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THE BANG-BANG PROPERTY OF TIME AND NORM OPTIMAL CONTROL PROBLEMS FOR PARABOLIC EQUATIONS WITH TIME-VARYING FRACTIONAL LAPLACIAN*

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Abstract. In this paper, we establish the bang-bang property of time and norm optimal control problems for parabolic equations governed by time-varying fractional Laplacian, evolved in a bounded domain of $\mathbb{R}^d$. We firstly get a quantitative unique continuation at one point in time for parabolic equations governed by time-varying fractional Laplacian. Then, we establish an observability inequality from measurable sets in time for solutions of the above-mentioned equations. Finally, with the aid of the observability inequality, the bang-bang property of time and norm optimal control problems can be obtained.

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1. Introduction

In the nature world, the usual physical laws are not always followed by all dynamics systems. They sometimes divert from the traditional integer-order dynamic laws to the fractional case, when the systems under consideration are extremely complex or can only be studied on a macroscopic scale. Many important phenomena, such as diffusion of a scalar tracer in an array of convection rolls, transport on fractal geometries, dynamics of a bead in a polymeric network and transport in viscoelastic materials, are described by fractional-order differential equations (see, *e.g.*, [6, 7, 13, 14]). We refer the reader to [22, 23] and references therein for the motivation and description, as well as mathematical analysis of these models.

Due to their importance for applications in various branches of applied sciences and engineering, extensive research has been devoted to the study of fractional-order differential equations. Fractional calculus includes various extensions of the usual derivative from integer to real order. For instance, one of the most popular is the fractional Laplacian introduced below, which is not a local operator when the fractional order is not an integer.

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Let Ω be a bounded domain in $\mathbb{R}^d$, $d \geq 1$, with a smooth boundary. We define an unbounded operator A in $L^2(\Omega)$ (called Dirichlet Laplacian) by

$$\begin{cases} D(A) = H^2(\Omega) \cap H_0^1(\Omega), \\ \quad Au = -\Delta u, \quad \forall u \in D(A). \end{cases}$$

Let $\{\lambda_n\}_{n=1}^{\infty}, 0 < \lambda_1 < \lambda_2 \leq \dots$, be the eigenvalues of A, and let $\{e_n\}_{n=1}^{\infty}$ be the corresponding eigenfunctions satisfying $\|e_n\|_{L^2(\Omega)} = 1$, $n = 1, 2, \dots$, which constitutes an orthonormal basis of $L^2(\Omega)$.

It is well-known that one can define the fractional power of order $\alpha > 0$ for the Dirichlet Laplacian. More precisely, the fractional-order operator $A^\alpha : D(A^\alpha) \subset L^2(\Omega) \longrightarrow L^2(\Omega)$ can be defined as follows

$$D(A^\alpha) = \Big\{ u \in L^2(\Omega);\ u = \sum_{n=1}^{\infty} a_n e_n, \sum_{n=1}^{\infty} |a_n|^2 \lambda_n^{2\alpha} < \infty \Big\},$$
$$A^\alpha u = \sum_{n=1}^{\infty} a_n \lambda^\alpha e_n, \quad \text{where } u = \sum_{n=1}^{\infty} a_n e_n \in D(A^\alpha).$$

The problem of null controllability for fractional-order partial differential equations has been investigated in a number of articles, for instance, [12, 15, 16] and references therein. In these works, the authors studied the problem of null controllability in which the objective is to drive the solution to rest, in other words, to the trivial null state, in any given finite time. In particular, the null controllability for the parabolic equation associated with the fractional Laplacian is well analyzed in [16]. It has been observed that the fractional order $\alpha = 1/2$ is the critical point for null controllability. More precisely, when $\alpha > 1/2$, the parabolic equation with α-order fractional Laplacian is null controllable. However, the result is negative for the case where $0 < \alpha \leq 1/2$.

The bang-bang property of time optimal control problem for the parabolic systems governed by the fractional Laplacian has been studied in [11]. There the proof for the bang-bang property heavily relies on an observability from measurable sets of positive measure in time, and the latter is based on a sharp observability estimate on the eigenfunctions of the Dirichlet Laplacian due to Lebeau and Robbiano [10]. Recently, observability from measurable sets attracts intense attention and several interesting results have been established, see [1, 2, 3, 26], etc.

In this paper, we shall consider the time and norm optimal control problems for the parabolic systems associated with the time-varying fractional-order Laplacian. The model of non-autonomous diffusion under consideration does not include fractional derivatives with respect to the time variable (*cf. e.g.* [12]).

We first formulate the time optimal control problem under consideration. Let ω be a nonempty open set of Ω. Define the linear and bounded control operator $Bf = \chi_\omega f$, $\forall f \in L^2(\Omega)$. Consider the evolution control system associated with time-varying fractional Laplacian

$$\begin{cases} \frac{\mathrm{d}u(t)}{\mathrm{d}t} + A^{\alpha(t)} u(t) = \chi_{(\tau,T)} Bf, \qquad 0 < t \leq T, \\ u(0) = u_0, \end{cases} \tag{1.1}$$

where $u_0 \in L^2(\Omega)$, $f \in L^\infty(0, T; L^2(\Omega))$, and $\alpha(t)$ is a continuous function on $[0, T]$. Moreover, we assume that there exists $\alpha_0 > 1/2$ such that $\alpha(t) \geq \alpha_0$ for $t \in [0, T]$. Here and in what follows, we shall use $u(t)$ to denote the value of the function u from $[0, T]$ to the space $L^2(\Omega)$ at time t if there is no risk to make any confusion.

We shall postpone discussing the well-posedness of the equation (1.1) until Section 2. Now, we assume that (1.1) admits a unique solution u in $C([0, T]; L^2(\Omega))$. For each $M > 0$, we define

$$\mathbf{F}_{ad} = \big\{ f \in L^\infty(0, T; L^2(\Omega));\ \|f\|_{L^\infty(0,T;L^2(\Omega))} \leq M \big\},$$

and

$$\mathcal{U}^M = \big\{(\tau, f) \in [0, T) \times \mathbf{F}_{ad};\ f(t) = 0 \text{ for a.e. } t \in (0, \tau) \text{ and } u(T) = 0 \quad \text{in } \Omega\big\},$$

where u is the solution of (1.1) corresponding to f.

The first main result of this paper is stated as follows.

Theorem 1.1. *Suppose that $\mathcal{U}^M \neq \emptyset$. If there exists a pair $(\tau^*, f^*) \in \mathcal{U}^M$ such that*

$$\tau^* = \max_{(\tau,f)\in\mathcal{U}^M} \tau, \tag{1.2}$$

then f^* has the following bang-bang property:

$$\|f^*(t)\|_{L^2(\Omega)} = M, \quad \text{for a.e. } t \in (\tau^*, T). \tag{1.3}$$

Moreover, such pair (τ^*, f^*) is unique.

Next, we introduce the norm optimal control problem. Let $\tau \in [0, T)$ be fixed. Consider the control system

$$\begin{cases} \frac{\mathrm{d}u(t)}{\mathrm{d}t} + A^{\alpha(t)}u(t) = \chi_{(\tau,T)}Bf, & 0 < t \le T, \\ u(0) = u_0, \end{cases} \tag{1.4}$$

where $f \in L^\infty(0, T; L^2(\Omega))$. Write

$$\mathcal{U}_\tau = \big\{f \in L^\infty(0, T; L^2(\Omega));\ f(t) = 0 \text{ for a.e. } t \in (0, \tau) \text{ and } u(T) = 0 \quad \text{in } \Omega\big\},$$

where u is the solution of (1.4) corresponding to f.

The second main result of this paper is contained in the following theorem.

Theorem 1.2. *There exists a unique $f^* \in \mathcal{U}_\tau$ such that*

$$\|f^*\|_{L^\infty(\tau,T;L^2(\Omega))} = \min_{f\in\mathcal{U}_\tau} \|f\|_{L^\infty(\tau,T;L^2(\Omega))}. \tag{1.5}$$

Furthermore, f^* has the bang-bang property:

$$\|f^*(t)\|_{L^2(\Omega)} = \|f^*\|_{L^\infty(\tau,T;L^2(\Omega))} \quad \text{for a.e. } t \in (\tau, T). \tag{1.6}$$

Theorems 1.1 and 1.2 extend the results established in [20] for the heat equation with bounded potentials to the parabolic equation with time-varying fractional Laplacian. The bang-bang property is a crucial topic in studies of time and norm optimal control problems in finite and infinite dimensional control systems. It has other applications in the field of control theory, for instance, the uniqueness of the optimal control (see *e.g.* [4, 24, 26]), the equivalence of several different kinds of optimal control problems (see *e.g.* [25, 29, 30]). The bang-bang property for the time and norm optimal control problem for parabolic equations has been studied in [1, 5, 8, 9, 17, 19, 21, 27, 28] and references therein. The main tools used to prove the bang-bang property in those works are usually the observability inequality from measurable sets, as well as the Pontryagin maximum principle associated with a unique continuation property.

