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STRONG STABILITY OF LINEAR PARABOLIC TIME-OPTIMAL CONTROL PROBLEMS[☆]

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Abstract. Sufficient conditions for strong stability of a class of linear time-optimal control problems with general convex terminal set are derived. Strong stability in turn guarantees qualified optimality conditions. The theory is based on a characterization of weak invariance of the target set under the controlled equation. An appropriate strengthening of the resulting Hamiltonian condition ensures strong stability and yields *a priori* bounds on the size of multipliers, independent of, *e.g.*, the initial point or the running cost. In particular, the results are applied to the control of the heat equation into an L^2-ball around a desired state.

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1. Introduction

We investigate the following class of time-optimal control problems, where u denotes the state, q the control, and T the terminal time:

$$\text{Minimize} \quad j(T,q) \coloneqq T + \int_0^T L(q(t))\,\mathrm{d}t,$$
$$\text{subject to} \quad \begin{cases} T > 0, \\ \partial_t u(t) + Au(t) = Bq(t), \quad t \in (0,T), \\ u(0) = u_0, \\ u(T) \in U, \\ q(t) \in Q_{ad}, \quad t \in (0,T). \end{cases} \tag{P}$$

Here, $A\colon V \to V^*$ (for a Gelfand triple $V \hookrightarrow H \hookrightarrow V^*$) is a linear, weakly coercive operator, $Q_{ad} \subset Q$ (for a Hilbert space Q) a bounded set of admissible controls, and B a control operator mapping Q into a subspace

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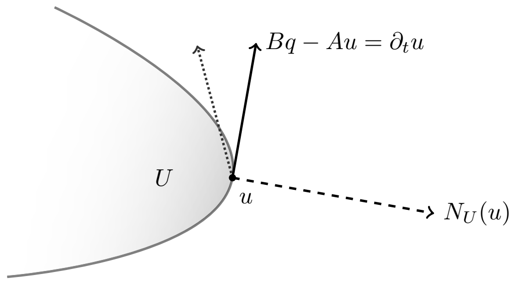

FIGURE 1. Geometric interpretation of the lower Hamiltonian condition (1.1) with strengthened condition (1.2) (*dotted*).

of V^*. In particular, if $A = -\Delta$ equipped with homogeneous Dirichlet boundary conditions on a bounded domain Ω, then we choose $V = H_0^1(\Omega)$, $H = L^2(\Omega)$, and $V^* = H^{-1}(\Omega)$. Note that this in particular allows for distributed ($Q = L^2(\omega)$ for $\omega \subset \Omega$ open) and Neumann boundary control ($Q = L^2(\omega)$ for $\omega \subset \partial\Omega$ open) of reaction diffusion equations; see Section 2 for the precise assumptions. The task is to steer an initial state $u_0 \in H$ into a closed convex target set $U \subset H$ by an appropriate choice of the control q (and the time horizon T), while minimizing a sum of T and the running cost $L\colon Q \to \mathbb{R}_+$ for the control (which, for convenience, we assume to be Lipschitz continuous and convex). In the case $L \equiv 0$ we obtain the pure time-optimal problem, where we are plainly interested to steer u_0 into U in the shortest time possible; see, *e.g.*, [16, 25, 31, 40, 42] and the overview given in Chapter 7 from [27]. A choice of L different from zero allows to include a cost for the control, for instance:

$$L(q) = \frac{\alpha}{2}\|q\|_Q^2 \quad \text{for some } \alpha > 0.$$

This is useful in applications where the control has some inherent cost or where bang-bang controls are not desirable (*cf.*, *e.g.*, [23]). Additionally it can be used as a regularization strategy for the pure time-optimal case (*cf.*, *e.g.*, [22, 24]).

Strong stability (also known as calmness [9, 36] or weak calmness [8]) quantifies the dependency of the optimal value function of (P) on small perturbations of the constraint. The concrete meaning of strong stability (as employed in this work) is given roughly as follows: The optimal value function of (P) (*i.e.* the minimal value of $j(\cdot,\cdot)$) depends Lipschitz continuously on perturbations of the target set U of the form $U_\delta = \{\, u \in H \mid d_U(u) \le \delta \,\}$ with $\delta \ge 0$, where $d_U(\cdot)$ denotes the distance to U measured in the norm of H. We refer to Section 4.1 for a precise definition. For instance, for the prototypical example

$$U = \{\, u \in H \mid \|u - u_d\|_H \le \delta_0 \,\} \quad \text{for some } \delta_0 \ge 0 \text{ and } u_d \in H,$$

the optimal value function is required to be Lipschitz continuous with respect to the radius δ_0. The main objective of this work is to derive conditions on the triple (A, U, BQ_{ad}) which guarantee that (P) is strongly stable for all optimal solutions. Although it is generally well-known that "almost all" problems are strongly stable, it remains a difficult task to verify strong stability of a particular problem; *cf.* Section 3 from [7]. To adress this, we show that strong stability follows from a strengthened Hamiltonian condition, which will be introduced below. Roughly speaking, this condition guarantees that for any point in the target set there exist trajectories that have a sufficiently wide angle with any direction normal to U; see Figure 1.

Mathematically, our approach relies on weak invariance of the terminal set. U is called weakly invariant under (A, BQ_{ad}) if for any $u_0 \in U$ there is a control such that the corresponding trajectory with initial value u_0 remains in U. The precise meaning of weak invariance used in this work is given in Definition 3.1; *cf.* also ([12], Sect. 12.1), [38] (denoted as holdability), and Section 1 from [13]. One of the main contributions of this article is the characterization of weak invariance by the conditions that the minimizing projection onto U in H

denoted P_U is stable in V, *i.e.* $P_U(V) \subseteq V$, and

$$h(u, \zeta) := \min_{q \in Q_{ad}} \langle Bq - Au, \zeta \rangle \leq 0 \quad \text{for all } u \in U \cap V,\ \zeta \in N_U(u) \cap V, \tag{1.1}$$

where $h \colon V \times V \to \mathbb{R}$ is the *lower Hamiltonian* and $N_U(u)$ is the normal cone in H to U at the point u; see Theorem 3.8. This extends results for invariance under semigroups, *i.e.* uncontrolled systems (see, *e.g.*, [32], Sect. 2.1), and results for optimal control of ordinary differential equations (see, *e.g.*, [12], Sect. 12.1).

Precisely, our main result can now be stated as follows: Assume that the projection P_U is stable in V and that the *strengthened Hamiltonian condition*,

$$h(u, \zeta) \leq -h_0 \|\zeta\|_H \quad \text{for all } u \in U \cap V,\ \zeta \in N_U(u) \cap V, \tag{1.2}$$

holds for some $h_0 > 0$ (independent of u and ζ). Then, strong stability is satisfied for *all solutions* of the time-optimal problem; see Theorem 4.5. In particular, strong stability guarantees that qualified optimality conditions hold. On top of this, condition (1.2) enables to derive Lipschitz continuity results of the value function for a variety of perturbations of the problem (P), not only in the target set. Note that this corresponds to an estimate for the optimal time for the pure time-optimal problem, which is of independent interest. These consequences will be discussed in detail below.

For (P), optimality conditions can be stated as follows: For any optimal solution $(T, \bar{q}, \bar{u})$, there exists a nontrivial $\bar{\mu} \in N_U(\bar{u}(T))$, a corresponding adjoint state $\bar{z}$ with

$$-\partial_t \bar{z}(t) + A^* \bar{z}(t) = 0, \quad t \in (0, T), \tag{1.3}$$

$$\bar{z}(T) = \bar{\mu}, \tag{1.4}$$

and a $\bar{\mu}_0 \in \{0, 1\}$, such that

$$0 = \langle B\bar{q}(t) - A\bar{u}(t), \bar{z}(t) \rangle + \bar{\mu}_0 [1 + L(\bar{q}(t))], \quad t \in (0, T), \tag{1.5}$$

$$\bar{q}(t) = \operatorname*{argmin}_{q \in Q_{ad}} \left[\langle Bq, \bar{z}(t) \rangle + \bar{\mu}_0 L(q) \right], \quad t \in (0, T). \tag{1.6}$$

This general form is fulfilled in any optimum of (P) if, *e.g.*, the target set U is of finite co-dimension in H. We give an independent proof of the general form of the optimality conditions for (P) in Theorem 4.13; *cf.* [12, 27, 35]. In the case that $\bar{\mu}_0 = 1$, the optimality conditions are called *qualified*. Assuming strong stability, the qualified form holds; see Theorem 4.12 (*cf.* also [35], Rem. 2.2). In detail, strong stability implies the existence of an exact penalty function, which in turn allows to derive qualified optimality conditions, where we use the approach due to Clarke [11] (here, the assumptions of Lipschitz continuity and convexity of L are used to apply Clarke's generalized subdifferential and to identify the solution to the minimization problem (1.6) with the solution of the corresponding subdifferential inclusion, respectively). We emphasize that Theorem 4.12 does not require any structural assumptions on U, such as finite co-dimension. Additionally, the *a priori* estimate $\|\bar{\mu}\|_H \leq c/h_0$ holds for some generic constant c. Note that in the examples considered in Section 5 we can explicitly determine the constant h_0 in terms of the problem data.

Furthermore, in the purely time-optimal case (where $L \equiv 0$), strong stability can be interpreted as a perturbation result of order one for the optimal time T with respect to perturbations in the terminal set. However, the strengthened Hamiltonian condition also allows for the derivation of such results with respect to different perturbations in the problem (P). In particular, we prove Lipschitz continuity of the optimal time with respect to perturbations of the initial state as considered in, *e.g.*, [10, 19]. We remark that the condition (1.2) is more general than the one of Theorem 4.1 from [10], which is given in the setting of a point target and $B = \mathrm{Id}$; *cf.* Proposition 5.1. Additionally, we extend the analysis to certain perturbations of the operator A as considered in [43]; *cf.* also [40] and the references therein. However, corresponding Lipschitz perturbation results for the optimal controls require additional structural assumptions on the control mechanism and the adjoint equation, and are outside of the scope of this article; see, *e.g.*, [40, 42, 43].

To the best of our knowledge, several of the applications of the sufficient conditions derived in this work yield new results for concrete problems. In particular, these conditions allow to derive qualified optimality conditions for several interesting scenarios, such as the control of the heat equation into L^2-balls around certain target sets. We will discuss these applications in Section 5. In the case of steering the system into a single point, *i.e.* $U = \{ u_d \}$, we can compare the results to those of Barbu Section 5.3 from [4], who derived the maximum principle for a nonlinear monotone equation using a quadratic penalty method; *cf.* also [5] for the Navier-Stokes equation or [24] for the linear wave equation. Note that the qualifying condition on the target state in Theorem 5.3.1 from [4], is essentially the same as the one obtained from (1.2) in the case $U = \{ u_d \}$; see Section 5.1. However, this condition holds in concrete applications only for controls which are acting everywhere in space. A different approach, which is based on controllability, has been proposed by Wang and Zuazua [42]. Here, the equivalence between time- and norm-optimality (see also [16]) is used in an essential way. In particular, the conditions (1.3), (1.4), and (1.6) (which are independent of $\bar{\mu}_0$ in this case) are obtained for the problem of steering the heat equation into zero with pointwise bounded controls restricted to an arbitrary subset of the underlying domain. In this case, the multiplier is obtained in a space of distributions, larger than L^2. However, this technique seems to be restricted to the case $L \equiv 0$ and yields a different condition instead of (1.5) to characterize the optimality of the time variable.

To further assess the applicability of the strengthened Hamiltonian condition (1.2) in the context of concrete examples, in Section 5 we discuss several cases when A is given by a general convection-diffusion operator on a bounded domain Ω. On the one hand, we find that (1.2) always holds for the control of, say, the heat equation into a $L^2(\Omega)$-ball centered at a sufficiently small u_d, assuming only that the zero control is admissible. We emphasize that this already includes the classical setting $u_d = 0$ considered in, *e.g.*, [40, 41, 43], without further assumptions. On the other hand, we find that it is fulfilled for more restrictive target sets or more general convection-diffusion operators only under additional assumptions on the form of the control operator and admissible set. We compare these requirements to established controllability assumptions (see, *e.g.*, [45]) and find that our conditions are stronger, in general. This can be connected to the fact that the cost of the controls resulting from controllability conditions (see [17]) grows exponentially if the length of the control horizon is decreased towards zero. However, for general A, we also give an example of a special target set where (1.2) follows directly from an established stabilizability assumption, based on the Fattorini criterion (which can be fulfilled even with finite-dimensional controls).

Clearly, as (1.2) implies (1.1), we implicitly only consider systems that are weakly invariant. This can also be justified from a practical point of view. Note first, that we only require the state to be inside the target set at the final time T in the mathematical formulation of the time optimal problem (P). However, in practice, time continues to advance afterwards and in many cases we are interested to remain inside of the target set. Therefore, it seems to be reasonable to restrict attention to systems where this is always possible. Otherwise, the optimal control might achieve $u(T) \in U$ with small cost, but every trajectory continuing from $u(T)$ might be forced to leave the target set again (possibly immediately).

We appreciate that (1.2) might not be fulfilled in all practically relevant cases. However, we anticipate that it is useful in many situations, where the objective is to steer the system "sufficiently close" to a weakly invariant, or even asymptotically stable state u_d; *cf.*, *e.g.*, [1, 14, 23]. Here, it could also help to guide the choice of appropriate target sets U, which guarantee both that the terminal state will be close to u_d, and that the resulting control problem will be strongly stable. We also note that, if the optimal trajectory $\bar{u}$ is assumed to be known and U has finite co-dimension with regular normal cone, condition (1.2) can be weakened to

$$h(\bar{u}(T), \zeta) \leq -h_0 \|\zeta\|_H \quad \text{for all } \zeta \in N_U(\bar{u}(T)), \tag{1.7}$$

while still implying the qualified form of the optimality conditions; see Proposition 4.17. Furthermore, if the normal cone contains only one element, this condition is already equivalent to the qualified optimality conditions (see Prop. 4.18), which further clarifies the role of the strengthened Hamiltonian condition.

Viewing (P) as an abstract constrained nonconvex optimization problem, one could also require a *constraint qualification* (CQ) to guarantee the qualified form of the optimality conditions. However, the concrete form of the standard CQs does not only depend on the parametrization of the constraint, but also on objects such as gradients, which require a proper (but in some sense arbitrary) parametrization of the time variable T; see Section 4.2. Therefore, strong stability appears to be the more straightforward tool in this context. Comparing CQs to the strengthened Hamiltonian condition (1.2), we remark that the latter qualifies all optimal solutions at once, whereas the other considers only one specific, but *a priori* unknown solution, similar to (1.7).

The article is organized as follows: In Section 2 we introduce some notation and state the main assumptions. Weak invariance is characterized in Section 3. The concept of strong stability is introduced in Section 4, where we discuss the time-optimal control problem and derive optimality conditions. Moreover, we show that strengthening of weak invariance implies strong stability as well as further perturbation results. Last, Section 5 is devoted to applications of the results in the context of convection-diffusion equations on a bounded domain. The text will be accompanied by the illustrative example $U = \{ u_d \}$ with fixed $u_d \in H$, to make ideas visible to the reader. However, we emphasize that it does not represent the main application.

2. Notation and main assumptions

For any two Banach spaces X and Y we use $Y \hookrightarrow X$ to denote the continuous embedding and $Y \hookrightarrow_c X$ for the continuous and compact embedding. The domain of a linear (possibly unbounded) operator A on X is denoted by $\mathcal{D}_X(A)$. Let V and H be real Hilbert spaces such that $V \hookrightarrow_c H \cong H^* \hookrightarrow V^*$ form a Gelfand triple. Without restriction suppose $\|v\|_V \geq \|v\|_H$ for all $v \in V$. In general, we abbreviate the duality pairing and the inner product and norm in H by

$$\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{V^*,V}, \quad (\cdot, \cdot) = (\cdot, \cdot)_H, \quad \|\cdot\| = \|\cdot\|_H.$$

Assumption 2.1. Let $a\colon V \times V \to \mathbb{R}$ be a continuous bilinear form, which satisfies the Gårding inequality (which is also referred to as weak coercivity): we assume there are constants $\alpha_0 > 0$ and $\omega_0 \geq 0$ such that

$$a(u,u) + \omega_0 \|u\|^2 \geq \alpha_0 \|u\|_V^2 \quad u \in V. \tag{2.1}$$

We denote by $A\colon V \subset V^* \to V^*$ the unique linear operator with

$$\langle Au, v \rangle = a(u,v) \quad \text{for all } v \in V.$$

It holds $\mathcal{D}_{V^*}(A) = V$; see, *e.g.*, Theorem 3.4 from [21]. Due to the Gårding inequality, the operator $-(A + \omega_0)$ generates an analytic semigroup on V^*; see, *e.g.*, Section 1.4 from [32]. We abbreviate $\omega_0 \,\mathrm{Id}$ by ω_0 to simplify the presentation, where Id is the identity operator on V^*. Due to (2.1), we can define fractional powers in the sense of Section 2.6 from [33]. For fixed $\theta \geq 0$, we abbreviate $X_\theta = \mathcal{D}_{V^*}((A+\omega_0)^\theta)$ and introduce the norm on X_θ as

$$\|\cdot\|_{X_\theta} \coloneqq \|(A+\omega_0)^\theta \cdot\|_{V^*}.$$

As usual, $(V^*, V)_{\theta,s}$, respectively $[V^*, V]_\theta$, stand for the real, respectively complex interpolation couple with $\theta \in (0,1)$ and $s \in (1, \infty)$. Since V is a Hilbert space (and thus V^* as well), the operator $(A + \omega_0)$ has bounded imaginary powers and it holds for $\theta \in (0,1)$ that $X_\theta = [V^*, V]_\theta = (V^*, V)_{\theta,2}$; see, *e.g.*, Section 1.15.3 from [39]. In particular, $X_{1/2} = H$; see, *e.g.,* Section 1.2.4 from [30]. Moreover,

$$X_\theta^* = [V^*, V]_\theta^* = [V^*, V]_{1-\theta} = X_{1-\theta};$$

see, *e.g.*, Theorems 1.9.3 b), 1.11.3 from [39]. Furthermore, using Theorems 1.9.3 b), 1.11.3, and 1.15.3 from [39], we find

$$X_{1-\theta} = [V^*, V]_{1-\theta} = [[V^*, V]_{1/2}, V]_{1-2\theta} = [H, V]_{1-2\theta}. \tag{2.2}$$

For any set $S \subset Y$ of a Banach space Y, let $d_S^Y(\cdot)$ denote the distance function

$$d_S^Y(y) \coloneqq \inf_{y' \in S} \|y - y'\|_Y.$$

Furthermore, if Y is a Hilbert space and S is closed and convex, we denote by $P_S^Y : Y \to S$ the *minimizing projection* to S. Note that P_S^Y is Lipschitz continuous in H (with Lipschitz constant one); see, *e.g.*, Proposition 4.8 from [6]. We denote by

$$N_S^Y(y) \coloneqq \{\, v \in Y^* \mid \langle v, y' - y\rangle_{Y^*,Y} \le 0 \quad \text{for all } y' \in S \,\}$$

the *normal cone* to S at the point $y \in S$. In the case $Y = H$ and $S = U$ (or if no ambiguity arises), we simply write $d_U(\cdot)$, P_U, and $N_U(\cdot)$.

