t \geq \tau, \\
\varepsilon_{xx}(\xi^-,s) = \varepsilon_{xx}(\xi^+,s), \quad 0 \leq s \leq t - \tau, \ t \geq \tau, \\
\varepsilon_{xs}(\xi^-,s) - \varepsilon_{xs}(\xi^+,s) = -k_1\varepsilon_{xx}(\xi,s), \quad 0 \leq s \leq t - \tau, \ t \geq \tau, \\
\varepsilon_{xxx}(\xi^+,s) - \varepsilon_{xxx}(\xi^-,s) = -k_2\varepsilon_s(\xi,s), \quad 0 \leq s \leq t - \tau, \ t \geq \tau, \\
\varepsilon(x,0) = \widehat{w}_0(x) - w_0(x), \quad \varepsilon_s(x,0) = \widehat{w}_1(x) - w_1(x), \quad 0 \leq x \leq 1.
\end{cases}
\tag{3.4}
$$

The system (3.4) can be written as:

$$
\frac{\mathrm{d}}{\mathrm{d}s}\begin{pmatrix} \varepsilon(\cdot,s) \\ \varepsilon_s(\cdot,s) \end{pmatrix} = \mathbb{B}\begin{pmatrix} \varepsilon(\cdot,s) \\ \varepsilon_s(\cdot,s) \end{pmatrix},
\tag{3.5}
$$

where $\mathbb{B} : D(\mathbb{B})(\subset \mathcal{H}_1) \to \mathcal{H}_1$ is defined as

$$
\begin{cases}
\mathbb{B}(f,g)^T = \left(g, -f^{(4)}\right)^T, \ \forall (f,g)^T \in D(\mathbb{B}), \\
D(\mathbb{B}) = \{(f,g)^T \in \mathcal{H}_1 \cap (H^4(0,\xi) \cap H^4(\xi,1)) \times (H^2(0,\xi) \cap \mathcal{H}^2(\xi,1)), \ f''(0) = f'''(1) = 0, f(\xi^-) = f(\xi^+), \\
g'(\xi^-) - g'(\xi^+) = -k_1f''(\xi), f''(\xi^-) = f''(\xi^+), f'''(\xi^+) - f'''(\xi^-) = -k_2g(\xi),
\end{cases}
\tag{3.6}
$$

and

$$
\mathcal{H}_1 = \{(f,g)^T \in (H^2(0,\xi) \cap H^2(\xi,1)) \times L^2(0,1) | f(0) = f'(1) = 0\}.
$$

From Theorem 3.1 of Appendix A it is known that when $\xi \in (0,1)$ is a rational number satisfying $\xi \neq 2l/(2m-1)$ for any integers l, m, the operator $\mathbb{B}$ generates an exponentially stable C_0-semigroup on $\mathcal{H}_1$, that is to say, for any $(w_0, w_1)^T \in \mathcal{H}_1$ and $(\widehat{w}_0, \widehat{w}_1)^T \in \mathcal{H}_1$, there exists a unique solution $(\varepsilon(\cdot,s), \varepsilon_s(\cdot,s))^T$ of (3.4)

such that

$$\| (\varepsilon(\cdot, s), \varepsilon_s(\cdot, s))^T \|_{\mathcal{H}_1} \leq M e^{-\omega s} \| (\widehat{w}_0 - w_0, \widehat{w}_1 - w_1)^T \|_{\mathcal{H}_1}, \quad \forall s \in [0, t - \tau], \ t > \tau, \tag{3.7}$$

for some positive constants M and ω.

The second step is to predict $\{(w(x, s), w_s(x, s))^T, s \in (t - \tau, t], t > \tau\}$ based on the state $\{(\widehat{w}(x, s), \widehat{w}_s(x, s))^T, s \in [0, t - \tau], t > \tau\}$, especially, the estimated initial value $\{(\widehat{w}(x, t - \tau), \widehat{w}_s(x, t - \tau))^T\}$ obtained from (3.2):

$$\begin{cases} \widehat{w}_{ss}(x, s, t) + \widehat{w}_{xxxx}(x, s, t) = 0, \quad 0 < x < \xi, \ \xi < x < 1, \ t - \tau < s \leq t, \ t > \tau, \\ \widehat{w}(0, s, t) = \widehat{w}_x(1, s, t) = \widehat{w}_{xx}(0, s, t) = \widehat{w}_{xxx}(1, s, t) = 0, \quad t - \tau \leq s \leq t, \ t \geq \tau, \\ \widehat{w}(\xi^-, s, t) = \widehat{w}(\xi^+, s, t), \quad t - \tau \leq s \leq t, \ t \geq \tau, \\ \widehat{w}_{xx}(\xi^-, s, t) = \widehat{w}_{xx}(\xi^+, s, t), \quad t - \tau \leq s \leq t, \ t \geq \tau, \\ \widehat{w}_{xs}(\xi^-, s, t) - \widehat{w}_{xs}(\xi^+, s, t) = u_1(s), \quad t - \tau \leq s \leq t, \ t \geq \tau, \\ \widehat{w}_{xxx}(\xi^+, s, t) - \widehat{w}_{xxx}(\xi^-, s, t) = u_2(s), \quad t - \tau \leq s \leq t, \ t \geq \tau, \\ \widehat{w}(x, t - \tau, t) = \widehat{w}(x, t - \tau), \quad \widehat{w}_s(x, t - \tau, t) = \widehat{w}_s(x, t - \tau), \quad 0 \leq x \leq 1, \ t \geq \tau. \end{cases} \tag{3.8}$$

Finally we get the estimated state variable by

$$\widetilde{w}(x, t) = \widehat{w}(x, t, t), \quad \widetilde{w}_t(x, t) = \widehat{w}_s(x, t, t), \quad t > \tau. \tag{3.9}$$

Theorem 3.1. *For any $t > \tau$, we have*

$$\| (w(\cdot, t) - \widetilde{w}(\cdot, t), w_t(\cdot, t) - \widetilde{w}_t(\cdot, t))^T \|_{\mathcal{H}} \leq M e^{-\omega(t - \tau)} \| (\widehat{w}_0 - w_0, \widehat{w}_1 - w_1)^T \|_{\mathcal{H}_1}, \tag{3.10}$$

where $(\widehat{w}_0, \widehat{w}_1)^T$ is the initial state of observer (3.2), $(w_0, w_1)^T$ is the initial state of original system (1.1), and M, ω are constants in (3.7).

Proof. Let

$$\varepsilon(x, s, t) = \widehat{w}(x, s, t) - w(x, s), \quad t - \tau \leq s \leq t, \ t > \tau. \tag{3.11}$$

Then $\varepsilon(x, s, t)$ satisfies

$$\begin{cases} \varepsilon_{ss}(x, s, t) + \varepsilon_{xxxx}(x, s, t) = 0, \quad 0 < x < 1, \ t - \tau < s \leq t, \ t > \tau, \\ \varepsilon(0, s, t) = \varepsilon_x(1, s, t) = \varepsilon_{xx}(0, s, t) = \varepsilon_{xxx}(1, s, t) = 0, \quad t - \tau \leq s \leq t, \ t \geq \tau, \\ \varepsilon(x, t - \tau, t) = \varepsilon(x, t - \tau), \quad \varepsilon_s(x, t - \tau, t) = \varepsilon_s(x, t - \tau), \quad 0 \leq x \leq 1, \ t \geq \tau, \end{cases} \tag{3.12}$$

which is a conservative system, that is,

$$\| (\varepsilon(\cdot, t, t), \varepsilon_s(\cdot, t, t))^T \|_{\mathcal{H}} = \| (\varepsilon(\cdot, t - \tau), \varepsilon_s(\cdot, t - \tau))^T \|_{\mathcal{H}_1}.$$