The structure of the paper is as follows. In Section 2, we first consider the well-posedness for the inhomogeneous evolution equation with time-varying fractional Laplacian. In Section 3, we establish the null controllability from measurable subsets in time with positive measure. Finally, in Section 4 we provide the proofs for Theorems 1.1 and 1.2.

2. Solutions for equations with time-varying fractional Laplacian

Let $0 \leq s < T$, $X = L^2(\Omega)$ and $f \in L^2(s,T;X)$. In this section, we only assume that $\alpha(\cdot) \in C([s,T];\mathbb{R})$ is a positive function. Consider the following inhomogeneous differential equation

$$\begin{cases} \frac{\mathrm{d}u(t)}{\mathrm{d}t} + A^{\alpha(t)}u(t) = f(t), \quad s < t \leq T, \\ u(s) = u_0 \in X, \end{cases} \tag{2.1}$$

where $A^{\alpha(t)}$ is the time-varying fractional Laplacian defined in the first section for every $t \in [s,T]$. This section is devoted to study the existence and uniqueness of solutions to (2.1). To this end, we give a definition of evolution system as follows.

Definition 2.1 (Pazy [18])**.** A two parameter family of bounded linear operators $U(t,s), 0 \leq s \leq t \leq T$, on X is called an evolution system if the following two conditions are satisfied:

(i) $U(s,s) = I, U(t,r)U(r,s) = U(t,s)$ for $0 \leq s \leq r \leq t \leq T$;
(ii) $(t,s) \longrightarrow U(t,s)$ is strongly continuous for $0 \leq s \leq t \leq T$.

Recall $\{\lambda_n\}_{n=1}^\infty$ is the set of the eigenvalues of the Dirichlet Laplacian $A = -\Delta$ defined in the first section, and $\{e_n\}_{n=1}^\infty$ is the set of the corresponding eigenfunctions. Let

$$G_n(t,s) = e^{-\int_s^t \lambda_n^{\alpha(\tau)}\mathrm{d}\tau} \quad \text{for } 0 \leq s \leq t, \tag{2.2}$$

and, then define a two parameter family of operators by

$$U(t,s)g = \sum_{n=1}^\infty c_n G_n(t,s) e_n \quad \text{for } 0 \leq s \leq t \leq T, \tag{2.3}$$

where $g = \sum_{n=1}^\infty c_n e_n \in X$.

Theorem 2.2. *$U(t,s), 0 \leq s \leq t \leq T$, defined by (2.3) is an evolution system on X.*

Proof. Let $g = \sum_{n=1}^\infty c_n e_n \in X$. For $0 \leq s \leq t \leq T$,

$$\begin{aligned} \|U(t,s)g\|^2 &= \sum_{n=1}^\infty c_n^2 G_n^2(t,s) \\ &= \sum_{n=1}^\infty c_n^2 e^{-2\int_s^t \lambda_n^{\alpha(\tau)}\mathrm{d}\tau} \\ &\leq \sum_{n=1}^\infty c_n^2 = \|g\|^2, \end{aligned}$$

where we use the fact that

$$-2\int_s^t \lambda_n^{\alpha(\tau)}\mathrm{d}\tau \leq 0, \quad \text{for } \lambda_n > 0, \alpha(t) > 0.$$

Thus, we get that for any $0 \le s \le t \le T$, $U(t,s)$ is a bounded linear operator on X and

$$\|U(t,s)\| \le 1. \tag{2.4}$$

By (2.2) and (2.3), we can see easily that $U(s,s) = I$. Moreover, when $0 \le s \le r \le t \le T$,

$$U(t,r)U(r,s)g = U(t,r)\sum_{n=1}^{\infty} c_n G_n(r,s)e_n = \sum_{n=1}^{\infty} c_n G_n(t,r)G_n(r,s)e_n. \tag{2.5}$$

Since

$$G_n(t,r)G_n(r,s) = e^{-\int_r^t \lambda_n^{\alpha(\tau)}\mathrm{d}\tau} e^{-\int_s^r \lambda_n^{\alpha(\tau)}\mathrm{d}\tau} = e^{-\int_s^t \lambda_n^{\alpha(\tau)}\mathrm{d}\tau},$$

we have

$$G_n(t,r)G_n(r,s) = G_n(t,s). \tag{2.6}$$

Thus, combining (2.5) and (2.6) yields that

$$U(t,r)U(r,s)g = \sum_{n=1}^{\infty} c_n G_n(t,s)e_n = U(t,s)g,$$

which implies that

$$U(t,r)U(r,s) = U(t,s), \quad \text{for } 0 \le s \le r \le t \le T.$$

Finally, the strong continuity of $U(t,s)$ for $0 \le s \le t \le T$ is not difficult to verify. Thus, we have shown that $U(t,s)$ is an evolution system on X. $\square$

Theorem 2.3. *The evolution system $U(t,s), 0 \le s \le t \le T$, defined by (2.3) on X satisfies the following statements:*

(i) $\|U(t,s)\| \le 1$ *for* $0 \le s \le t \le T$;
(ii) $U(t,s) : X \longrightarrow D(A^{\alpha(t)})$ *for* $0 \le s < t \le T$;
(iii) For $0 \le s < t \le T$, $t \longrightarrow U(t,s)$ *is strongly continuously differentiable in* X *and* $\frac{\partial}{\partial t}U(t,s) \in \mathcal{L}(X)$;
(iv) For any $g \in X$,

$$\frac{\partial}{\partial t}U(t,s)g + A^{\alpha(t)}U(t,s)g = 0, \quad 0 \le s < t \le T.$$

(v) For any $g \in D(A^{\alpha(s)})$,

$$\frac{\partial}{\partial s}U(t,s)g - U(t,s)A^{\alpha(s)}g = 0, \quad 0 \le s < t \le T.$$

Proof. The first statement follows from (2.4) directly. For any $g \in X$ with $g = \sum_{n=1}^{\infty} c_n e_n$, we have

$$U(t,s)g = \sum_{n=1}^{\infty} c_n G_n(t,s)e_n.$$

For any given s,t, satisfying $0\le s<t\le T$, there exists $\tau\in(s,t)$ such that

$$\sum_{n=1}^{\infty}\lambda_n^{2\alpha(t)}c_n^2G_n^2(t,s)=\sum_{n=1}^{\infty}\lambda_n^{2\alpha(t)}c_n^2e^{-2\int_s^t\lambda_n^{\alpha(\sigma)}\mathrm{d}\sigma}=\sum_{n=1}^{\infty}\lambda_n^{2\alpha(t)}c_n^2e^{-2\lambda_n^{\alpha(\tau)}(t-s)}. \tag{2.7}$$

For $\lambda_n<1$, we have

$$\lambda_n^{2\alpha(t)}e^{-2\lambda_n^{\alpha(\tau)}(t-s)}\le 1. \tag{2.8}$$

For $\lambda_n=1$, we have

$$\lambda_n^{2\alpha(t)}e^{-2\lambda_n^{\alpha(\tau)}(t-s)}=e^{-2(t-s)}. \tag{2.9}$$

For $\lambda_n>1$, we can choose a constant $c\ge 1$, which is independent of n, such that

$$\lambda_n^{2\alpha(t)}e^{-2\lambda_n^{\alpha(\tau)}(t-s)}\le c.$$

This, together with (2.7), (2.8) and (2.9), yields that

$$\sum_{n=1}^{\infty}\lambda_n^{2\alpha(t)}c_n^2G_n^2(t,s)\le c\sum_{n=1}^{\infty}c_n^2=c\|g\|^2, \tag{2.10}$$

for any given $0\le s<t\le T$, which implies that $U(t,s)g\in D(A^{\alpha(t)})$. Hence we complete the proof of (ii).

By some simple calculations, we can get

$$\frac{\partial}{\partial t}U(t,s)g=\sum_{n=1}^{\infty}c_n\frac{\partial}{\partial t}G_n(t,s)e_n=-\sum_{n=1}^{\infty}\lambda_n^{\alpha(t)}c_nG_n(t,s)e_n, \tag{2.11}$$

is continuous for $0\le s<t\le T$. Moreover, by (2.10), we have

$$\|\frac{\partial}{\partial t}U(t,s)g\|^2=\sum_{n=1}^{\infty}\lambda_n^{2\alpha(t)}c_n^2G_n^2(t,s)\le c\|g\|^2,$$

which implies that $\frac{\partial}{\partial t}U(t,s)\in\mathcal{L}(X)$ for $0\le s<t\le T$. Thus, we get (iii).

