Γ-CONVERGENCE AND RELAXATIONS FOR GRADIENT FLOWS IN METRIC SPACES: A MINIMIZING MOVEMENT APPROACH

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Abstract. We present new abstract results on the interrelation between the minimizing movement scheme for gradient flows along a sequence of *Γ*-converging functionals and the gradient flow motion for the corresponding limit functional, in a general metric space. We are able to allow a relaxed form of minimization in each step of the scheme, and so we present new relaxation results too.

1. Introduction

For a sequence of *Γ*-converging functionals $\phi_\epsilon \xrightarrow{\Gamma} \phi$, one can consider the minimizing movement scheme for gradient flows along $(\phi_\epsilon)_{\epsilon>0}$, in which every time step τ is associated with a parameter $\epsilon = \epsilon(\tau)$, simultaneously passing to the limit in the time steps $\tau \to 0$ and parameters $\epsilon \to 0$.

The aim of the paper is to introduce and study an abstract condition concerning the choice $\epsilon = \epsilon(\tau)$ in order to obtain curves of maximal slope for the limit functional ϕ. Moreover, we want to allow a relaxed form of minimization in each step of the scheme.

Curves of maximal slope and *Γ*-convergence. The notion of curves of maximal slope goes back to [12], with further developments in [14, 16].

Let $(\mathscr{S}, d)$ be a complete metric space. Curves of maximal slope for an extended real functional $\phi : \mathscr{S} \to (-\infty, +\infty]$ with respect to its relaxed slope are described by the energy dissipation inequality (EDI(ϕ))

$$\phi(u(s)) - \phi(u(t)) \geq \frac{1}{2} \int_s^t |\partial^-\phi|^2(u(r)) \, \mathrm{d}r + \frac{1}{2} \int_s^t |u'|^2(r) \, \mathrm{d}r$$

for $\mathscr{L}^1$-a.e. s, t, $s \leq t$, in which $|\partial^-\phi|$ denotes the relaxed slope and $|u'|$ the metric derivative (Sect. 2.1).

In the finite dimensional and smooth setting, this corresponds to the gradient flow equation

$$u'(t) = -\nabla\phi(u(t)).$$

The term gradient flow is also common for curves of maximal slope.

Let a sequence of energy functionals $(\phi_\epsilon)_{\epsilon>0}$ Γ-converging to a limit functional ϕ be given. A natural question arises:

If u_ϵ are curves of maximal slope for ϕ_ϵ and they converge to a curve u, is u then a curve of maximal slope for ϕ? $(\star)$

(1) The λ-convex case. Since the pioneering work on G-convergence for differential operators [13, 24], the following statement is well-known: If the functionals are λ-convex with equi-compact sublevel sets and $\mathscr{S}$ is a Hilbert space, and the initial data converge, then the gradient flows u_ϵ converge to the limit gradient flow (in fact, Mosco convergence is sufficient in this case [3]). A corresponding result can be proved for metric spaces as well [10, 15].

(2) The Serfaty$-$Sandier approach. The considerations by Sandier and Serfaty in [22, 23] are motivated by the convergence of the Ginzburg-Landau heat flow and the question of a general underlying structure in the cases in which a positive answer can be given to $(\star)$. Their main assumption (revisited by other authors, *e.g.* in [8]) is that the upper gradients $|\partial^-\phi_\epsilon|$, $|\partial^-\phi|$ satisfy

$$u_\epsilon \to u \;\Rightarrow\; \liminf_{\epsilon \to 0} |\partial^-\phi_\epsilon|(u_\epsilon) \geq |\partial^-\phi|(u). \tag{1.1}$$

A related problem for generalized gradient systems and rate-independent evolution is studied in [17–19].

In the λ-convex case, the following condition on the local slopes can be proved [20]:

$$u_\epsilon \to u \;\Rightarrow\; \liminf_{\epsilon \to 0} |\partial\phi_\epsilon|(u_\epsilon) \geq |\partial\phi|(u) \tag{1.2}$$

(see Sect. 2.1 for the definition of local slope).

So in both situations (1) and (2), the slopes satisfy a Γ-liminf condition. In general, such a condition does not hold.

(3) The general case. In general, the answer to our question $(\star)$ is **no**. Even in the finite dimensional and smooth setting (as in Ex. 1.1), the limit of a sequence of gradient flows u_ϵ for ϕ_ϵ is in general no solution to the gradient flow equation for the limit functional ϕ.

For illustrative purposes, we give a simple example.

Example 1.1. We consider $f_\epsilon(x) = x^2 + \rho_\epsilon \cos^2\left(\frac{x}{\epsilon}\right)$ $(\epsilon << \rho_\epsilon \to 0)$ and $f(x) = x^2$ $(x \in \mathbb{R})$. Let $u_\epsilon : [0, +\infty) \to \mathbb{R}$ satisfy

$$\begin{aligned} u_\epsilon'(t) &= -f_\epsilon'(u_\epsilon(t)), \\ u_\epsilon(0) &= u_\epsilon^0 \end{aligned}$$

with initial values $u_\epsilon^0 \to u_0 \neq 0$. Then $(u_\epsilon)_{\epsilon>0}$ converges pointwise to the constant curve $u \equiv u_0$ which does not solve the gradient flow equation for f.

The notion of Γ-convergence allows for a wide range of perturbations ϕ_ϵ of ϕ so that a passage to the limit $\epsilon \to 0$ in the energy dissipation inequality $(\text{EDI}(\phi_\epsilon)) \longrightarrow (\text{EDI}(\phi))$ is in general not possible for lack of control over the upper gradients $|\partial^-\phi_\epsilon|$ of ϕ_ϵ.

Our approach aims to study the gradient flow motion along $(\phi_\epsilon)_{\epsilon>0}$ on the level of the minimizing movement schemes and to establish a connection with the gradient flow motion of the limit functional ϕ.

Minimizing movements. At the beginning of the 90's, Ennio De Giorgi introduced the notion of minimizing movements [11] as "natural meeting point" of many different research fields in mathematics.

The minimizing movement scheme for gradient flows is given by

$$\psi(u_\tau^n) + \frac{1}{2\tau}d^2(u_\tau^n, u_\tau^{n-1}) \;=\; \min_{v \in \mathscr{S}}\left\{\psi(v) + \frac{1}{2\tau}d^2(v, u_\tau^{n-1})\right\}$$

with $n \in \mathbb{N}$ and time steps $\tau > 0$. It is closely related to the notion of curves of maximal slope. Under suitable assumptions, the piecewise constant interpolations of the discrete values $(u_\tau^n)_{n\in\mathbb{N}}$ converge (up to a subsequence) to a curve of maximal slope for ψ (as $\tau \to 0$) [2]. The scheme mimics the gradient flow motion along the functional ψ on a discrete level (Sect. 2.3).

For functionals $\phi_\epsilon \xrightarrow{\Gamma} \phi$, one can consider the minimizing movement scheme along $(\phi_\epsilon)_{\epsilon>0}$, given by

$$\phi_\epsilon(u_{\tau,\epsilon}^n) + \frac{1}{2\tau}d^2(u_{\tau,\epsilon}^n, u_{\tau,\epsilon}^{n-1}) \;=\; \min_{v \in \mathscr{S}}\left\{\phi_\epsilon(v) + \frac{1}{2\tau}d^2(v, u_{\tau,\epsilon}^{n-1})\right\}$$

for $\tau, \epsilon > 0$ (with well-prepared initial values $u_{\tau,\epsilon}^0$), define the piecewise constant interpolations $u_{\tau,\epsilon}(t) \equiv u_{\tau,\epsilon}^n$ for $t \in ((n-1)\tau, n\tau]$, and let τ and ϵ tend to 0 simultaneously.

We are interested in the cases in which the limit curves of the minimizing movement scheme along $(\phi_\epsilon)_{\epsilon>0}$ are curves of maximal slope for ϕ.

In ([5], Chap. 8) such minimization problems are examined for concrete examples of functionals $\phi_\epsilon \xrightarrow{\Gamma} \phi$, highlighting the dependence of the asymptotic behaviour on the interaction between ϵ and τ.

Moreover, under suitable assumptions on ϕ_ϵ, the following statement can be proved ([5], Thm. 8.1) by means of the Fundamental Theorem of Γ-convergence and a diagonal argument:

If discrete values $(u_{\tau,\epsilon}^n)_{n\in\mathbb{N}_0}$ $(\tau, \epsilon > 0)$ are given in accordance to the scheme above, then

- there exists $\epsilon = \epsilon(\tau)$ such that any limit curve of the corresponding piecewise constant interpolations $(u_{\tau,\epsilon(\tau)})_{\tau>0}$ can also be obtained by the minimizing movement scheme along the single functional ϕ,
- there exists $\tau = \tau(\epsilon)$ such that any limit curve of $(u_{\tau(\epsilon),\epsilon})_{\epsilon>0}$ is the limit of a sequence of curves of maximal slope for ϕ_ϵ.