This together with (3.7) and (3.11) gives (3.10). $\qquad\square$

4. Stabilization of the closed-loop system

Since the feedback

$$u_1(t) = -k_3 w_{xx}(\xi, t), u_2(t) = -k_4 w_t(\xi, t)(k_3 > 0, k_4 > 0)$$

stabilizes exponentially the system (1.1), and we have the estimation $\widetilde{w}_{xt}(\xi, t)$ of $w_{xt}(\xi, t)$ and $\widetilde{w}_t(\xi, t)$ of $w_t(\xi, t)$ respectively, naturally the estimated state feedback control law is designed as following (Figs. 1 and 2):

$$u_1^*(t) = \begin{cases} 0, & t \in [0, \tau], \\ -k_3 \widehat{w}_{xx}(\xi, t, t), & t > \tau, \ k_3 > 0, \end{cases} \tag{4.1}$$

and

$$u_2^*(t) = \begin{cases} 0, & t \in [0, \tau], \\ -k_4 \widehat{w}_s(\xi, t, t), & t > \tau, \ k_4 > 0, \end{cases} \tag{4.2}$$

under which, the closed-loop system becomes a system of partial differential equations (4.3)–(4.5):

$$\begin{cases} w_{tt}(x, t) + w_{xxxx}(x, t) = 0, & 0 < x < \xi, \ \xi < x < 1, \ t > 0, \\ w(0, t) = w_x(1, t) = w_{xx}(0, t) = w_{xxx}(1, t) = 0, & t \geq 0, \\ w(\xi^-, t) = w(\xi^+, t), & t \geq 0, \\ w_{xx}(\xi^-, t) = w_{xx}(\xi^+, t), & t \geq 0, \\ w_{xt}(\xi^-, t) - w_{xt}(\xi^+, t) = u_1^*(t), & t \geq 0, \\ w_{xxx}(\xi^+, t) - w_{xxx}(\xi^-, t) = u_2^*(t), & t \geq 0, \\ w(x, 0) = w_0(x), \quad w_t(x, 0) = w_1(x), & 0 \leq x \leq 1, \end{cases} \tag{4.3}$$

$$\begin{cases} \widehat{w}_{ss}(x, s) + \widehat{w}_{xxxx}(x, s) = 0, & 0 < x < \xi, \ \xi < x < 1, \ 0 < s \leq t - \tau, \ t > \tau, \\ \widehat{w}(0, s) = \widehat{w}_x(1, s) = \widehat{w}_{xx}(0, s) = \widehat{w}_{xxx}(1, s) = 0, & 0 \leq s \leq t - \tau, \ t \geq \tau, \\ \widehat{w}(\xi^-, s) = \widehat{w}(\xi^+, s), & 0 \leq s \leq t - \tau, \ t \geq \tau, \\ \widehat{w}_{xx}(\xi^-, s) = \widehat{w}_{xx}(\xi^+, s), & 0 \leq s \leq t - \tau, \ t \geq \tau, \\ \widehat{w}_{xs}(\xi^-, s) - \widehat{w}_{xs}(\xi^+, s) = u_1^*(s) - k_1[\widehat{w}_{xx}(\xi, s) - y_1(s + \tau)], & 0 \leq s \leq t - \tau, \ t \geq \tau, \\ \widehat{w}_{xxx}(\xi^+, s) - \widehat{w}_{xxx}(\xi^-, s) = u_2^*(s) - k_2[\widehat{w}_s(\xi, s) - y_2(s + \tau)], & 0 \leq s \leq t - \tau, \ t \geq \tau, \\ \widehat{w}(x, 0) = \widehat{w}_0(x), \quad \widehat{w}_s(x, 0) = \widehat{w}_1(x), & 0 \leq x \leq 1, \end{cases} \tag{4.4}$$

and

$$\begin{cases} \widehat{w}_{ss}(x, s, t) + \widehat{w}_{xxxx}(x, s, t) = 0, & 0 < x < \xi, \ \xi < x < 1, \ t - \tau < s \leq t, \ t > \tau, \\ \widehat{w}(0, s, t) = \widehat{w}_x(1, s, t) = \widehat{w}_{xx}(0, s, t) = \widehat{w}_{xxx}(1, s, t) = 0, & t - \tau \leq s \leq t, \ t \geq \tau, \\ \widehat{w}(\xi^-, s, t) = \widehat{w}(\xi^+, s, t), & t - \tau \leq s \leq t, \ t \geq \tau, \\ \widehat{w}_{xx}(\xi^-, s, t) = \widehat{w}_{xx}(\xi^+, s, t), & t - \tau \leq s \leq t, \ t \geq \tau, \\ \widehat{w}_{xs}(\xi^-, s, t) - \widehat{w}_{xs}(\xi^+, s, t) = u_1^*(s), & t - \tau \leq s \leq t, \ t \geq \tau, \\ \widehat{w}_{xxx}(\xi^+, s, t) - \widehat{w}_{xxx}(\xi^-, s, t) = u_2^*(s), & t - \tau \leq s \leq t, \ t \geq \tau, \\ \widehat{w}(x, t - \tau, t) = \widehat{w}(x, t - \tau), \quad \widehat{w}_s(x, t - \tau, t) = \widehat{w}_s(x, t - \tau), & 0 \leq x \leq 1, \ t \geq \tau. \end{cases} \tag{4.5}$$

The systems (4.3)–(4.5) are considered in the state space $X = \mathcal{H}_1^3$. Obviously (4.3)–(4.5) are equivalent to (4.6)–(4.8) when $t > \tau$ provided that $u_1^* \in L_{loc}^2(0, \infty)$ and $u_2^* \in L_{loc}^2(0, \infty)$ which will be clarified by
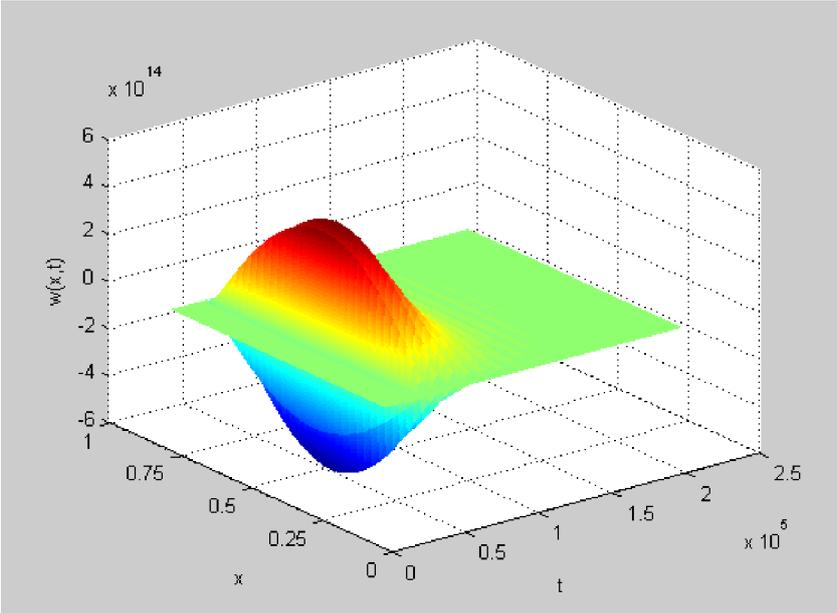
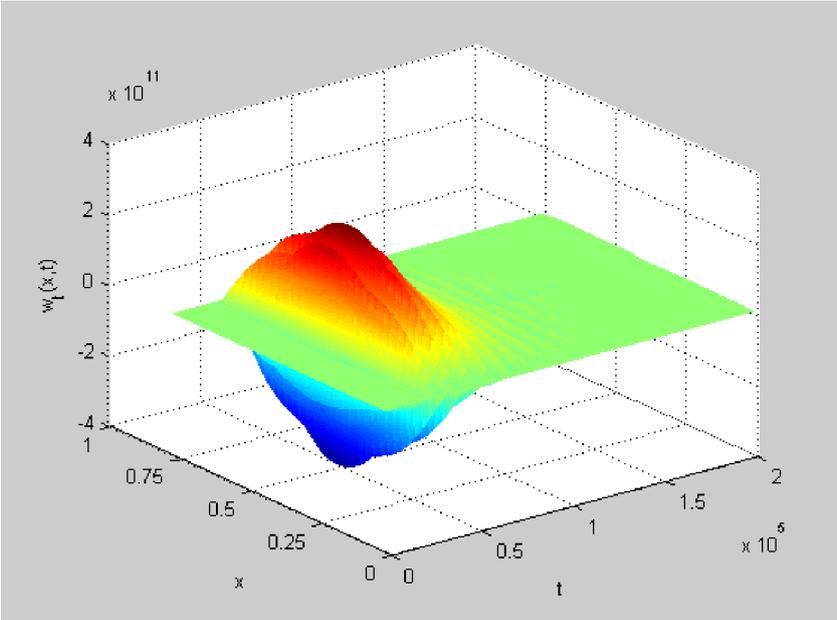FIGURE 1. State of the closed-loop system. (Color online.)