By making use of the definition of the fractional Laplacian $A^{\alpha(t)}$ and (2.11), we have for any $g\in X$ and $0\le s<t\le T$,

$$A^{\alpha(t)}U(t,s)g=A^{\alpha(t)}\sum_{n=1}^{\infty}c_nG_n(t,s)e_n=\sum_{n=1}^{\infty}\lambda_n^{\alpha(t)}c_nG_n(t,s)e_n=-\frac{\partial}{\partial t}U(t,s)g,$$

from which (iv) follows. Similarly, for any $g\in D(A^{\alpha(s)})$ and $0\le s<t\le T$,

$$U(t,s)A^{\alpha(s)}g=U(t,s)\sum_{n=1}^{\infty}c_n\lambda_n^{\alpha(s)}e_n=\sum_{n=1}^{\infty}\lambda_n^{\alpha(s)}c_nG_n(t,s)e_n=\frac{\partial}{\partial s}U(t,s)g,$$

which implies that (v) holds. Thus, the proof of this theorem is completed. □

Definition 2.4. A X-valued function $u : [s,T] \longrightarrow X$ is a (classical) solution of (2.1) if $u \in C([s,T];X) \cap C^1((s,T];X)$, $u(t) \in D(A^{\alpha(t)})$ for $0 \le s < t \le T$ and satisfies (2.1).

Corollary 2.5. *For any $u_0 \in X$, the inhomogeneous differential equation (2.1) has at most one solution. If it has a solution, this solution is given by*

$$u(t) = U(t,s)u_0 + \int_s^t U(t,\sigma)f(\sigma)\mathrm{d}\sigma, \quad \textit{for } s \le t \le T. \tag{2.12}$$

Proof. Assume that u is the solution of (2.1). Then the X-valued function $\sigma \longrightarrow U(t,\sigma)u(\sigma)$ is differentiable on $(s,T]$. Moreover, by Definition 2.4 and the property (v) of Theorem 2.3,

$$\begin{aligned}\frac{\partial}{\partial\sigma}(U(t,\sigma)u(\sigma)) &= U(t,\sigma)A^{\alpha(\sigma)}u(\sigma) + U(t,\sigma)(f(\sigma) - A^{\alpha(\sigma)}u(\sigma)) \\ &= U(t,\sigma)f(\sigma).\end{aligned} \tag{2.13}$$

Integrating (2.13) from s to t yields (2.12), which implies that the proof is completed. □

Remark 2.6. By Theorem 2.2, we see obviously that $U(t,s)u_0$ is the classical solution of the corresponding homogeneous equation, *i.e.*, the equation (2.1) with $f \equiv 0$. Moreover, by Corollary 2.5, the solution $U(t,s)u_0$ is unique.

Motivated by Corollary 2.5, it is natural to consider the formula (2.12) as a generalized solution of (2.1) even if it is not differentiable and does not strictly satisfy the equation in the sense of Definition 2.4 for every $u_0 \in X, f \in L^2(s,T;X)$. Note that for each $u_0 \in X$, $f \in L^2(s,T;X)$ and evolution system $U(\cdot,\cdot)$, the formula on the right hand side of (2.12) gives a continuous function in X. Therefore, we can give the following definition.

Definition 2.7. Let $u_0 \in X$ and $f \in L^2(s,T;X)$. The function $u \in C([s,T];X)$ is called a mild solution of the inhomogeneous differential equation (2.1) on $[s,T]$ if

$$u(t) = U(t,s)u_0 + \int_s^t U(t,\sigma)f(\sigma)\mathrm{d}\sigma, \quad \text{for } s \le t \le T.$$

Remark 2.8. By the similar methods in [18], the mild solution u defined by (2.12) can be proved to be a classical solution in the sense of Definition 2.4, provided that some additional conditions on f and u_0 are imposed.

Remark 2.9. By the above discussion, we know that the controlled equations (1.1) and (1.4) admit a unique mild solution $u \in C([0,T];L^2(\Omega))$ given by

$$u(t) = U(t,0)u_0 + \int_0^t U(t,\sigma)\chi_{(\tau,T)}Bf(\sigma)\mathrm{d}\sigma$$

for each $u_0 \in L^2(\Omega), f \in L^\infty(0,T;L^2(\Omega))$.

3. Observation from measurable sets in time

Throughout the rest of this paper, we assume that $T > 0$, $\alpha(\cdot) \in C([0,T];\mathbb{R})$ and there exists $\alpha_0 > \frac{1}{2}$ such that $\alpha(t) \geq \alpha_0$ for all $t \in [0,T]$. Let $u(\cdot) \in C([0,T]; L^2(\Omega))$ be the unique mild solution to the homogenous equation

$$\begin{cases} \frac{\mathrm{d}u(t)}{\mathrm{d}t} + A^{\alpha(t)}u(t) = 0, \quad 0 < t \leq T, \\ u(0) = u_0. \end{cases} \tag{3.1}$$

First of all, by (i) of Theorem 2.3, we have the following estimate for the solution to the equation (3.1).

Lemma 3.1. *For any $u_0 \in L^2(\Omega)$ and $0 \leq s \leq t$,*

$$\|u(t)\|_{L^2(\Omega)} \leq \|u(s)\|_{L^2(\Omega)}, \tag{3.2}$$

where u is the solution of (3.1).

We next establish an interpolation estimate at one point in time.

Theorem 3.2. *For any $\theta \in (0,1)$, there exists $N = N(\Omega, \omega, \alpha_0, \theta) > 0$ such that*

$$\|u(t)\|_{L^2(\Omega)} \leq \left(N e^{N(t-s)^{\frac{1}{1-2\alpha_0}}} \|u(t)\|_{L^2(\omega)} \right)^{1-\theta} \|u(s)\|^{\theta}_{L^2(\Omega)}, \tag{3.3}$$

where $0 \leq s < t \leq T, t - s \leq 1$.

Proof. Let $\lambda \geq 1$, which will be determined later. Let $u_0 = \sum_{n=1}^{\infty} c_n e_n$ be the initial data of (3.1) and $a_n(t) = c_n G_n(t,0)$. For any $t > 0$, we have

$$u(t) = \sum_{\lambda_n^{\alpha_0} \leq \lambda} a_n(t) e_n + \sum_{\lambda_n^{\alpha_0} > \lambda} a_n(t) e_n, \tag{3.4}$$

where by the spectral inequality (see [1, 3, 10]), there exists a positive constant N_1, depending only on Ω and ω, such that

$$\begin{aligned} \Big\| \sum_{\lambda_n^{\alpha_0} \leq \lambda} a_n(t) e_n \Big\|_{L^2(\Omega)} &= \Big\| \sum_{\lambda_n \leq \lambda^{\frac{1}{\alpha_0}}} a_n(t) e_n \Big\|_{L^2(\Omega)} \\ &\leq N_1 e^{N_1 \lambda^{\frac{1}{2\alpha_0}}} \Big\| \sum_{\lambda_n \leq \lambda^{\frac{1}{\alpha_0}}} a_n(t) e_n \Big\|_{L^2(\omega)} \\ &= N_1 e^{N_1 \lambda^{\frac{1}{2\alpha_0}}} \Big\| u(t) - \sum_{\lambda_n^{\alpha_0} > \lambda} a_n(t) e_n \Big\|_{L^2(\omega)} \\ &\leq N_1 e^{N_1 \lambda^{\frac{1}{2\alpha_0}}} \left(\|u(t)\|_{L^2(\omega)} + \Big\| \sum_{\lambda_n^{\alpha_0} > \lambda} a_n(t) e_n \Big\|_{L^2(\Omega)} \right). \end{aligned}$$

This, together with (3.4), indicates that

$$
\begin{aligned}
\|u(t)\|_{L^2(\Omega)} &\le \Big\| \sum_{\lambda_n^{\alpha_0} \le \lambda} a_n(t) e_n \Big\|_{L^2(\Omega)} + \Big\| \sum_{\lambda_n^{\alpha_0} > \lambda} a_n(t) e_n \Big\|_{L^2(\Omega)} \\
&\le N_1 e^{N_1 \lambda^{\frac{1}{2\alpha_0}}} \left(\|u(t)\|_{L^2(\omega)} + \Big\| \sum_{\lambda_n^{\alpha_0} > \lambda} a_n(t) e_n \Big\|_{L^2(\Omega)} \right) + \Big\| \sum_{\lambda_n^{\alpha_0} > \lambda} a_n(t) e_n \Big\|_{L^2(\Omega)} \\
&\le (N_1 + 1) e^{N_1 \lambda^{\frac{1}{2\alpha_0}}} \left(\|u(t)\|_{L^2(\omega)} + \Big\| \sum_{\lambda_n^{\alpha_0} > \lambda} a_n(t) e_n \Big\|_{L^2(\Omega)} \right). \qquad (3.5)
\end{aligned}
$$

In the following, we will give an estimate of the last term of (3.5). For any λ_n satisfying $\lambda_n^{\alpha_0} > \lambda$, it follows from $\lambda \ge 1$ that $\lambda_n > 1$. Thus $\lambda_n^{\alpha(\tau)} \ge \lambda_n^{\alpha_0}$ for $\tau \in [0, T]$, which implies that

$$
G_n(t,s) = e^{-\int_s^t \lambda_n^{\alpha(\tau)} \mathrm{d}\tau} \le e^{-\lambda_n^{\alpha_0}(t-s)} \le e^{-\lambda(t-s)} \quad \text{for } \lambda_n^{\alpha_0} > \lambda. \qquad (3.6)
$$