Concerning the problem (P), the terminal set $U \subset H$ is assumed to be nonempty, closed, and convex and the initial state satisfies $u_0 \in H$.

Assumption 2.2. Let Q be a Hilbert space, and Q_{ad} be a closed convex subset. We assume the control operator $B\colon Q \to X_{\theta_0} \hookrightarrow V^*$ for some $\theta_0 \in (0, 1/2]$ to be linear and continuous. In addition, we assume Q_{ad} to be bounded in Q, and define $C_{Q_{ad}} = \max_{q \in Q_{ad}} \|q\|_Q$. Furthermore, the functional $L\colon Q \to \mathbb{R}_+$ is Lipschitz continuous on Q_{ad} and convex.

In addition, for $T > 0$ we define $Q(0,T) \coloneqq L^2((0,T); Q)$ and

$$Q_{ad}(0,T) = \{\, q \in Q(0,T) \mid q(t) \in Q_{ad} \quad \text{a.e. } t \in (0,T) \,\} \subset L^\infty((0,T); Q).$$

Moreover, for $T > 0$ we use the symbol $W(0,T)$ to abbreviate $H^1((0,T); V^*) \cap L^2((0,T); V)$, endowed with the canonical norm and inner product. The symbol $i_T\colon W(0,T) \to H$ denotes the trace mapping $i_T u = u(T)$.

3. Weak invariance

We first introduce the notion of weak invariance.

Definition 3.1. The set $U \subset H$ is said to be *weakly invariant* under (A, BQ_{ad}), if for every $u_0 \in U$ there exists a control $q \in [0, \infty) \to Q_{ad}$ such that the solution u to

$$\partial_t u + Au = Bq, \quad u(0) = u_0,$$

satisfies $u(t) \in U$ for all $t \ge 0$. If ambiguity is not to be expected, we simply say U is weakly invariant.

Remark 3.2. Different terms for weak invariance are being used in the literature, such as *holdability* or *viability*; *cf.* [38] and Section 1 from [13].

The structure of this section is as follows: We first discuss stability of the minimizing projection P_U in V. This is then needed to characterize weak invariance in terms of the lower Hamiltonian.

3.1. Stability of the projection to the target set

We call the minimizing projection P_U in H onto U stable in V, if $P_U(V) \subset V$. In general, stability of P_U in V is a non-trivial assumption. However, in the uncontrolled case, it is known that invariance of U under A

(*i.e.*, the property $\mathrm{e}^{-tA}U \subset U$ for all $t \geq 0$, with e^{-tA} the semigroup generated by $-A$) implies the stability of P_U; see, *e.g.*, Theorem 2.2 from [32], (*cf.* also Sect. II.6.3 from [2], for the nonautonomous case). In the following we generalize this known sufficient condition for stability of P_U in V to controlled systems. This will be a prerequisite for the characterization of weak invariance of U under (A, BQ_{ad}).

As an illustrative example, we consider the set $U = \{ u_d \}$. The projection P_U is given by $P_U(u) = u_d$. Clearly, P_U is stable in V if and only if $u_d \in V$. We will verify later that weak invariance holds if and only if $Au_d \in BQ_{ad}$; see Proposition 5.1. Due to Assumption 2.2, this implies $Au_d \in X_{\theta_0} \hookrightarrow V^*$, which in turn leads to $u_d \in V$, in accordance with the results of this section. Additionally, invariance of U under A corresponds to weak invariance with the trivial choice $Q_{ad} = \{ 0 \}$, which holds only for $Au_d = 0$.

The proof is divided into two steps. Roughly speaking, we first prove that for a weakly invariant set U, the scaled resolvent of A does not map points in U too far outside of U. We define for any $u \in H$

$$E_\lambda u \coloneqq \lambda(\lambda + A)^{-1}u = (1 + A/\lambda)^{-1}u.$$

Provided that $\lambda \geq \omega_0$, where ω_0 was defined in (2.1), we find that $E_\lambda u \in X_1 = V$ is well defined for any $u \in X_0 = V^*$. Additionally, using a resolvent identity and the interpolation inequality, there holds the estimate $\|E_\lambda u - u\|_{V^*} = \lambda^{-1}\|AE_\lambda u\|_{V^*} \leq c\,\lambda^{-1/2}\|u\|$ for all $u \in H = X_{1/2}$. For $u \in U$, an improved estimate for the distance of $E_\lambda u$ to U can be obtained under weak invariance.

Proposition 3.3. *Suppose that U is weakly invariant under (A, BQ_{ad}) and let θ_0 be the constant from Assumption 2.2. Then, for all $u \in U$ and $\gamma \in [0, 1/2]$ it holds*

$$d_U^{X_\gamma}(E_\lambda u) \leq c\,\lambda^{-1+(\gamma-\theta_0)_+}, \quad \lambda \geq \omega_0,$$

where $(\cdot)_+ = \max\{ \cdot, 0 \}$ denotes the positive part, and the constant c depends only on γ, θ_0, A, and Q_{ad}.

Proof. By assumption, there is a control such that the state $\breve{u}$ with initial value u stays in U for all $t \geq 0$. Now, we can estimate the distance of $\mathrm{e}^{-tA}u$ to U in X_γ by the distance of $\breve{u}(t)$, and obtain

$$d_U^{X_\gamma}(\mathrm{e}^{-tA}u) \leq \|\mathrm{e}^{-tA}u - \breve{u}(t)\|_{X_\gamma} \leq c\,t^{1-(\gamma-\theta_0)_+},$$

where the last inequality is an application of Proposition A.1 (iii) with $\theta = \min\{ \gamma, \theta_0 \}$. Indeed, the variable $w(t) = \mathrm{e}^{-tA}u - \breve{u}(t)$ solves a parabolic equation with right-hand side in $L^\infty(0, \infty; X_\theta)$ and $w(0) = 0$. Since the resolvent is the Laplace transform of the semigroup it holds

$$E_\lambda u = \lambda(\lambda + A)^{-1}u = \int_0^\infty \lambda \mathrm{e}^{-\lambda t}\mathrm{e}^{-tA}u\,\mathrm{d}t.$$

Note, that due to $u \in U \subset H = X_{1/2}$ and $\lambda \geq \omega_0$, the integral is defined with values in X_γ for all $\gamma \leq 1/2$; *cf.* Section 1.7 from [33]. Finally, we apply the distance function on both sides of the equation, and we derive

$$d_U^{X_\gamma}(E_\lambda u) \leq \int_0^\infty \lambda \mathrm{e}^{-\lambda t} d_U^{X_\gamma}(\mathrm{e}^{-tA}u)\,\mathrm{d}t \leq c\int_0^\infty \lambda \mathrm{e}^{-\lambda t} t^{1-(\gamma-\theta_0)_+}\,\mathrm{d}t = c\,\Gamma(2 + \theta - \gamma)\,\lambda^{-1-(\gamma-\theta_0)_+},$$

with $\int_0^\infty \lambda \mathrm{e}^{-\lambda t} = 1$, convexity of the distance function, and a generalized Jensen's inequality (see, *e.g.*, [34], Thm. 3.10 (ii)). ☐

Remark 3.4. Note that for the result of Proposition 3.3, we only used the assumption that BQ_{ad} is a bounded set in X_{θ_0} (using Assumption 2.2). All the results from this section remain valid under this modified assumption.

Lemma 3.5. *If U is weakly invariant under (A, BQ_{ad}), then the projection P_U is stable in V, i.e. $P_U(V) \subseteq V$.*

Proof. Let $v \in V$ be fix and set $u = P_U(v) \in H$. We first prove that $u \in X_{(n-1)/n}$ with $n = 2^m$ for all $m \geq 1$. Since $u \in H = X_{1/2}$, the assertion holds for $m = 1$. Proceeding by induction, we assume it holds for all $1 \leq m' \leq m$ and show it for $2n = 2^{m+1}$. Since $AE_\lambda u = \lambda(u - E_\lambda u)$, we compute

$$\begin{aligned}\langle AE_\lambda u, E_\lambda u\rangle &= \langle AE_\lambda u, E_\lambda u - u\rangle + \langle AE_\lambda u, u\rangle \\ &= \lambda(u - E_\lambda u, E_\lambda u - u) + \langle AE_\lambda u, u\rangle = -\lambda\|u - E_\lambda u\|^2 + \langle AE_\lambda u, u\rangle.\end{aligned}$$

Now, we take for any λ a $u'_\lambda \in U$ with $\|u'_\lambda - E_\lambda u\|_{X_{1/n}} \leq 2\, d_U^{X_{1/n}}(E_\lambda u)$. Moreover, since $X_\theta^* = X_{1-\theta} \hookrightarrow V^*$, it holds $\langle \varphi, \psi\rangle \leq \|\varphi\|_{[V^*,V]_{1-\theta}} \|\psi\|_{[V^*,V]_\theta}$ for $\varphi \in X_{1-\theta}$ and $\psi \in V$. Thus, for $v \in V$ with $u = P_U(v)$ from the beginning of the proof it holds

$$\begin{aligned}\langle AE_\lambda u, E_\lambda u\rangle + \lambda\|u - E_\lambda u\|^2 &= \langle AE_\lambda u, u - v\rangle + \langle AE_\lambda u, v\rangle \\ &= \lambda(u - u'_\lambda, u - v) + \lambda(u'_\lambda - E_\lambda u, u - v) + \langle AE_\lambda u, v\rangle \\ &\leq 0 + \lambda\|u'_\lambda - E_\lambda u\|_{X_{1/n}} \|u - v\|_{X_{(n-1)/n}} + c\|E_\lambda u\|_V \|v\|_V \\ &\leq c\,\lambda^{(1/n-\theta_0)_+} \|u - v\|_{X_{(n-1)/n}} + c\|E_\lambda u\|_V \|v\|_V, \end{aligned} \tag{3.1}$$

where we have used $(u - u'_\lambda, u - v) = (u - u'_\lambda, P_U(v) - v) \leq 0$, the estimate $d_U^{X_{1/n}}(E_\lambda u) \leq c\lambda^{-1+(1/n-\theta_0)_+}$ (from Prop. 3.3 with $\gamma = 1/n$), and the continuity of A. Consequently, with Young's inequality, we arrive at

$$\langle AE_\lambda u, E_\lambda u\rangle + \lambda\|u - E_\lambda u\|^2 \leq c\,\lambda^{(1/n-\theta_0)_+} \|u - v\|_{X_{(n-1)/n}} + \frac{\alpha_0}{2}\|E_\lambda u\|_V^2 + c\|v\|_V^2,$$

and the Gårding inequality (2.1) yields

$$\frac{\alpha_0}{2}\|E_\lambda u\|_V^2 + \lambda\|u - E_\lambda u\|^2 \leq c\,\lambda^{(1/n-\theta_0)_+} \|u - v\|_{X_{(n-1)/n}} + c\|v\|_V^2 + \omega_0\|E_\lambda u\|^2.$$

With $\|E_\lambda u\| \leq c\|u\| \leq c\|v\|$ we obtain constants c_1 and c_2 (depending on the norms of $v \in V$ and $u \in X_{(n-1)/n}$, by the induction hypothesis) such that for all $\lambda \geq \omega_0$ it holds

$$\|E_\lambda u\|_V + \lambda^{1/2}\|u - E_\lambda u\| \leq c_1\lambda^{(1/n-\theta_0)_+/2} + c_2.$$

Recall the functional of the K-method of real interpolation, see, *e.g.*, Section 1.3 from [39],

$$K(u, t, V, H) = \inf_{\tilde{u} \in V} \left[\|\tilde{u}\|_V + t\|u - \tilde{u}\|\right].$$

By inserting for each $t \geq t_{\min} := \max\{1, \sqrt{\omega_0}\}$ the values $\tilde{u} = E_\lambda u$ for $\lambda = t^2$, we obtain the estimate $K(u, t, V, H) \leq c_1 t^{(1/n-\theta_0)_+} + c_2$. Moreover, inserting $\tilde{u} = 0$ yields $K(u, t, V, H) \leq t\|u\| \leq ct$. Thereby, we obtain

$$\begin{aligned}\|u\|^2_{(V,H)_{1/n,2}} &= \int_0^\infty \left(t^{-1/n} K(u, t, V, H)\right)^2 t^{-1}\,\mathrm{d}t \\ &\leq c\int_0^{t_{\min}} t^{1-2/n}\,\mathrm{d}t + \int_{t_{\min}}^\infty \left(c_1 \max\{t^{-1/n}, t^{-\theta_0}\} + c_2 t^{-1/n}\right)^2 t^{-1}\,\mathrm{d}t < \infty.\end{aligned}$$

As in (2.2) with $\theta = 1/(2n)$, we find $(V, H)_{1/n,2} = X_{1-1/(2n)}$. Therefore, $u \in X_{(2n-1)/2n}$ and we have shown the assertion for $2n = 2^{m+1}$.

Finally, let $n \in \mathbb{N}$ such that $1/n \leq \theta_0$. Then, in the last step of (3.1) we obtain that

$$\|E_\lambda u\|_V^2 \leq c\|u - v\|_{X_{1-(n-1)/n}} + c\|v\|_V^2.$$

Thus, $E_\lambda u$ is uniformly bounded in V. As $E_\lambda u \to u$ in H, we conclude $u \in V$. □

Corollary 3.6. *Under the assumptions of Lemma 3.5, there exist constants $c_1, c_2 > 0$, such that*

$$\|P_U(v)\|_V \leq c_1 + c_2\|v\|_V. \tag{3.2}$$

Proof. Let $v \in V$. Then, $u = P_U(v) \in U \cap V$ due to Lemma 3.5. As in the last step of Lemma 3.5, we derive

$$\|E_\lambda u\|_V^2 \leq c_1 \|u - v\|_V + c_2 \|E_\lambda u\|_V \|v\|_V \leq c_1 (\|u\|_V + \|v\|_V) + c_2 \|E_\lambda u\|_V \|v\|_V.$$

Recall that $E_\lambda u = \lambda(\lambda + A)^{-1}u$. Since $u \in V$, it holds $E_\lambda u \to u$ in V. Passing to the limit in the inequality above yields

$$\|u\|_V^2 \leq c_1 (\|u\|_V + \|v\|_V) + c_2 \|u\|_V \|v\|_V.$$

Dividing by $\|u\|_V$, we conclude $\|P_U(v)\|_V \leq \max\{1, c_1(1 + \|v\|_V) + c_2\|v\|_V\}$ and the assertion follows for appropriately modified constants c_1, c_2. □

3.2. Characterization of invariance

Using the result on the stability of the projection, weak invariance can be characterized by conditions involving either the projection or the normal cone. In the following, we will make repeated use of the following basic identification.

Proposition 3.7 (see Prop. 6.46 from [6])**.** *Let $u \in U$. Then*

$$N_U(u) = \{ v - u \mid v \in H \quad \text{with } P_U(v) = u \}.$$

In particular, it holds $v - P_U(v) \in N_U(P_U(v))$ for all $v \in H$, and $P_U(u + \zeta) = u$ for all $u \in U$ and $\zeta \in N_U(u)$.

Following Section 12.1 from [12], we define the *lower Hamiltonian* as

$$h(u, \zeta) = \min_{q \in Q_{ad}} \langle Bq - Au, \zeta \rangle \quad \text{for } u \in V,\ \zeta \in V.$$

Analogous to the corresponding theory for ordinary differential equations, we can now characterize weak invariance in terms of the lower Hamiltonian.