FIGURE 2. Velocity of the closed-loop system. (Color online.)

(4.26) and (4.27).

$$
\begin{cases}
w_{tt}(x,t) + w_{xxxx}(x,t) = 0, & 0 < x < \xi,\ \xi < x < 1,\ t > \tau, \\
w(0,t) = w_x(1,t) = w_{xx}(0,t) = w_{xxx}(1,t) = 0, & t > \tau, \\
w(\xi^-,t) = w(\xi^+,t), & t > \tau, \\
w_{xx}(\xi^-,t) = w_{xx}(\xi^+,t), & t > \tau, \\
w_{xt}(\xi^-,t) - w_{xt}(\xi^+,t) = -k_3[w_{xx}(\xi,t) + \varepsilon_{xx}(\xi,t,t)], & t > \tau, \\
w_{xxx}(\xi^+,t) - w_{xxx}(\xi^-,t) = -k_4[w_t(\xi,t) + \varepsilon_s(\xi,t,t)], & t > \tau, \\
w(x,\tau) = w(x,\tau), \quad w_t(x,\tau) = w_t(x,\tau), & 0 \le x \le 1,
\end{cases}
\tag{4.6}
$$

$$
\begin{cases}
\varepsilon_{ss}(x,s) + \varepsilon_{xxxx}(x,s) = 0, & 0 < x < \xi,\ \xi < x < 1,\ 0 < s \le t-\tau, t > \tau, \\
\varepsilon(0,s) = \varepsilon_x(1,s) = \varepsilon_{xx}(0,s) = \varepsilon_{xxx}(1,s) = 0, & 0 \le s \le t-\tau,\ t > \tau, \\
\varepsilon(\xi^-,s) = \varepsilon(\xi^+,s), & 0 \le s \le t-\tau,\ t > \tau, \\
\varepsilon_{xx}(\xi^-,s) = \varepsilon_{xx}(\xi^+,s), & 0 \le s \le t-\tau,\ t > \tau, \\
\varepsilon_{xs}(\xi^-,s) - \varepsilon_{xs}(\xi^+,s) = -k_1\varepsilon_{xx}(\xi,s), & 0 \le s \le t-\tau,\ t > \tau, \\
\varepsilon_{xxx}(\xi^+,s) - \varepsilon_{xxx}(\xi^-,s) = -k_2\varepsilon_s(\xi,s), & 0 \le s \le t-\tau,\ t > \tau, \\
\varepsilon(x,0) = \widehat{w}_0(x) - w_0(x), \quad \varepsilon_s(x,0) = \widehat{w}_1(x) - w_1(x), & 0 \le x \le 1,
\end{cases}
\tag{4.7}
$$

$$
\begin{cases}
\varepsilon_{ss}(x,s,t) + \varepsilon_{xxxx}(x,s,t) = 0, & 0 < x < 1, t-\tau < s \le t,\ t > \tau, \\
\varepsilon(0,s,t) = \varepsilon_x(1,s,t) = \varepsilon_{xx}(0,s,t) = \varepsilon_{xxx}(1,s,t) = 0, & t-\tau \le s \le t,\ t > \tau, \\
\varepsilon(x,t-\tau,t) = \varepsilon(x,t-\tau), \quad \varepsilon_s(x,t-\tau,t) = \varepsilon_s(x,t-\tau), & 0 \le x \le 1,\ t > \tau,
\end{cases}
\tag{4.8}
$$

where $\varepsilon(x,s)$ and $\varepsilon(x,s,t)$ are given by (3.3) and (3.11), respectively.

Theorem 4.1. *Let $k_1 > 0, k_2 > 0, k_3 >, k_4 > 0,\ t > \tau$, and $\xi \in (0,1)$ is a rational number satisfying $\xi \ne 2l/(2m-1)$ for any integers l, m, then for any $(w_0,w_1)^T \in \mathcal{H}$ and $(\widehat{w}_0,\widehat{w}_1)^T \in \mathcal{H}_1$, there exists a unique solution of systems (4.6)–(4.8) such that $(w(\cdot,t),w_t(\cdot,t))^T \in C(\tau,\infty;\mathcal{H})$, $(\varepsilon(\cdot,s),\varepsilon_s(\cdot,s))^T \in C(0,t-\tau;\mathcal{H})$ and $(\varepsilon(\cdot,s,t),\varepsilon_s(\cdot,s,t))^T \in C([t-\tau,t] \times (\tau,\infty);\mathcal{H})$. Moreover, for any*

$$
(\widehat{w}_0 - w_0, \widehat{w}_1 - w_1)^T \in D(\mathbb{B}),
\tag{4.9}
$$

where $\mathbb{B}$ is defined by (3.6), the system (4.6) decays exponentially in the sense that

$$
\|(w(\cdot,t),w_t(\cdot,t))^T\|_{\mathcal{H}} \le M_0 e^{-\omega_0(t-\tau)}\|(w_0,w_1)^T\|_{\mathcal{H}} + L_0 M M_0 e^{\omega_0\tau}\sqrt{(c_3^2 + c_4^2)/2\omega}
$$
$$
\cdot \left(e^{-\omega_0 t/2} + e^{\omega\tau} \cdot e^{-\omega t/2}\right) \cdot \|\mathbb{B}(\widehat{w}_0 - w_0, \widehat{w}_1 - w_1)^T\|_{\mathcal{H}_1}.
$$

where M, ω and M_0, ω_0 are defined in (3.7) and (4.36) respectively.

Proof. For any $(w_0,w_1)^T \in \mathcal{H}$ and $(\widehat{w}_0,\widehat{w}_1)^T \in \mathcal{H}_1$, since when $\xi \in (0,1)$ is a rational number satisfying $\xi \ne 2l/(2m-1)$ for any integers l, m, the operator $\mathbb{B}$ defined by (3.6) generates an exponentially stable C_0-semigroup on $\mathcal{H}_1$, there exists a unique solution $(\varepsilon(\cdot,s),\varepsilon_s(\cdot,s))^T \in C(0,t-\tau;\mathcal{H})$ of (4.7) such that (3.7) holds true.

Next for any given time $t > \tau$, write (4.8) as:

$$\frac{\mathrm{d}}{\mathrm{d}s}\begin{pmatrix} \varepsilon(\cdot, s, t) \\ \varepsilon_s(\cdot, s, t) \end{pmatrix} = \mathbb{A}\begin{pmatrix} \varepsilon(\cdot, s, t) \\ \varepsilon_s(\cdot, s, t) \end{pmatrix}, \tag{4.10}$$

where $\mathbb{A} : D(\mathbb{A})(\subset \mathcal{H}) \to \mathcal{H}$ is defined as

$$\begin{cases} \mathbb{A}(f, g)^T = \left(g, -f^{(4)}\right)^T, & \forall (f, g)^T \in D(\mathbb{A}), \\ D(\mathbb{A}) = \{(f, g)^T \in \mathcal{H} \cap (H^4(0, 1) \times H^2_e(0, 1)) | f''(0) = f'''(1) = 0\}. \end{cases} \tag{4.11}$$