Hence, by (3.6),

$$
\begin{aligned}
\Big\| \sum_{\lambda_n^{\alpha_0} > \lambda} a_n(t) e_n \Big\|^2_{L^2(\Omega)} &= \sum_{\lambda_n^{\alpha_0} > \lambda} c_n^2 G_n^2(t,0) \\
&= \sum_{\lambda_n^{\alpha_0} > \lambda} c_n^2 G_n^2(t,s) G_n^2(s,0) \\
&\le e^{-2\lambda(t-s)} \sum_{\lambda_n^{\alpha_0} > \lambda} c_n^2 G_n^2(s,0) \\
&\le e^{-2\lambda(t-s)} \sum_{n=1}^{\infty} c_n^2 G_n^2(s,0) \\
&= e^{-2\lambda(t-s)} \|u(s)\|^2_{L^2(\Omega)}.
\end{aligned}
$$

For any $\theta \in (0,1)$, combining this with (3.5) yields

$$
\begin{aligned}
\|u(t)\|_{L^2(\Omega)} &\le (N_1+1) e^{N_1 \lambda^{\frac{1}{2\alpha_0}}} \left(\|u(t)\|_{L^2(\omega)} + e^{-\lambda(t-s)} \|u(s)\|_{L^2(\Omega)} \right) \\
&= (N_1+1) e^{N_1 \lambda^{\frac{1}{2\alpha_0}} - \theta\lambda(t-s)} \left(e^{\theta\lambda(t-s)} \|u(t)\|_{L^2(\omega)} + e^{-(1-\theta)\lambda(t-s)} \|u(s)\|_{L^2(\Omega)} \right). \qquad (3.7)
\end{aligned}
$$

Let

$$
f(\lambda) = N_1 \lambda^{\frac{1}{2\alpha_0}} - \theta\lambda(t-s).
$$

Noting that $\alpha_0 > \frac{1}{2}$, we have

$$
f^{''}(\lambda) = \frac{N_1(1-2\alpha_0)}{4\alpha_0^2} \lambda^{\frac{1-4\alpha_0}{2\alpha_0}} < 0,
$$

which implies that f is a strictly concave function. Let λ_0 be the unique maximum point of f. Thus, by

$$f^{'}(\lambda_0) = \frac{N_1}{2\alpha_0}\lambda_0^{\frac{1-2\alpha_0}{2\alpha_0}} - \theta(t-s) = 0,$$

we can obtain that $\lambda_0 = (\frac{2\alpha_0\theta(t-s)}{N_1})^{\frac{2\alpha_0}{1-2\alpha_0}}$. Then for any $\lambda \geq 1$,

$$\max_{\lambda\geq 1} f(\lambda) \leq N_1^{\frac{-2\alpha_0}{1-2\alpha_0}}\theta^{\frac{1}{1-2\alpha_0}}\left((2\alpha_0)^{\frac{1}{1-2\alpha_0}} - (2\alpha_0)^{\frac{2\alpha_0}{1-2\alpha_0}}\right)(t-s)^{\frac{1}{1-2\alpha_0}}. \tag{3.8}$$

Here, again due to $\alpha_0 > \frac{1}{2}$, we have $(2\alpha_0)^{\frac{1}{1-2\alpha_0}} - (2\alpha_0)^{\frac{2\alpha_0}{1-2\alpha_0}} > 0$. Combining (3.7) and (3.8), we can see that

$$\begin{aligned}\|u(t)\|_{L^2(\Omega)} &\leq (N_1+1)e^{N_1^{\frac{-2\alpha_0}{1-2\alpha_0}}\theta^{\frac{1}{1-2\alpha_0}}\left((2\alpha_0)^{\frac{1}{1-2\alpha_0}} - (2\alpha_0)^{\frac{2\alpha_0}{1-2\alpha_0}}\right)(t-s)^{\frac{1}{1-2\alpha_0}}} \\ &\quad\times\left(e^{\theta\lambda(t-s)}\|u(t)\|_{L^2(\omega)} + e^{-(1-\theta)\lambda(t-s)}\|u(s)\|_{L^2(\Omega)}\right) \\ &\leq N_2 e^{N_2(t-s)^{\frac{1}{1-2\alpha_0}}}\left(e^{\theta\lambda(t-s)}\|u(t)\|_{L^2(\omega)} + e^{-(1-\theta)\lambda(t-s)}\|u(s)\|_{L^2(\Omega)}\right),\end{aligned} \tag{3.9}$$

where

$$N_2 \geq \max\{N_1+1, N_1^{\frac{-2\alpha_0}{1-2\alpha_0}}\theta^{\frac{1}{1-2\alpha_0}}\left((2\alpha_0)^{\frac{1}{1-2\alpha_0}} - (2\alpha_0)^{\frac{2\alpha_0}{1-2\alpha_0}}\right)\} > 1.$$

By Lemma 3.1,

$$\|u(t)\|_{L^2(\Omega)} \leq \|u(s)\|_{L^2(\Omega)}, \quad \forall\ 0 \leq s < t.$$

Then

$$\|u(t)\|_{L^2(\omega)} \leq \|u(t)\|_{L^2(\Omega)} \leq \|u(s)\|_{L^2(\Omega)},$$

which implies that

$$\frac{\|u(s)\|_{L^2(\Omega)}}{\|u(t)\|_{L^2(\omega)}} \geq 1 \quad \text{and} \quad \ln\frac{\|u(s)\|_{L^2(\Omega)}}{\|u(t)\|_{L^2(\omega)}} \geq 0.$$

By taking

$$\lambda = \frac{\ln\frac{\|u(s)\|_{L^2(\Omega)}}{\|u(t)\|_{L^2(\omega)}}}{t-s} + N_2(t-s)^{\frac{2\alpha_0}{1-2\alpha_0}},$$

where $\alpha_0 > \frac{1}{2}$, $N_2 > 1$ and $t-s \leq 1$, we have $\lambda \geq 1$. Moreover, we see that

$$e^{\theta\lambda(t-s)} = e^{\theta N_2(t-s)^{\frac{1}{1-2\alpha_0}}}\left(\frac{\|u(s)\|_{L^2(\Omega)}}{\|u(t)\|_{L^2(\omega)}}\right)^{\theta}$$

and

$$e^{-(1-\theta)\lambda(t-s)} = e^{-(1-\theta)N_2(t-s)^{\frac{1}{1-2\alpha_0}}} \left(\frac{\|u(t)\|_{L^2(\omega)}}{\|u(s)\|_{L^2(\Omega)}}\right)^{1-\theta}.$$

Hence, by (3.9),

$$\begin{aligned}
\|u(t)\|_{L^2(\Omega)} \leq\ & N_2 e^{N_2(t-s)^{\frac{1}{1-2\alpha_0}}} \Bigg[e^{\theta N_2(t-s)^{\frac{1}{1-2\alpha_0}}} \left(\frac{\|u(s)\|_{L^2(\Omega)}}{\|u(t)\|_{L^2(\omega)}}\right)^{\theta} \|u(t)\|_{L^2(\omega)} \\
& + e^{-(1-\theta)N_2(t-s)^{\frac{1}{1-2\alpha_0}}} \left(\frac{\|u(t)\|_{L^2(\omega)}}{\|u(s)\|_{L^2(\Omega)}}\right)^{1-\theta} \|u(s)\|_{L^2(\Omega)} \Bigg] \\
=\ & N_2 e^{(1+\theta)N_2(t-s)^{\frac{1}{1-2\alpha_0}}} \|u(s)\|^{\theta}_{L^2(\Omega)} \|u(t)\|^{1-\theta}_{L^2(\omega)} \\
& + N_2 e^{\theta N_2(t-s)^{\frac{1}{1-2\alpha_0}}} \|u(s)\|^{\theta}_{L^2(\Omega)} \|u(t)\|^{1-\theta}_{L^2(\omega)} \\
\leq\ & 2N_2 e^{(1+\theta)N_2(t-s)^{\frac{1}{1-2\alpha_0}}} \|u(s)\|^{\theta}_{L^2(\Omega)} \|u(t)\|^{1-\theta}_{L^2(\omega)} \\
\leq\ & \left(N e^{N(t-s)^{\frac{1}{1-2\alpha_0}}} \|u(t)\|_{L^2(\omega)}\right)^{1-\theta} \|u(s)\|^{\theta}_{L^2(\Omega)},
\end{aligned}$$

where $N \geq \max\{(2N_2)^{\frac{1}{1-\theta}}, \frac{1+\theta}{1-\theta}N_2\}$. Thus, we complete the proof of this theorem. □

Remark 3.3. In Theorem 3.2, the assumption $t-s \leq 1$ can be dropped. In fact, if $t-s>1$, it follows from (3.3) that

$$\begin{aligned}
\|u(t)\|_{L^2(\Omega)} &\leq \left(N e^{N(t-(t-1))^{\frac{1}{1-2\alpha_0}}} \|u(t)\|_{L^2(\omega)}\right)^{1-\theta} \|u(t-1)\|^{\theta}_{L^2(\Omega)} \\
&= \left(N e^{N} \|u(t)\|_{L^2(\omega)}\right)^{1-\theta} \|u(t-1)\|^{\theta}_{L^2(\Omega)}.
\end{aligned}$$

Noting $s < t-1$, by Lemma 3.1, we have

$$\begin{aligned}
\|u(t)\|_{L^2(\Omega)} &\leq \left(N e^{N} \|u(t)\|_{L^2(\omega)}\right)^{1-\theta} \|u(s)\|^{\theta}_{L^2(\Omega)} \\
&\leq \left(N e^{N} e^{N(t-s)^{\frac{1}{1-2\alpha_0}}} \|u(t)\|_{L^2(\omega)}\right)^{1-\theta} \|u(s)\|^{\theta}_{L^2(\Omega)}.
\end{aligned}$$

For each measurable set $E \subset \mathbb{R}$, we denote by $|E|$ the Lebesgue measure of E in $\mathbb{R}$. The following useful lemma is quoted from [20].