Theorem 3.8. *The following conditions are equivalent:*

(i) U is weakly invariant,
(ii) P_U is stable in V and $h(u, \zeta) \leq 0$ for all $u \in U \cap V$ and $\zeta \in N_U(u) \cap V$,
(iii) P_U is stable in V and $h(P_U(v), v - P_U(v)) \leq 0$ for all $v \in V$.

For the proof of Theorem 3.8 we need an estimate of the distance to the target set for the controlled system, which is given next. For later use, we prove it in a more general form, including both the strengthened condition (1.2) as well as the weaker condition (1.1) (which is the special case for $h_0 = 0$).

Lemma 3.9. *Suppose that P_U is stable in V and that there is $h_0 \geq 0$ such that:*

$$\text{for all } v \in V \quad \text{it holds} \quad h(u,\zeta) \leq -h_0\|\zeta\|, \quad \text{where } u = P_U(v),\ \zeta = v - u. \tag{3.3}$$

Then, for each $u_0 \in H$ with $d_U(u_0)\,\omega_0 \leq h_0$ there exists a control $q\colon [0,\infty) \to Q_{ad}$ such that the solution u to

$$\partial_t u + Au = Bq, \quad u(0) = u_0,$$

satisfies

$$d_U(u(t)) \leq \max\{\,0,\ d_U(u_0) + (d_U(u_0)\,\omega_0 - h_0)\,t\,\} \quad \text{for } t \geq 0.$$

To prove this result, we construct a sequence of feedback controls which have approximately the desired property, and then we go to the limit. We start with an auxiliary result.

Proposition 3.10. *The squared distance function $d_U^2\colon H \to \mathbb{R}$ is differentiable, and it holds*

$$\nabla d_U^2(u) = 2(u - P_U(u)).$$

Moreover, if P_U is stable in V, then ∇d_U^2 is continuous from V to $X_{1-\theta_0}$.

Proof. For the differentiability of the squared distance function, we refer to Corollary 12.30 from [6]. Using the expression of the derivative, we infer that ∇d_U^2 is Lipschitz continuous on H with Lipschitz constant two, and stable on V due to stability of P_U in V; see Corollary 3.6. The interpolation inequality Theorem 1.9.3 f) from [39], yields

$$\begin{aligned}\frac{1}{2}\|\nabla d_U^2(u) - \nabla d_U^2(v)\|_{[H,V]_{1-2\theta_0}} &\leq \frac{1}{2}\|\nabla d_U^2(u) - \nabla d_U^2(v)\|_V^{1-2\theta_0}\|\nabla d_U^2(u) - \nabla d_U^2(v)\|^{2\theta_0}\\ &\leq \left[2c_1 + (1+c_2)(\|v\|_V + \|u\|_V)\right]^{1-2\theta_0}\|u - v\|^{2\theta_0},\end{aligned}$$

where c_1, c_2 are from estimate (3.2). Hence, ∇d_U^2 is continuous from V to $[H,V]_{1-2\theta_0} = X_{1-\theta_0}$; see (2.2). □

We now construct the desired sequence of approximate feedback controls.

Proposition 3.11. *Let $u_0 \in H$, $\gamma > 0$ and $T > 0$. Then the equation*

$$\begin{aligned}\partial_t u_\gamma + Au_\gamma &= Bq_\gamma,\\ q_\gamma &= P_{Q_{ad}}\left(-\gamma^{-1}B^*(u_\gamma - P_U(u_\gamma))\right),\\ u_\gamma(0) &= u_0,\end{aligned} \tag{3.4}$$

possesses a solution $u_\gamma \in W(0,T) \cap C((0,T);V) \cap C^1((0,T);V^*)$ and $q_\gamma \in C((0,T);Q)$.

Proof. Consider the mapping $\mathcal{F}\colon Q(0,T) \to Q(0,T)$ defined by

$$\mathcal{F}(q) \coloneqq P_{Q_{ad}}\left(-(2\gamma)^{-1}B^*\left[\nabla d_U^2\left(\mathcal{S}(u_0, Bq)\right)\right]\right),$$

where $\mathcal{S}\colon H \times L^2((0,T);X_{\theta_0}) \to W(0,T)$ denotes the solution operator of the parabolic equation with initial value u_0 and right-hand side Bq. According to Proposition 3.10, the function ∇d_U^2 is continuous from V into $X_{1-\theta_0}$. Moreover, since $X_{\theta_0}^* = X_{1-\theta_0}$, and B is supposed to be continuous from Q to X_{θ_0}, we infer continuity of B^* from $X_{1-\theta_0}$ to $Q^* = Q$. Continuity of $P_{Q_{ad}}$ on Q leads to continuity of $\mathcal{F}$ from $Q(0,T)$ into itself. Using compactness of $q \mapsto \mathcal{S}(u_0, Bq)$ into $L^2((0,T);V)$ according to Proposition A.2, we deduce that $\mathcal{F}(Q_{ad}(0,T))$ is contained in a compact subset of $Q(0,T)$.

Finally, Schauder's fixed point theorem (see, *e.g.*, [44], Thm. 2.A) yields the existence of a fixed point $\mathcal{F}(q_\gamma) = q_\gamma$. Setting $u_\gamma = \mathcal{S}(u_0, q_\gamma)$ proves the existence of a solution to (3.4). According to Proposition A.1, u_γ is continuous on $(0, T]$ with values in V. Now, the continuity of the projection $P_{Q_{ad}}$ on Q yields the improved regularity of q_γ. Furthermore, from $\partial_t u_\gamma = Bq_\gamma - Au_\gamma$ we deduce that u_γ is continuously differentiable on $(0, T)$ with values in V^*. □

Next, we observe that the feedback control q_γ is close to the minimizing argument of the lower Hamiltonian.

Proposition 3.12. *For any $\zeta, u \in V$ and $q_\gamma = P_{Q_{ad}}\left(-\gamma^{-1}B^*\zeta\right)$ it holds*

$$\langle Bq_\gamma - Au, \zeta\rangle \leq h(u, \zeta) + c\gamma, \tag{3.5}$$

where c solely depends on Q_{ad}.

Proof. Consider for $\gamma \geq 0$ the family of functions defined by

$$h_\gamma(u, \zeta) = \min_{q \in Q_{ad}} \left[\langle Bq - Au, \zeta\rangle + \frac{\gamma}{2}\|q\|_Q^2\right]. \tag{3.6}$$

Clearly, h_0 is the lower Hamiltonian h. Denote the minimizers of (3.6) by q_γ. Then, we estimate

$$\langle Bq_\gamma - Au, \zeta\rangle \leq h_\gamma(u, \zeta) \leq \langle Bq_0 - Au, \zeta\rangle + \frac{\gamma}{2}\|q_0\|_Q^2 \leq h_0(u, \zeta) + \frac{\gamma}{2}C_{Q_{ad}}^2.$$

Furthermore, for $\gamma > 0$, from the optimality conditions for (3.6) we infer that the minimizer q_γ is given by $q_\gamma = P_{Q_{ad}}\left(-\gamma^{-1}B^*\zeta\right)$. □

Now we prove the main result of this section.

Proof of Lemma 3.9. Clearly, it suffices to show the result for $t \in (0, T)$ for some arbitrary but fixed $T > 0$. Let $u_0 \in H$ be given, let u_γ for $\gamma > 0$ denote the corresponding solution to (3.4), and define $d_\gamma(t) = d_U(u_\gamma(t))$. Then, for any $0 < t < T$ we infer

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}d_\gamma^2(t) &= \langle\partial_t u_\gamma(t), \nabla d_U^2(u_\gamma(t))\rangle = \langle Bq_\gamma(t) - Au_\gamma(t), \nabla d_U^2(u_\gamma(t))\rangle \\ &= \langle Bq_\gamma(t) - AP_U(u_\gamma(t)), \nabla d_U^2(u_\gamma(t))\rangle + \langle AP_U(u_\gamma(t)) - Au_\gamma(t), \nabla d_U^2(u_\gamma(t))\rangle,\end{aligned}$$

where we have used (3.4). For the last term, the Gårding inequality yields

$$\begin{aligned}\langle AP_U(u_\gamma(t)) - Au_\gamma(t), \nabla d_U^2(u_\gamma(t))\rangle &= -\frac{1}{2}\langle A\nabla d_U^2(u_\gamma(t)), \nabla d_U^2(u_\gamma(t))\rangle \\ &\leq \frac{\omega_0}{2}\|\nabla d_U^2(u_\gamma(t))\|^2 - \frac{\alpha_0}{2}\|\nabla d_U^2(u_\gamma(t))\|_V^2 \leq \frac{\omega_0}{2}\|\nabla d_U^2(u_\gamma(t))\|^2.\end{aligned}$$

Employing (3.5), the Hamiltonian condition (3.3) and $\|\nabla d_U^2(u_\gamma(t))\| = 2d_U(u_\gamma(t)) = 2d_\gamma(t)$ we infer

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}d_\gamma^2(t) &\leq h(P_U(u_\gamma(t)), \nabla d_U^2(u_\gamma(t))) + c\gamma + \frac{\omega_0}{4}\|\nabla d_U^2(u_\gamma(t))\|^2 \\ &\leq -h_0 d_\gamma(t) + c\gamma + \omega_0 d_\gamma^2(t).\end{aligned} \tag{3.7}$$

Using the fact that $\frac{\mathrm{d}}{\mathrm{d}t}d_\gamma^2(t) = 2d_\gamma'(t)d_\gamma(t)$, we obtain from (3.7) that

$$d_\gamma'(t) \leq \omega_0 d_\gamma(t) + c\gamma/d_\gamma(t) - h_0 \quad \text{on } \{\, t \mid d_\gamma(t) > 0 \,\}.$$

According to Proposition C.1 the differential inequality implies

$$d_\gamma(t) \le \max\left\{ \sqrt{\gamma}, (d_U(u_0) + \sqrt{\gamma})\mathrm{e}^{\omega_0 t} + (c\sqrt{\gamma} - h_0)\phi(t) \right\} =: D_\gamma(t), \tag{3.8}$$

where $\phi(t) = \omega_0^{-1}(\mathrm{e}^{\omega_0 t} - 1)$, if $\omega_0 > 0$, and $\phi(t) = t$ otherwise.

For $\gamma \to 0$ we now choose suitable subsequences such that $q_\gamma \rightharpoonup q$ in $Q(0,T)$ and $u_\gamma \rightharpoonup u$ in $W(0,T)$. Clearly, the weak limits satisfy

$$\partial_t u + Au = Bq, \quad u(0) = u_0.$$

Thus, with $W(0,T) \hookrightarrow C([0,T]; H)$ we have $u_\gamma(t) \rightharpoonup u(t)$ in H for all $t \in [0,T]$. Using weak lower semicontinuity of the distance function $d_U(\cdot)$ and (3.8), we obtain

$$d_U(u(t)) \le \liminf_{\gamma\to 0} d_U(u_\gamma(t)) \le \lim_{\gamma \to 0} D_\gamma(t) = \max\left\{ 0,\ d_U(u_0)\mathrm{e}^{\omega_0 t} - h_0\phi(t) \right\}.$$

Now, using the supposition $d_U(u_0)\omega_0 \le h_0$, the definition of ϕ, and the fact that $\phi(t) \ge t$, we obtain

$$d_U(u(t)) \le (d_U(u_0) + (\omega_0 d_U(u_0) - h_0)\phi(t))_+ \le (d_U(u_0) + (\omega_0 d_U(u_0) - h_0)\, t)_+$$

concluding the proof. □

Finally, we show the charactization of weak invariance by means of the lower Hamiltonian.

Proof of Theorem 3.8. We separately prove three implications.

(i) ⇒ (ii). The stability of P_U in V follows with Lemma 3.5. For the second condition, let $u_0 \in U \cap V$ be arbitrary. Then, with weak invariance, there is a control $q \in Q_{ad}(0,\infty)$ such that the corresponding state satisfies $u(0) = u_0$ and $u(t) \in U$ for all $t \ge 0$. Additionally, $u(t) \in V$ for all $t \ge 0$ follows by Proposition A.1 (i). Let further $\zeta \in N_U(u_0) \cap V$. It holds $\partial_t u = Bq - Au$ in $L^2((0,s); V^*)$ for any $s > 0$, and we have

$$0 \ge \frac{1}{s}\langle u(s) - u_0, \zeta\rangle = \left\langle \frac{1}{s}\int_0^s [Bq(t) - Au(t)]\,\mathrm{d}t, \zeta \right\rangle. \tag{3.9}$$

Define the temporal averages $\bar q_s = (1/s)\int_0^s q(t)\,\mathrm{d}t$ and $\bar u_s = (1/s)\int_0^s u(t)\,\mathrm{d}t$. Due to $u \in C([0,1]; V)$, it holds $\bar u_s \to u_0$ in V for $s \to 0$. Furthermore, with $q(t) \in Q_{ad}$ for all t, it follows $\bar q_s \in Q_{ad}$ (see, *e.g.*, ([12], Ex. 2.44)) and we can select a sequence $s_n \to 0$ and a $q_0 \in Q$ such that $\bar q_{s_n} \rightharpoonup q_0$ in Q for $n \to \infty$. By weak closedness of Q_{ad} we have $q_0 \in Q_{ad}$. Going to the limit in (3.9), we obtain

$$0 \ge \langle Bq_0 - Au_0, \zeta\rangle \ge h(u_0, \zeta),$$

using boundedness of $B\colon Q \to V^*$ and $A\colon V \to V^*$. Since u_0 and ζ were arbitrary, we finish the proof.

(ii) ⇒ (iii). This follows directly from the fact that $u = P_U(v) \in U \cap V$ and $v - P_U(v) \in N_U(u) \cap V$ for all $v \in V$ with the stability of the projection.

(iii) ⇒ (i). The last implication is consequence of Lemma 3.9 (with $h_0 = 0$). □

4. Time-optimal control problem

We now turn to the time-optimal control problem. In the following, we use the notation $u[q]$ for the solution of the state equation $\partial_t u + Au = Bq$ and $u(0) = u_0$ for a given control q. Furthermore, to exclude the trivial case with zero optimal time, we assume that $u_0 \in H \setminus U$. Problem (P) can then be restated as:

$$\inf_{T>0,\ q\in Q_{ad}(0,T)} j(T,q) \quad \text{subject to} \quad u[q](T) \in U.$$

First, we consider the question of existence of optimal controls. We show that if there exists a feasible pair $(T,q) \in \mathbb{R}_+ \times Q_{ad}(0,T)$, the problem is well-posed:

Proposition 4.1. *Suppose there exists a finite time $T>0$ and a feasible control $q \in Q_{ad}(0,T)$ such that the corresponding state satisfies $u[q](T) \in U$. Then, problem (P) admits at least one optimal solution $(T,\bar{q}) \in \mathbb{R}_+ \times Q_{ad}(0,T)$.*

Proof. The proof is done by standard arguments (the direct method); *cf.*, *e.g.*, Section III.17 from [29]. We use in particular the boundedness of j for bounded T due to boundedness of the admissible set Q_{ad}, $j(T,q) \geq T$ and that j is weakly lower semicontinuous in q for fixed T. Furthermore, we use the $W(0,T)$ regularity of the solution to the state equation, the continuity of the trace mapping i_T, and the convexity of U. □

Remark 4.2. In view of the preceding result, the question of existence reduces to the question of *controllability* under constraints. We exemplary state situations where feasible controls exist. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary and $A = -\Delta$ be the usual Laplace operator equipped with homogeneous Dirichlet boundary conditions. Moreover, for fixed $u_d \in L^2(\Omega)$ and $\delta_0 \geq 0$, suppose the terminal set to be given by $U = \{\, u \in L^2(\Omega) \colon \|u - u_d\| \leq \delta_0 \,\}$.

(i) In case of distributed control on an open subset $\omega \subset \Omega$, the resulting equation is known to be approximately controllable; see, *e.g.*, [37, 45], *i.e.* for all $T>0$, $u_d \in L^2(\Omega)$, and $\delta_0 > 0$, there exists a control $q \in Q(0,T)$ such that $u[q](T) \in U$. Clearly, for sufficiently large control constraints, feasible controls exist. For estimates concerning the controls, we refer to [17].

(ii) The semigroup generated by Δ is exponentially stable in $L^2(\Omega)$. Hence, if $u_d \equiv 0$ and $0 \in Q_{ad}(0,1)$, then for any $\delta_0 > 0$ there exist feasible controls (simply taking $T>0$ sufficiently large).

(iii) Furthermore, Lemma 3.9 provides a sufficient condition for existence of feasible points, under the assumption $d_U(u_0)\omega_0 < h_0$ (which is clearly true for $\omega_0 = 0$ or the initial state u_0 sufficiently close to U). Note that Lemma 3.9 generalizes the argument of (ii), since $\omega_0 = 0$ in case of homogeneous Dirichlet conditions due to the Poincaré inequality.

4.1. Strong stability

We now introduce the strong stability condition on the objective functional with respect to small perturbations of the terminal constraint set. This will allow for exact penalization of the constraints which in turn leads to optimality conditions in qualified form. For $\delta \geq 0$, define the perturbed control problem

$$\inf_{T>0,\ q\in Q_{ad}(0,T)} j(T,q) \quad \text{subject to} \quad u[q](T) \in U_\delta, \tag{P_δ}$$

where U is replaced with $U_\delta = U + \overline{\mathcal{B}_\delta(0)} = \{\, u \in H \mid d_U(u) \leq \delta \,\}$. Evidently, (P_0) is equal to (P). We define the corresponding value function $v \colon \mathbb{R}_+ \to \mathbb{R}_+$ by

$$v(\delta) = \inf\,(P_\delta).$$

Clearly, v is a monotonously decreasing function with $v(d_U(u_0)) = 0$.