It is known that $\mathbb{A}$ is skew-adjoint in $\mathcal{H}$ and hence generates a conservative C_0-semigroup on $\mathcal{H}$, that is, for any $(\varepsilon(\cdot, t - \tau), \varepsilon_s(\cdot, t - \tau))^T \in \mathcal{H}$ determined by (4.7), there exists a unique solution of (4.8) such that

$$\| (\varepsilon(\cdot, s, t), \varepsilon_s(\cdot, s, t))^T \|_{\mathcal{H}} = \| (\varepsilon(\cdot, t - \tau), \varepsilon_s(\cdot, t - \tau))^T \|_{\mathcal{H}_1}, \ \forall s \in [t - \tau, t], t > \tau. \tag{4.12}$$

So, $(\varepsilon(\cdot, s, t), \varepsilon_s(\cdot, s, t))^T \in C([t - \tau, t] \times [\tau, \infty); \mathcal{H})$. Moreover, since $\mathbb{A}$ is skew-adjoint with compact resolvent, the solution of (4.8) can be represented as

$$\begin{pmatrix} \varepsilon(\cdot, s, t) \\ \varepsilon_s(\cdot, s, t) \end{pmatrix} = \sum_{n=1}^{\infty} a_n(t) e^{i(n-1/2)^2 \pi^2 s} \begin{pmatrix} -i(n - 1/2)^{-2} \pi^{-2} \psi_n(\cdot) \\ \psi_n(\cdot) \end{pmatrix}$$
$$+ \sum_{n=1}^{\infty} b_n(t) e^{-i(n-1/2)^2 \pi^2 s} \begin{pmatrix} i(n - 1/2)^{-2} \pi^{-2} \psi_n(\cdot) \\ \psi_n(\cdot) \end{pmatrix}, \tag{4.13}$$

in terms of s where $\left\{ \left(\pm i(n - 1/2)^{-2} \pi^{-2} \psi_n(\cdot), \psi_n(\cdot)\right)^T \right\}_{n=1}^{\infty}$ is a set of the maximal ω-linearly independent sequence of eigenfunctions of $\mathbb{A}$ corresponding to eigenvalues $\pm i(n - 1/2)^2 \pi^2 (n \in \mathbb{N})$. It forms an orthogonal basis in $\mathcal{H}$ and

$$\psi_n(x) = \sin \left((n - 1/2)\pi x\right), \quad (n \in \mathbb{N}). \tag{4.14}$$

Therefore, from (4.13),

$$\varepsilon_s(\xi, t, t) = \sum_{n=1}^{\infty} \left[a_n(t) e^{i(n-1/2)^2 \pi^2 t} + b_n(t) e^{-i(n-1/2)^2 \pi^2 t} \right] \sin \left((n - 1/2)\pi \xi\right), \tag{4.15}$$

and

$$\varepsilon_{xx}(\xi, t, t) = \sum_{n=1}^{\infty} i \left[a_n(t) e^{i(n-1/2)^2 \pi^2 t} - b_n(t) e^{-i(n-1/2)^2 \pi^2 t} \right] \sin \left((n - 1/2)\pi \xi\right), \tag{4.16}$$

where

$$c_1 \le \ell_n = \left\| \left(i(n - 1/2)^{-2} \pi^{-2} \psi_n(\cdot), \psi_n(\cdot)\right)^T \right\|_{\mathcal{H}}^2 \le c_2, \quad \forall n \in \mathbb{N}, \tag{4.17}$$

with some constants $c_1, c_2 > 0$.

Now suppose $(\varepsilon(x, 0), \varepsilon_s(x, 0))^T \in D(\mathbb{B})$. Then by C_0-semigroup theory, $(\varepsilon(x, t - \tau), \varepsilon_s(x, t - \tau))^T \in D(\mathbb{B})$. That is to say, $\varepsilon_s(\cdot, t - \tau) \in \{f \in H^2(0, \xi) \cap H^2(\xi, 1) | f(0) = f'(1) = 0\}, \varepsilon(\cdot, t - \tau) \in \{f \in H^4(0, \xi) \cap H^4(\xi, 1) | f(0) = f'(1) = 0\}$ for any $t > \tau$.

From (4.13), we have that

$$
\begin{aligned}
\ell_n a_n(t) e^{i(n-1/2)^2\pi^2(t-\tau)} &= \left\langle \begin{pmatrix} \varepsilon(\cdot, t-\tau) \\ \varepsilon_s(\cdot, t-\tau) \end{pmatrix}, \begin{pmatrix} -i(n-1/2)^{-2}\pi^{-2}\psi_n(\cdot) \\ \psi_n(\cdot) \end{pmatrix} \right\rangle_{\mathcal{H}_1} \\
&= (n-1/2)^{-2}\pi^{-2}\big\{ [ik_2\varepsilon_s(\xi, t-\tau) - k_1\varepsilon_{xx}(\xi, t-\tau)]\sin\left((n-1/2)\pi\xi\right) \\
&\quad - \int_0^1 (i\varepsilon_{xxxx}(x, t-\tau) + \varepsilon_{sxx}(x, t-\tau))\sin\left((n-1/2)\pi x\right)\mathrm{d}x \big\},
\end{aligned}
\tag{4.18}
$$

and

$$
\begin{aligned}
\ell_n b_n(t) e^{-i(n-1/2)^2\pi^2(t-\tau)} &= \left\langle \begin{pmatrix} \varepsilon(\cdot, t-\tau) \\ \varepsilon_s(\cdot, t-\tau) \end{pmatrix}, \begin{pmatrix} i(n-1/2)^{-2}\pi^{-2}\psi_n(\cdot) \\ \psi_n(\cdot) \end{pmatrix} \right\rangle_{\mathcal{H}_1} \\
&= (n-1/2)^{-2}\pi^{-2}\big\{ [-ik_2\varepsilon_s(\xi, t-\tau) - k_1\varepsilon_{xx}(\xi, t-\tau)]\sin\left((n-1/2)\pi\xi\right) \\
&\quad + \int_0^1 (i\varepsilon_{xxxx}(x, t-\tau) - \varepsilon_{sxx}(x, t-\tau))\sin\left((n-1/2)\pi x\right)\mathrm{d}x \big\}.
\end{aligned}
\tag{4.19}
$$

Since

$$
\begin{cases}
|\varepsilon_s(\xi, t-\tau)| = \dfrac{1}{2}\left| \displaystyle\int_0^\xi \varepsilon_{sx}(x, t-\tau)\mathrm{d}x - \int_\xi^1 \varepsilon_{sx}(x, t-\tau)\mathrm{d}x \right| \\[4pt]
\leq \dfrac{1}{2}\left(\displaystyle\int_0^\xi |\varepsilon_{sx}(x, t-\tau)|^2\mathrm{d}x \right)^{1/2} + \dfrac{1}{2}\left(\int_\xi^1 |\varepsilon_{sx}(x, t-\tau)|^2\mathrm{d}x \right)^{1/2} \\[4pt]
\leq \left(\displaystyle\int_0^1 |\varepsilon_{sx}(x, t-\tau)|^2\mathrm{d}x \right)^{1/2} \leq \left(\int_0^1 |\varepsilon_{sxx}(x, t-\tau)|^2\mathrm{d}x \right)^{1/2}, \\[4pt]
|\varepsilon_{xx}(\xi, t-\tau)| = \dfrac{1}{2}\left| \displaystyle\int_0^\xi \varepsilon_{xxx}(x, t-\tau)\mathrm{d}x - \int_\xi^1 \varepsilon_{xxx}(x, t-\tau)\mathrm{d}x \right| \\[4pt]
\leq \dfrac{1}{2}\left(\displaystyle\int_0^\xi |\varepsilon_{xxx}(x, t-\tau)|^2\mathrm{d}x \right)^{1/2} + \dfrac{1}{2}\left(\int_\xi^1 |\varepsilon_{xxx}(x, t-\tau)|^2\mathrm{d}x \right)^{1/2} \\[4pt]
\leq \left(\displaystyle\int_0^1 |\varepsilon_{xxx}(x, t-\tau)|^2\mathrm{d}x \right)^{1/2} \leq \left(\int_0^1 |\varepsilon_{xxxx}(x, t-\tau)|^2\mathrm{d}x \right)^{1/2},
\end{cases}
\tag{4.20}
$$

we get that

$$
|\ell_n a_n(t)| \leq (n-1/2)^{-2}\pi^{-2}(k_1 + k_2 + 2)\|\mathbb{B}(\varepsilon(\cdot, t-\tau), \varepsilon_s(\cdot, t-\tau))^T\|_{\mathcal{H}_1},
\tag{4.21}
$$

and

$$
|\ell_n b_n(t)| \leq (n-1/2)^{-2}\pi^{-2}(k_1 + k_2 + 2)\|\mathbb{B}(\varepsilon(\cdot, t-\tau), \varepsilon_s(\cdot, t-\tau))^T\|_{\mathcal{H}_1}.
\tag{4.22}
$$

By (4.15), collecting (4.17)–(4.22) gives

$$
|\varepsilon_s(\xi, t, t)| \leq c_3 \|\mathbb{B}(\varepsilon(\cdot, t-\tau), \quad \varepsilon_s(\cdot, t-\tau))^T\|_{\mathcal{H}_1},
\tag{4.23}
$$

for some constant $c_3 > 0$ independent of t.