In view of this result and the examples in [5], we notice that the interaction between the time step τ and the parameter ϵ should be crucial in order to achieve convergence of the scheme to a curve of maximal slope for ϕ. This goes hand in hand with our considerations on why the passage to the limit in the energy dissipation inequality is in general not possible.

A special case in which the desired convergence of the scheme holds independent of the interaction between ϵ and τ is considered by Ortner in [20], by extending the arguments of [2]. His main assumption is the Γ-liminf condition (1.2) on the local slopes which can be viewed as discrete counterpart of the Serfaty−Sandier condition (1.1).

Aim of our paper. This paper aims to derive a general condition concerning the choice $\epsilon = \epsilon(\tau)$ in order to achieve convergence of the minimizing movement scheme along $(\phi_\epsilon)_{\epsilon>0}$ to curves of maximal slope for ϕ. Note that we interpret the gradient flow motion along $(\phi_\epsilon)_{\epsilon>0}$ as joint steepest descent movement instead of considering the single energy dissipation inequalities (EDI(ϕ_ϵ)): We pass to the limits $\tau \to 0$ and $\epsilon \to 0$ simultaneously in the minimizing movement scheme, with a suitable choice $\epsilon = \epsilon(\tau)$. By comparison, the study of the limit behaviour of (EDI(ϕ_ϵ)) (as in [22,23]) is related to first passing to the limit $\tau \to 0$ in the scheme for fixed $\epsilon > 0$ and only then passing to the limit $\epsilon \to 0$.

A special feature of our theory is that we are able to *relax* the minimizing movement scheme along $(\phi_\epsilon)_{\epsilon>0}$ by allowing *approximate minimizers* in each step. We do not need the existence of exact solutions to the minimization problems. In particular, we do not need to require any lower semicontinuity or compactness property of the single functionals ϕ_ϵ.

As a particular case, the paper also deals with a relaxation of the classical minimizing movement scheme along a single functional.

Let us introduce a relaxed minimizing movement scheme along $(\phi_\epsilon)_{\epsilon>0}$. We associate every time step $\tau > 0$ with a parameter $\epsilon = \epsilon(\tau)$ in such a way that $\epsilon(\tau)$ converges to 0 as $\tau \to 0$.

Relaxed minimizing movement scheme along $(\phi_\epsilon)_{\epsilon>0}$. For every time step $\tau > 0$, find a sequence $(u_\tau^n)_{n\in\mathbb{N}}$ by the following scheme. zz The sequence of initial values $u_\tau^0 \to u^0 \in D(\phi)$ satisfies $\phi_{\epsilon(\tau)}(u_\tau^0) \to \phi(u^0)$ and $u_\tau^n \in \mathscr{S}$ $(n \in \mathbb{N})$ satisfies

$$\phi_{\epsilon(\tau)}(u_\tau^n) + \frac{1}{2\tau}d^2(u_\tau^n, u_\tau^{n-1}) \;\leq\; \inf_{v\in\mathscr{S}}\left\{\phi_{\epsilon(\tau)}(v) + \frac{1}{2\tau}d^2(v, u_\tau^{n-1})\right\} + \gamma_\tau\tau \tag{1.3}$$

with some $\gamma_\tau > 0$, $\gamma_\tau \to 0$ as $\tau \to 0$.

The reader might be interested in the question of whether we may allow an error $\gamma_\tau^{(n)}$ depending on $n \in \mathbb{N}$ (instead of $\gamma_\tau\tau$) in the approximate minimization problems (1.3). Indeed, a generalization of our theory to a non-uniform distribution of the error is possible.

A general condition concerning the right choice $\epsilon = \epsilon(\tau)$ in (1.3).

Our main assumption (see Ass. 3.3) is as follows: we suppose that for all $u_\tau, u \in \mathscr{S}$ with $u_\tau \to u$, $\sup_\tau \phi_{\epsilon(\tau)}(u_\tau) < +\infty$ it holds that

$$\liminf_{\tau\to 0} \frac{\phi_{\epsilon(\tau)}(u_\tau) - \mathcal{Y}_\tau\phi_{\epsilon(\tau)}(u_\tau)}{\tau} \;\geq\; \frac{1}{2}|\partial^-\phi|^2(u), \tag{1.4}$$

in which

$$\mathcal{Y}_\tau\phi_{\epsilon(\tau)}(u_\tau) := \inf_{v\in\mathscr{S}}\left\{\phi_{\epsilon(\tau)}(v) + \frac{1}{2\tau}d^2(v, u_\tau)\right\}$$

denotes the Moreau−Yosida approximation. Our condition (1.4) relates the diagonal steepest descent movement along the sequence $(\phi_\epsilon)_{\epsilon>0}$ with the relaxed slope of ϕ.

Under suitable natural coercivity assumptions, we prove

- if condition (1.4) holds for $\epsilon = \epsilon(\tau)$, then any limit curve of our relaxed minimizing movement scheme along $(\phi_\epsilon)_{\epsilon>0}$ with choice $\epsilon = \epsilon(\tau)$ is a curve of maximal slope for ϕ (see Thm. 3.4, Prop. 3.7).

Existence of a right choice $\epsilon = \epsilon(\tau)$. If $(\mathscr{S}, d)$ is separable, we can prove that

- there always exists a sequence $(\epsilon_\tau)_{\tau>0}$ with $\epsilon_\tau > 0$ such that our main assumption (1.4) is satisfied for all choices $(\epsilon(\tau))_{\tau>0}$ with $\epsilon(\tau) \leq \epsilon_\tau$ ($\epsilon(\tau) \to 0$) (see Thm. 6.1).

We notice that this choice $\epsilon = \epsilon(\tau)$ solely depends on the velocity of Γ-convergence $\phi_\epsilon \xrightarrow{\Gamma} \phi$.

A general example for the choice $\epsilon = \epsilon(\tau)$. We prove under suitable natural coercivity assumptions that if the local boundedness condition

$$\limsup_{n\to+\infty} \sup_{v:d(v,u)\leq\rho_n} \left|\frac{\phi(v) - \phi_{\epsilon_n}(v)}{\epsilon_n}\right| \;<\; +\infty \tag{1.5}$$

holds for every $u \in \mathscr{S}$, $\rho_n \to 0$ ($\rho_n > 0$), then (1.4) is satisfied for every choice $\epsilon = \epsilon(\tau)$ with $\frac{\epsilon(\tau)}{\tau} \to 0$ as $\tau \to 0$ (see Prop. 8.1).

Plan of the paper and further results. In Section 2, we give basic definitions. In 2.2, we specify our topological assumptions (which allow for metric topological spaces $(\mathscr{S}, d, \sigma)$ where the topology σ does not necessarily coincide with the one induced by the metric d).

In Section 3, we introduce the relaxed minimizing movement scheme (1.3) along $(\phi_\epsilon)_{\epsilon>0}$ and we prove the convergence of the scheme to the gradient flow motion of the limit functional ϕ under the main assumption (1.4). We discuss a generalization of our theory to a non-uniform distribution of the error in (1.3).

In Section 4, we focus on the special case in which $\phi_\epsilon = \psi$ is independent of $\epsilon > 0$. Applying the results of Section 3, we show that one may allow a relaxed form of minimization in the clasical scheme along a single functional. Moreover, we notice that the error order $o(\tau)$, characterizing the asymptotic behaviour of $\gamma_\tau \tau$ in the definition of the relaxed minimizing movement scheme (1.3), is optimal. We also obtain a lower semicontinuous envelope relaxation result.

In Section 5, we consider the case that $(\phi_\epsilon)_{\epsilon>0}, \phi$ satisfy the Serfaty$-$Sandier condition (1.1) or its discrete counterpart (1.2). We prove that in this situation, condition (1.4) is satisfied for every choice $\epsilon = \epsilon(\tau)$.

In Section 6, we prove, for the case that $(\mathscr{S}, d)$ is separable, the existence of a right choice $\epsilon = \epsilon(\tau)$ with regard to condition (1.4).

In Sections 7 and 8, we illustrate general methods to determine $\epsilon = \epsilon(\tau)$ with regard to (1.4).

In 8.1, we determine $\epsilon = \epsilon(\tau)$ for the case that the general condition (1.5) holds.

In 8.2, we show explicitly that the deciding factor in the whole theory is the local behaviour of the Γ-converging sequence of functionals.