Definition 4.3. The problem (P_δ) is called *strongly stable (on the right)* if there exist $\varepsilon > 0$ and $\eta_0 > 0$ such that

$$v(\delta) - v(\delta') \leq \eta_0(\delta' - \delta) \quad \text{for all } \delta' \in [\delta, \delta+\varepsilon]. \tag{4.1}$$

Remark 4.4. (i) In the case that $\delta > 0$, we can also define *stability on the left* in an analogous way. In this work, we only consider stability on the right, which is meaningful also for the important case $\delta = 0$.

(ii) Strong stability is satisfied almost everywhere. Precisely, if (P) has feasible controls, then (P_δ) is strongly stable for all $\delta \in \mathbb{R}_+$ except on a set of Lebesgue measure zero; see, *e.g.*, Proposition 3.2 from [7]. However, since we consider the terminal set U to be a given datum, we are interested in conditions assuring strong stability on the right at $\delta = 0$.

(iii) Strong stability is also referred to as calmness, *cf.* [9], Chapter 8.F from [36], or weak calmness, *cf.* Definition 3.114 from [8].

We now prove one of the main results of the paper, which guarantees strong stability under a condition which is a direct strengthening of the necessary condition for weak invariance from Theorem 3.8. We require that there exists a $h_0 > 0$ such that

$$h(u, \zeta) \leq -h_0 \|\zeta\| \quad \text{for all } u \in U \cap V,\ \zeta \in N_U(u) \cap V. \tag{4.2}$$

Recall that weak invariance of (A, U, BQ_{ad}) corresponds to the same condition with $h_0 = 0$; see Theorem 3.8. In the case $h_0 > 0$, strong stability of (P_δ) holds for all small enough $\delta \geq 0$ (which includes the important case $\delta = 0$).

Theorem 4.5 (Strong stability)**.** *Let P_U be stable in V and suppose that condition* (4.2) *holds for some constant $h_0 > 0$. Then, for all $\delta \geq 0$ such that $\omega_0 \delta < h_0/2$ the problem* (P_δ) *is strongly stable on the right with $\eta_0 \leq c/h_0$, where the constant c only depends on the concrete choice of L and Q_{ad}.*

Proof. Fix $\varepsilon > 0$ such that $\omega_0(\delta + \varepsilon) \leq h_0/2$. Then, let $\delta' \in [\delta, \delta + \varepsilon]$ be arbitrary and fix a solution (T', q', u') to ($P_{\delta'}$). Consider the auxiliary problem $\partial_t \breve{u} + A\breve{u} = B\breve{q}$ with initial condition $\breve{u}(0) = u'(T')$ and an auxiliary control $\breve{q} \colon [0, \infty) \to Q_{ad}$. Employing Lemma 3.9 we can chose $\breve{q}$ such that it holds

$$d_U(\breve{u}(t)) \leq \max\{\, 0,\ \delta' + (\delta'\omega_0 - h_0)t \,\} \quad \text{for } t \geq 0,$$

considering that $d_U(\breve{u}(0)) = d_U(u'(T')) = \delta'$. Clearly, it follows that $d_U(\breve{u}(\delta T)) \leq \delta$ for the choice $\delta T = (\delta' - \delta)/(h_0 - \delta'\omega_0)$. Thus, $q \in Q_{ad}(0, T' + \delta T)$ defined by

$$q(t) = \begin{cases} q'(t) & \text{if } t \leq T', \\ \breve{q}(t - T') & \text{if } t > T', \end{cases}$$

is admissible for (P_δ) and we find

$$v(\delta) = \inf(P_\delta) \leq j(T' + \delta T, q) = j(T', q') + \int_{T'}^{T' + \delta T} [1 + L(\breve{q}(t - T'))] \, \mathrm{d}t \leq v(\delta') + \delta T\, (1 + L_\infty),$$

where $L_\infty = \max_{q \in Q_{ad}} L(q)$. Using $\omega_0 \delta' \leq h_0/2$, we obtain that $\delta T \leq 2(\delta' - \delta)/h_0$, which results in (4.1) with a choice of $\eta_0 = 2(1 + L_\infty)/h_0$. This concludes the proof. □

4.2. Change of variable

In this subsection, we discuss the implications of strong stability on optimality conditions for (P_δ). To derive optimality conditions we first transform the time interval to the reference interval $(0, 1)$ (*cf.* Prop. 4.2 in [23], Prop. 4.1 in [35]). Consider the set of admissible scaling functions

$$N_{ad} \coloneqq \left\{ \nu \in L^\infty(0,1) \colon \operatorname*{ess\,inf}_{\tau \in (0,1)} \nu(\tau) > 0 \right\} = \{ \nu \in L^\infty(0,1) \colon \nu \geq 0 \quad \text{and } 1/\nu \in L^\infty(0,1) \}$$

and define a family of transformations

$$T_\nu \colon [0,1] \to \mathbb{R}_+, \quad T_\nu(t) = \int_0^t \nu(\tau)\,\mathrm{d}\tau.$$

For $\nu \in N_{ad}$ and any mapping $u\colon (0,1) \to V$ we define the transformed elliptic operator

$$(\nu A u)(t) = \nu(t) A u(t),$$

and, by a change of variables, we obtain the transformed state equation

$$\partial_t u + \nu A u = \nu B q, \quad u(0) = u_0.$$

By standard results, for each right-hand side in $L^2((0,1);V^*)$ the transformed equation possesses a unique solution $u \in W(0,1)$ (see, *e.g.*, [15], Thm. 2, Chap. XVIII, Sect. 3). We introduce the control-to-state mapping as

$$S\colon N_{ad} \times Q_{ad}(0,1) \subset L^\infty(0,1) \times Q(0,1) \to W(0,1), \quad S(\nu,q) = u.$$

The transformed optimal control problem is then given by

$$\inf_{\nu \in N_{ad},\ q \in Q_{ad}(0,1)} j(\nu,q) \quad \text{subject to} \quad i_1 S(\nu,q) \in U, \tag{$\hat{P}$}$$

where the objective function is defined as

$$j(\nu,q) \coloneqq \int_0^1 \nu(t)\,(1 + L(q(t)))\,\mathrm{d}t.$$

Since no ambiguity arises, we do not rename variables. The definition of the set of admissible controls Q_{ad} transfers to the transformed problem, because the control constraints do not depend on time. In fact, both problems ($\hat{P}$) and (P) are equivalent in the following sense.

Proposition 4.6. *If (ν, q) is admissible for ($\hat{P}$) and $u = S(\nu,q)$, then*

$$(T_\nu(1), q \circ T_\nu, u \circ T_\nu)$$

is admissible for (P) and $j(\nu, q \circ T_\nu) = j(T,q)$. If (T,q,u) is admissible for (P), then for every $\nu \in N_{ad}$ such that $T_\nu(1) = T$,

$$(\nu, q \circ T_\nu^{-1})$$

is admissible for ($\hat{P}$) and $j(\nu, q \circ T_\nu^{-1}) = j(T,q)$.

Considering ν as an additional control variable, we obtain by standard arguments the following differentiability result.

Proposition 4.7. *The control-to-state mapping S is (infinitely often) continuously Fréchet-differentiable. In particular, $\delta u = S'(\nu,q)(\delta\nu,\delta q) \in W(0,1)$ is the unique solution to*

$$\partial_t \delta u + \nu A \delta u = \delta\nu (Bq - Au) + \nu B \delta q, \quad \delta u(0) = 0,$$

for $(\delta\nu, \delta q) \in L^\infty(0,1) \times Q(0,1)$.

By the previous result and the continuity of the trace mapping i_1, the parameter-to-observation mapping $i_1 S(\nu, q)\colon (\nu, q) \mapsto u(1)$ is differentiable. Furthermore, for any fixed $\mu \in H$, the gradient of the functional $(\nu, q) \mapsto (i_1 S(\nu, q), \mu)$, which is given by the expression $(i_1 S'(\nu, q))^* \mu$, can be characterized by an adjoint equation.

Proposition 4.8. *Let $\nu \in N_{ad}$ and $q \in Q(0,1)$. For any $\mu \in H$ we have*

$$(i_1 S'(\nu, q))^* \mu = \begin{pmatrix} \langle Bq - Au, z\rangle \\ \nu B^* z \end{pmatrix} \in L^1(0,1) \times L^2((0,1); Q),$$

where $z \in W(0,1)$ is the unique solution to the adjoint equation

$$-\partial_t z + \nu A^* z = 0, \quad z(1) = \mu,$$

where A^* denotes the adjoint operator of A.

Proof. Using Proposition 4.7, integration by parts, and the definition of z we observe

$$\begin{aligned}(\mu, i_1 S'(\nu, q)(\delta\nu, \delta q)) = (\delta u(1), \mu) = (\delta u(1), z(1)) - (\delta u(0), z(0)) = \int_0^1 \langle \partial_t \delta u, z\rangle + \int_0^1 \langle \partial_t z, \delta u\rangle \\ = \int_0^1 \langle \partial_t \delta u, z\rangle + \int_0^1 \langle \nu A \delta u, z\rangle = \int_0^1 \langle \delta\nu (Bq - Au) + \nu B \delta q, z\rangle,\end{aligned}$$

where $\delta u = S'(\nu, q)(\delta\nu, \delta q)$. Furthermore, we identify the partial derivative with respect to ν, *i.e.* $\delta\nu \mapsto \int_0^1 \delta\nu \langle Bq - Au, z\rangle$, with the function $\langle Bq - Au, z\rangle \in L^1(0,1)$. □

The transformed perturbed problems $(\hat{P}_\delta)$ for $\delta \geq 0$ are defined analogously:

$$\inf_{\nu \in N_{ad},\ q \in Q_{ad}(0,1)} j(\nu, q) \quad \text{subject to} \quad i_1 S(\nu, q) \in U_\delta. \tag{$\hat{P}_\delta$}$$

The notion of strong stability for $(\hat{P}_\delta)$ and (P_δ) are obviously equivalent, since the value function v is identical. We will derive optimality conditions by adding the terminal constraint as a penalty term to the objective functional. Under a strong stability assumption the resulting functional is exact.

Definition 4.9. Let $\delta \geq 0$ and (ν, q) be a local minimum of (P_δ). The functional

$$j_\eta(\cdot) = j(\cdot) + \eta\, d_{U_\delta}(i_1 S(\cdot))$$

is called an *exact penalty function* for (P_δ) at (ν, q), if there is $\eta \geq 0$ such that (ν, q) is a local minimizer of j_η.

Proposition 4.10. *Let $\delta \geq 0$ and $(\bar\nu, \bar q)$ be a solution to (P_δ) and let (P_δ) be strongly stable on the right with constant $\eta_0 > 0$. Then, j_η is an exact penalty function for (P_δ) at $(\bar\nu, \bar q)$ for any $\eta \geq \eta_0$.*

Proof. We give a proof of this well-known result for convenience of the reader: Let $\eta \geq \eta_0$ and (ν, q) be a local minimizer of j_η in a suitable small neighborhood of $(\bar\nu, \bar q)$ (such that $d_{U_\delta}(i_1 S(\nu, q)) \leq \varepsilon$), and set $\delta' = d_{U_\delta}(i_1 S(\nu, q))$. Due to feasibility of $(\bar\nu, \bar q)$ for (P_δ) and strong stability on the right, we obtain

$$\begin{aligned} j_\eta(\bar\nu, \bar q) = j(\bar\nu, \bar q) &\leq \inf(P_{\delta'}) + \eta(\delta' - \delta) \leq j(\nu, q) + \eta(\delta' - \delta) \\ &= j(\nu, q) + \eta\, d_{U_\delta}(i_1 S(\nu, q)) = j_\eta(\nu, q), \end{aligned}$$

where we have used optimality of (ν, q) for j_η in the last step. Whence, $(\bar\nu, \bar q)$ is a local minimizer for j_η. □

Remark 4.11. The constraint in ($\hat{P}$) can be written as $g(\nu, q) = i_1 S(\nu, q) \in U_\delta$ and g is differentiable. If a constraint qualification such as Robinson's CQ holds,

$$0 \in \operatorname{int}\{\, g(\bar\nu, \bar q) + g'(\bar\nu, \bar q)(N_{ad} - \bar\nu, Q_{ad}(0,1) - \bar q) - U_\delta \,\} \subset H,$$

then j_η is an exact penalty function for (P_δ); see, *e.g.*, Theorem 2.87, Proposition 3.111 from [8]. This presents an alternative approach to obtain qualified optimality conditions. We expect that the sufficient conditions from Section 4.4 are related to Robinson's CQ, but are unable to prove this in the general setting.

4.3. Optimality conditions

We define for any $\mu_0 \in \mathbb{R}_+$ the *Hamiltonian* $H_{\mu_0}\colon Q \times V \times V \to \mathbb{R}$ by

$$H_{\mu_0}(q, u, z) = \langle Bq - Au, z\rangle + \mu_0 \left[1 + L(q)\right].$$

Based on strong stability, qualified optimality conditions can be derived.

Theorem 4.12. *Let $\delta \geq 0$ and (P_δ) be strongly stable on the right (with constant $\eta > 0$). If $(\bar\nu, \bar q)$ is a solution of (P_δ) with $\bar u = S(\bar\nu, \bar q)$, then there exist $\bar\mu \in N_{U_\delta}(\bar u(1))$, $\bar\mu \neq 0$, $\|\bar\mu\| \leq \eta$, and a corresponding adjoint state $\bar z \in W(0,1)$ with*

$$-\partial_t \bar z + \bar\nu A^* \bar z = 0, \quad \bar z(1) = \bar\mu, \tag{4.3}$$

such that

$$\min_{q \in Q_{ad}} H_1(q, \bar u(t), \bar z(t)) = H_1(\bar q(t), \bar u(t), \bar z(t)) = 0, \quad a.e. \;\; t \in (0,1). \tag{4.4}$$

The first equality in (4.4) can be equivalently expressed by

$$0 \in \partial L(\bar q(t)) + B^* \bar z(t) + N_{Q_{ad}}(\bar q(t)), \quad a.e. \; t \in (0,1), \tag{4.5}$$

where ∂L denotes the convex subdifferential of L.

Proof. The proof is based on the minimization of the exact penalty function. Using Proposition 4.10, $(\bar\nu, \bar q)$ also is a minimizer of the penalty function j_η. Since $\bar\nu \in N_{ad}$, which is open, we may restrict the minimization to some neighborhood and neglect the constraints on ν in the following. We note that $j_\eta\colon L^\infty(0,1) \times Q(0,1) \to \mathbb{R}$ is locally Lipschitz continuous and derive the stationary conditions by Fermat's rule; see Proposition 10.36 from [12]. We obtain

$$\begin{aligned} 0 &\in \partial_{\mathrm{C}} j_\eta(\bar\nu, \bar q) + N_{L^\infty(0,1) \times Q_{ad}(0,1)}(\bar\nu, \bar q) \\ &\subseteq \partial_{\mathrm{C}} j(\bar\nu, \bar q) + \eta\, \partial_{\mathrm{C}}\big[d_{U_\delta}(i_1 S(\bar\nu, \bar q))\big] + \{\, 0 \,\} \times N_{Q_{ad}(0,1)}(\bar q), \end{aligned} \tag{4.6}$$

where ∂_{C} denotes the generalized subdifferential due to Clarke; see, *e.g.*, Chapter 10 from [12]. Using Proposition B.2 and Theorem 10.8 from [12], we find

$$\partial_{\mathrm{C}} j(\bar\nu, \bar q) \subseteq \{1 + L(\bar q)\} \times \bar\nu\, \partial_{\mathrm{C}} L(\bar q) = \{1 + L(\bar q)\} \times \bar\nu\, \partial L(\bar q),$$

because j is continuously differentiable with respect to ν and convex and Lipschitz continuous with respect to q due to the corresponding assumptions on L. Concerning the second term, we employ the chain rule Theorem 10.19 from [12] and obtain

$$\partial_{\mathrm{C}}\left[d_{U_\delta}(i_1 S(\bar\nu,\bar q))\right] \subseteq (i_1 S'(\bar\nu,\bar q))^* \left[\partial_{\mathrm{C}} d_{U_\delta}(i_1 S(\bar\nu,\bar q))\right]. \tag{4.7}$$

The gradient $(i_1 S'(\bar\nu,\bar q))^*$ was computed in Proposition 4.8. Furthermore, the set $\partial_{\mathrm{C}} d_{U_\delta}(\cdot)$ can be identified with the ordinary convex subdifferential (see [12], Thm. 10.8) and

$$\partial_{\mathrm{C}} d_{U_\delta}(v) = \partial d_{U_\delta}(v) = \{\, \mu \in N_{U_\delta}(v) \mid \|\mu\| \le 1 \,\},$$

for all $v \in U_\delta$; see, *e.g.*, Proposition 18.22 from [6]. Therefore, from (4.6) and (4.7) we obtain that there exists a $\bar\mu \in N_{U_\delta}(\bar u(1))$ with $\|\bar\mu\| \le \eta$, a $\bar\xi \in \partial L(\bar q)$, and a $\bar\zeta \in N_{Q_{ad}(0,1)}(\bar q)$, such that

$$0 = \begin{pmatrix} 1 + L(\bar q) + \langle B\bar q - A\bar u, \bar z\rangle \\ \bar\nu(\bar\xi + B^*\bar z + \bar\zeta) \end{pmatrix},$$

where $\bar z$ solves the corresponding adjoint equation (4.3). The first component of this equation is the second equality in (4.4). Pointwise inspection of the second component for $t \in (0,1)$ and $\bar\nu(t) > 0$ implies (4.5). Now, we observe that (4.5) is the necessary and sufficient optimality condition for $\bar q(t)$ to be the solution of a convex optimization problem, namely

$$\bar q(t) = \operatorname*{argmin}_{q\in Q_{ad}} \left[L(q) + \langle Bq, \bar z(t)\rangle\right] = \operatorname*{argmin}_{q\in Q_{ad}} H_1(q, \bar u(t), \bar z(t)).$$

Finally, assume that $\bar\mu = 0$. This implies $\bar z = 0$ by unique solvability of the adjoint equation. Using the Hamiltonian condition (4.4) we infer $1 + L(\bar q) = 0$ almost everywhere in $(0,1)$. This contradicts $L \ge 0$, and we conclude $\bar\mu \ne 0$. ☐

Without strong stability, under a structural assumption on only the constraint set, the generalized form of the optimality conditions can be derived. In the following theorem, we assume that U_δ is of finite co-dimension; see Definition 4.1.5 from [27]. Note that this is always fulfilled in the case $\delta > 0$, since then $\overline{\operatorname{span}\{\, U_\delta - u' \,\}} = H$ for all $u' \in U$.