Similarly, together with (4.16) and (4.17)–(4.22) gives

$$|\varepsilon_{xx}(\xi, t, t)| \le c_4 \|\mathbb{B}(\varepsilon(\cdot, t - \tau), \varepsilon_s(\cdot, t - \tau))^T\|_{\mathcal{H}_1}. \tag{4.24}$$

for some constant $c_4 > 0$ independent of t.

On the other hand, from (3.7) and C_0-semigroup theory, we have

$$\|\mathbb{B}(\varepsilon(\cdot, t - \tau), \varepsilon_s(\cdot, t - \tau))^T\|_{\mathcal{H}_1} \le M e^{-\omega(t-\tau)} \|\mathbb{B}(\widehat{w}_0 - w_0, \widehat{w}_1 - w_1)^T\|_{\mathcal{H}_1}, \tag{4.25}$$

for $\forall t \in [\tau, \infty)$ where M, ω are defined in (3.7).

Finally based on the equations above (4.23)–(4.25),

$$|\varepsilon_s(\xi, t, t)| \le c_3 M e^{-\omega(t-\tau)} \|\mathbb{B}(\widehat{w}_0 - w_0, \widehat{w}_1 - w_1)^T\|_{\mathcal{H}_1}, \tag{4.26}$$

and

$$|\varepsilon_{xx}(\xi, t, t)| \le c_4 M e^{-\omega(t-\tau)} \|\mathbb{B}(\widehat{w}_0 - w_0, \widehat{w}_1 - w_1)^T\|_{\mathcal{H}_1}. \tag{4.27}$$

Next we consider the equation (4.6) which is written as

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} w(\cdot, t) \\ w_t(\cdot, t) \end{pmatrix} = \mathcal{A}_0 \begin{pmatrix} w(\cdot, t) \\ w_t(\cdot, t) \end{pmatrix} + \mathcal{B}_0 \begin{pmatrix} \varepsilon_{xx}(\xi, t, t) \\ \varepsilon_s(\xi, t, t) \end{pmatrix}, \tag{4.28}$$

where

$$\begin{cases} \mathcal{A}_0(f, g)^T = (g, -f^{(4)})^T, \ \forall (f, g)^T \in D(\mathcal{A}_0), \\ D(\mathcal{A}_0) = \{(f, g)^T \in \mathcal{H}_1 \cap (H^4(0, \xi) \cap H^4(\xi, 1)) \times (H^2(0, \xi) \cap H^2(\xi, 1)) | f''(0) = f'''(1) = 0, f(\xi^-) = f(\xi^+), \\ f''(\xi^-) = f''(\xi^+), g'(\xi^-) - g'(\xi^+) = -k_3 f''(\xi), f'''(\xi^+) - f'''(\xi^-) = -k_4 g(\xi)\}. \end{cases} \tag{4.29}$$

Simple computations give that $\mathcal{A}_0^* = -\mathcal{A}_0$ and

$$\mathcal{B}_0^* = \begin{pmatrix} \langle \cdot, -k_3 \delta''(x - \xi) \rangle & 0 \\ 0 & \langle \cdot, -k_4 \delta(x - \xi) \rangle \end{pmatrix}. \tag{4.30}$$

A direct computation shows

$$\mathcal{B}_0^* \mathcal{A}_0^{*-1}(f, g)^T = (f'(\xi^-) - f'(\xi^+), k_4 f(\xi))^T, \quad \forall (f, g)^T \in \mathcal{H}_1, \tag{4.31}$$

which means $\mathcal{B}_0^* \mathcal{A}_0^{*-1}$ is bounded.

For the energy $E_0(t)$ of the system (4.6), we have

$$\dot{E}_0(t) = -k_3 w_{xx}^2(\xi, t) - k_4 w_t^2(\xi, t), \tag{4.32}$$

which means

$$\int_0^T \left(k_3 |w_{xx}(\xi, t)|^2 + k_4 |w_t(\xi, t)|^2 \right) \mathrm{d}t \le E_0(0), \tag{4.33}$$

for any $T > 0$. This inequality together with (4.31) illustrates that $\mathcal{B}_0$ is admissible for $e^{\mathcal{A}_0 t}$. Therefore, there exists a unique solution of (4.28) such that $(w(\cdot, t), w_t(\cdot, t))^T \in C(\tau, \infty; \mathcal{H})$. Since $\mathcal{A}_0$ generates an exponentially stable C_0-semigroup, it follows from Proposition 2.5 of [30], (4.26), and (4.27) that

$$\left\| \int_\tau^{t/2} e^{\mathcal{A}_0(t/2-s)} \mathcal{B}_0 \begin{pmatrix} \varepsilon_{xx}(\xi, s, s) \\ \varepsilon_s(\xi, s, s) \end{pmatrix} \mathrm{d}s \right\|_{\mathcal{H}} \leq L_0 \left\| \begin{pmatrix} \varepsilon_{xx}(\xi, s, s) \\ \varepsilon_s(\xi, s, s) \end{pmatrix} \right\|_{L^2(\tau, t/2)}$$

$$\leq L_0 M \sqrt{(c_3^2 + c_4^2)/2\omega} \, \| \mathbb{B}(\widehat{w}_0 - w_0, \widehat{w}_1 - w_1)^T \|_{\mathcal{H}_1},$$

and

$$\left\| \int_{t/2}^t e^{\mathcal{A}_0(t-s)} \mathcal{B}_0 \begin{pmatrix} \varepsilon_{xx}(\xi, s, s) \\ \varepsilon_s(\xi, s, s) \end{pmatrix} \mathrm{d}s \right\|_{\mathcal{H}} \leq \left\| \int_0^t e^{\mathcal{A}_0(t-s)} \mathcal{B}_0 \begin{pmatrix} 0 \underset{t/2}{\Diamond} \varepsilon_{xx}(\xi, s, s) \\ 0 \underset{t/2}{\Diamond} \varepsilon_s(\xi, s, s) \end{pmatrix} \mathrm{d}s \right\|_{\mathcal{H}}$$

$$\leq L_0 \left\| \begin{pmatrix} \varepsilon_{xx}(\xi, s, s) \\ \varepsilon_s(\xi, s, s) \end{pmatrix} \right\|_{L^2(t/2, t)}$$

$$\leq L_0 M e^{\omega\tau} \sqrt{(c_3^2 + c_4^2)/2\omega} \, \| \mathbb{B}(\widehat{w}_0 - w_0, \widehat{w}_1 - w_1)^T \|_{\mathcal{H}_1} \cdot e^{-\omega t/2}, \quad \forall t \geq 0,$$

for some constant $L_0 > 0$ that is independent of $\varepsilon_{xx}(\xi, s, s)$ and $\varepsilon_s(\xi, s, s)$, and

$$(u \underset{\tau}{\Diamond} v)(t) = \begin{cases} u(t), & 0 \leq t \leq \tau, \\ v(t), & t > \tau. \end{cases} \tag{4.34}$$