In 8.3, we consider time-space discretizations for a single functional, *i.e.* we set $\phi_\epsilon = \phi + \mathbb{I}_{\mathscr{W}_\epsilon}$ and we derive conditions on $\mathscr{W}_{\epsilon(\tau)} \subset \mathscr{S}$ such that condition (1.4) holds.

In Section 9, we mention possible generalizations of our theory.

2. PRELIMINARIES

2.1. Gradient flows in metric spaces

For an introduction to gradient flows in metric spaces we refer to the fundamental book by Ambrosio, Gigli and Savaré [2]. In this section we give a brief overview of some of the basic definitions which can be found in detail in ([2], Chaps. 1 and 2).

Absolutely continuous curves, relaxed slope, curves of maximal slope. Let (a, b) be an open (possibly unbounded) interval of $\mathbb{R}$.

In the finite dimensional case, the classical gradient flow equation

$$u'(t) = -\nabla f(u(t)), \ t \in (a, b)$$

for a C^1-function $f : \mathbb{R}^d \to \mathbb{R}$ can be equivalently expressed by the equation

$$(f \circ u)'(t) = -\frac{1}{2}|u'|^2(t) - \frac{1}{2}\left|\nabla f(u(t))\right|^2, \ t \in (a, b).$$

This equivalent formulation constitutes a heuristic starting point for the introduction of gradient flows in general metric spaces $(\mathscr{S}, d)$, the so-called curves of maximal slope, for extended real functionals $\psi : \mathscr{S} \to (-\infty, +\infty]$.

Absolutely continuous curves, relaxed slope.

Definition 2.1. Let $(\mathscr{S}, d)$ be a complete metric space. We say that a curve $v : (a, b) \to \mathscr{S}$ is absolutely continuous and write $v \in AC(a, b; \mathscr{S})$ if there exists $m \in L^1(a, b)$ such that

$$d(v(s), v(t)) \leq \int_s^t m(r) \, \mathrm{d}r \ \ \forall a < s \leq t < b.$$

In this case, the limit

$$|v'|(t) := \lim_{s \to t} \frac{d(v(s), v(t))}{|s - t|}$$

exists for $\mathscr{L}^1$-a.e. $t \in (a, b)$, the function $t \mapsto |v'|(t)$ belongs to $L^1(a, b)$ and is called the metric derivative of v. The metric derivative is $\mathscr{L}^1$-a.e. the smallest admissible function m in the definition above.

Let $\psi : \mathscr{S} \to (-\infty, +\infty]$ be an extended real functional with proper effective domain

$$D(\psi) := \{\psi < +\infty\} \neq \emptyset.$$

For $v \in D(\psi)$ we define

$$|\partial \psi|(v) := \limsup_{w \xrightarrow{d} v} \frac{(\psi(v) - \psi(w))^+}{d(v, w)}.$$

Following [2], we consider a slight modification of the sequentially σ-lower semicontinuous envelope of $|\partial \psi|$ with respect to a suitable topology σ on $\mathscr{S}$ (see Sect. 2.2 for the topological assumptions).

Definition 2.2. The relaxed slope $|\partial^- \psi| : \mathscr{S} \to [0, +\infty]$ of ψ is defined by

$$|\partial^- \psi|(u) := \inf \left\{ \liminf_{n \to \infty} |\partial \psi|(u_n) : \ u_n \xrightarrow{\sigma} u, \ \sup_n \{d(u_n, u), \psi(u_n)\} < +\infty \right\}.$$

The concept of strong and weak upper gradients can be viewed as a weak counterpart of the modulus of the gradient in a general metric and non-smooth setting.

Definition 2.3. The relaxed slope $|\partial^- \psi|$ is a strong upper gradient for the functional ψ if for every absolutely continuous curve $v \in AC(a, b; \mathscr{S})$ the function $|\partial^- \psi| \circ v$ is Borel and

$$|\psi(v(t)) - \psi(v(s))| \leq \int_s^t |\partial^- \psi|(v(r))|v'|(r) \, \mathrm{d}r \quad \forall a < s \leq t < b.$$

In particular, if $|\partial^- \psi| \circ v|v'| \in L^1(a, b)$ then $\psi \circ v$ is absolutely continuous and

$$|(\psi \circ v)'(t)| \leq |\partial^- \psi|(v(t))|v'|(t) \text{ for } \mathscr{L}^1\text{-a.e. } t \in (a, b).$$

Definition 2.4. The relaxed slope $|\partial^- \psi|$ is a weak upper gradient for the functional ψ if for every absolutely continuous curve $v \in AC(a, b; \mathscr{S})$ such that $\psi \circ v$ is $\mathscr{L}^1$-a.e. equal in (a, b) to a function φ with finite pointwise variation in (a, b) and $|\partial^- \psi| \circ v|v'| \in L^1(a, b)$, it holds that

$$|\varphi'(t)| \leq |\partial^- \psi|(v(t))|v'|(t) \text{ for } \mathscr{L}^1-a.e.t \in (a, b).$$

Now, we have all the ingredients to define curves of maximal slope.

Definition 2.5. Let $|\partial^- \psi|$ be a strong or weak upper gradient for the functional $\psi : \mathscr{S} \to (-\infty, +\infty]$.

A locally absolutely continuous curve $u : (a, b) \to \mathscr{S}$ is called a curve of maximal slope for ψ with respect to its upper gradient $|\partial^- \psi|$ if the following energy dissipation inequality is satisfied for $\mathscr{L}^1$-a.e. $s, t \in (a, b)$, $s \leq t$:

$$\psi(u(s)) - \psi(u(t)) \ \geq \ \frac{1}{2} \int_s^t |\partial^- \psi|^2(u(r)) \, \mathrm{d}r \ + \ \frac{1}{2} \int_s^t |u'|^2(r) \, \mathrm{d}r. \tag{2.1}$$

Under additional assumptions, the existence of curves of maximal slope with respect to the relaxed slope can be proved [2]. Their proof is based on the minimizing movement scheme for gradient flows which we explain in Section 2.3.

Before outlining the connection between curves of maximal slope and the notion of minimizing movements, we specify the assumptions on the metric space $(\mathscr{S}, d)$ and the topology σ on it.

2.2. Topological assumptions

Throughout this paper

$$(\mathscr{S}, d) \text{ will be a given complete metric space} \tag{2.2}$$

and

$$\sigma \text{ will be an Hausdorff topology on } \mathscr{S} \text{ compatible with } d:$$

$$(u_n, v_n) \xrightarrow{\sigma} (u, v) \;\Rightarrow\; \liminf_{n \to \infty} d(u_n, v_n) \geq d(u, v) \;, \tag{2.3}$$

$$d(u_n, v_n) \to 0, \; u_n \xrightarrow{\sigma} u \;\Rightarrow\; v_n \xrightarrow{\sigma} v. \tag{2.4}$$

These are exactly the same topological assumptions as in [2]. The reasons for which they are made are explained in [2] and can also be observed in this paper.

2.3. Minimizing movement scheme for gradient flows

The notion of minimizing movements was established by Ennio de Giorgi [11] at the beginning of the 90's, who got his inspiration from the paper [1].

Definition 2.6. Let $F : (0, 1) \times \mathscr{S} \times \mathscr{S} \to [-\infty, +\infty]$ be given and initial values $u_\tau^0 \xrightarrow{\sigma} u^0$ ($\tau \to 0$). If we can define $\overline{u}_\tau : [0, +\infty) \to \mathscr{S}$, $\overline{u}_\tau(0) = u_\tau^0$, in the following way

$$\overline{u}_\tau(t) \equiv u_\tau^n \text{ for } t \in ((n-1)\tau, n\tau]$$

and

$$u_\tau^n \text{ is a minimizer for } F(\tau, u_\tau^{n-1}, \cdot), \; n \in \mathbb{N},$$

and if (a subsequence of) $(\overline{u}_\tau)_{\tau>0}$ σ-converges pointwise in $[0, +\infty)$ to a curve $u : [0, +\infty) \to \mathscr{S}$, then u is called a (generalized) minimizing movement for F to the initial value u^0.

If we apply the minimizing movement scheme to

$$F(\tau, u, v) := f(v) + \frac{1}{2\tau}|u - v|^2 \; (u, v \in \mathbb{R}^d)$$

with $f : \mathbb{R}^d \to \mathbb{R}$ a Lipschitz continuous C^1-function, then the necessary condition of first order leads to a discrete version of the gradient flow equation for f, and indeed, every (generalized) minimizing movement for F is a gradient flow for f.

With this in mind, for a given functional $\psi : \mathscr{S} \to (-\infty, +\infty]$ we define

$$F_\psi : (0, 1) \times \mathscr{S} \times \mathscr{S} \to [-\infty, +\infty], \; F_\psi(\tau, u, v) := \psi(v) + \frac{1}{2\tau}d^2(v, u)$$

which seems to be only natural in order to construct steepest descent curves for ψ.