Theorem 4.13. *Assume that U_δ is of finite co-dimension (or $\delta > 0$); see Definition 4.1.5 from [27]. Let $(\bar\nu,\bar q)$ be a solution of (P_δ), $\bar u = S(\bar\nu,\bar q)$. Then there exist $\bar\mu \in N_{U_\delta}(\bar u(1))$, $\bar\mu \ne 0$, $\bar\mu_0 \in \{\, 0, 1 \,\}$ and a corresponding adjoint state $\bar z \in W(0,1)$ which fulfills (4.3), such that*

$$\min_{q\in Q_{ad}} H_{\bar\mu_0}(q, \bar u(t), \bar z(t)) = H_{\bar\mu_0}(\bar q(t), \bar u(t), \bar z(t)) = 0, \quad \text{a.e. } t \in (0,1). \tag{4.8}$$

Proof. We only give a short outline of the proof. It combines the one of Theorem 10.47 from [12], with the one of Theorem 4.1 from [35]. As before, since N_{ad} is open, we may restrict the minimization to some neighborhood and neglect the constraints on ν in the following. Define the function

$$\phi^\varepsilon(\nu, q) = \sqrt{\max\{\, 0, j(\nu,q) - j(\bar\nu,\bar q) + \varepsilon \,\}^2 + d_U(i_1 S(\nu,q))^2}.$$

Ekeland's variational principle with $\lambda = \sqrt{\varepsilon}$ yields a sequence $\nu_\varepsilon \in N_{ad}$, $q_\varepsilon \in Q_{ad}(0,1)$ such that $(\nu_\varepsilon, q_\varepsilon) \to (\bar\nu,\bar q)$ for $\varepsilon \to 0$ and the function

$$(\nu, q) \mapsto \phi^\varepsilon(\nu,q) + \sqrt{\varepsilon}\|\nu - \nu_\varepsilon\| + \sqrt{\varepsilon}\|q - q_\varepsilon\|$$

attains a strict (local) minimum at $(\nu_\varepsilon, q_\varepsilon)$ over $L^\infty(0,1) \times Q_{ad}(0,1)$; see, *e.g.*, Theorem 5.19 from [12]. Employing Theorem 10.31 from [12], there exists a constant K solely depending on the Lipschitz constant of $\phi^\varepsilon(\nu,q) + \sqrt{\varepsilon}\|\nu - \nu_\varepsilon\| + \sqrt{\varepsilon}\|q - q_\varepsilon\|$, that in turn can be chosen to be independent of ε, such that

$$(\nu, q) \mapsto \phi^\varepsilon(\nu,q) + \sqrt{\varepsilon}\|\nu - \nu_\varepsilon\| + \sqrt{\varepsilon}\|q - q_\varepsilon\| + K d_{Q_{ad}(0,1)}(q)$$

has a local minimum at $(\nu_\varepsilon, q_\varepsilon)$. Nonsmooth calculus as in Theorem 4.12 yields

$$\gamma_\varepsilon \in \partial_{\mathrm{C}}\phi^\varepsilon(\nu_\varepsilon, q_\varepsilon) + \{\,0\,\} \times N_{Q_{ad}(0,1)}(\nu_\varepsilon, q_\varepsilon) \tag{4.9}$$

with $\gamma_\varepsilon \to 0$ in $L^\infty(0,1)^* \times Q(0,1)$ as $\varepsilon \to 0$.

Now, we define $\lambda_\varepsilon \in \mathbb{R}^2_+$ by

$$\begin{aligned}
\lambda_{\varepsilon,1} &= \max\{\,0, j(\nu_\varepsilon, q_\varepsilon) - j(\bar\nu, \bar q) + \varepsilon\,\}/\phi^\varepsilon(\nu_\varepsilon, q_\varepsilon),\\
\lambda_{\varepsilon,2} &= d_U(i_1 S(\nu_\varepsilon, q_\varepsilon))/\phi^\varepsilon(\nu_\varepsilon, q_\varepsilon).
\end{aligned}$$

Clearly, it holds $\lambda_{\varepsilon,1}^2 + \lambda_{\varepsilon,2}^2 = 1$. By computing the subdifferential $\partial_{\mathrm{C}}\phi^\varepsilon$ (combining the arguments of ([12], Thm. 10.47) and Thm. 4.12), we obtain sequences of $\mu_\varepsilon \in N_{U_\delta}(u_\varepsilon(1))$ with $\|\mu_\varepsilon\| \le 1$, $\xi_\varepsilon \in \partial L(q_\varepsilon)$, $\zeta_\varepsilon \in N_{Q_{ad}(0,1)}(q_\varepsilon)$, and $\|\zeta_\varepsilon\| \le C$ such that

$$\gamma_\varepsilon = \begin{pmatrix} \lambda_{\varepsilon,1}\,[1 + L(q_\varepsilon)] + \lambda_{\varepsilon,2}\langle Bq_\varepsilon - Au_\varepsilon, z_\varepsilon\rangle \\ \nu_\varepsilon(\lambda_{\varepsilon,1}\xi_\varepsilon + \lambda_{\varepsilon,2}B^* z_\varepsilon + \zeta_\varepsilon) \end{pmatrix}, \tag{4.10}$$

where z_ε solves the corresponding adjoint equation (4.3). Now, we go to the limit. Due to boundedness of the sequence $(\mu, \xi, \zeta, \lambda)_\varepsilon \in H \times Q(0,1) \times Q(0,1) \times \mathbb{R}^2$, we can go to a weak limit on a subsequence $(\mu, \xi, \zeta, \lambda)_n \rightharpoonup (\hat\mu, \hat\xi, \hat\zeta, \hat\lambda)$ for $n \to \infty$. Moreover, by combining the general result from Proposition 10.10 from [12], with the continuity of the solution mapping S we can go to the limit in the inclusion (4.9) and obtain $\hat\mu \in N_{U_\delta}(\bar u(1))$ with $\|\hat\mu\| \le 1$, $\hat\xi \in \partial L(\bar q)$, $\hat\zeta \in N_{Q_{ad}(0,1)}(\bar q)$, and $\hat\lambda \in \mathbb{R}^2_+$, $\hat\lambda_1^2 + \hat\lambda_2^2 = 1$.

Now, we distinguish two cases: In the case $\hat\lambda_1 > 0$, we set $(\bar\mu, \bar\xi, \bar\zeta) = (\hat\lambda_2\hat\mu, \hat\xi, \hat\zeta)/\hat\lambda_1$, and we can derive the conditions for $\mu_0 = 1$ as in Theorem 4.12. As before, the nontriviality of $\bar\mu$ follows. Note that the case $\hat\lambda_2 = 0$ cannot occur, since from the first equation of (4.10) we would deduce $0 = 1 + L(\bar q)$.

In case $\hat\lambda_1 = 0$, it follows $\hat\lambda_2 = 1$, and we obtain the desired set of conditions with $(\bar\mu, \bar\xi, \bar\zeta) = (\hat\mu, \hat\xi, \hat\zeta)$. It remains to verify $\bar\mu \neq 0$. Since $\hat\lambda_{n,2} \to 1$, we obtain $u_n(1) = i_1 S(\nu_n, q_n) \notin U_\delta$ and $\mu_n = (u_n(1) - P_{U_\delta}(u_n(1)))/d_{U_\delta}(u_n(1))$, *i.e.*, $\|\mu_n\| = 1$, for n sufficiently large. Moreover, as $\mu_n \in N_{U_\delta}(u_n(1))$ we find for all $u' \in U_\delta$ that

$$(\mu_n, u' - \bar u(1)) \le (\mu_n, u_n(1) - \bar u(1)) \le \|\mu_n\|\|\bar u(1) - u_n(1)\| \to 0.$$

Finally, we use the fact that U_δ has finite co-dimension with Lemma 4.3.6 from [27], to conclude that $0 \neq \bar\mu = \hat\mu = \operatorname{weak\,lim}_{n\to\infty} \mu_n$. ☐

Remark 4.14. As an example, consider the choice $L(q) = (\alpha/2)\|q\|^2$ for $\alpha \ge 0$. In the qualified case, condition (4.5) reduces to the variational inequality

$$(\alpha\bar q(t) + B^*\bar z(t), q - \bar q(t)) \ge 0 \quad \text{for all } q \in Q_{ad},$$

which implies the projection formula $\bar q(t) = P_{Q_{ad}}(-(1/\alpha)B^*\bar z(t))$ for almost all $t \in (0,1)$. In contrast, in the unqualified case $\bar\mu_0 = 0$ the condition (4.8) is *independent* of the cost parameter α, and we obtain that

$$(B^*\bar z(t), q - \bar q(t)) \ge 0 \quad \text{for all } q \in Q_{ad}.$$

In this case, an unqualified stationary point for any $\alpha > 0$ corresponds to a stationary point for the pure time-optimal problem with $\alpha = 0$. Moreover, if $B^*\bar{z}(t) \neq 0$ for almost every $t \in (0,1)$, the control always assumes an extreme value in Q_{ad}, *i.e.*, it is bang-bang.

4.4. The Hamiltonian condition and qualified optimality conditions

In this subsection we investigate connections between the strengthened Hamiltonian condition and qualified optimality conditions. We fist give the main result, which is a direct consequence of the previous results.

Corollary 4.15. *Let P_U be stable in V and suppose that the Hamiltonian condition (4.2) holds for some constant $h_0 > 0$. Then, the optimality conditions (1.3) – (1.6) hold for any optimal solution of (P) in the qualified form (with $\bar{\mu}_0 = 1$), and additionally $\|\bar{\mu}\| \leq c/h_0$.*

Proof. This is a consequence of Theorem 4.5, Theorem 4.12, and the equivalence of the transformed problem ($\hat{P}$) and the original problem (P). ☐

The Hamiltonian condition (4.2) is required to hold for all $u \in U \cap V$. Certainly, only elements of $\partial U \cap V$ are relevant; the condition is trivially fulfilled otherwise. However, if the terminal value $\bar{u}(T) \in \partial U \cap V$ of the optimal solutions to (P_δ) is assumed to be known, it appears desirable to weaken (4.2) to a local condition. In fact, at least in case of finite co-dimension of U and regular normal cones, it is sufficient to require the strengthened Hamiltonian condition only at the optimal terminal value $\bar{u}(T)$ to obtain qualified optimality conditions. We give an auxiliary lemma before the result.

Lemma 4.16. *The lower Hamiltonian $h\colon V \times V \to \mathbb{R}$ is continuous.*

Proof. We introduce the support function of Q_{ad} as $h_{Q_{ad}}(\cdot) = \sup_{q \in Q_{ad}}(q, \cdot)_Q$. Then it holds

$$h(u,\zeta) = -h_{Q_{ad}}(-B^*\zeta) - \langle Au, \zeta \rangle.$$

Employing the facts that support functions are convex and that $h_{Q_{ad}}$ is finite ($h_{Q_{ad}}(\zeta) \leq C_{Q_{ad}} \|B\|_{\mathcal{L}(Q,V^*)} \|\zeta\|_V$ for all $\zeta \in Q$), we infer that $h\colon V \times V \to \mathbb{R}$ is continuous, since convex functions are locally Lipschitz continuous; see, *e.g.*, Theorem 2.34 from [12]. ☐

Proposition 4.17. *Suppose that U has finite co-dimension and an optimal solution $(\bar{q}, T, \bar{u})$ of (P) is given with $N_U(\bar{u}(T)) \subset V$ and*

$$h(\bar{u}(T), \zeta) \leq -h_0 \|\zeta\| \quad \textit{for all } \zeta \in N_U(\bar{u}(T)), \tag{4.11}$$

for some constant $h_0 > 0$. Then, the optimality conditions (1.3)–(1.5) hold in the qualified form (with $\bar{\mu}_0 = 1$), and additionally $\|\bar{\mu}\| \leq c/h_0$.

Proof. We argue by contradiction. Let the conditions of Theorem 4.13 hold with $\bar{\mu}_0 = 0$. Then, $\bar{u} \in C((0,T];V)$, $\bar{z} \in C([0,T];V)$ according to Proposition A.1, and

$$h(\bar{u}(t), \bar{z}(t)) = \min_{q \in Q_{ad}} \langle Bq - A\bar{u}(t), \bar{z}(t) \rangle = \langle B\bar{q}(t) - A\bar{u}(t), \bar{z}(t) \rangle = 0$$

for almost all $t \in (0,T)$. However, since $t \mapsto h(\bar{u}(t), \bar{z}(t))$ is continuous on $(0,T]$ due to Lemma 4.16, this leads to a contradiction, because $h(\bar{u}(T), \bar{z}(T)) = h(\bar{u}(T), \bar{\mu}) \leq -h_0\|\bar{\mu}\| < 0$. Thus, $\bar{\mu}_0 = 1$, and inspection of the Hamiltonian optimality condition yields

$$\begin{aligned} -h_0\|\bar{\mu}\| \geq h(\bar{u}(T), \bar{z}(T)) &= \min_{q \in Q_{ad}} \left[H_1(q, \bar{u}(T), \bar{z}(T)) - (1 + L(q)) \right] \\ &\geq \min_{q \in Q_{ad}} H_1(q, \bar{u}(T), \bar{z}(T)) + \min_{q \in Q_{ad}} -(1 + L(q)) = -(1 + \max_{q \in Q_{ad}} L(q)) = -L_\infty, \end{aligned}$$

which implies the estimate for $\bar{\mu}$. □

Clearly, (4.11) is a weaker assumption than (4.2) (given the requirements on the terminal set U and the normal cone). Additionally, if $N_U(\bar{u}(T))$ contains just one direction, condition (4.11) is already equivalent to the qualified optimality conditions.

Proposition 4.18. *Let the qualified optimality conditions (as in Cor. 4.15) hold and assume that the normal cone $N_U(\bar{u}(T)) \subset V$ has dimension one. Then, the condition (4.11) holds with $h_0 = \|\bar{\mu}\|^{-1}$.*

Proof. First, we note that $N_U(\bar{u}(T)) = \{\, \lambda\bar{\mu} \mid \lambda \geq 0 \,\}$ (since $0 \neq \bar{\mu} \in N_U(\bar{u}(T))$), and thus also $\bar{\mu} \in V$. Condition (4.4) implies

$$0 = \min_{q \in Q_{ad}} H_1(q, \bar{u}(t), \bar{z}(t)) \geq \min_{q \in Q_{ad}} \langle Bq - A\bar{u}(t), \bar{z}(t) \rangle + 1 + \min_{q \in Q_{ad}} L(q)$$

and, since $L(q) \geq 0$, we obtain

$$h(\bar{u}(t), \bar{z}(t)) = \min_{q \in Q_{ad}} \langle Bq - A\bar{u}(t), \bar{z}(t) \rangle \leq -1, \quad \text{a.e. } t \in [0, T]. \tag{4.12}$$

Recall that the lower Hamiltonian $h\colon V \times V \to \mathbb{R}$ is continuous; see Lemma 4.16. Moreover, according to Proposition A.1 with $\bar{z}(T) = \bar{\mu} \in V$ we find that $u \in C((0, T]; V)$ and $z \in C([0, T]; V)$. Thus, we can evaluate the expression (4.12) at $t = T$ and arrive at

$$h(\bar{u}(T), \bar{\mu}) = \min_{q \in Q_{ad}} \langle Bq - A\bar{u}(T), \bar{\mu} \rangle \leq -1.$$

We finish the proof by multiplying both sides by $\lambda = \|\bar{\mu}\|^{-1}\|\zeta\| \geq 0$ and using the positive homogeneity of the terms on the left and right. □

4.5. Further perturbation results

Up to this point, we have studied the sensitivity of the objective functional with respect to perturbations of the terminal constraint. In this subsection, as another consequence of the theory of Section 3, we study perturbations with respect to the initial state u_0 (*cf.* [10, 19]) and the operator A (*cf.* [40, 43]) of problem (P). In particular, we restrict attention to the classical case $L \equiv 0$. In view of the fact that the choice $L \equiv 0$ results in $j(T, q) = T$, an estimate for the optimal value function corresponds to a perturbation estimate for the optimal time T, which is of independent interest. In the following, we introduce a perturbation parameter $\varepsilon > 0$ (to be made concrete later) and derive estimates for $T - T_\varepsilon$, where $T = T_0$ and T_ε denote the optimal times for the original and the perturbed problem, respectively. Moreover, $c > 0$ is a generic constant that may have different values at different appearances.