On the other hand, the solution of the systems (4.6) can be represented as

$$\begin{pmatrix} w(\cdot, t) \\ w_t(\cdot, t) \end{pmatrix} = e^{\mathcal{A}_0(t-\tau)} \begin{pmatrix} w(\cdot, \tau) \\ w_t(\cdot, \tau) \end{pmatrix} + \int_\tau^t e^{\mathcal{A}_0(t-s-\tau)} \mathcal{B}_0 \begin{pmatrix} \varepsilon_{xx}(\xi, s, s) \\ \varepsilon_s(\xi, s, s) \end{pmatrix} \mathrm{d}s$$

$$= e^{\mathcal{A}_0(t-\tau)} \begin{pmatrix} w(\cdot, \tau) \\ w_t(\cdot, \tau) \end{pmatrix} + e^{\mathcal{A}_0(t/2-\tau)} \int_\tau^{t/2} e^{\mathcal{A}_0(t/2-s)} \mathcal{B}_0 \begin{pmatrix} \varepsilon_{xx}(\xi, s, s) \\ \varepsilon_s(\xi, s, s) \end{pmatrix} \mathrm{d}s$$

$$+ e^{-\mathcal{A}_0 \tau} \int_{t/2}^t e^{\mathcal{A}_0(t-s)} \mathcal{B}_0 \begin{pmatrix} \varepsilon_{xx}(\xi, s, s) \\ \varepsilon_s(\xi, s, s) \end{pmatrix} \mathrm{d}s. \tag{4.35}$$

Since $\mathcal{A}_0$ generates an exponentially stable C_0-semigroup, there exists two positive constants M_0, ω_0 such that

$$\| e^{\mathcal{A}_0 t} \| \leq M_0 e^{-\omega_0 t}, \tag{4.36}$$

which together with (4.35) and the conservative property of the system (4.3) when $u_1^*(t) = 0$ and $u_2^*(t) = 0$ lead to the inequality below

$$\| (w(\cdot, t), w_t(\cdot, t))^T \|_{\mathcal{H}} \leq M_0 e^{-\omega_0(t-\tau)} \| (w_0, w_1)^T \|_{\mathcal{H}} + L_0 M M_0 e^{\omega_0 \tau} \sqrt{(c_3^2 + c_4^2)/2\omega}$$

$$\cdot \left(e^{-\omega_0 t/2} + e^{\omega\tau} \cdot e^{-\omega t/2} \right) \cdot \| \mathbb{B}(\widehat{w}_0 - w_0, \widehat{w}_1 - w_1)^T \|_{\mathcal{H}_1}.$$

$\square$

5. Numerical simulation

In this section, we use the finite difference method in the time and space domain to give some numerical simulation results for the closed-loop system (2.16)–(2.18). Here we choose the space grid size $N = 40$, time step $dt = 0.0001$ and time span $[0, 10]$. Parameters and coefficients respectively are chosen to be $\xi = 0.5, \tau = 0.5, k_1 = k_2 = k_3 = k_4 = 1$. For the initial values:

$$\begin{cases} w_0(x) = x^2(x-1)^2, & 0 \le x \le 1, \\ w_1(x) = x(x-1), & 0 \le x \le 1, \end{cases} \quad \begin{cases} \varepsilon(x,0) = x^2(x-1)^2, & 0 \le x \le 1, \\ \varepsilon_s(x,0) = x(x-1), & 0 \le x \le 1, \end{cases} \tag{5.1}$$

we plot the displacement $w(x,t)$ and velocity $w_t(x,t)$ in the figure above. It is seen that the displacement of the string is almost at rest after $t = 5$. That is to say, the predictor-observer based scheme is also useful to make the pointwise control system exponentially stable for the beam equation in which observation is subjected to a time delay.

Appendix A

When $\tau = 0$, after proposing both of the controls below

$$\begin{cases} u_1(t) = -k_1 y_1(t) = -k_1 w_{xx}(\xi, t), & k_1 > 0, \\ u_2(t) = -k_2 y_2(t) = -k_2 w_t(\xi, t), & k_2 > 0, \end{cases}$$

the system (1.1) becomes

$$\begin{cases} w_{tt}(x,t) + w_{xxxx}(x,t) = 0, & 0 < x < \xi, \ \xi < x < 1, \ t > 0, \\ w(0,t) = w_x(1,t) = w_{xx}(0,t) = w_{xxx}(1,t) = 0, & t \ge 0, \\ w(\xi^-, t) = w(\xi^+, t), & t \ge 0, \\ w_{xx}(\xi^-, t) = w_{xx}(\xi^+, t), & t \ge 0, \\ w_{xt}(\xi^-, t) - w_{xt}(\xi^+, t) = -k_1 w_{xx}(\xi, t), & t \ge 0, \\ w_{xxx}(\xi^+, t) - w_{xxx}(\xi^-, t) = -k_2 w_t(\xi, t), & t \ge 0, \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), & 0 \le x \le 1, \end{cases} \tag{A.1}$$

where w is the state, $(w_0, w_1)^T$ is the initial state, $\forall \xi \in (0, 1)$.

In the state space defined as (2.1), the system (A.1) can be written as that:

$$\frac{d}{dt} \begin{pmatrix} w(\cdot, t) \\ w_t(\cdot, t) \end{pmatrix} = A \begin{pmatrix} w(\cdot, t) \\ w_t(\cdot, t) \end{pmatrix}, \tag{A.2}$$

where

$$\begin{cases} A(f,g)^T = (g, -f^{(4)})^T, \ \forall (f,g)^T \in D(A), \\ D(A) = \{(f,g)^T \in \left(H^4(0,\xi) \cap H^4(\xi,1)\right) \times \left(H^2(0,\xi) \cap H^2(\xi,1)\right) \\ | f(0) = f'(1) = f''(0) = f'''(1) = 0, f(\xi^-) = f(\xi^+), f''(\xi^-) = f''(\xi^+), \\ g'(\xi^-) - g'(\xi^+) = -k_1 f''(\xi), f'''(\xi^+) - f'''(\xi^-) = -k_2 g(\xi) \}. \end{cases} \tag{A.3}$$

Theorem A.1. *There is a family of eigenvalues $\lambda_n = \{i\omega_n^2, -i\overline{\omega}_n^2\}$ of A for all sufficiently large positive integer n satisfying the following asymptotic expression*

$$\lambda_n = -(k_1 + k_2)\sin^2((n-1/2)\xi\pi) + i(n-1/2)^2\pi^2 + \mathcal{O}(n^{-1}). \tag{A.4}$$

Proof. The characteristic equations of the operator A are given as following:

$$\begin{cases} \lambda^2\phi(x) + \phi^{(4)}(x) = 0, & 0 < x < \xi, \ \xi < x < 1, \\ \phi(0) = \phi'(1) = \phi''(0) = \phi'''(1) = 0, \\ \phi(\xi^-) = \phi(\xi^+), \\ \phi''(\xi^+) = \phi''(\xi^-), \\ \lambda\left[\phi'(\xi^-) - \phi'(\xi^+)\right] = -k_1\phi''(\xi), \\ \phi'''(\xi^+) - \phi'''(\xi^-) = -k_2\lambda\phi(\xi). \end{cases} \tag{A.5}$$

Firstly, let $\lambda = i\omega^2$, the solution of the equations

$$\begin{cases} \phi^{(4)}(x) = \omega^4\phi(x), & 0 < x < \xi, \ \xi < x < 1, \\ \phi(0) = \phi'(1) = \phi''(0) = \phi'''(1) = 0, \end{cases} \tag{A.6}$$

is

$$\phi(x) = \begin{cases} c_1 sh\omega x + c_2\sin\omega x, & 0 \le x \le \xi, \\ d_1 ch\omega(1-x) + d_2\cos\omega(1-x), & \xi \le x \le 1, \end{cases} \tag{A.7}$$

which together with the boundary condition

$$\phi(\xi^-) = \phi(\xi^+), \tag{A.8}$$

gives that

$$c_1 sh\omega\xi + c_2\sin\omega\xi = d_1 ch\omega(1-\xi) + d_2\cos\omega(1-\xi), \tag{A.9}$$

while $\phi''(\xi^+) = \phi''(\xi^-)$ yields

$$c_1 sh\omega\xi - c_2\sin\omega\xi = d_1 ch\omega(1-\xi) - d_2\cos\omega(1-\xi). \tag{A.10}$$