Lemma 3.4. *Let $E \subset (0,T)$ be a measurable set with a positive measure, and let l be a density point for E. Then for each $z>1$, there exists $l_1 \in (l,T)$ (depending on z and E), such that the sequence $\{l_m\}_{m\geq 1}$ given by*

$$l_{m+1} = l + z^{-m}(l_1 - l), \quad m = 1, 2, ...$$

satisfies

$$|E \cap (l_{m+1}, l_m)| \geq \frac{l_m - l_{m+1}}{3}, \quad \forall m \geq 1.$$

Now we obtain the following observability inequality from a measurable set in time.

Theorem 3.5. *Let $E \subset (0,T)$ be a measurable set with a positive measure. Then there is a positive constant $C = C(\Omega, \omega, \alpha_0, E)$ such that*

$$\|u(T)\|_{L^2(\Omega)} \leq C \int_E \|u(t)\|_{L^2(\omega)} \mathrm{d}t, \quad \forall\, u_0 \in L^2(\Omega). \tag{3.10}$$

Proof. Let $l > 0$ be a density point of E and $q \in (0,1)$ be a constant, which will be fixed later. By lemma 3.4, there exists a decreasing sequence $\{l_m\}_{m\geq 1}$ satisfying

$$\lim_{m\to\infty} l_m = l,$$

$$|E \cap (l_{m+1}, l_m)| \geq \frac{l_m - l_{m+1}}{3}, \ \forall m \geq 1 \tag{3.11}$$

and

$$\frac{l_m - l_{m+1}}{l_{m+1} - l_{m+2}} = \frac{1}{q}, \ m \in \mathbb{N} \tag{3.12}$$

Obviously, we can choose a positive integer m_0 such that

$$0 < l_m - l_{m+1} \leq 1, \ \forall m \geq m_0. \tag{3.13}$$

In the following, we assume that $m \geq m_0$. Define

$$\tau_m = l_{m+1} + \frac{l_m - l_{m+1}}{6}.$$

Thus, by (3.11), we have

$$\begin{aligned} |E \cap (\tau_m, l_m)| &= |E \cap ((l_{m+1}, l_m) \setminus (l_{m+1}, \tau_m))| \\ &\geq |E \cap (l_{m+1}, l_m)| - |(l_{m+1}, \tau_m)| \\ &\geq \frac{l_m - l_{m+1}}{3} - \frac{l_m - l_{m+1}}{6} = \frac{l_m - l_{m+1}}{6} \end{aligned} \tag{3.14}$$

and by (3.2), we have

$$\|u(l_m)\| \leq \|u(t)\|, \quad \forall t \in [\tau_m, l_m). \tag{3.15}$$

Moreover, by (3.3) and (3.13), for $m \geq m_0$ and $\theta = \frac{1}{2}$,

$$\begin{aligned}\|u(t)\|_{L^2(\Omega)} &\leq \left(Ne^{N(t-l_{m+1})^{\frac{1}{1-2\alpha_0}}}\|u(t)\|_{L^2(\omega)}\right)^{\frac{1}{2}}\|u(l_{m+1})\|_{L^2(\Omega)}^{\frac{1}{2}}\\ &\leq \left(Ne^{N(\tau_m-l_{m+1})^{\frac{1}{1-2\alpha_0}}}\|u(t)\|_{L^2(\omega)}\right)^{\frac{1}{2}}\|u(l_{m+1})\|_{L^2(\Omega)}^{\frac{1}{2}}\\ &= \left(Ne^{N\left(\frac{l_m-l_{m+1}}{6}\right)^{\frac{1}{1-2\alpha_0}}}\|u(t)\|_{L^2(\omega)}\right)^{\frac{1}{2}}\|u(l_{m+1})\|_{L^2(\Omega)}^{\frac{1}{2}}, \quad \forall t \in [\tau_m, l_m).\end{aligned}$$

It follows from the Young inequality that

$$\|u(t)\|_{L^2(\Omega)} \leq \epsilon\|u(l_{m+1})\|_{L^2(\Omega)} + \epsilon^{-1}Ne^{N\left(\frac{l_m-l_{m+1}}{6}\right)^{\frac{1}{1-2\alpha_0}}}\|u(t)\|_{L^2(\omega)}, \quad \forall \epsilon > 0, t \in [\tau_m, l_m).$$

Integrating the inequality over $E \cap (\tau_m, l_m)$, by (3.15), we have

$$\begin{aligned}|E \cap (\tau_m, l_m)|\|u(l_m)\|_{L^2(\Omega)} &\leq \int_{\tau_m}^{l_m} \chi_E\|u(t)\|_{L^2(\Omega)}\mathrm{d}t\\ &\leq \epsilon|E \cap (\tau_m, l_m)|\|u(l_{m+1})\|_{L^2(\Omega)} + \epsilon^{-1}Ne^{N\left(\frac{l_m-l_{m+1}}{6}\right)^{\frac{1}{1-2\alpha_0}}}\int_{\tau_m}^{l_m}\chi_E\|u(t)\|_{L^2(\omega)}\mathrm{d}t\\ &\leq \epsilon|E \cap (\tau_m, l_m)|\|u(l_{m+1})\|_{L^2(\Omega)} + \epsilon^{-1}Ne^{N\left(\frac{l_m-l_{m+1}}{6}\right)^{\frac{1}{1-2\alpha_0}}}\int_{l_{m+1}}^{l_m}\chi_E\|u(t)\|_{L^2(\omega)}\mathrm{d}t,\end{aligned}$$

which, along with (3.14), deduces that

$$\begin{aligned}\|u(l_m)\|_{L^2(\Omega)} &\leq \epsilon\|u(l_{m+1})\|_{L^2(\Omega)} + \epsilon^{-1}\frac{N}{|E \cap (\tau_m, l_m)|}e^{N\left(\frac{l_m-l_{m+1}}{6}\right)^{\frac{1}{1-2\alpha_0}}}\int_{\tau_m}^{l_m}\chi_E\|u(t)\|_{L^2(\omega)}\mathrm{d}t\\ &\leq \epsilon\|u(l_{m+1})\|_{L^2(\Omega)} + \epsilon^{-1}\frac{6N}{l_m-l_{m+1}}e^{N\left(\frac{l_m-l_{m+1}}{6}\right)^{\frac{1}{1-2\alpha_0}}}\int_{l_{m+1}}^{l_m}\chi_E\|u(t)\|_{L^2(\omega)}\mathrm{d}t. \quad (3.16)\end{aligned}$$

In the following, we will carry out the remaining proof by two cases.
Case 1. We first consider the case for $\frac{1}{2} < \alpha_0 < 1$.