4.5.1. Perturbations of the initial state u_0

For $T > 0$, we use $u[q, u_0]$ to denote the solution to the state equation with control $q \in Q(0, T)$ and initial state $u_0 \in H$. Consider the time-optimal control problems with perturbed initial values $u_0^\varepsilon \in H$ defined as

$$\inf_{T>0,\ q \in Q_{ad}(0,T)} T \quad \text{subject to } u[q, u_0^\varepsilon](T) \in U. \tag{4.13}$$

We suppose that the initial values converge to u_0 at a rate ε, *i.e.* there is $c > 0$ such that

$$\|u_0^\varepsilon - u_0\| \leq c\varepsilon, \quad \varepsilon > 0. \tag{4.14}$$

Using similar arguments as in the proof of Theorem 4.5 we obtain the following perturbation result.

Theorem 4.19. *Suppose that P_U is stable in V, the strengthened Hamiltonian condition (3.3) holds, and the perturbed initial condition fufills (4.14). Then, there exists an $\varepsilon_0 > 0$ such that problem (4.13) has solutions for $\varepsilon \le \varepsilon_0$. Moreover, it holds*

$$|T - T_\varepsilon| \le c\varepsilon, \quad 0 < \varepsilon \le \varepsilon_0,$$

where T is the optimal time to (P) and T_ε is the optimal time to (4.13).

Proof. Let $(T, \bar{q})$ be an optimal solution of (P). Since the semigroup e^{-tA} is strongly continuous, for all $T' > 0$ there is $c > 0$ such that $\|\mathrm{e}^{-tA}\|_{\mathcal{L}(H)} \le c$ for all $t \in [0, T']$. Thus, setting $\breve{u}_T = u[\bar{q}, u_0^\varepsilon](T)$ we find $c > 0$ such that

$$d_U(\breve{u}_T) \le \|u[\bar{q}, u_0^\varepsilon](T) - u[q, u_0](T)\| = \|\mathrm{e}^{-TA}(u_0^\varepsilon - u_0)\| \le c\varepsilon,$$

because $u[\bar{q}, u_0](T) \in U$. For $\varepsilon > 0$ sufficiently small, we may apply Lemma 3.9 to obtain a control $\breve{q}\colon [0, \infty) \to Q_{ad}$ such that the corresponding trajectory with initial value $\breve{u}_T = u[\bar{q}, u_0^\varepsilon](T)$ satisfies

$$d_U(u[\breve{q}, \breve{u}_T](t)) \le \max\{\, 0, d_U(u_T) + (d_U(\breve{u}_T)\omega_0 - h_0)\, t \,\} \le \max\{\, 0, c\varepsilon + (c\varepsilon\omega_0 - h_0)\, t \,\}$$

for all $t \ge 0$. Setting $\delta T = c\varepsilon/(h_0 - c\varepsilon\omega_0)$ and

$$q'(t) = \begin{cases} \bar{q}(t) & \text{if } t \le T, \\ \breve{q}(t - T) & \text{if } t > T, \end{cases}$$

the pair $(T + \delta T, q')$ is feasible for (4.13). This implies that there exists an optimal solution $(T_\varepsilon, \bar{q}_\varepsilon)$ of (4.13). Furthermore, by optimality of T_ε, we obtain

$$T_\varepsilon \le T + \delta T = T + \frac{c\varepsilon}{h_0 - c\varepsilon\omega_0} \le T + c\varepsilon.$$

In particular, this implies that T_ε is uniformly bounded. Hence, we can find an uniform estimate for $\|\mathrm{e}^{-\cdot A}\|_{\mathcal{L}(H)}$ on $[0, T_\varepsilon] \subseteq [0, T + c\varepsilon]$ and the same arguments as before (exchanging the roles of $(T, \bar{q})$ and $(T_\varepsilon, \bar{q}_\varepsilon)$) yield the estimate $T \le T_\varepsilon + c\varepsilon$. □

Note that the previous result is essentially a generalization of Theorem 4.1 from [10], where a sufficient condition for the Hamiltonian condition in a specific setting is assumed to hold.

4.5.2. Perturbation of the operator A

Next, we consider perturbations of the operator A. Let $A_\varepsilon\colon V \to V^*$ be a family of linear operators such that for each $\varepsilon > 0$ the general assumptions from Section 2 are fulfilled and $A_0 = A$. Moreover, let $u_\varepsilon[q]$ denote the solution to the associated perturbed state equation for $q \in Q(0, T)$ and fixed $u_0 \in H$. We define the corresponding perturbed optimization problem as

$$\inf_{T>0,\ q \in Q_{ad}(0,T)} T \quad \text{subject to } u_\varepsilon[q](T) \in U. \tag{4.15}$$

Suppose that for every $T' > 0$ there exists $c > 0$ such that

$$\|u_\varepsilon[q](t) - u[q](t)\| \le c\varepsilon, \quad 0 \le t \le T', \quad q \in Q_{ad}(0, T'), \quad \varepsilon > 0. \tag{4.16}$$

Moreover, suppose that P_U is stable in V and the strengthened Hamiltonian condition (3.3) holds uniformly with respect to ε, *i.e.* for all $\varepsilon > 0$ sufficiently small there exists $h_0 > 0$ such that:

$$\text{for all } v \in V \quad \text{it holds} \quad h_\varepsilon(u,\zeta) \coloneqq \min_{q\in Q_{ad}} \langle Bq - A_\varepsilon u, \zeta\rangle \le -h_0\|\zeta\|, \quad \text{where } u = P_U(v),\ \zeta = v - u. \tag{4.17}$$

These assumptions lead to the following error estimate.

Proposition 4.20. *Let (4.16) and (4.17) hold. Then, there exists a $\varepsilon_0 > 0$ such that problem (4.15) has solutions for $\varepsilon \le \varepsilon_0$. Moreover, it holds*

$$|T - T_\varepsilon| \le c\varepsilon, \quad 0 < \varepsilon \le \varepsilon_0,$$

where T is the optimal time to (P) and T_ε is the optimal time to (4.15).

Proof. This result is shown along the lines of the proof of Theorem 4.19, where we use the supposition (4.16) instead of (4.14) as well as (4.17). □

We conclude with some comments on the assumptions of the preceding result. In particular, we show that they are always fulfilled for bounded perturbations of the operator. Concretely, assume that the perturbation is of the form:

$$A_\varepsilon = A + \delta A_\varepsilon, \quad \text{where } \|\delta A_\varepsilon\|_{\mathcal{L}(H)} \le c\varepsilon. \tag{4.18}$$

We obtain the following result.

Theorem 4.21. *Let U be bounded in H. Suppose that P_U is stable in V, the strengthened Hamiltonian condition (3.3) holds, and the perturbed operator is of the form (4.18). Then, the result of Proposition 4.20 holds true.*

Proof. We verify the conditions of Proposition 4.20: Concerning (4.17), we obtain for all $u \in U \cap V$ and $\zeta \in N_U(u) \cap V$ that

$$h_\varepsilon(u,\zeta) = \min_{q\in Q_{ad}} \langle Bq - A_\varepsilon u, \zeta\rangle = \min_{q\in Q_{ad}} \langle Bq - Au, \zeta\rangle - (\delta A_\varepsilon u, \zeta) \le -h_0\|\zeta\| + c\varepsilon\|u\|\|\zeta\|.$$

Thus, for $\varepsilon > 0$ sufficiently small, condition (4.17) holds uniformly in ε.

Concerning (4.16), consider $u^0 = u[q]$ and $u^\varepsilon = u_\varepsilon[q]$, and fix some arbitrary $T' > 0$. By straightforward calculations we verify that, for ε small enough, A_ε still satisfies the Gårding inequality (2.1) with slightly modified constants. Thus, by standard energy estimates we have the estimate $\|u^\varepsilon\|_{L^2((0,T');V)} \le c(\|u_0\| + \|q\|_{L^2((0,T');Q)})$ with a constant c independent of ε, u_0, and q; see, *e.g.*, Chapter XVIII, §3 from [15]. Clearly, the perturbation $\delta u^\varepsilon = u^0 - u^\varepsilon$ solves

$$\partial_t \delta u^\varepsilon + A\,\delta u^\varepsilon = \delta A_\varepsilon\, u^\varepsilon, \quad \delta u_\varepsilon(0) = 0.$$

Hence, we find that

$$\|\delta u^\varepsilon(t)\| \le c\|\delta A_\varepsilon u^\varepsilon\|_{L^2((0,T');V^*)} \le c\varepsilon\|u^\varepsilon\|_{L^2((0,T');V)} \le c\varepsilon(\|u_0\| + \|q\|_{L^\infty((0,T');Q)}), \quad t \in [0,T'],$$

with a constant c independent of u_0 and q. This shows (4.16). □

In Theorem 4.21, we have focused on the fundamental case of a bounded perturbation of the operator. Note that this includes the perturbation of a reaction diffusion equation in the lowest order term; see Section 5. In particular, this fully covers the setting considered in [43]. The uniform Hamiltonian condition (4.17) is automatically fulfilled there, since the perturbed operators are uniformly coercive ($\omega_0 = 0$), and the target set is a L^2-ball around zero (*cf.* Proposition 5.3).

Different scenarios are also of interest; see, *e.g.*, [40]. Let us briefly comment on possible generalizations of Theorem 4.21. Clearly, for the verification of (4.16) it suffices that $\|\delta A_\epsilon\|_{\mathcal{L}(V,V')} \leq c\varepsilon$ (which is still more restrictive than [40], but allows for perturbations even in the main part of the operator). Additionally, we have to verify the uniform Hamiltonian condition (4.17). Even though it cannot simply be derived from the corresponding condition for $\varepsilon = 0$, as in the proof of Theorem 4.21, it can be done directly in concrete scenarios for the terminal set U. For instance, if the operators are uniformly coercive for small ε, the terminal set is the H ball around zero, and $0 \in Q_{ad}$, then (4.17) holds uniformly for any perturbation; *cf.* Proposition 5.3.

5. Applications

In this section we derive criteria for strong stability for a controlled convection-diffusion equation. Let $\Omega \subset \mathbb{R}^d$ with $d \in \{2, 3\}$ be a bounded domain with Lipschitz boundary. Define $H = L^2(\Omega)$, and for a closed subset $\Gamma \subset \partial\Omega$, define $V = H^1_\Gamma(\Omega)$ as the space of $H^1(\Omega)$ functions with zero trace on Γ. It is constructed in the usual way as the closure of the restriction of the C^∞ functions supported on $\mathbb{R}^d \setminus \Gamma$ in the Sobolev space $H^1(\Omega)$; *cf.*, *e.g.*, Chapter 4 from [32]. The operator A is given by the bilinear form

$$a(u, \varphi) = \int_\Omega [\kappa \nabla u \cdot \nabla \varphi + b \cdot \nabla u \varphi + c_0 u \varphi] \,\mathrm{d}x + \int_{\partial\Omega\setminus\Gamma} c_1 u \varphi \,\mathrm{d}s, \quad u, \varphi \in H^1_\Gamma(\Omega),$$

for $\kappa \in L^\infty(\Omega, \mathbb{R}^d)$ with $\inf_{x\in\Omega,\xi\in\mathbb{R}^d,|\xi|=1} \xi \cdot \kappa(x)\xi = \alpha_0 > 0$, $b \in W^{1,\infty}(\Omega, \mathbb{R}^d)$, $c_0 \in L^\infty(\Omega)$, and $c_1 \in L^\infty(\Gamma)$. We note that the first term is the weak formulation of a convection-diffusion-reaction equation and the second term allows for either the Robin boundary conditions ($\kappa\partial u/\partial n + c_1 u = 0$, where n is the outer normal to Ω) or homogeneous Neumann boundary conditions by setting $c_1 = 0$. By standard arguments, using the generalized Poincaré-Friedrichs inequality, the Gårding inequality (2.1) holds for α_0 and a suitable $\omega_0 \geq 0$.

Concerning the control operator, we are in particular interested in distributed control on a subset and Neumann boundary control. In the first case, for an open subset $\omega \subset \Omega$ we define the control space as $Q = L^2(\omega)$ and the control operator is given by the extension operator $L^2(\omega) \to L^2(\Omega) = X_{1/2}$. In case of boundary control, we choose $\omega \subset \partial\Omega \setminus \Gamma$, set $Q = L^2(\omega)$, and the control operator is defined by the adjoint of the trace operator $X_{1-\theta_0} \to L^2(\omega)$ that is continuous for $\theta_0 \in (0, 1/4)$ (which can be verified with the Sobolev trace and embedding theorems). In this case, the boundary conditions will read as $\kappa\partial u/\partial n + c_1 u = q$ in strong formulation. Furthermore, purely time dependent controls are of independent interest in control theory and applications. Given functions $b_n \in X_{\theta_0}$, we define the control operator by

$$Bq \coloneqq \sum_{n=1}^{N} q_n b_n, \quad q = (q_1, \ldots, q_n)^\top \in \mathbb{R}^N$$

with control space $Q = \mathbb{R}^N$ endowed with the Euclidean inner product. Possible choices for the set of admissible controls are

$$Q_{ad} \coloneqq \{\, q \in Q \mid q_a \leq q \leq q_b \text{ almost everywhere in } \omega \,\},$$

for two fixed elements $q_a, q_b \in L^\infty(\omega)$ for distributed and Neumann boundary control and

$$Q_{ad} \coloneqq \{\, q \in Q \mid q_{a,n} \leq q_n \leq q_{b,n},\ n = 1, \ldots, N \,\},$$

with two fixed elements $q_a, q_b \in \mathbb{R}^N$ in case of finite dimensional control. A different choice is

$$Q_{ad} \coloneqq \{ \, q \in Q \mid \|q\|_Q \le M \, \} ,$$

with some fixed $M > 0$. Since in the following we will not rely on the concrete form of the set of admissible controls, we do not explicitly distinguish the different settings, but instead detail the concrete assumptions in each of the following results. Note that all the results hold for general control operator B and general control set Q_{ad} satisfying Assumption 2.2, unless otherwise indicated.

This section is organized as follows: First, we discuss the illustrative example $U = \{ \, u_d \, \}$ and observe that this leads to rather restrictive conditions. Significantly weaker conditions can be derived for the case of a $L^2(\Omega)$-ball around u_d if the operator A is coercive. In the general case, which includes unstable systems, we discuss a finite approximate controllability constraint that stabilizes the system around the zero point. The resulting conditions turn out to require at least as many controls as there are unstable modes. Finally, we only require a standard stabilizability assumption to hold, and show that there always exist target sets around zero such that the resulting optimization problem is strongly stable.

5.1. Point target and pointwise constraint

We first consider the example of steering the system in minimal time into a single point u_d, which has been extensively studied in the literature; see, *e.g.*, [4, 16]. Defining U to be the singleton $U = \{ \, u_d \, \}$ with $u_d \in V$ we obtain the following result.

Proposition 5.1. *Suppose that $U = \{ \, u_d \, \}$ with $Au_d \in \operatorname{ran}(B)$ and for some $h_0 > 0$ it holds*

$$Au_d + \mathcal{B}_{h_0}(0) \subset BQ_{ad}. \tag{5.1}$$

Then (P) is strongly stable on the right for all $\delta \ge 0$.

Proof. Clearly, $P_U(u) = u_d$. Due to Proposition 3.7 it holds

$$N_U(u_d) = \{ \, \lambda(u' - u_d) \mid \lambda \ge 0, \, u' \in V \, \} = V.$$

We now take $u = u_d$ and $\zeta \in V$. Then

$$h(u, \zeta) = \min_{q \in Q_{ad}} \langle Bq - Au_d, \zeta \rangle \le \min_{v \in Au_d + \mathcal{B}_{h_0}(0)} \langle v - Au_d, \zeta \rangle = \min_{v \in \mathcal{B}_{h_0}(0)} \langle v, \zeta \rangle = -h_0 \|\zeta\|.$$

Now, Theorem 4.5 yields the assertion. ∎

We point out that (5.1) is essentially the condition which is used in Theorem 5.3.1 from [4], to guarantee existence of (qualified) multipliers in a similar setting; *cf.* also Theorem 4.1 from [10], for Lipschitz continuity of the minimal time function with respect to the initial value. From an application point of view, it is rather restrictive. It is essentially only fulfilled in settings where $Q = H = L^2(\Omega)$, B is the identity, and Q_{ad} contains a sufficiently large $L^2(\Omega)$-ball. For settings with pointwise bounded control action ($BQ_{ad} \subset L^\infty(\Omega)$), controls restricted to some $\omega \subset \Omega$, or finite dimensional controls, it is not fulfilled. In this regard we also mention [42] for the pure time-optimal control (*i.e.* $L \equiv 0$) of the heat equation into zero with pointwise bounded controls active only on a subset of Ω. Therein, the authors obtain Lagrange multipliers in a larger space than $L^2(\Omega)$ (containing distributions) using essentially the exact null controllability of the heat equation.