Adding (A.9) and $\pm$(A.10) separately gives

$$c_1 sh\omega\xi = d_1 ch\omega(1-\xi), \quad c_2\sin\omega\xi = d_2\cos\omega(1-\xi), \tag{A.11}$$

that is,

$$c_1 = \frac{ch\omega(1-\xi)}{sh\omega\xi}d_1, \quad c_2 = \frac{\cos\omega(1-\xi)}{\sin\omega\xi}d_2. \tag{A.12}$$

The boundary condition $\lambda\left[\phi'(\xi^-) - \phi'(\xi^+)\right] = -k_1\phi''(\xi)$ shows that:

$$sh\omega(1-\xi)\cdot d_1 - \sin\omega(1-\xi)\cdot d_2 + \left[ch\omega\xi - i\omega^{-1}k_1 sh\omega\xi\right]\cdot c_1 + \left[\cos\omega\xi + i\omega^{-1}k_1\sin\omega\xi\right]\cdot c_2 = 0, \tag{A.13}$$

while the boundary condition $\phi'''(\xi^+) - \phi'''(\xi^-) = -k_2\lambda\phi(\xi)$ illustrates that:

$$[ch\omega\xi - ik_2\omega^{-1}sh\omega\xi] \cdot c_1 - [\cos\omega\xi + ik_2\omega^{-1}\sin\omega\xi] \cdot c_2 + sh\omega(1-\xi) \cdot d_1 + \sin\omega(1-\xi) \cdot d_2 = 0. \tag{A.14}$$

It follows from (A.11), (A.13) and (A.14) that $\Delta(\omega)(c_1, c_2, d_1, d_2)^T = 0$, where

$$\Delta(\omega) = \begin{bmatrix} sh\omega\xi & 0 & -ch\omega(1-\xi) & 0 \\ 0 & \sin\omega\xi & 0 & -\cos\omega(1-\xi) \\ ch\omega\xi - ik_1\omega^{-1}sh\omega\xi & \cos\omega\xi + ik_1\omega^{-1}\sin\omega\xi & sh\omega(1-\xi) & -\sin\omega(1-\xi) \\ ch\omega\xi - ik_2\omega^{-1}sh\omega\xi & -\cos\omega\xi - ik_2\omega^{-1}\sin\omega\xi & sh\omega(1-\xi) & \sin\omega(1-\xi) \end{bmatrix}. \tag{A.15}$$

So,

$$\det\Delta(\omega) = \begin{vmatrix} sh\omega\xi & 0 & -ch\omega(1-\xi) & 0 \\ 0 & \sin\omega\xi & 0 & -\cos\omega(1-\xi) \\ ch\omega\xi - ik_1\omega^{-1}sh\omega\xi & \cos\omega\xi + ik_1\omega^{-1}\sin\omega\xi & sh\omega(1-\xi) & -\sin\omega(1-\xi) \\ i(k_1 - k_2)\omega^{-1}sh\omega\xi & -2\cos\omega\xi - i(k_1 + k_2)\omega^{-1}\sin\omega\xi & 0 & 2\sin\omega(1-\xi) \end{vmatrix}. \tag{A.16}$$

Noting that $\lambda \in (\pi/2, 3\pi/2)$ is equivalent to $\omega \in (0, \pi/2)$. Moreover, the eigenvalues are symmetric with real axis, so we only consider $\omega \in (0, \pi/4)$, where for $0 < \xi < 1$,

$$e^{\omega\xi}, e^{\omega(1-\xi)} \to \infty, \quad e^{-\omega\xi}, e^{-\omega(1-\xi)} \to 0.$$

Therefore,

$$\frac{1}{2}e^{-\omega\xi}e^{-\omega(1-\xi)}\det(\Delta(\omega))$$

$$= \begin{vmatrix} 1 + \mathcal{O}(e^{-2\omega\xi}) & 0 & -1 + \mathcal{O}(e^{-2\omega(1-\xi)}) & 0 \\ 0 & \sin\omega\xi & 0 & -\cos\omega(1-\xi) \\ 1 - i\frac{1}{2}(k_1 + k_2)\omega^{-1} + \mathcal{O}(e^{-2\omega\xi}) & i\frac{1}{2}(k_1 - k_2)\omega^{-1}\sin\omega\xi & 1 + \mathcal{O}(e^{-2\omega(1-\xi)}) & 0 \\ i\frac{1}{2}(k_1 - k_2)\omega^{-1} + \mathcal{O}(\omega^{-1}e^{-2\omega\xi}) & -\cos\omega\xi - i\frac{1}{2}(k_1 + k_2)\omega^{-1}\sin\omega\xi & 0 & \sin\omega(1-\xi) \end{vmatrix}$$

$$= -2\cos\omega - i(k_1 + k_2)\omega^{-1}\sin\omega\xi\cos\omega(1-\xi) + i\frac{1}{2}(k_1 + k_2)\omega^{-1}\cos\omega + \mathcal{O}(\omega^{-2}).$$

By $\det(\Delta(\omega)) = 0$, we have

$$-2\cos\omega - i(k_1 + k_2)\omega^{-1}\sin\omega\xi\cos\omega(1-\xi) + i\frac{1}{2}(k_1 + k_2)\omega^{-1}\cos\omega + \mathcal{O}(\omega^{-2}) = 0, \tag{A.17}$$

or

$$\mathcal{O}(\omega^{-2}) - 2\cos\omega - i\frac{1}{2}(k_1 + k_2)\omega^{-1}[\sin\omega - \cos\omega + \sin\omega(2\xi - 1)] = 0. \tag{A.18}$$

Let $\tilde{\omega}_n = (n - 1/2)\pi$ and $\omega_n = \tilde{\omega}_n + \mathcal{O}(n^{-1})$, from (A.18) we have

$$\mathcal{O}(n^{-1}) = i\frac{1}{2}(k_1 + k_2)\tilde{\omega}_n^{-1}\sin^2(\tilde{\omega}_n\xi) + \mathcal{O}(n^{-2}). \tag{A.19}$$

Hence

$$\begin{aligned}
\lambda_n = i\omega_n^2 &= i\left[\tilde{\omega}_n + \mathcal{O}(n^{-1})\right]^2 \\
&= -(k_1+k_2)\sin^2((n-1/2)\xi\pi) + i(n-1/2)^2\pi^2 + \mathcal{O}(n^{-1}).
\end{aligned} \tag{A.20}$$

$\square$

Theorem A.2. *For any $k_1, k_2 > 0$,*
(i) *the system (A.1) is asymptotically stable;*
(ii) *the system (A.1) is exponentially stable if and only if $\xi \in (0,1)$ is a rational number satisfying $\xi \neq 2l/(2m-1)$ for any integers l, m.*

Proof. (i) It is well known that A is a dissipative and discrete operator for any $k_1, k_2 > 0$, so the system (A.1) is asymptotically stable if and only if $\mathcal{R}e\lambda \neq 0$ for any $\lambda \in \sigma(A)$. Let $\lambda = i\omega^2 (\omega \in \mathbb{R})$, multiplying $\overline{\phi(x)}$ on both sides of the first equation in (A.5) and integrating from 0 to 1 with respect to x, we have

$$\omega^4 \int_0^1 |\phi(x)|^2 \mathrm{d}x - \int_0^1 |\phi''(x)|^2 \mathrm{d}x + i\omega^2 \left(k_2|\phi(\xi)|^2 - |\phi'(\xi^-) - \phi'(\xi^+)|/k_1\right) = 0. \tag{A.21}$$

Comparing the real and imaginary parts gives

$$\begin{cases}
\omega^4 \int_0^1 |\phi(x)|^2 \mathrm{d}x = \int_0^1 |\phi''(x)|^2 \mathrm{d}x, \\
|\phi'(\xi^-) - \phi'(\xi^+)| = k_1 k_2 |\phi(\xi)|^2.
\end{cases} \tag{A.22}$$