In this case, $\frac{1}{2\alpha_0-1} > 1$. Since $\frac{6}{l_m-l_{m+1}} > 1$ by (3.13), we have

$$\frac{6}{l_m-l_{m+1}} \leq \left(\frac{6}{l_m-l_{m+1}}\right)^{\frac{1}{2\alpha_0-1}}.$$

Therefore

$$\begin{aligned}\frac{6N}{l_m-l_{m+1}}e^{N\left(\frac{l_m-l_{m+1}}{6}\right)^{\frac{1}{1-2\alpha_0}}} &\leq N\left(\frac{6}{l_m-l_{m+1}}\right)^{\frac{1}{2\alpha_0-1}}e^{N\left(\frac{6}{l_m-l_{m+1}}\right)^{\frac{1}{2\alpha_0-1}}}\\ &\leq e^{2N\left(\frac{6}{l_m-l_{m+1}}\right)^{\frac{1}{2\alpha_0-1}}},\end{aligned}$$

where we used $e^{2x} \geq xe^x$ for $x \geq 0$. This, along with (3.16), indicates that

$$\|u(l_m)\|_{L^2(\Omega)} \leq \epsilon \|u(l_{m+1})\|_{L^2(\Omega)} + \epsilon^{-1} e^{2N\left(\frac{6}{l_m - l_{m+1}}\right)^{\frac{1}{2\alpha_0 - 1}}} \int_{l_{m+1}}^{l_m} \chi_E \|u(t)\|_{L^2(\omega)} \mathrm{d}t.$$

Multiplying the above inequality by $\epsilon e^{-2N\left(\frac{6}{l_m - l_{m+1}}\right)^{\frac{1}{2\alpha_0 - 1}}}$, we obtain that

$$\begin{aligned} \epsilon e^{-2N\left(\frac{6}{l_m - l_{m+1}}\right)^{\frac{1}{2\alpha_0 - 1}}} \|u(l_m)\|_{L^2(\Omega)} &\leq \epsilon^2 e^{-2N\left(\frac{6}{l_m - l_{m+1}}\right)^{\frac{1}{2\alpha_0 - 1}}} \|u(l_{m+1})\|_{L^2(\Omega)} \\ &\quad + \int_{l_{m+1}}^{l_m} \chi_E \|u(t)\|_{L^2(\omega)} \mathrm{d}t. \end{aligned}$$

Taking $\epsilon = e^{-\left(\frac{1}{l_m - l_{m+1}}\right)^{\frac{1}{2\alpha_0 - 1}}}$ in the above inequality, we have

$$\begin{aligned} &e^{\left(\frac{1}{l_m - l_{m+1}}\right)^{\frac{1}{2\alpha_0 - 1}} \left(-2N6^{\frac{1}{2\alpha_0 - 1}} - 1\right)} \|u(l_m)\|_{L^2(\Omega)} - e^{\left(\frac{1}{l_m - l_{m+1}}\right)^{\frac{1}{2\alpha_0 - 1}} \left(-2N6^{\frac{1}{2\alpha_0 - 1}} - 2\right)} \|u(l_{m+1})\|_{L^2(\Omega)} \\ &\leq \int_{l_{m+1}}^{l_m} \chi_E \|u(t)\|_{L^2(\omega)} \mathrm{d}t. \end{aligned} \tag{3.17}$$

Now, we choose $\left(\frac{2N6^{\frac{1}{2\alpha_0 - 1}} + 1}{2N6^{\frac{1}{2\alpha_0 - 1}} + 2}\right)^{2\alpha_0 - 1} = q$. It is easy to see that $q \in (0, 1)$. Moreover, by (3.12),

$$\begin{aligned} \left(\frac{1}{l_m - l_{m+1}}\right)^{\frac{1}{2\alpha_0 - 1}} \left(-2N6^{\frac{1}{2\alpha_0 - 1}} - 2\right) &= \frac{-2N6^{\frac{1}{2\alpha_0 - 1}} - 2}{2N6^{\frac{1}{2\alpha_0 - 1}} + 1} \frac{2N6^{\frac{1}{2\alpha_0 - 1}} + 1}{(l_m - l_{m+1})^{\frac{1}{2\alpha_0 - 1}}} \\ &= -\left(\frac{1}{q}\right)^{\frac{1}{2\alpha_0 - 1}} \frac{2N6^{\frac{1}{2\alpha_0 - 1}} + 1}{(l_m - l_{m+1})^{\frac{1}{2\alpha_0 - 1}}} \\ &= -\left(\frac{l_m - l_{m+1}}{l_{m+1} - l_{m+2}}\right)^{\frac{1}{2\alpha_0 - 1}} \frac{2N6^{\frac{1}{2\alpha_0 - 1}} + 1}{(l_m - l_{m+1})^{\frac{1}{2\alpha_0 - 1}}} \\ &= \left(\frac{1}{l_{m+1} - l_{m+2}}\right)^{\frac{1}{2\alpha_0 - 1}} \left(-2N6^{\frac{1}{2\alpha_0 - 1}} - 1\right), \end{aligned}$$

which, combined with (3.17), derives that

$$\begin{aligned} &e^{\left(\frac{1}{l_m - l_{m+1}}\right)^{\frac{1}{2\alpha_0 - 1}} \left(-2N6^{\frac{1}{2\alpha_0 - 1}} - 1\right)} \|u(l_m)\|_{L^2(\Omega)} - e^{\left(\frac{1}{l_{m+1} - l_{m+2}}\right)^{\frac{1}{2\alpha_0 - 1}} \left(-2N6^{\frac{1}{2\alpha_0 - 1}} - 1\right)} \|u(l_{m+1})\|_{L^2(\Omega)} \\ &\leq \int_{l_{m+1}}^{l_m} \chi_E \|u(t)\|_{L^2(\omega)} \mathrm{d}t. \end{aligned}$$

Summing the above inequality from $m = m_0$ to $+\infty$, we have

$$e^{\left(\frac{1}{l_{m_0}-l_{m_0+1}}\right)^{\frac{1}{2\alpha_0-1}}\left(-2N6^{\frac{1}{2\alpha_0-1}}-1\right)}\|u(l_{m_0})\|_{L^2(\Omega)} - \lim_{m\to\infty} e^{\left(\frac{1}{l_{m+1}-l_{m+2}}\right)^{\frac{1}{2\alpha_0-1}}\left(-2N6^{\frac{1}{2\alpha_0-1}}-1\right)}\|u(l_{m+1})\|_{L^2(\Omega)}$$
$$\leq \int_0^T \chi_E \|u(t)\|_{L^2(\omega)} \mathrm{d}t. \tag{3.18}$$

Since by (3.2)

$$\sup_{t\in[0,T]} \|u(t)\|_{L^2(\Omega)} \leq \|u_0\|_{L^2(\Omega)} < \infty,$$

and

$$\lim_{m\to\infty} e^{\left(\frac{1}{l_{m+1}-l_{m+2}}\right)^{\frac{1}{2\alpha_0-1}}\left(-2N6^{\frac{1}{2\alpha_0-1}}-1\right)} = 0,$$

we obtain that

$$\lim_{m\to\infty} e^{\left(\frac{1}{l_{m+1}-l_{m+2}}\right)^{\frac{1}{2\alpha_0-1}}\left(-2N6^{\frac{1}{2\alpha_0-1}}-1\right)}\|u(l_{m+1})\|_{L^2(\Omega)} = 0. \tag{3.19}$$

Moreover, by (3.2)

$$\|u(T)\| \leq \|u(l_{m_0})\|.$$

This, together with (3.18) and (3.19), implies that

$$\|u(T)\|_{L^2(\Omega)} \leq e^{\left(\frac{1}{l_{m_0}-l_{m_0+1}}\right)^{\frac{1}{2\alpha_0-1}}\left(2N6^{\frac{1}{2\alpha_0-1}}+1\right)} \int_0^T \chi_E \|u(t)\|_{L^2(\omega)} \mathrm{d}t. \tag{3.20}$$

Case 2. We finally consider the second case where $\alpha_0 \geq 1$.

In this case,

$$\frac{1}{2\alpha_0 - 1} \leq 1, i.e., \frac{1}{1-2\alpha_0} \geq -1.$$

Due to $\frac{l_m - l_{m+1}}{6} < 1$, so $\left(\frac{l_m-l_{m+1}}{6}\right)^{\frac{1}{1-2\alpha_0}} \leq \left(\frac{l_m-l_{m+1}}{6}\right)^{-1}$. Thus

$$\begin{aligned}
\frac{6N}{l_m - l_{m+1}} e^{N\left(\frac{l_m-l_{m+1}}{6}\right)^{\frac{1}{1-2\alpha_0}}} &\leq \frac{6N}{l_m - l_{m+1}} e^{N\left(\frac{l_m-l_{m+1}}{6}\right)^{-1}} \\
&= \frac{6N}{l_m - l_{m+1}} e^{\frac{6N}{l_m-l_{m+1}}} \\
&\leq e^{\frac{12N}{l_m-l_{m+1}}},
\end{aligned}$$

where we note $e^{2x} \geq xe^x$ for $x \geq 0$ again. This together with (3.16) yields that

$$\|u(l_m)\|_{L^2(\Omega)} \leq \epsilon\|u(l_{m+1})\|_{L^2(\Omega)} + \epsilon^{-1}e^{\frac{12N}{l_m-l_{m+1}}}\int_{l_{m+1}}^{l_m}\chi_E\|u(t)\|_{L^2(\omega)}\mathrm{d}t.$$

Multiplying the above inequality by $\epsilon e^{\frac{-12N}{l_m-l_{m+1}}}$, we obtain

$$\epsilon e^{\frac{-12N}{l_m-l_{m+1}}}\|u(l_m)\|_{L^2(\Omega)} \leq \epsilon^2 e^{\frac{-12N}{l_m-l_{m+1}}}\|u(l_{m+1})\|_{L^2(\Omega)} + \int_{l_{m+1}}^{l_m}\chi_E\|u(t)\|_{L^2(\omega)}\mathrm{d}t.$$