A related object of study is the Moreau−Yosida approximation defined below. We will repeatedly use the Moreau−Yosida approximation and the following notation.

Moreau−Yosida approximation. Let $\tau > 0$ be given. The Moreau−Yosida approximation $\mathcal{Y}_\tau \psi$ of a functional $\psi : \mathscr{S} \to (-\infty, +\infty]$ is defined as

$$\mathcal{Y}_\tau \psi(u) := \inf_{v \in \mathscr{S}} \left\{ \psi(v) + \frac{1}{2\tau}d^2(v, u) \right\}, \; u \in \mathscr{S}. \tag{2.5}$$

We define $J_\tau \psi[u]$ as the corresponding set of minimizers, i.e.

$$u_\tau \in J_\tau \psi[u] \;:\Leftrightarrow\; \psi(u_\tau) + \frac{1}{2\tau}d^2(u_\tau, u) = \mathcal{Y}_\tau \psi(u). \tag{2.6}$$

Existence of curves of maximal slope. Let $\psi : \mathscr{S} \to (-\infty, +\infty]$ be given. We apply the minimizing movement scheme to F_ψ and initial values $u^0_\tau \xrightarrow{\sigma} u^0 \in D(\psi)$ $(\tau \to 0)$ with $\sup_\tau d(u^0_\tau, u^0) < +\infty$, $\psi(u^0_\tau) \to \psi(u^0)$.

The following existence result is proved in ([2], Chaps. 2 and 3):

If ψ satisfies Assumption 2.7, then the set of generalized minimizing movements for F_ψ is non-empty and for every generalized minimizing movement $u : [0, +\infty) \to \mathscr{S}$ the energy inequality

$$\psi(u(0)) - \psi(u(t)) \ \geq \ \frac{1}{2} \int_0^t |\partial^- \psi|^2(u(r)) \ \mathrm{d}r \ + \ \frac{1}{2} \int_0^t |u'|^2(r) \ \mathrm{d}r \tag{2.7}$$

holds for all $t > 0$.

Assumption 2.7. We suppose that there exist $A, B > 0$, $u_\star \in \mathscr{S}$ such that

$$\psi(\cdot) \geq -A - Bd^2(\cdot, u_\star). \tag{2.8}$$

Moreover, we assume that the following holds:

$$\sup_{n,m} d(u_n, u_m) < +\infty, \ u_n \xrightarrow{\sigma} u \ \Rightarrow \ \liminf_{n \to \infty} \psi(u_n) \geq \psi(u) \tag{2.9}$$

and

$$\sup_{n,m} \{d(u_n, u_m), \psi(u_n)\} < +\infty \ \Rightarrow \ \exists \, n_k \uparrow +\infty, u \in \mathscr{S} : \ u_{n_k} \xrightarrow{\sigma} u. \tag{2.10}$$

In particular, if $|\partial^- \psi|$ is a strong upper gradient, equality holds in (2.7) and u is a curve of maximal slope for ψ with respect to $|\partial^- \psi|$.

If $|\partial^- \psi|$ is only a weak upper gradient for ψ and we assume in addition that ψ satisfies the following continuity condition

$$\sup_{n \in \mathbb{N}} \{|\partial\psi|(v_n), d(v_n, v), \psi(v_n)\} < +\infty, \ v_n \xrightarrow{\sigma} v \ \Rightarrow \ \psi(v_n) \to \psi(v), \tag{2.11}$$

then the proof of the energy inequality (2.7) can be extended and the energy dissipation inequality (2.1) be obtained.

Strategies to prove the conditions on the relaxed slope to be an upper gradient and the continuity condition (2.11) respectively and examples in which they are satisfied are expounded *e.g.* in [2, 21].

2.4. Γ-convergence

Γ-convergence was introduced by Ennio de Giorgi in the early 70's.

Definition 2.8. Let X be a topological space. A sequence $f_j : X \to [-\infty, +\infty]$ sequentially Γ-converges in X to $f_\infty : X \to [-\infty, +\infty]$ if for all $x \in X$, $x_j \to x$ the following liminf-inequality holds

$$f_\infty(x) \leq \liminf_{j \to \infty} f_j(x_j)$$

and if for all $x \in X$ there exists a recovery sequence $\tilde{x}_j \to x$ such that

$$f_\infty(x) = \lim_{j \to \infty} f_j(\tilde{x}_j).$$

For its various applications, see for example the introductory book by Braides [4].

3. Main theorem

We systematically study the gradient flow motion along a sequence of functionals $\phi_\epsilon : \mathscr{S} \to (-\infty, +\infty]$ on the level of the minimizing movement scheme in each step of which we are able to allow a relaxed form of minimization. The functionals ϕ_ϵ are associated with a limit functional $\phi : \mathscr{S} \to (-\infty, +\infty]$ through a weakened $\Gamma(\sigma)$-liminf-inequality and a recovery sequence of initial values.

3.1. Minimizing movement: Γ-convergence, relaxations

Let $\phi, \phi_\epsilon : \mathscr{S} \to (-\infty, +\infty]$, $D(\phi), D(\phi_\epsilon) \neq \emptyset$ ($\epsilon > 0$) be given.

We are dealing with the following assumptions on $(\phi_\epsilon)_{\epsilon>0}$ and the relation between $(\phi_\epsilon)_{\epsilon>0}$ and ϕ:

Assumption 3.1. We suppose that there exist $A, B > 0$, $u_\star \in \mathscr{S}$ such that

$$\phi_\epsilon(\cdot) \geq -A - Bd^2(\cdot, u_\star) \text{ for all } \epsilon > 0. \tag{3.1}$$

Moreover, we assume that for $\epsilon_n \to 0$ ($\epsilon_n > 0$) it holds that

$$\sup_{n,m} \{\phi_{\epsilon_n}(u_{\epsilon_n}), d(u_{\epsilon_n}, u_{\epsilon_m})\} < +\infty \;\Rightarrow\; \exists\, n_k \uparrow +\infty, u \in \mathscr{S} : \; u_{\epsilon_{n_k}} \overset{\sigma}{\to} u. \tag{3.2}$$

Assumption 3.2. We suppose that $(\phi_\epsilon)_{\epsilon>0}$ and ϕ are connected through a weakened $\Gamma(\sigma)$-liminf-inequality, i.e. for $\epsilon_n \to 0$ ($\epsilon_n > 0$) it holds that

$$\sup_{n,m} d(u_{\epsilon_n}, u_{\epsilon_m}) < +\infty, \; u_{\epsilon_n} \overset{\sigma}{\to} u \;\Rightarrow\; \liminf_{n\to\infty} \phi_{\epsilon_n}(u_{\epsilon_n}) \geq \phi(u). \tag{3.3}$$

In view of the theory developed in [2] (see Sect. 2.3 in this paper), our Assumptions 3.1 and 3.2 arise quite naturally.

Let us define a minimizing movement scheme for the gradient flow motion along the sequence $(\phi_\epsilon)_{\epsilon>0}$ with a relaxed form of minimization in each step and time steps $\tau \to 0$.

We associate every time step $\tau > 0$ in the scheme with $\epsilon = \epsilon(\tau) > 0$ in such a way that $\epsilon(\tau)$ converges to 0 as $\tau \to 0$. This choice $\epsilon = \epsilon(\tau)$ is crucial, as we will see.

Relaxed minimizing movement scheme along $(\phi_\epsilon)_{\epsilon>0}$. For every time step $\tau > 0$, find a sequence $(u_\tau^n)_{n\in\mathbb{N}}$ by the following scheme.

We suppose that the sequence of initial values $u_\tau^0 \overset{\sigma}{\to} u^0 \in D(\phi)$ ($\tau \to 0$) satisfies

$$\sup_\tau d(u_\tau^0, u^0) < +\infty, \; \phi_{\epsilon(\tau)}(u_\tau^0) \to \phi(u^0) \tag{3.4}$$

and we successively choose $u_\tau^n \in \mathscr{S}$ ($n \in \mathbb{N}$) such that

$$\phi_{\epsilon(\tau)}(u_\tau^n) + \frac{1}{2\tau}d^2(u_\tau^n, u_\tau^{n-1}) \;\leq\; \inf_{v\in\mathscr{S}} \left\{ \phi_{\epsilon(\tau)}(v) + \frac{1}{2\tau}d^2(v, u_\tau^{n-1}) \right\} + \gamma_\tau \tau \tag{3.5}$$

with some $\gamma_\tau > 0$, $\gamma_\tau \to 0$ as $\tau \to 0$.