Next, we turn to point-wise terminal constraints that are of independent interest in applications; *cf.* [25]. As an example, we consider

$$U = \{ \, u \in H \mid |u - u_d| \le u_{\max} \quad \text{a.e. in } \Omega \, \} , \tag{5.2}$$

where $u_d \in V$ and $u_{\max} \in \mathbb{R}$, $u_{\max} > 0$. For simplicity, we consider only an illustrative special case for A.

Proposition 5.2. *Let A be defined with $b = c_1 = c_0 = 0$. Suppose that U is defined as in (5.2) with $Au_d \in \operatorname{ran}(B)$ and (5.1) holds for some $h_0 > 0$. Then (P) is strongly stable on the right for all $\delta \geq 0$.*

Proof. We will verify the supposition of Theorem 4.5. Clearly, it holds

$$P_U(v) = v - (v - u_d - u_{\max})_+ + (v + u_d - u_{\max})^-.$$

Due to Proposition 3.7 we infer

$$N_U(u) = \{\, (u' - u_d - u_{\max})_+ - (u' - u_d - u_{\max})^- \mid u' \in V,\, u = P_U(u') \,\}.$$

Take $u' \in V$ with $P_U(u') = u$ and set $\zeta = (u' - u_d - u_{\max})_+ - (u' - u_d - u_{\max})^-$. Then

$$\begin{aligned} h(u, \zeta) = \min_{q \in Q_{ad}} \langle Bq - Au, \zeta \rangle &\leq \min_{v \in Au_d + \mathcal{B}_{h_0}(0)} \langle v, \zeta \rangle - \int_\Omega [\kappa \nabla P_U(u') \cdot \nabla \zeta] \\ &\leq -h_0 \|\zeta\| + \langle Au_d, \zeta \rangle - \int_{\{\, x \in \Omega \mid \zeta \neq 0 \,\}} [\kappa \nabla u_d \cdot \nabla \zeta] = -h_0 \|\zeta\|. \end{aligned}$$

Finally, Theorem 4.5 yields the assertion. □

Again we remark that (5.1) is rather restrictive. However, note that for pointwise constraints one typically searches for Lagrange multipliers in a space of regular Borel measures (*cf.*, *e.g.*, [35]), whereas under assumption (5.1), we obtain multipliers in $H = L^2(\Omega)$. A corresponding extension of the above theory to include multipliers in spaces of measures (under potentially weaker conditions) is outside of the scope of this article.

However, it seems that in applications it is often sufficient to steer the system close to a desired point u_d. In the subsequent subsections we will derive significantly weaker conditions guaranteeing strong stability for this type of terminal constraint.

5.2. L^2-norm constraint

Let $u_d \in V$ and $\delta_0 > 0$ be given and consider the set

$$U = \{\, u \in H \mid \|u - u_d\| \leq \delta_0 \,\}.$$

We emphasize that $u_d \in V$ (instead of just $u_d \in H$, $u_d \notin V$) is required for the minimizing projection P_U to be stable in V, which is necessary for weak invariance; see Lemma 3.5.

If the operator A is coercive (*i.e.* $\omega_0 = 0$) we can easily verify the strengthened Hamiltonian condition assuming only the existence of one control $\breve{q} \in Q_{ad}$ such that $B\breve{q}$ is sufficiently close to Au_d in V^*. This condition can be interpreted as the requirement that u_d lies sufficiently close to an asymptotically stable state of the system with fixed control $\breve{q}$. Note that this always holds for sufficiently small $u_d \in V$ and $0 \in Q_{ad}$.

Proposition 5.3. *Let (2.1) hold with $\omega_0 = 0$. If there exists $\breve{q} \in Q_{ad}$ such that $\|B\breve{q} - Au_d\|_{V^*} < \alpha_0 \delta_0$, then ($P$) is strongly stable on the right for all $\delta \geq 0$.*

Proof. Let $u \in U \cap V$. If $\|u - u_d\| < \delta_0$, we have $N_U(u) = \{\, 0 \,\}$, and nothing to show. Therefore, let $\|u - u_d\| = \delta_0$. Due to Corollary 10.44 from [12], it holds

$$N_U(u) = \{\, \tau(u - u_d) \mid \tau \geq 0 \,\}.$$

Without restriction, we can therefore consider $\zeta = u - u_d$. We calculate

$$\begin{aligned} h(u,\zeta) &= \min_{q\in Q_{ad}} \langle Bq - Au, \zeta\rangle = \langle Au_d - Au, u - u_d\rangle_{V^*,V} + \min_{q\in Q_{ad}} \langle Bq - Au_d, \zeta\rangle \\ &\le -\alpha_0 \|u - u_d\|_V^2 + \langle B\breve{q} - Au_d, \zeta\rangle \\ &\le -\alpha_0 \|u - u_d\| \|u - u_d\|_V + \|B\breve{q} - Au_d\|_{V^*} \|\zeta\|_V = (-\alpha_0\delta_0 + \|B\breve{q} - Au_d\|_{V^*}) \|\zeta\|_V. \end{aligned}$$

Due to the supposition there is $h_0 > 0$ such that $h(u,\zeta) \le -h_0\|\zeta\|_V \le -h_0\|\zeta\|$ and we can apply Theorem 4.5 to guarantee strong stability on the right. □

However, in case $\omega_0 > 0$, the control has to counteract unstable modes of A. We will discuss this situation in the following example.

5.3. Finite-approximate controllability constraint

Motivated by the concept of finite-approximate controllability (see, *e.g.*, [45]), we consider the constraint

$$U = \{ u \in H \mid \|u\| \le \delta_0 \text{ and } Fu = 0 \}. \tag{5.3}$$

Concretely, let $\{ f_1, \ldots, f_M \} \subset V$ be pairwise orthonormal in H and set

$$Fu = \sum_{i=1}^{M} (f_i, u) f_i, \quad u \in H.$$

In this subsection, we will investigate weak invariance in the particular case that $\operatorname{ran} F \coloneqq \operatorname{span}\{ f_1, \ldots, f_M \}$ is an invariant subspace of A^*. Concretely, we require that

$$A^* f_i \subset \operatorname{ran} F, \quad i = 1, \ldots, M. \tag{5.4}$$

A particularly interesting example is to choose the functions f_i as a basis of the unstable subspace of A^* (the real span of all eigenvalues with negative real part). A target set of the form $U = \ker F$ is then motivated by the desire to steer the system into a stable subspace; *cf.* [18]. From an application point of view, it might be desirable not just to steer the system into a stable subspace but also into a sufficiently small stable state. In this case, the terminal set is given by (5.3).

First, for the sake of clarity, we will investigate (5.3) with $\delta_0 = \infty$, *i.e.*, we will consider $U = \ker(F)$. The minimizing projection onto $\ker(F)$ is given by $P_{\ker(F)} = \operatorname{Id} - F$. By virtue of Proposition 3.7 for $u \in U$ we have

$$N_U(u) = \{ Fu' \mid u' \in H,\, u = u' - Fu' \}.$$

Proposition 5.4. *If* $0 \in Q_{ad}$ *and* (5.4) *holds, then* $U = \ker(F)$ *is weakly invariant under* (A, BQ_{ad})*. Moreover, if there is* $h_0 > 0$ *such that for all* $u' \in V$ *there is* $\breve{q} \in Q_{ad}$ *such that*

$$\langle \breve{q}, B^* F u'\rangle \le -h_0 \|Fu'\|, \tag{5.5}$$

then (P) *with* $U = \ker(F)$ *is strongly stable on the right for all* $\delta \ge 0$.

Condition (5.5) implies that $\ker(B^*) \cap \operatorname{ran}(F) = \{ 0 \}$. In particular, we require at least as many controls as $\dim \operatorname{ran}(F) = M$. Hence, this condition is in general stronger than approximate controllability (or stabilizability), where the necessary number of controls is given by the largest geometric multiplicity of the eigenvalues (resp. the unstable eigenvalues); *cf.* Section 3.4 from [3]. We can also give examples where (5.5) holds: For instance, if the control acts in an arbitrary open subset $\omega \subset \Omega$, then (5.5) is satisfied (under certain smoothness assumptions on the coefficients of A and the domain), since the eigenfunctions of A^* restricted to ω are linearly independent; see Theorem 4.1 from [18].

Proof of Proposition 5.4. Let $u' \in V$ such that $u = u' - Fu'$ and set $\zeta = Fu'$. Then

$$h(u,\zeta) = \min_{q \in Q_{ad}} \langle Bq - Au, \zeta\rangle \le -\langle u' - Fu', A^* Fu'\rangle = 0,$$

since $A^* Fu' \in \operatorname{ran}(F)$. Theorem 3.8 yields the first assertion. Moreover, the strengthened Hamiltonian condition is equivalent to (5.5) due to the calculation above proving the second assertion. $\square$

Next, we turn to the general case of (5.3) with $\delta_0 < \infty$.

Proposition 5.5. *Assume $0 \in Q_{ad}$ and let (5.4) and (5.5) hold. Moreover, suppose that $\{ f_1, \ldots, f_M \}$ is chosen such that for all $\varphi \in \ker(F)$ it holds $\langle A\varphi, \varphi\rangle \ge \omega_1 \|\varphi\|^2$ with $\omega_1 > 0$. Then (P) with $U = \mathcal{B}_{\delta_0}(0) \cap \ker(F)$ is strongly stable on the right for all $\delta \ge 0$.*

Proof. First, we will show the following formula for the minimizing projection P_U:

$$P_U(u) = \min\{ 1, \delta_0/\|u - Fu\| \}\,(u - Fu) =: \gamma(u)\,(u - Fu).$$

Let $u \in H$. If $\|u - Fu\| \le \delta_0$, then for all $u' \in U$ we calculate

$$(u - P_U(u), u' - P_U(u)) = (Fu, u' - u + Fu) = (u, Fu') - (Fu, u - Fu) = 0.$$

In the other case $\|u - Fu\| > \delta_0$ set $\gamma = \gamma(u)$ and we obtain for all $v \in U$ that

$$\begin{aligned}(u - P_U(u), v - P_U(u)) &= (1-\gamma)(u, v - \gamma(u - Fu)) + \gamma(Fu, v) - \gamma^2(Fu, u - Fu)\\ &= (1-\gamma)(u - Fu, v) - (1-\gamma)\gamma\|u - Fu\|^2\\ &\le (1-\gamma)\|u - Fu\|\|v\| - (1-\gamma)\delta_0\|u - Fu\| \le 0,\end{aligned}$$

where we have used again that $(Fu, v) = (u, Fv) = 0$ and $(Fu, u - Fu) = 0$ in the second step, and $\|v\| \le \delta_0$ in the last step. By virtue of Proposition 3.7 for $u \in U$ we infer that

$$\begin{aligned}N_U(u) &= \{ (1 - \gamma(u'))u' + \gamma(u')Fu' \mid u' \in H,\ u = P_U(u') \}\\ &= \{ (1 - \gamma(u'))(u' - Fu') + Fu' \mid u' \in H,\ u = P_U(u') \}.\end{aligned}$$

Consider the single terms of the Hamiltonian for $u' \in V$ and set $\gamma = \gamma(u')$. We consider the case $\gamma < 1$, only; the other case is analogous to Proposition 5.4. Then

$$\langle Bq, u' - P_U(u')\rangle = (1-\gamma)\langle Bq, u' - Fu'\rangle + \langle Bq, Fu'\rangle$$

and, since $\langle A(u' - Fu'), Fu'\rangle = 0$, we find

$$\begin{aligned}-\langle AP_U(u'), u' - P_U(u')\rangle &= -\gamma(1-\gamma)\langle A(u' - Fu'), u' - Fu'\rangle - \gamma\langle A(u' - Fu'), Fu'\rangle\\ &= -\gamma(1-\gamma)\langle A(u' - Fu'), u' - Fu'\rangle.\end{aligned}$$

Due to the supposition $\langle A\varphi, \varphi\rangle \ge \omega_1\|\varphi\|^2$ for all $\varphi \in \ker(F)$ we infer

$$-\langle AP_U(u'), u' - P_U(u')\rangle \le -\gamma(1-\gamma)\omega_1\|u' - Fu'\|^2 = -(1-\gamma)\omega_1\delta_0\|u' - Fu'\|$$

from the calculation above. Combining the previous estimates, we obtain

$$\begin{aligned}\langle Bq - AP_U(u'), u' - P_U(u')\rangle &\le (1-\gamma)\left[\langle Bq, u' - Fu'\rangle - \omega_1\delta_0\|u' - Fu'\|\right] + \langle Bq, Fu'\rangle \\ &\le (1-\gamma)(\|B\|\|q\| - \omega_1\delta_0)\|u' - Fu'\| + \langle Bq, Fu'\rangle.\end{aligned}$$

Assuming that $0 \in Q_{ad}$, choosing $q = \lambda\check{q}$, $\lambda = \min\{\,1, (\omega_1\delta_0)/(2\|B\|\|\check{q}\|)\,\}$, where $\check{q}$ is the control to realize (5.5), we obtain the strengthened Hamiltonian condition (with a suitably modified constant h_0). □

5.4. Stabilization with finite dimensional control

We have seen that the criteria for strong stability of systems with general A and U require certain assumptions, which are somewhat restrictive. In this section, we will show that there exist neighbourhoods U of zero such that the resulting problem is strongly stable, assuming only stabilizability (controllability of the unstable modes).

Here, we suppose that the control is finite dimensional, $Q = \mathbb{R}^M$. The set of admissible controls contains a neighborhood of zero, *e.g.*, $Q_{ad} = \{\, q \in \mathbb{R}^M : q \in [-K, K]^M \,\}$ for some fixed $K > 0$. We are interested to bring the system into a small neighborhood of the stationary state zero. Note that we could more generally consider weakly invariant states u_d, *i.e.* $\{\, u_d \,\}$ is weakly invariant under (A, BQ_{ad}). A short computation based on Theorem 3.8 reveals that $Au_d \in BQ_{ad}$. However, this case follow directly from the case $u_d = 0$ by an affine change of variables, and we omit it for simplicity of notation.

To ensure that admissible controls for (P) exist, we can employ the concept of stabilizability, which is widely accepted in the control literature. Concretely, we assume in the following that $(-A, B)$ should be *stabilizable*, which can be verified with the Fattorini criterion; see [3] and the references therein. This means that

$$A^*\zeta = \lambda\zeta, \quad \Re\lambda \le 0, \quad B^*\zeta = 0 \quad \Longrightarrow \quad \zeta = 0.$$

It is known that this implies the existence of a stabilizing feedback law, such that $\|u(t)\| \le M_0 \exp(-\gamma_0 t)\|u_0\|$ for some $\gamma_0 > 0$, which in turn guarantees existence for (P) (given u_0 sufficiently small or Q_{ad} sufficiently large). Additionally, we will show that it is possible to choose some appropriate neighborhood U of zero, such that the criterion for strong stability (and thus weak invariance) is guaranteed.

First, we consider the infinite horizon optimization problem

$$\min_{q \in L^2((0,\infty);\mathbb{R}^M)} \int_0^\infty \left[\|u[q,u'](t)\|^2 + \|q(t)\|^2_{\mathbb{R}^M}\right] \mathrm{d}t, \tag{5.6}$$

where $u[q, u']$ is the solution to the state equation on $(0, \infty)$ with control $q \in L^2((0, \infty); \mathbb{R}^M)$ and initial condition $u' \in H$. This defines a linear, bounded, self-adjoint and nonnegative operator $\Pi\colon H \to H$ such that $(\Pi u', u')$ is the minimal value of (5.6) and Π satisfies the following algebraic Riccati equation

$$-\langle A^*\Pi\varphi, \psi\rangle - \langle \Pi A\varphi, \psi\rangle + (\varphi, \psi) = (B^*\Pi\varphi, B^*\Pi\psi)_{\mathbb{R}^M}, \tag{5.7}$$

for all $\varphi, \psi \in V$; see, *e.g.*, Theorem 2.2.1 (a_2), (a_4) from [26]. Furthermore, Π maps H into $X_{1-\theta_0}$, hence Π is compact on H; see Theorem 2.2.1 (a_3) from [26].

Define the norm $\|\cdot\|_\Pi = (\Pi\cdot, \cdot)^{1/2}$ induced by the operator Π. Let the terminal constraint be given by

$$U = \{\, u \in H \mid \|u\|_\Pi \le \delta_0 \,\}. \tag{5.8}$$

Thus, $u' \in U$ corresponds to a constraint on the optimal value function of (5.6) with initial value u'. Since Π is self-adjoint, according to Corollary 10.44 from [12], for all $u \in \partial U$ we have

$$N_U(u) = \{\, \lambda\Pi u \mid \lambda \ge 0 \,\} \subset V.$$

Inserting the optimal feedback law $\breve{q} = -B^*\varPi u$ we estimate

$$\begin{aligned} h(u,\zeta) = h(u,\varPi u) &= \inf_{q\in Q_{ad}} (q, B^*\varPi u)_{\mathbb{R}^M} - \frac{1}{2}\langle u, (A^*\varPi + \varPi A)u\rangle \\ &\le -(B^*\varPi u, B^*\varPi u)_{\mathbb{R}^M} - \frac{1}{2}\langle u, (A^*\varPi + \varPi A)u\rangle. \end{aligned}$$

This is valid as long as $\breve{q} = -B^*\varPi u \in Q_{ad}$. Since $\|B^*\varPi u\|_{\mathbb{R}^M} \le \|B^*\varPi^{1/2}\|_{\mathcal{L}(H,\mathbb{R}^M)}\|u\|_{\varPi} = \|B^*\varPi^{1/2}\|_{\mathcal{L}(H,\mathbb{R}^M)}\delta_0$, this can be achieved by a sufficiently small choice of δ_0. Now we use (5.7) to obtain

$$h(u,\varPi u) \le -\frac{1}{2}(B^*\varPi u, B^*\varPi u)_{\mathbb{R}^M} - \frac{1}{2}\|u\|^2 \le 0 - \frac{1}{2\|\varPi\|\|\varPi^{1/2}\|}\|\varPi u\|\|u\|_{\varPi} \le -h_0\|\varPi u\|,$$

where $h_0 = \delta_0/(2\|\varPi\|^{3/2}_{\mathcal{L}(H)})$. Thus, strong stability of (P) is guaranteed by Theorem 4.5, assuming only stabilizability (approximate controllability of the unstable modes).