From the fifth equation $\lambda\left[\phi'(\xi^-) - \phi'(\xi^+)\right] = -k_1\phi''(\xi)$ in (A.5), we have that

$$\left|\phi'(\xi^-) - \phi'(\xi^+)\right|^2 = \left|-\frac{k_1}{i\omega^2}\phi''(\xi)\right|^2 = -\frac{k_1^2}{\omega^4}\left|\phi''(\xi)\right|^2$$

which together with the second equation of (A.5) shows that $\phi(\xi) = \phi''(\xi) = 0$.
So, (A.5) becomes that

$$\begin{cases}
\phi^{(4)}(x) = \omega^4\phi(x), \;\; 0 < x < \xi, \xi < x < 1, \\
\phi(0) = \phi'(1) = \phi''(0) = \phi'''(1) = 0, \\
\phi(\xi) = \phi''(\xi) = 0.
\end{cases} \tag{A.23}$$

Since $\phi(0) = \phi(\xi) = 0$ and Rolle Theorem, there exists $\xi_1 \in (0, \xi)$ such that $\phi'(\xi_1) = 0$. Together with $\phi'(\xi_1) = \phi'(1) = 0$ and Rolle Theorem tell us that there exists $\xi_2 \in (\xi_1, 1)$ such that $\phi''(\xi_2) = 0$. Since $\phi''(0) = \phi''(\xi_2) = \phi''(\xi) = 0$ (it might assume that $0 < \xi_2 < \xi < 1$) and Rolle Theorem, there exists $\xi_3 \in (0, \xi_2)$ such that $\phi'''(\xi_3) = 0$, and there exists $\xi_4 \in (\xi_2, \xi)$ such that $\phi'''(\xi_4) = 0$. Similarly since $\phi'''(\xi_3) = \phi'''(\xi_4) = \phi'''(1) = 0$ and Rolle Theorem, there exists $\xi_5 \in (\xi_3, \xi_4)$ such that $\phi^{(4)}(\xi_5) = 0$ and there exists $\xi_6 \in (\xi_4, 1)$ such that $\phi^{(4)}(\xi_6) = 0$. Thus, $\phi(\xi_5) = \phi(\xi_6) = 0$, that is, there is at least one zero point of ϕ in the interval $(0,1)$, for example ξ_5, ξ_6.

Now we show that if there are n numbers of different zeros of ϕ in the interval $(0,1)$, there are at least $n+1$ numbers of different zeros of ϕ in the interval $(0,1)$.

Suppose

$$0 < \xi_1 < \xi_2 < \cdots < \xi_n < 1, \phi(\xi_i) = 0, i = 1, 2, \ldots, n,$$

since $\phi(0) = 0$ and the Rolle Theorem it is naturally shown that there exist $\eta_i (i = 1, 2, \ldots, n)$ such that

$$0 < \eta_1 < \xi_1 < \eta_2 < \xi_2 < \cdots < \eta_n < \xi_n < 1,$$

with $\phi'(\eta_i) = 0 (i = 1, 2, \ldots, n)$.

By $\phi'(1) = \phi'(\eta_i) = 0 (i = 1, 2, \ldots, n)$ and Rolle Theorem, there exist $\alpha_i (i = 1, 2, \ldots, n)$ such that

$$0 < \eta_1 < \alpha_1 < \eta_2 < \cdots < \eta_n < \alpha_n < 1,$$

such that $\phi''(\alpha_i) = 0 (i = 1, 2, \ldots, n)$.

By $\phi''(0) = \phi''(\xi) = \phi''(\alpha_i) = 0 (i = 1, 2, \ldots, n)$ (it might assume that $0 < \xi < \alpha_1 < \alpha_2 < \cdots < \alpha_n < 1$) and Rolle Theorem, there exists $\beta_i (i = 1, 2, \ldots, n+1)$ such that

$$0 < \beta_1 < \xi < \beta_2 < \alpha_1 < \beta_3 < \cdots < \beta_{n+1} < \alpha_n < 1,$$

with $\phi'''(\beta_i) = 0 (i = 1, 2, \ldots, n+1)$.

Similarly since $\phi'''(1) = \phi'''(\beta_i) = 0 (i = 1, 2, \ldots, n+1)$ and Rolle Theorem, there exist $\theta_i (i = 1, 2, \ldots, n+1)$ such that

$$0 < \beta_1 < \theta_1 < \beta_2 < \theta_2 < \cdots < \beta_{n+1} < \theta_{n+1} < 1$$

with $\phi^{(4)}(\theta_i) = 0 (i = 1, 2, \ldots, n+1)$, so $\phi(\theta_i) = 0 (i = 1, 2, \ldots, n+1)$.

Therefore, $\phi \equiv 0$ by the uniqueness of the solution of linear ordinary differential equations. (See the proof of Theorem 4.2 in the page 388 of the reference [18]) This contradicts that $\lambda = i\omega^2 (\omega \in \mathbb{R}) \in \sigma(A)$. Therefore, for any $k_1, k_2 > 0$, the system (A.1) is asymptotically stable.

(ii) Since the system (A.1) is asymptotically stable but not exponentially stable if and only if

$$\sin\left((n - 1/2)\xi\pi\right) \to 0, \tag{A.24}$$

for some positive integer sequences $\{n \to \infty\}$, the second part of the theorem has been then obtained by Theorem A.3 and Corollary 1 of the reference [22]. $\qquad\square$

Especially when $\xi = 0.5$, (A.1) becomes that

$$\begin{cases} w_{tt}(x,t) + w_{xxxx}(x,t) = 0, \quad 0 < x < 0.5, \ 0.5 < x < 1, \ t > 0, \\ w(0,t) = w_x(1,t) = w_{xx}(0,t) = w_{xxx}(1,t) = 0, \quad t \geq 0, \\ w(0.5^-,t) = w(0.5^+,t), \quad t \geq 0, \\ w_{xx}(0.5^-,t) = w_{xx}(0.5^+,t), \quad t \geq 0, \\ w_{xt}(0.5^-,t) - w_{xt}(0.5^+,t) = -k_1 w_{xx}(0.5,t), \quad t \geq 0, \\ w_{xxx}(0.5^+,t) - w_{xxx}(0.5^-,t) = -k_2 w_t(0.5,t), \quad t \geq 0, \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), \quad 0 \leq x \leq 1, \end{cases} \tag{A.25}$$

where w is the state, $(w_0, w_1)^T$ is the initial state, $\forall \xi \in (0, 1)$.

The system (A.25) can be written as that:

$$\frac{d}{dt}\begin{pmatrix} w(\cdot,t) \\ w_t(\cdot,t) \end{pmatrix} = A_0 \begin{pmatrix} w(\cdot,t) \\ w_t(\cdot,t) \end{pmatrix}, \tag{A.26}$$

where

$$
\begin{cases}
A_0(f,g)^T = (g, -f^{(4)})^T, \ \ \forall (f,g)^T \in D(A_0), \\
D(A_0) = \{(f,g)^T \in \left(H^4(0,0.5) \cap H^4(0.5,1)\right) \times \left(H^2(0,0.5) \cap H^2(0.5,1)\right) \\
|f(0) = f'(1) = f''(0) = f'''(1) = 0, f(0.5^-) = f(0.5^+), f''(0.5^-) = f''(0.5^+), \\
g'(0.5^-) - g'(0.5^+) = -k_1 f''(0.5), f'''(0.5^+) - f'''(0.5^-) = -k_2 g(0.5)\},
\end{cases}
\tag{A.27}
$$

in the state space defined as (2.1).

From Theorems A.1 and A.2, we naturally have the two theorems below.

Theorem A.3. *There is a family of eigenvalues $\lambda_n = \{i\omega_n^2, -i\overline{\omega}_n^2\}$ of A_0 for all sufficiently large positive integer n satisfying the following asymptotic expression*

$$
\lambda_n = -(k_1 + k_2)/2 + i(n - 1/2)^2 \pi^2 + \mathcal{O}(n^{-1}).
\tag{A.28}
$$

Theorem A.4. *For any $k_1, k_2 > 0$, the system (A.25) is exponentially stable.*

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