Let $\overline{u}_\tau : [0, +\infty) \to \mathscr{S}$ be the corresponding piecewise constant interpolation, i.e.

$$\overline{u}_\tau(t) \equiv u_\tau^n \text{ if } t \in ((n-1)\tau, n\tau], \; n \in \mathbb{N}, \tag{3.6}$$
$$\overline{u}_\tau(0) = u_\tau^0. \tag{3.7}$$

Main assumption. The next assumption on the interrelation between the relaxed slope $|\partial^- \phi|$ of ϕ and the relaxed minimizing movement scheme along $(\phi_\epsilon)_{\epsilon>0}$ is playing a central role throughout this paper.

Assumption 3.3. We suppose that for all $u, u_\tau \in \mathscr{S}$ ($\tau > 0$) such that

$$u_\tau \overset{\sigma}{\to} u, \; \sup_\tau \left\{ \phi_{\epsilon(\tau)}(u_\tau), d(u_\tau, u) \right\} < +\infty$$

it holds that

$$\liminf_{\tau \to 0} \frac{\phi_{\epsilon(\tau)}(u_\tau) - \mathcal{Y}_\tau \phi_{\epsilon(\tau)}(u_\tau)}{\tau} \geq \frac{1}{2}|\partial^- \phi|^2(u). \tag{3.8}$$

As defined in Section 2.3, $\mathcal{Y}_\tau \phi_{\epsilon(\tau)}$ denotes the Moreau−Yosida approximation of the functional $\phi_{\epsilon(\tau)}$. Now, our theorem reads as follows.

Theorem 3.4. *Let the Assumptions* 3.1, 3.2 *and* 3.3 *be satisfied and construct* $\overline{u}_\tau : [0, +\infty) \to \mathscr{S}$ $(\tau > 0)$ *according to* (3.4)−(3.7).

Then there exist a locally absolutely continuous curve $u : [0, +\infty) \to \mathscr{S}$ *and a subsequence* $(\overline{u}_{\tau_k})_{k \in \mathbb{N}}$ *such that*

$$\overline{u}_{\tau_k}(t) \xrightarrow{\sigma} u(t) \text{ for all } t \geq 0 \tag{3.9}$$

and u *satisfies the initial condition*

$$u(0) = u^0 \tag{3.10}$$

and the energy inequality

$$\phi(u(0)) - \phi(u(t)) \geq \frac{1}{2}\int_0^t |\partial^- \phi|^2(u(s)) \, \mathrm{d}s + \frac{1}{2}\int_0^t |u'|^2(s) \, \mathrm{d}s \tag{3.11}$$

for all $t \geq 0$.

In particular, if the relaxed slope of ϕ *is a strong upper gradient for* ϕ, *then equality holds in* (3.11) *and*

$$u \text{ is a curve of maximal slope for } \phi \text{ with respect to } |\partial^- \phi|. \tag{3.12}$$

Some comments on our assumptions. Please be aware of the fact that we do not require any lower semi-continuity or compactness property of the single functionals ϕ_ϵ. Since we are allowing a relaxed form of minimization (3.5) in our scheme, the coercivity (3.1) of every functional ϕ_ϵ is sufficient to guarantee the existence of solutions to (3.5) for small $\tau > 0$.

Some equi-coercivity and combined compactness assumption on $(\phi_\epsilon)_{\epsilon > 0}$ such as (3.1) and (3.2) in Assumption 3.1 are needed in order to prove the existence of a subsequence $(\overline{u}_{\tau_k})_{k \in \mathbb{N}}$ pointwise σ-converging to a locally absolutely continuous curve.

The energy inequality (3.11) can then be proved by using (3.4) and Assumptions 3.2 and 3.3 which make the passage to ϕ and $|\partial^- \phi|$ possible. Some kind of assumption appropriately connecting the relaxed slope of ϕ with the energy driven motion along the sequence $(\phi_{\epsilon(\tau)})_{\tau > 0}$ with time steps $\tau > 0$ is necessary and our Assumption 3.3 will turn out to be a good choice.

3.2. Proof

We prove Theorem 3.4.

Proof. The proof divides into three main steps.

Existence of u. The proof of the existence of a subsequence $(\overline{u}_{\tau_k})_{k \in \mathbb{N}}$ pointwise σ-converging to a locally absolutely continuous curve u follows similar arguments as the proof of the existence of a generalized minimizing movement in the classical scheme along a single functional ψ ([2], Chap. 3).

We have already noticed that for $n \in \mathbb{N}$ and small $\tau > 0$ there exist $u_\tau^n \in \mathscr{S}$ satisfying (3.5). Moreover, for every $\delta > 0$ it holds that

$$
\frac{1}{2} d^2(u_\tau^n, u_\star) - \frac{1}{2} d^2(u_\tau^0, u_\star) = \sum_{j=1}^n \left(\frac{1}{2} d^2(u_\tau^j, u_\star) - \frac{1}{2} d^2(u_\tau^{j-1}, u_\star) \right)
$$

$$
\leq \sum_{j=1}^n d(u_\tau^j, u_\tau^{j-1}) d(u_\tau^j, u_\star)
$$

$$
\leq \delta \sum_{j=1}^n \frac{d^2(u_\tau^j, u_\tau^{j-1})}{2\tau} + \frac{1}{2\delta} \sum_{j=1}^n \tau d^2(u_\tau^j, u_\star)
$$

$$
\leq \delta \sum_{j=1}^n \left(\phi_{\epsilon(\tau)}(u_\tau^{j-1}) - \phi_{\epsilon(\tau)}(u_\tau^j) + \gamma_\tau \tau \right) + \frac{1}{2\delta} \sum_{j=1}^n \tau d^2(u_\tau^j, u_\star)
$$

$$
\leq \delta(\phi_{\epsilon(\tau)}(u_\tau^0) + A + B d^2(u_\tau^n, u_\star) + \gamma_\tau n\tau) + \frac{1}{2\delta} \sum_{j=1}^n \tau d^2(u_\tau^j, u_\star).
$$

Choose $\delta := \frac{1}{4B}$. Then we have

$$
d^2(u_\tau^n, u_\star) \leq 2 d^2(u_\tau^0, u_\star) + \frac{1}{B} \phi_{\epsilon(\tau)}(u_\tau^0) + \frac{A}{B} + \frac{\gamma_\tau n\tau}{B} + \frac{B}{2} \sum_{j=1}^n \tau d^2(u_\tau^j, u_\star).
$$

By applying the discrete version of Gronwall lemma stated below we obtain that for every $T > 0$ there exists a constant $C > 0$ such that

$$
d^2(u_\tau^n, u_\star) \leq C
$$

whenever $n\tau \leq T$, $\tau \leq \frac{1}{B}$.

Lemma 3.5 (A discrete version of Gronwall lemma, [2]). *Let $A_1, \alpha \in [0, +\infty)$ and, for $n \geq 1$, let $a_n, \tau_n \in [0, +\infty)$ be satisfying*

$$
a_n \leq A_1 + \alpha \sum_{j=1}^n \tau_j a_j \ \forall n \geq 1, \ m := \sup_{n \in \mathbb{N}} \alpha \tau_n < 1.
$$

Then, setting $\beta = \frac{\alpha}{1-m}$, $A_2 := \frac{A_1}{1-m}$ and $\tau_0 = 0$, we have

$$
a_n \leq A_2 e^{\beta \sum_{i=0}^{n-1} \tau_i} \ \forall n \geq 1.
$$

In addition, the following estimates hold

$$
\phi_{\epsilon(\tau)}(u_\tau^n) \leq \gamma_\tau n\tau + \phi_{\epsilon(\tau)}(u_\tau^0)
$$

and

$$
\sum_{j=1}^n \frac{d^2(u_\tau^j, u_\tau^{j-1})}{2\tau} \leq \phi_{\epsilon(\tau)}(u_\tau^0) - \phi_{\epsilon(\tau)}(u_\tau^n) + \gamma_\tau n\tau
$$

$$
\leq \phi_{\epsilon(\tau)}(u_\tau^0) + A + B d^2(u_\tau^n, u_\star) + \gamma_\tau n\tau.
$$

We define $|U_\tau'| : [0, +\infty) \to [0, +\infty)$ by

$$
|U_\tau'|(t) := \frac{d(u_\tau^j, u_\tau^{j-1})}{\tau} \text{ if } t \in ((j-1)\tau, j\tau].
$$

The last estimate shows that there exist $\tau_k \downarrow 0$, $\mathcal{A} \in L^2_{loc}([0, +\infty))$ such that

$$|U'_{\tau_k}| \text{ converges weakly in } L^2(0, T) \text{ to } \mathcal{A} \text{ for all } T > 0.$$