From a practical point of view, the choice of the target set (5.8) can be interpreted as follows: Since the norm $\|u\|_{\varPi}$ corresponds to the optimal value of (5.6), we have in particular the estimates $\|\breve{u}(t)\|_{L^2((0,\infty);H)} \le \|u\|_{\varPi}$ and $\|\breve{q}(t)\|_{L^2((0,\infty);\mathbb{R}^M)} \le \|u\|_{\varPi}$ where $\breve{u}(t)$ is the trajectory starting at $\breve{u}(0) = u$ with control given by the feedback law $\breve{q}(t) = -B^*\varPi\breve{u}(t)$. Thus, we aim to enter a neighboorhood of zero that contains only states which can be stabilized at low cost. After the end of the optimization horizon T, the control can be chosen by the optimal feedback law to keep the trajectory stable.

Appendix A. Regularity of the state equation

Let e^{-tA} denote the semigroup generated by $-A$.

Proposition A.1. *Let $T > 0$, $\theta \in (0, 1/2]$, $f \in L^\infty((0,T); X_\theta)$, $u_0 \in V^*$. Consider the solution u to*

$$\partial_t u + Au = f, \quad u(0) = u_0.$$

Then it holds:

(i) If $u_0 \in V$, then u is continuous from $[0,T]$ into V,
(ii) If $u_0 \in H$, then u is continuous from $(0,T]$ into V,
(iii) If $u_0 = 0$ and $\gamma \in [\theta, 1]$, then

$$\|u(t)\|_{X_\gamma} \le c\|f\|_{L^\infty((0,T);X_\theta)}\, t^{1+\theta-\gamma}, \quad 0 \le t \le T,$$

with $c > 0$ depending on θ, γ, but independent of f.

Proof. The unique solution is given by the variation of constants formula

$$u(t) = \mathrm{e}^{-tA}u_0 + \int_0^t \mathrm{e}^{-(t-s)A} f(s)\,\mathrm{d}s, \quad t \in [0,T]. \tag{A.1}$$

According to Theorem 2.6.13 c) in [33], for $\theta > 0$ there is a constant $M_\theta > 0$ such that it holds

$$\mathrm{e}^{-\omega_0 t}\|\mathrm{e}^{-tA}v\|_{X_\theta} = \|(A+\omega_0)^\theta \mathrm{e}^{-t(A+\omega_0)}v\|_{V^*} \le M_\theta t^{-\theta}\|v\|_{V^*} \tag{A.2}$$

for all $v \in V^*$ and $t > 0$.

(iii): Employing (A.2) we obtain

$$\|u(t)\|_{X_\gamma} = \|\int_0^t \mathrm{e}^{-(t-s)A} f(s)\,\mathrm{d}s\|_{X_\gamma} \le \int_0^t \|(A+\omega_0)^{\gamma-\theta}\mathrm{e}^{-(t-s)A}(A+\omega_0)^\theta f(s)\|_{V^*}\,\mathrm{d}s$$
$$\le M_{\gamma-\theta}\mathrm{e}^{t\omega_0}\|(A+\omega_0)^\theta f\|_{L^\infty((0,t);V^*)}\int_0^t s^{\theta-\gamma}\,\mathrm{d}s \le c\,t^{1+\theta-\gamma}\|f\|_{L^\infty((0,t);X_\theta)}.$$

(i), (ii): If $u_0 \in V$, it holds $(A+\omega_0)\mathrm{e}^{-tA}u_0 = \mathrm{e}^{-tA}(A+\omega_0)u_0$; see, *e.g.*, Theorem 2.6.13 b) from [33]. Whence, continuity of $t \mapsto \mathrm{e}^{-tA}u_0$ from $[0,T]$ into V follows from Corollary 1.2.3 from [33]. If $u_0 \in H$, we find for any $t, \tau > 0$ that

$$\|\left(\mathrm{e}^{-(t+\tau)A} - \mathrm{e}^{-tA}\right)u_0\|_V = \|\mathrm{e}^{-tA}\left(\mathrm{e}^{-\tau A}-1\right)u_0\|_V \le M_{1/2}\mathrm{e}^{\omega_0 t}t^{-1/2}\|\left(\mathrm{e}^{-\tau A}-1\right)u_0\|_{X_{1/2}}.$$

This proves continuity of $t \mapsto \mathrm{e}^{-tA}u_0$ in V for $t > 0$, using that $-A$ induces a continuous semigroup also on $H = X_{1/2}$.

Now we turn to the second term of (A.1). Since A exhibits maximal parabolic regularity, both on V^* and H, it also possesses maximal regularity on the interpolation space X_θ; see Lemma 5.3 from [20]. Hence, for $f \in L^r((0,T),X_\theta)$, the function $\check{u}(t) = \int_0^t \mathrm{e}^{-(t-s)A}f(s)\,\mathrm{d}s$ has the regularity $\check{u} \in W^{1,r}((0,T);X_\theta) \cap L^r((0,T);X_{1+\theta})$ for any $r \in (1,\infty)$. Furthermore, by the trace theorem, there holds the embedding

$$W^{1,r}((0,T);X_\theta) \cap L^r((0,T);X_{1+\theta}) \hookrightarrow C([0,T];(X_\theta,X_{1+\theta})_{1-1/r,r});$$

see, *e.g.*, Theorem III.4.10.2 from [2]. Choose $r > 1/\theta$, which is equivalent to $1-\theta < 1-1/r$. Thus,

$$(X_\theta,X_{1+\theta})_{1-1/r,r} \hookrightarrow (X_\theta,X_{1+\theta})_{1-\theta,1} \hookrightarrow [X_\theta,X_{1+\theta}]_{1-\theta} = \mathcal{D}_{V^*}(A+\omega_0) = V$$

due to Theorems 1.3.3 e), 1.15.2 d) and 1.15.3 from [39]. In summary, we conclude the proof of (i) and (ii). ☐

Proposition A.2. *Let $T > 0$ and $u_0 \in H$. The solution operator $f \mapsto u$ with*

$$\partial_t u + Au = f, \quad u(0) = u_0,$$

is continuous and compact from $L^2((0,T);X_{\theta_0})$ into $L^2((0,T);V)$.

Proof. Let $\mathcal{S}$ denote the solution operator of the parabolic state equation, *i.e.* $u = \mathcal{S}(u_0,f)$ satisfies $\partial_t u + Au = f$, $u(0) = u_0$. Since A exhibits maximal parabolic regularity, both on V^* and H, it also possesses maximal regularity on the interpolation space X_{θ_0}; see, *e.g.*, Lemma 5.3 from [20]. Hence, $f \mapsto \mathcal{S}(0,f)$ is continuous from $L^2((0,T);X_{\theta_0})$ into $H^1((0,T);X_{\theta_0}) \cap L^2((0,T);X_{1+\theta_0})$, where we have used the identification $\mathcal{D}_{X_{\theta_0}}(A) = X_{1+\theta_0}$. Clearly, $X_{1+\theta_0} \hookrightarrow \mathcal{D}_{V^*}(A) = V \hookrightarrow_c H \hookrightarrow X_{\theta_0}$. Employing Theorem I.2.11.1 from [2], we deduce $X_{1+\theta_0} \hookrightarrow_c [X_{\theta_0},X_{1+\theta_0}]_{1-\theta_0} = V$, where we have used Theorem 1.15.3 from [39], in the last step. Therefore, the Aubin-Lions Lemma (see, *e.g.*, [28], Thm. I.5.1) yields the compact injection

$$H^1((0,T);X_{\theta_0}) \cap L^2((0,T);X_{1+\theta_0}) \hookrightarrow_c L^2((0,T);V).$$

Furthermore, $\mathcal{S}(u_0,0) \in W(0,T) \hookrightarrow L^2((0,T);V)$. Whence, the assertion follows from the splitting $\mathcal{S}(u_0,f) = \mathcal{S}(u_0,0) + \mathcal{S}(0,f)$. ☐

Appendix B. Clarke's generalized subdifferential

The *generalized directional derivative* at x from a Banach space X for any function $f\colon X \to \mathbb{R}$ that is Lipschitz near x is given by Section 10.1 from [12].

$$f^\circ(x;v) \coloneqq \limsup_{y\to x,\ \tau\downarrow 0} \tau^{-1}\left[f(y+\tau v) - f(y)\right]. \tag{B.1}$$

Then $\zeta \in X^*$ belongs to the *generalized gradient* $\partial_{\mathrm{C}} f(x)$ if and only if $f^\circ(x;v) \geq \langle \zeta, v\rangle$ for all $v \in X$.

Let X_1, X_2 be Banach spaces and $f\colon X_1 \times X_2 \to \mathbb{R}$ Lipschitz near $x_1 \in X_1$ and $x_2 \in X_2$. We define the partial generalized directional derivatives and partial generalized gradients $f^\circ_{x_1}$, $f^\circ_{x_2}$, $\partial_{\mathrm{C},x_1} f$, and $\partial_{\mathrm{C},x_2} f$ analogously to (B.1).

Proposition B.1. *If $f^\circ_{x_1}(x_1,x_2;v_1) = f^\circ(x_1,x_2;v_1,0)$ and $f^\circ_{x_2}(x_1,x_2;v_2) = f^\circ(x_1,x_2;0,v_2)$ for all $v_1 \in X_1$ and $v_2 \in X_2$, then*

$$\partial_{\mathrm{C}} f(x_1,x_2) \subset \partial_{\mathrm{C},x_1} f(x_1,x_2) \times \partial_{\mathrm{C},x_2} f(x_1,x_2).$$

Proof. $\zeta \in \partial_{\mathrm{C}} f(x_1,x_2)$ if and only if $f^\circ(x_1,x_2;v_1,v_2) \geq \langle \zeta_1, v_1\rangle + \langle \zeta_2, v_2\rangle$ for all $v_1 \in X_1$ and $v_2 \in X_2$. Taking $v_1 = 0$ and $v_2 = 0$ implies $f^\circ(x_1,x_2;v_1,0) \geq \langle \zeta_1, v_1\rangle$ for all $v_1 \in X_1$ and $f^\circ(x_1,x_2;0,v_2) \geq \langle \zeta_2, v_2\rangle$ for all $v_2 \in X_2$. Using the suppositions on $f^\circ_{x_1}$ and $f^\circ_{x_2}$ we finish the proof. □

Proposition B.2. *For j from problem ($\hat{P}$), it holds*

$$\partial_{\mathrm{C}} j(\bar\nu,\bar q) \subset \partial_{\mathrm{C},\nu} j(\bar\nu,\bar q) \times \partial_{\mathrm{C},q} j(\bar\nu,\bar q).$$

Proof. In our case the assumptions of the preceding proposition are satisfied for j. Regarding the differentials with respect to ν, we obtain for all $\delta\nu \in L^\infty(0,1)$ that

$$\begin{aligned} j^\circ(\bar\nu,\bar q;\delta\nu,0) &= \limsup_{\nu\to\bar\nu,\ q\to\bar q,\ \tau\downarrow 0} \tau^{-1}\left[j(\nu+\tau\delta\nu,q) - j(\nu,q)\right] = \limsup_{q\to\bar q} \int_0^1 \delta\nu(1+L(q))\,\mathrm{d}t \\ &= \int_0^1 \delta\nu(1+L(\bar q))\,\mathrm{d}t = j^\circ_\nu(\bar\nu,\bar q;\delta\nu), \end{aligned}$$

using the fact that j is linear in ν in the first and last step. In the other case, we estimate

$$\begin{aligned} j^\circ_q(\bar\nu,\bar q;\delta q) &= \limsup_{q\to\bar q,\ \tau\downarrow 0} \tau^{-1}\left[j(\bar\nu,q+\tau\delta q) - j(\bar\nu,q)\right] \leq j^\circ(\bar\nu,\bar q;0,\delta q) \\ &= \limsup_{\nu\to\bar\nu,\ q\to\bar q,\ \tau\downarrow 0} \tau^{-1}\int_0^1 \nu\left[L(q+\tau\delta q) - L(q)\right]\mathrm{d}t \\ &\leq j^\circ_q(\bar\nu,\bar q;\delta q) + \limsup_{\nu\to\bar\nu,\ q\to\bar q,\ \tau\downarrow 0} \tau^{-1}\int_0^1 \left[\nu-\bar\nu\right]\left[L(q+\tau\delta q) - L(q)\right]\mathrm{d}t \\ &\leq j^\circ_q(\bar\nu,\bar q;\delta q) + \limsup_{\nu\to\bar\nu} c_L \int_0^1 |\nu-\bar\nu|\|\delta q\|_Q\,\mathrm{d}t = j^\circ_q(\bar\nu,\bar q;\delta q), \end{aligned}$$

for all $\delta q \in Q(0,1)$, where c_L is the Lipschitz constant of L. □

Appendix C. Comparison principle

For any $\omega_0 \geq 0$, define $\phi\colon \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\phi(t) = \omega_0^{-1}(\mathrm{e}^{\omega_0 t} - 1), \text{ if } \omega_0 > 0, \text{ and } \phi(t) = t, \text{ if } \omega_0 = 0.$$

We easily verify that $\phi(t) \geq t$ for all $t \geq 0$.

Proposition C.1. *Let $c, \gamma > 0$ and $\omega_0, h_0 \geq 0$. Moreover, let d_γ be continuously differentiable on $(0, \infty)$ and continuous on $[0, \infty)$ with $d_\gamma \geq 0$ such that*

$$d_\gamma'(t) \leq \omega_0 d_\gamma(t) + c\gamma/d_\gamma(t) - h_0 \quad \text{on } \{\, t \mid d_\gamma(t) > 0 \,\}. \tag{C.1}$$

Then it holds

$$d_\gamma(t) \leq \max\{\, \sqrt{\gamma}, (d_\gamma(0) + \sqrt{\gamma})\mathrm{e}^{\omega_0 t} + (c\sqrt{\gamma} - h_0)\phi(t) \,\} =: D_\gamma(t). \tag{C.2}$$

Proof. We argue by contradiction: Suppose that (C.2) is not satisfied and let t_0 be the first time such that $d_\gamma(t_0) = D_\gamma(t_0)$ and $d_\gamma(t) > D_\gamma(t)$ for $t \in (t_0, t_1)$. This implies $d_\gamma(t) > \sqrt{\gamma}$ and therefore from (C.1) we infer $d_\gamma'(t) \leq \omega_0 d_\gamma(t) + c\sqrt{\gamma} - h_0$ for $t \in (t_0, t_1)$.

The unique solution of $z'(t) = \omega_0 z(t) + c\sqrt{\gamma} - h_0$ with $z(t_0) = d_\gamma(t_0)$ is given by

$$z(t) = d_\gamma(t_0)\mathrm{e}^{\omega_0(t-t_0)} + (c\sqrt{\gamma} - h_0)\phi(t - t_0).$$

The comparison principle yields $d_\gamma(t) \leq z(t)$ for $t \in [t_0, t_1)$. Now we distinguish two cases: If $d_\gamma(t_0) = D_\gamma(t_0) = (d_U(u_0) + \sqrt{\gamma})\mathrm{e}^{\omega_0 t_0} + (c\sqrt{\gamma} - h_0)\phi(t - t_0)$, we obtain

$$d_\gamma(t) \leq z(t) = (d_U(u_0) + \sqrt{\gamma})\mathrm{e}^{\omega_0 t} + (c\sqrt{\gamma} - h_0)\phi(t) \leq D_\gamma(t) < d_\gamma(t),$$

for $t \in (t_0, t_1)$, yielding a contradiction. Otherwise, it holds

$$\sqrt{\gamma} = D_\gamma(t_0) > (d_U(u_0) + \sqrt{\gamma})\mathrm{e}^{\omega_0 t} + (c\sqrt{\gamma} - h_0)\phi(t) = d_U(u_0) + \sqrt{\gamma} + ((c + \omega_0)\sqrt{\gamma} + \omega_0 d_U(u_0) - h_0)\,\phi(t)$$

and we necessarily must have $((c + \omega_0)\sqrt{\gamma} + \omega_0 d_U(u_0) - h_0) < 0$. Thus, we have

$$\sqrt{\gamma} < d_\gamma(t) \leq z(t) = \sqrt{\gamma} + ((c + \omega_0)\sqrt{\gamma} - h_0)\phi(t - t_0) < \sqrt{\gamma},$$

also yielding a contradiction. □

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