Taking $\epsilon = e^{\frac{-1}{l_m-l_{m+1}}}$, we get that

$$e^{\frac{-12N-1}{l_m-l_{m+1}}}\|u(l_m)\|_{L^2(\Omega)} - e^{\frac{-12N-2}{l_m-l_{m+1}}}\|u(l_{m+1})\|_{L^2(\Omega)} \leq \int_{l_{m+1}}^{l_m}\chi_E\|u(t)\|_{L^2(\omega)}\mathrm{d}t.$$

Choosing $\frac{12N+1}{12N+2} = q \in (0,1)$ and noting that

$$\begin{aligned}\frac{-12N-2}{l_m-l_{m+1}} &= \frac{-12N-2}{12N+1}\frac{12N+1}{l_m-l_{m+1}} = \frac{-1}{q}\frac{12N+1}{l_m-l_{m+1}}\\ &= -\frac{l_m-l_{m+1}}{l_{m+1}-l_{m+2}}\frac{12N+1}{l_m-l_{m+1}} = \frac{-12N-1}{l_{m+1}-l_{m+2}},\end{aligned}$$

we have

$$e^{\frac{-12N-1}{l_m-l_{m+1}}}\|u(l_m)\|_{L^2(\Omega)} - e^{\frac{-12N-1}{l_{m+1}-l_{m+2}}}\|u(l_{m+1})\|_{L^2(\Omega)} \leq \int_{l_{m+1}}^{l_m}\chi_E\|u(t)\|_{L^2(\omega)}\mathrm{d}t.$$

Summing the above inequality from $m = m_0$ to $+\infty$ can yield that

$$e^{\frac{-12N-1}{l_{m_0}-l_{m_0+1}}}\|u(l_{m_0})\|_{L^2(\Omega)} - \lim_{m\to\infty} e^{\frac{-12N-1}{l_{m+1}-l_{m+2}}}\|u(l_{m+1})\|_{L^2(\Omega)} \leq \int_0^T\chi_E\|u(t)\|_{L^2(\omega)}\mathrm{d}t.$$

Since $\|u(T)\|_{L^2(\Omega)} \leq \|u(l_{m_0})\|_{L^2(\Omega)}$, $\lim_{m\to\infty} e^{\frac{-12N-1}{l_{m+1}-l_{m+2}}} = 0$ and $\sup_{t\in[0,T]}\|u(t)\| < \infty$, we have

$$\|u(T)\|_{L^2(\Omega)} \leq e^{\frac{12N+1}{l_{m_0}-l_{m_0+1}}}\int_0^T\chi_E\|u(t)\|_{L^2(\omega)}\mathrm{d}t. \tag{3.21}$$

Combining (3.20) and (3.21), we show that there exists a constant $C = C(\Omega, \omega, \alpha_0, E) > 0$ such that (3.10) holds. This completes the proof. $\square$

Remark 3.6. From the proof of Theorem 3.5, we see that the constant C in (3.10) can be taken as

$$C = \begin{cases} e^{\left(\frac{1}{l_{m_0}-l_{m_0+1}}\right)^{\frac{1}{2\alpha_0-1}}\left(2N6^{\frac{1}{2\alpha_0-1}}+1\right)} & \text{for } \frac{1}{2} < \alpha_0 < 1,\\ e^{\frac{12N+1}{l_{m_0}-l_{m_0+1}}} & \text{for } \alpha_0 \geq 1,\end{cases}$$

where N is given in (3.3).

By the equivalence of the null controllability and observability inequality from measurable sets in time (*cf.*, *e.g.*, [20] or [24]), we have

Theorem 3.7. *Let $E \subset (0,T)$ be a subset of positive measure. For each $u_0 \in L^2(\Omega)$, there is a control $f \in L^\infty(0,T;L^2(\Omega))$ such that*

$$\begin{cases} \frac{\mathrm{d}u(t)}{\mathrm{d}t} + A^{\alpha(t)}u(t) = \chi_E Bf, \qquad 0 < t \le T, \\ u(0) = u_0, \\ u(T) = 0. \end{cases} \tag{3.22}$$

Moreover,

$$\|f\|_{L^\infty(0,T;L^2(\Omega))} \le C\|u_0\|_{L^2(\Omega)}$$

with some positive constant $C = C(\Omega, \omega, \alpha_0, E)$ independent of initial data u_0.

4. Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1. We first prove the bang-bang property (1.3). To prove (1.3), by contraction, we suppose that there exists a pair $(\tau^*, f^*) \in \mathcal{U}^M$ satisfying (1.2) such that (1.3) does not hold. Thus, there would be an $\epsilon > 0$ and a measurable set $E^* \subset (\tau^*, T)$ with $|E^*| > 0$ such that

$$\|f^*(t)\|_{L^2(\Omega)} \le M - \epsilon, \quad \forall t \in E^*. \tag{4.1}$$

It is easy to see that there is a $\delta_0 \in (0,1)$ such that the measurable set

$$E = E^* \cap (\tau^* + \delta_0, T)$$

has a positive measure. Moreover, by (4.1), we have

$$\|f^*(t)\|_{L^2(\Omega)} \le M - \epsilon, \quad \forall t \in E. \tag{4.2}$$

Consider the following equation

$$\begin{cases} \frac{\mathrm{d}z(t)}{\mathrm{d}t} + A^{\alpha(t)}z(t) = -\chi_{(\tau^*, \tau^*+\delta)}Bf^*(t), \quad 0 < t \le \tau^* + \delta, \\ z(0) = 0, \end{cases} \tag{4.3}$$

where $\delta \in (0, \delta_0)$ will be determined later. Let $U(t,s), 0 \le s \le t \le T$, be the evolution system on $L^2(\Omega)$ defined by (2.3). By noting $\|U(t,s)\| \le 1$ and solving (4.3), we have

$$\begin{aligned} \|z(\tau^*+\delta)\|_{L^2(\Omega)} &= \left\| \int_0^{\tau^*+\delta} U(t,s)\chi_{(\tau^*,\tau^*+\delta)}Bf^*(s)\mathrm{d}s \right\|_{L^2(\Omega)} \\ &\le \int_{\tau^*}^{\tau^*+\delta} \|f^*(s)\|_{L^2(\Omega)}\mathrm{d}s \le M\delta. \end{aligned} \tag{4.4}$$

Consider another equation

$$\begin{cases} \frac{\mathrm{d}\phi(t)}{\mathrm{d}t} + A^{\alpha(t)}\phi(t) = \chi_E Bv(t), \qquad \tau^* + \delta < t \le T, \\ \phi(\tau^*+\delta) = z(\tau^*+\delta), \end{cases} \tag{4.5}$$

where $E = E^* \cap (\tau^* + \delta_0, T)$ with $|E| > 0$. By Theorem 3.7, there is a control $v \in L^\infty(\tau^* + \delta, T; L^2(\Omega))$ such that the solution $\phi(\cdot)$ to the equation (4.5) satisfies

$$\phi(T) = 0 \quad \text{in } L^2(\Omega),$$

and

$$\|v\|_{L^\infty(\tau^*+\delta,T;L^2(\Omega))} \le k\|z(\tau^* + \delta)\|_{L^2(\Omega))},$$

where $k > 0$ is a constant, which depends only on $\Omega, \omega, \alpha_0, E$ and is independent of δ. This, together (4.4), yields that

$$\|v\|_{L^\infty(\tau^*+\delta,T;L^2(\Omega))} \le kM\delta. \tag{4.6}$$

Define

$$w(t) = \begin{cases} z(t), \text{ if } t \in [0, \tau^* + \delta], \\ \phi(t), \text{ if } t \in (\tau^* + \delta, T]. \end{cases}$$

Clearly, $w(0) = 0$ and $w(T) = 0$ in $L^2(\Omega)$. Let u^* be the solution of (1.1) with (τ^*, f^*). Then $u^*(T) = 0$ in $L^2(\Omega)$. Further, we can check that the function $u = u^* + w$ solves the following equation

$$\begin{cases} \frac{\mathrm{d}u(t)}{\mathrm{d}t} + A^{\alpha(t)}u(t) = \chi_{(\tau^*+\delta,T)}Bf, \qquad 0 < t \le T, \\ u(0) = u_0, \\ u(T) = 0 \end{cases} \tag{4.7}$$

with

$$f(t) = \begin{cases} 0, & \text{if } t \in [0, \tau^* + \delta], \\ f^*(t) + \chi_E v(t), & \text{if } t \in (\tau^* + \delta, T]. \end{cases} \tag{4.8}$$

Now, we can choose $\delta \in (0, \delta_0)$ such that $kM\delta \le \epsilon$. Then, by (4.2), (4.6) and (4.8), for a.e. $t \in (\tau^* + \delta, T]$,

$$\begin{aligned} \|f(t)\|_{L^2(\Omega)} &= \|f^*(t) + \chi_E v(t)\|_{L^2(\Omega)} \\ &\le \begin{cases} M - \epsilon + kM\delta, & t \in E \cap (\tau^* + \delta, T], \\ M, & t \in (\tau^* + \delta, T] \setminus E, \end{cases} \\ &\le M. \end{aligned} \tag{4.9}$$

Moreover,

$$\|f(t)\|_{L^2(\Omega)} = 0, \quad \forall t \in [0, \tau^* + \delta].$$

This, together with (4.9) yields that

$$\|f(t)\|_{L^2(\Omega)} \le M, \quad \text{for a.e. } t \in [0, T].$$

Thus, we have shown that there is a pair $(\tau^* + \delta, f) \in \mathcal{U}^M$ such that (4.7) holds, which contradicts (1.2). Thus, the bang-bang property (1.3) holds.