Now, we can apply the refined version of Ascoli−Arzelà theorem stated below to conclude from the preceding estimates and from (3.2) that there exist a further subsequence again denoted by $(\tau_k)_{k \in \mathbb{N}}$ and a curve $u : [0, +\infty) \to \mathscr{S}$ such that

$$\overline{u}_{\tau_k}(t) \xrightarrow{\sigma} u(t) \text{ for all } t \geq 0.$$

Doing so we choose

$$\omega(s, t) := \int_s^t \mathcal{A}(r) \, \mathrm{d}r$$

since for $0 \leq s \leq t$ we have

$$\limsup_{k \to \infty} d(\overline{u}_{\tau_k}(s), \overline{u}_{\tau_k}(t)) \ \leq \ \limsup_{k \to \infty} \int_s^{t+\tau_k} |U'_{\tau_k}|(r) \, \mathrm{d}r \ = \ \int_s^t \mathcal{A}(r) \, \mathrm{d}r.$$

Lemma 3.6 (A refined version of Ascoli−Arzelà theorem, [2])**.** *Let $T > 0$, let $K \subset \mathscr{S}$ be a sequentially compact set w.r.t. σ, and let $u_n : [0, T] \to \mathscr{S}$ be curves such that*

$$u_n(t) \in K \ \forall n \in \mathbb{N}, \ t \in [0, T],$$
$$\limsup_{n \to \infty} d(u_n(s), u_n(t)) \leq \omega(s, t) \ \forall s, t \in [0, T],$$

for a (symmetric) function $\omega : [0, T] \times [0, T] \to [0, +\infty)$, such that

$$\lim_{(s,t) \to (r,r)} \omega(s, t) = 0 \ \forall r \in [0, T] \setminus \mathscr{C},$$

where $\mathscr{C}$ is an (at most) countable subset of $[0, T]$.
   Then there exist $n_k \uparrow +\infty$, $u : [0, T] \to \mathscr{S}$ such that

$$u_{n_k}(t) \xrightarrow{\sigma} u(t) \ \forall t \in [0, T], \ u \text{ is } d\text{-continuous in } [0, T] \setminus \mathscr{C}.$$

It remains to prove that u is locally absolutely continuous. Using the σ-lower semicontinuity of the distance d we obtain for all $0 \leq s \leq t$

$$d(u(s), u(t)) \ \leq \ \liminf_{k \to \infty} d(\overline{u}_{\tau_k}(s), \overline{u}_{\tau_k}(t)) \ \leq \ \int_s^t \mathcal{A}(r) \, \mathrm{d}r.$$

Thus, u is locally absolutely continuous with $|u'| \leq \mathcal{A}$ $\mathscr{L}^1$-a.e. and

$$\int_0^t |u'|^2(s) \, \mathrm{d}s \ \leq \ \liminf_{k \to \infty} \int_0^t |U'_{\tau_k}|^2(s) \, \mathrm{d}s \tag{3.13}$$

for all $t \geq 0$.

Energy inequality. We prove (3.11). Let $t > 0$ be given.
   For $\tau > 0$ fixed, we choose $N_\tau \in \mathbb{N}$ such that $t \in ((N_\tau - 1)\tau, N_\tau \tau]$. Then we have

$$\phi_{\epsilon(\tau)}(u^0_\tau) - \phi_{\epsilon(\tau)}(\overline{u}_\tau(t)) \ = \ \sum_{j=1}^{N_\tau} \big(\phi_{\epsilon(\tau)}(u^{j-1}_\tau) - \phi_{\epsilon(\tau)}(u^j_\tau)\big)$$

$$\geq \sum_{j=1}^{N_\tau} \big(\phi_{\epsilon(\tau)}(u^{j-1}_\tau) - \mathcal{Y}_\tau \phi_{\epsilon(\tau)}(u^{j-1}_\tau) - \gamma_\tau \tau\big) \ + \ \sum_{j=1}^{N_\tau} \frac{1}{2\tau} d^2(u^j_\tau, u^{j-1}_\tau)$$

$$= \sum_{j=1}^{N_\tau} \left(\int_0^\tau \frac{\phi_{\epsilon(\tau)}(u^{j-1}_\tau) - \mathcal{Y}_\tau \phi_{\epsilon(\tau)}(u^{j-1}_\tau)}{\tau} \, \mathrm{d}\eta + \frac{1}{2} \int_{(j-1)\tau}^{j\tau} |U'_\tau|^2(r) \, \mathrm{d}r - \gamma_\tau \tau \right).$$

Since for $j \geq 2$ it holds that

$$u_\tau^{j-1} = \overline{u}_\tau(\underbrace{(j-2)\tau + \eta}_{=:s}) \text{ for } 0 < \eta < \tau,$$

we obtain

$$\int_0^\tau \frac{\phi_{\epsilon(\tau)}(u_\tau^{j-1}) - \mathcal{Y}_\tau \phi_{\epsilon(\tau)}(u_\tau^{j-1})}{\tau} \, \mathrm{d}\eta = \int_{(j-2)\tau}^{(j-1)\tau} \frac{\phi_{\epsilon(\tau)}(\overline{u}_\tau(s)) - \mathcal{Y}_\tau \phi_{\epsilon(\tau)}(\overline{u}_\tau(s))}{\tau} \, \mathrm{d}s$$

for $2 \leq j \leq N_\tau$.

Thus, the following inequality holds

$$\phi_{\epsilon(\tau)}(u_\tau^0) - \phi_{\epsilon(\tau)}(\overline{u}_\tau(t)) \geq \int_0^{(N_\tau-1)\tau} \frac{\phi_{\epsilon(\tau)}(\overline{u}_\tau(s)) - \mathcal{Y}_\tau \phi_{\epsilon(\tau)}(\overline{u}_\tau(s))}{\tau} \, \mathrm{d}s$$
$$+ \frac{1}{2} \int_0^t |U_\tau'|^2(s) \, \mathrm{d}s - \gamma_\tau N_\tau \tau.$$

Now, let $\tau_k \downarrow 0$, $u : [0, +\infty) \to \mathscr{S}$ satisfy (3.9) (for details see the first part of the proof). Assumption 3.2 and (3.4) imply that

$$\phi(u(0)) - \phi(u(t)) \geq \limsup_{k \to \infty} \left(\phi_{\epsilon(\tau_k)}(u_{\tau_k}^0) - \phi_{\epsilon(\tau_k)}(\overline{u}_{\tau_k}(t)) \right).$$

All in all we obtain

$$\phi(u(0)) - \phi(u(t)) \geq \liminf_{k \to \infty} \int_0^{(N_{\tau_k}-1)\tau_k} \frac{\phi_{\epsilon(\tau_k)}(\overline{u}_{\tau_k}(s)) - \mathcal{Y}_{\tau_k} \phi_{\epsilon(\tau_k)}(\overline{u}_{\tau_k}(s))}{\tau_k} \, \mathrm{d}s$$
$$+ \liminf_{k \to \infty} \frac{1}{2} \int_0^t |U_{\tau_k}'|^2(s) \, \mathrm{d}s.$$

We used the fact that $N_\tau \tau \to t$ and thus $\gamma_\tau N_\tau \tau \to 0$ ($\tau \to 0$).

The energy inequality (3.11)

$$\phi(u(0)) - \phi(u(t)) \geq \frac{1}{2} \int_0^t |\partial^- \phi|^2(u(s)) \, \mathrm{d}s + \frac{1}{2} \int_0^t |u'|^2(s) \, \mathrm{d}s$$

follows by applying (3.13), Fatou's lemma and Assumption 3.3.

u is a curve of maximal slope. Let u be a locally absolutely continuous curve satisfying (3.9)−(3.11) and let $|\partial^- \phi|$ be a strong upper gradient for ϕ. Then, by definition, for all $t > 0$ it holds that

$$\phi(u(0)) - \phi(u(t)) \leq \int_0^t |\partial^- \phi|(u(r))|u'|(r) \, \mathrm{d}r$$
$$\leq \frac{1}{2} \int_0^t |\partial^- \phi|^2(u(r)) \, \mathrm{d}r + \frac{1}{2} \int_0^t |u'|^2(r) \, \mathrm{d}r.$$

Hence, since by (3.11) also the reverse inequality holds, we obtain

$$\phi(u(s)) - \phi(u(t)) = \frac{1}{2} \int_s^t |\partial^- \phi|^2(u(r)) \, \mathrm{d}r + \frac{1}{2} \int_s^t |u'|^2(r) \, \mathrm{d}r \qquad (3.14)$$

for all $0 \leq s \leq t$.

The proof of Theorem 3.4 is complete. $\qquad \square$

3.3. Supplement to Theorem 3.4

Convergence condition. The following convergence condition (3.15) in case $|\partial^- \phi|$ is only a weak upper gradient for ϕ is the counterpart of the continuity condition (2.11) in Section 2.3:

$$v_n \xrightarrow{\sigma} v, \ \sup_{n \in \mathbb{N}} \{\chi_{\epsilon_n, \tau_n}(v_n), d(v_n, v), \phi_{\epsilon_n}(v_n)\} < +\infty \ \Rightarrow \ \phi_{\epsilon_n}(v_n) \to \phi(v) \tag{3.15}$$

for $\tau_n \to 0$ ($\tau_n > 0$), with $\chi_{\epsilon, \tau}(\cdot) := \frac{\phi_\epsilon(\cdot) - \mathcal{Y}_\tau \phi_\epsilon(\cdot)}{\tau}$ and $\epsilon_n := \epsilon(\tau_n)$.

Proposition 3.7. *Let the Assumptions* 3.1, 3.2 *and* 3.3, *and in addition the convergence condition* (3.15) *be satisfied. Let* $u : [0, +\infty) \to \mathscr{S}$ *be constructed according to* (3.9) *in Theorem* 3.4. *Then the energy dissipation inequality*

$$\phi(u(s)) - \phi(u(t)) \ \geq \ \frac{1}{2} \int_s^t |\partial^- \phi|^2(u(r)) \ \mathrm{d}r \ + \ \frac{1}{2} \int_s^t |u'|^2(r) \ \mathrm{d}r \tag{3.16}$$

holds for all $t \geq 0$ *and a.e.* $s \in (0, t)$.

In particular, if $|\partial^- \phi|$ *is a weak upper gradient for* ϕ, *then* u *is a curve of maximal slope for* ϕ *with respect to* $|\partial^- \phi|$.

Proof. Note that the calculations in the second step of the proof of Theorem 3.4 show that for every $t > 0$ and for $\mathscr{L}^1$-a.e. $s \in (0, t)$ there exists a further subsequence $(\overline{u}_{\tau_{k_l}})_{l \in \mathbb{N}}$ such that

$$\sup_{l \in \mathbb{N}} \chi_{\epsilon(\tau_{k_l}), \tau_{k_l}}(\overline{u}_{\tau_{k_l}}(s)) < +\infty.$$

Then, by using (3.15) and the same arguments as in the proof of the energy inequality (3.11), we can conclude. $\square$

Generalization to a non-uniform distribution of the error. The proof of Theorem 3.4 shows that we may replace the error $\gamma_\tau \tau$ in the approximate minimization problem (3.5) by a more general $\gamma_\tau^{(n)}$ depending on $n \in \mathbb{N}$. Indeed, the convergence

$$\sum_{j=1}^{N_\tau} \gamma_\tau^{(j)} \to 0 \text{ as } \tau \to 0 \tag{3.17}$$

for $N_\tau \in \mathbb{N}$ as in the second step of the proof, is sufficient for our purposes.

Hence, we can extend our theory to a non-uniform distribution of the error and to the following relaxed minimizing movement scheme along $(\phi_\epsilon)_{\epsilon > 0}$:

The sequence of initial values $u_\tau^0 \xrightarrow{\sigma} u^0 \in D(\phi)$ ($\tau \to 0$) satisfies

$$\sup_\tau d(u_\tau^0, u^0) < +\infty, \ \phi_{\epsilon(\tau)}(u_\tau^0) \to \phi(u^0) \tag{3.18}$$

and u_τ^n ($\tau > 0, n \in \mathbb{N}$) satisfies

$$\phi_{\epsilon(\tau)}(u_\tau^n) + \frac{1}{2\tau} d^2(u_\tau^n, u_\tau^{n-1}) \ \leq \ \inf_{v \in \mathscr{S}} \left\{ \phi_{\epsilon(\tau)}(v) + \frac{1}{2\tau} d^2(v, u_\tau^{n-1}) \right\} + \gamma_\tau^{(n)} \tag{3.19}$$

with some $(\gamma_\tau^{(n)})_{n \in \mathbb{N}}$, $\gamma_\tau^{(n)} > 0$ and $\sum_{j=1}^{N_\tau} \gamma_\tau^{(j)} \to 0$ as $\tau \to 0$ whenever $(N_\tau \tau)_{\tau > 0}$ is bounded.

For the sake of clear presentation, we come back to considering a uniform distribution of the error in the remaining part of the paper. Note that all the results in this paper can be obtained for a non-uniform distribution of the error as well.

4. The case $\phi_\epsilon = \phi_{\tilde{\epsilon}}$ for all $\epsilon, \tilde{\epsilon} > 0$

The special case in which we have $\phi_\epsilon = \phi_{\tilde{\epsilon}}$ for all $\epsilon, \tilde{\epsilon} > 0$ is worth considering in order to better understand Theorem 3.4 and our main Assumption 3.3. Moreover, we will obtain interesting relaxation results.

4.1. Assumption 3.3 with regard to a single functional

We give a positive answer to the question if there is an interrelation corresponding to Assumption 3.3 between the (relaxed) minimizing movement scheme along a single functional $\psi : \mathscr{S} \to (-\infty, +\infty]$ and its relaxed slope.

Proposition 4.1. *Let $\psi : \mathscr{S} \to (-\infty, +\infty]$, $D(\psi) \neq \emptyset$, satisfy the following assumptions:*

$$\psi \text{ is } d\text{-lower semicontinuous}, \tag{4.1}$$

and there exists $\tau_\star > 0$ such that

$$J_\tau \psi[u] \neq \emptyset \text{ for all } u \in \mathscr{S}, \ \tau \in (0, \tau_\star). \tag{4.2}$$

Then whenever $u, u_\tau \in \mathscr{S}$ $(\tau > 0)$ satisfy

$$u_\tau \overset{\sigma}{\rightharpoonup} u, \ \sup_\tau \{\psi(u_\tau), d(u_\tau, u)\} < +\infty$$

it holds that

$$\liminf_{\tau \to 0} \frac{\psi(u_\tau) - \mathcal{Y}_\tau \psi(u_\tau)}{\tau} \geq \frac{1}{2} |\partial^- \psi|^2(u). \tag{4.3}$$

Proof. Note that (4.2) implies that there exist $A, B > 0$, $u_\star \in \mathscr{S}$ such that

$$\psi(\cdot) \geq -A - B d^2(\cdot, u_\star).$$

Let $u, u_\tau \in \mathscr{S}$ $(\tau > 0)$ satisfy

$$u_\tau \overset{\sigma}{\rightharpoonup} u, \ \sup_\tau \{\psi(u_\tau), d(u_\tau, u)\} < +\infty.$$

For $\pi \in (0, 1)$ we set $v_{\pi \cdot \tau} \in J_{\pi \cdot \tau} \psi[u_\tau]$. We have

$$d^2(v_{\pi \cdot \tau}, u_\tau) \leq 2\pi\tau(\psi(u_\tau) - \psi(v_{\pi \cdot \tau}))$$
$$\leq 2\pi\tau(\psi(u_\tau) + A + B d^2(v_{\pi \cdot \tau}, u_\star))$$

leading to

$$d^2(v_{\pi \cdot \tau}, u_\tau)[1 - 2\pi\tau B] - 4\pi\tau B d(u_\tau, u_\star) d(v_{\pi \cdot \tau}, u_\tau)$$
$$\leq 2\pi\tau\psi(u_\tau) + 2\pi\tau A + 2\pi\tau B d^2(u_\tau, u_\star).$$

We can conclude that

$$d(v_{\pi \cdot \tau}, u_\tau) \to 0 \text{ as } \tau \to 0$$

and $(v_{\pi \cdot \tau})_{\tau > 0}$ is an admissible test sequence in the definition of $|\partial^- \psi|(u)$.