To show the uniqueness of the pair (τ^*, f^*), by contradiction, we assume that there are two pair (τ_1^*, f_1^*) and $(\tau_2^*, f_2^*) \in \mathcal{U}^M$ satisfying (1.2) and the bang-bang property (1.3). Thus, we can easily observe that $\tau_1^* = \tau_2^* = \tau^*$ by (1.2). Moreover, the pair $(\tau^*, \frac{f_1^*+f_2^*}{2}) \in \mathcal{U}^M$ and $\frac{f_1^*+f_2^*}{2}$ is a time optimal control. If there exists a set $E \subset (\tau^*, T)$ of positive measure such that $f_1^*(t) \neq f_2^*(t)$ in E. Then, noting $\|f_1^*(t)\|_{L^2(\Omega)} = \|f_2^*(t)\|_{L^2(\Omega)} = M$ for a.e. $t \in E$, by the strictly convex property of $L^2(\Omega)$, we have

$$\frac{1}{2}\|f_1^*(t) + f_2^*(t)\|_{L^2(\Omega)} < M \quad \text{ for a.e. } t \in E,$$

which contradicts the bang-bang property (1.3). The proof of this theorem is completed. □

Proof of Theorem 1.2. We first prove the existence of minimal norm optimal control. Let $\rho = \inf_{f\in\mathcal{U}_\tau} \|f\|_{L^\infty(\tau,T;L^2(\Omega))}$. By Theorem 3.7, we see that $\mathcal{U}_\tau \neq \emptyset$. Thus, $0 \le \rho < \infty$ and there exists sequence $\{f_n\}_{n\ge1} \subset \mathcal{U}_\tau$ such that

$$\begin{gathered}\rho_n = \|f_n\|_{L^\infty(\tau,T;L^2(\Omega))} \longrightarrow \rho, \\ \rho_1 \ge \rho_2 \ge \cdots \ge \rho_n \ge \cdots .\end{gathered}$$

Moreover, the solution u_n to (1.4) with $f = f_n$ is

$$u_n(t) = U(t,0)u_0 + \int_0^t U(t,s)\chi_{(\tau,T)}Bf_n(s)\mathrm{d}s \tag{4.10}$$

and satisfies

$$u_n(T) = 0 \ \text{ for all } n, \tag{4.11}$$

where $U(t,s)$ is the evolution system defined by (2.3). Since the sequence $\{f_n\}_{n\ge1} \subset \mathcal{U}_\tau$ is bounded, there exist a subsequence named in the same way and $f^* \in L^\infty(0,T;L^2(\Omega))$ such that

$$f_n \longrightarrow f^* \quad \text{weakly star in } L^\infty(0,T;L^2(\Omega)), \text{as } n \to \infty$$

Due to the fact that $f_n^*(t) = 0$ for a.e. $t \in (0,\tau)$, one can easily observe that

$$f^*(t) = 0 \text{ for a.e. } t \in (0,\tau). \tag{4.12}$$

Moreover, by the property of weakly star convergence, we have

$$\|f^*\|_{L^\infty(\tau,T;L^2(\Omega))} \le \rho. \tag{4.13}$$

Thus, by (4.10) and (4.11), for any $g \in L^2(\Omega)$,

$$\begin{aligned}(g, U(T,0)u_0) &= -(g, \int_0^T U(T,s)\chi_{(\tau,T)}Bf_n(s)\mathrm{d}s)\\ &= -\int_0^T (g, U(T,s)Bf_n(s))\mathrm{d}s\\ &= -\int_0^T (B^*U^*(T,s)g, f_n(s))\mathrm{d}s\\ &\to -(g, \int_0^T U(T,s)Bf^*(s)\mathrm{d}s)\\ &= -(g, \int_0^T U(T,s)\chi_{(\tau,T)}Bf^*(s)\mathrm{d}s) \qquad (4.14)\end{aligned}$$

as $n \longrightarrow \infty$. We write u^* for the solution to (1.4) corresponding to the control $f = f^*$. Noting that $g \in L^2(\Omega)$ is arbitrary, we have by (4.14)

$$u^*(T) = U(T,0)u_0 + \int_0^T U(T,s)\chi_{(\tau,T)}Bf^*(s)\mathrm{d}s = 0. \qquad (4.15)$$

Thus, combing (4.12), (4.13) and (4.15) can deduce that $f^* \in \mathcal{U}_\tau$ satisfying (1.5) is the norm optimal control.

Next, we will prove the bang-bang property. Seeking a contradiction, we suppose that there exists $f^* \in \mathcal{U}_\tau$ satisfying (1.5) such that (1.6) does not hold. Then there was an $\epsilon \in (0,1)$ and a measurable set $E \subset (\tau, T)$ of positive measure such that

$$\|f^*(t)\|_{L^2(\Omega)} \le \rho - \epsilon, \quad \forall t \in E, \qquad (4.16)$$

where $\rho = \inf_{f\in\mathcal{U}_\tau} \|f\|_{L^\infty(\tau,T;L^2(\Omega))}$. Let $\delta \in (0,1)$, which will be determined later. By Theorem 3.7 there is a control $v_\delta \in L^\infty(0,T;L^2(\Omega))$ such that the solution $\phi_\delta(\cdot)$ to the equation

$$\begin{cases} \frac{\mathrm{d}\phi_\delta(t)}{\mathrm{d}t} + A^{\alpha(t)}\phi_\delta(t) = \chi_E Bv_\delta, & \text{in } (0,T),\\ \phi_\delta(0) = \delta u_0, \end{cases}$$

satisfies $\phi_\delta(T) = 0$ in Ω and there exists a constant $k > 0$ independent of δ such that

$$\|v_\delta\|_{L^\infty(0,T;L^2(\Omega))} \le k\delta\|u_0\|_{L^2(\Omega))}. \qquad (4.17)$$

Define

$$f_\delta = (1-\delta)f^* + \chi_E v_\delta. \qquad (4.18)$$

One can easily check that $f_\delta(t) = 0$ for a.e. $t \in (0,\tau)$. By taking $\delta = \epsilon/(k\|u_0\|_{L^2(\Omega))} + \epsilon)$, we can get that $\|v_\delta\|_{L^\infty(0,T;L^2(\Omega))} \le (1-\delta)\epsilon$. Then by (4.16) and (4.17),

$$\|f_\delta(t)\|_{L^2(\Omega)} \le (1-\delta)\rho \quad \text{for a.e. } t \in (\tau, T). \qquad (4.19)$$

Moreover, the function $u_\delta = (1-\delta)u + \phi_\delta$ is the solution of the following equation

$$\begin{cases} \frac{\mathrm{d}u_\delta(t)}{\mathrm{d}t} + A^{\alpha(t)}u_\delta(t) = \chi_{(\tau,T)}Bf_\delta, & \text{in } (0,T),\\ u_\delta(0) = u_0, \\ u_\delta(T) = 0, \end{cases}$$

where f_δ is given by (4.18). Thus, $f_\delta \in \mathcal{U}_\tau$. By noting (4.19), we see that the existence of (f_δ, u_δ) contradicts the definition of ρ, which implies that the bang-bang property (1.6) holds.

Finally, we will show the uniqueness of the norm optimal control. Let $f_1, f_2 \in \mathcal{U}_\tau$ be two norm optimal controls satisfying (1.5) and (1.6). Then we see easily that $(f_1+f_2)/2 \in \mathcal{U}_\tau$ and

$$\left\|\frac{f_1+f_2}{2}\right\|_{L^\infty(\tau,T;L^2(\Omega))} \leq \frac{1}{2}(\|f_1\|_{L^\infty(\tau,T;L^2(\Omega))} + \|f_2\|_{L^\infty(\tau,T;L^2(\Omega))}) = \rho,$$

where $\rho = \inf_{f\in\mathcal{U}_\tau}\|f\|_{L^\infty(\tau,T;L^2(\Omega))}$, which implies that $(f_1+f_2)/2$ is a norm optimal control as well. If there exists a set $E \in (\tau, T)$ of positive measure such that

$$f_1(t) \neq f_2(t), \ \forall t \in E,$$

by the strictly convex property of $L^2(\Omega)$, we have

$$\frac{1}{2}\|f_1^*(t) + f_2^*(t)\|_{L^2(\Omega)} < \rho \quad \text{ for a.e. } t \in E,$$

which contradicts the bang-bang property (1.6).

The proof of this theorem is completed. □

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