Now, it holds that

$$\frac{1}{2} |\partial^- \psi|^2(u) = \frac{1}{2} \int_0^1 |\partial^- \psi|^2(u) \, d\pi \leq \frac{1}{2} \int_0^1 \liminf_{\tau \to 0} |\partial\psi|^2(v_{\pi \cdot \tau}) \, d\pi$$

$$\leq \frac{1}{2} \int_0^1 \liminf_{\tau \to 0} \frac{d^2(v_{\pi \cdot \tau}, u_\tau)}{(\pi \cdot \tau)^2} \, d\pi \leq \liminf_{\tau \to 0} \frac{1}{2} \int_0^1 \frac{d^2(v_{\pi \cdot \tau}, u_\tau)}{(\pi \cdot \tau)^2} \, d\pi$$

$$= \liminf_{\tau \to 0} \frac{1}{\tau} \int_0^\tau \frac{d^2(v_\varsigma, u_\tau)}{2\varsigma^2} \, d\varsigma = \liminf_{\tau \to 0} \frac{\psi(u_\tau) - \mathcal{Y}_\tau \psi(u_\tau)}{\tau}.$$

We applied Fatou's Lemma and the following lemma with collected statements from ([2], Chap. 3):

Lemma 4.2 ([2]). *Let $\psi : \mathscr{S} \to (-\infty, +\infty]$ be given. For all $w \in \mathscr{S}$, $\tau > 0$ and $w_\tau \in J_\tau \psi[w]$ it holds that*

$$|\partial \psi|(w_\tau) \ \leq \ \frac{d(w_\tau, w)}{\tau}. \tag{4.4}$$

We set

$$d_\tau^+(w) := \sup_{w_\tau \in J_\tau[w]} d(w_\tau, w), \ d_\tau^-(w) := \inf_{w_\tau \in J_\tau[w]} d(w_\tau, w) \ (w \in \mathscr{S}, \ \tau > 0).$$

Now, let ψ be d-lower semicontinuous and satisfy (4.2) for all $\tau \in (0, \tau_\star)$ and let $w \in D(\psi)$. Then there exists an (at most) countable set $\mathscr{N}_w \subset (0, \tau_\star)$ such that

$$d_\tau^-(w) \ = \ d_\tau^+(w) \ for \ all \ \tau \in (0, \tau_\star) \setminus \mathscr{N}_w$$

and the map $\tau \mapsto \frac{d_\tau^\pm(w)}{\tau}$ has finite pointwise variation in (τ_0, τ_1) for every $0 < \tau_0 < \tau_1 < \tau_\star$. Moreover, it holds for all $\tau \in (0, \tau_\star)$ and $w_\tau \in J_\tau \psi[w]$ that

$$\frac{d^2(w_\tau, w)}{2\tau} \ + \ \int_0^\tau \frac{(d_r^\pm(w))^2}{2r^2} \ \mathrm{d}r \ = \ \psi(w) - \psi(w_\tau). \tag{4.5}$$

The proof of Proposition 4.1 is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4.2. Relaxation results

Relaxed form of minimization. We come back to the classical minimizing movement scheme for gradient flows (Sect. 2.3) along a single functional used to construct curves of maximal slope [2]. Let us introduce a relaxed form of minimization in each step of the scheme. Still we obtain curves of maximal slope. Let $\psi : \mathscr{S} \to (-\infty, +\infty]$ be given.

For every time step $\tau > 0$, find a sequence $(u_\tau^n)_{n \in \mathbb{N}}$ by the following scheme.

The sequence of initial values $u_\tau^0 \overset{\sigma}{\to} u^0 \in D(\psi)$ $(\tau \to 0)$ satisfies

$$\sup_\tau d(u_\tau^0, u^0) < +\infty, \ \psi(u_\tau^0) \to \psi(u^0) \tag{4.6}$$

and $u_\tau^n \in \mathscr{S}$ $(n \in \mathbb{N})$ satisfies

$$\psi(u_\tau^n) + \frac{1}{2\tau} d^2(u_\tau^n, u_\tau^{n-1}) \ \leq \ \inf_{v \in \mathscr{S}} \left\{ \psi(v) + \frac{1}{2\tau} d^2(v, u_\tau^{n-1}) \right\} + \gamma_\tau \tau \tag{4.7}$$

with some $\gamma_\tau > 0$, $\gamma_\tau \to 0$ as $\tau \to 0$.

Let $\overline{u}_\tau : [0, +\infty) \to \mathscr{S}$ be the corresponding piecewise constant interpolation, *i.e.*

$$\overline{u}_\tau(t) \equiv u_\tau^n \ \text{if} \ t \in ((n-1)\tau, n\tau], \ n \in \mathbb{N}, \tag{4.8}$$

$$\overline{u}_\tau(0) = u_\tau^0. \tag{4.9}$$

Then the following holds.

Theorem 4.3. *Let ψ satisfy Assumption 2.7 and $\overline{u}_\tau : [0, +\infty) \to \mathscr{S}$ $(\tau > 0)$ be constructed according to (4.6)−(4.9). Then there exist a locally absolutely continuous curve $u : [0, +\infty) \to \mathscr{S}$ and a subsequence $(\overline{u}_{\tau_k})_{k \in \mathbb{N}}$ such that*

$$\overline{u}_{\tau_k}(t) \overset{\sigma}{\to} u(t) \ for \ all \ t \geq 0 \tag{4.10}$$

and u satisfies the initial condition

$$u(0) = u^0 \tag{4.11}$$

and the energy inequality (2.7) for all $t \geq 0$. In particular, if $|\partial^- \psi|$ is a strong upper gradient for ψ, then

$$u \ is \ a \ curve \ of \ maximal \ slope \ for \ \psi \ with \ respect \ to \ |\partial^- \psi|. \tag{4.12}$$

Proof. We apply Theorem 3.4 with $\phi_\epsilon = \phi = \psi$ for all $\epsilon > 0$. Assumption 2.7 corresponds to the Assumptions 3.1 and 3.2. It can be proved ([2], Chap. 2) that Assumption 2.7 implies that (4.2) holds for $\tau > 0$ small enough. Thus, Proposition 4.1 shows that Assumption 3.3 is also satisfied in this case. □

There is some possibility that the relaxed minimizing movement scheme produces more curves of maximal slope than the classical scheme. We give a simple example.

Example 4.4. We apply our scheme (4.6)−(4.7) to $\psi(x) = -\frac{1}{\alpha}|x|^\alpha$ with $1 < \alpha < 2$, $\psi : \mathbb{R} \to \mathbb{R}$, and initial values $u_\tau^0 = u^0 = 0$ for all $\tau > 0$. Doing so we also obtain the trivial solution $u \equiv 0$ to the corresponding gradient flow equation since the choice $u_\tau^n = 0$ ($\tau > 0, n \in \mathbb{N}$) is admissible in (4.7). This solution cannot be reached by the classical scheme.

The error order $o(\tau)$ in (3.5) and (4.7) is optimal. The following example shows that we cannot expect to obtain curves of maximal slope if we allow an error greater than of order $o(\tau)$.

Example 4.5. We consider $\psi : \mathbb{R} \to \mathbb{R}$, $\psi(x) = \frac{1}{2}x^2$ with initial values $u_\tau^0 = u^0 = 0$ for all $\tau > 0$. Let us choose $u_\tau^1 = \tau$ and $u_\tau^n = \frac{n\tau}{(1+\tau)^{n-1}}$ for $n \geq 2$. Then we have

$$\psi(u_\tau^n) + \frac{1}{2\tau}d^2(u_\tau^n, u_\tau^{n-1}) \leq \mathcal{Y}_\tau \psi(u_\tau^{n-1}) + \tau$$

for all $0 < \tau < 1$ and $n \in \mathbb{N}$, and the limit curve $u : [0, +\infty) \to \mathbb{R}$, $u(t) = te^{-t}$ does not solve the corresponding gradient flow equation.

Lower semicontinuous envelope relaxation. Now, we consider a functional $\psi : \mathscr{S} \to (-\infty, +\infty]$ satisfying the coercivity and compactness conditions (2.8) and (2.10), but for which the minimization problems in the classical minimizing movement scheme for gradient flows (Sect. 2.3) need not have any solutions due to missing lower semicontinuity (2.9).

However, the approximate minimization problems (4.7) are solvable and we can follow the steepest descent movements in our relaxed scheme along the functional ψ. They are closely related to the gradient flow motion of the lower semicontinuous envelope $\psi_{sc} : \mathscr{S} \to (-\infty, +\infty]$ of ψ.

To be more precise, let $\psi_{sc} : \mathscr{S} \to (-\infty, +\infty]$ be both the d-lower semicontinuous and a weakened form of the σ-lower semicontinuous envelope of the functional ψ, *i.e.*

$$\sup_{n,m} d(u_n, u_m) < +\infty, \ u_n \overset{\sigma}{\rightharpoonup} u \ \Rightarrow \ \liminf_{n \to \infty} \psi(u_n) \geq \psi_{sc}(u) \tag{4.13}$$

and for all $u \in D(\psi_{sc})$ there exists $(v_n)_{n \in \mathbb{N}} \subset \mathscr{S}$ such that

$$v_n \overset{d}{\to} u, \ \psi(v_n) \to \psi_{sc}(u). \tag{4.14}$$

Moreover, let initial values $u_\tau^0 \overset{\sigma}{\rightharpoonup} u^0 \in D(\psi_{sc})$ ($\tau \to 0$) be given satisfying

$$\sup_\tau d(u_\tau^0, u^0) < +\infty, \ \psi(u_\tau^0) \to \psi_{sc}(u^0). \tag{4.15}$$

Theorem 4.6. *We assume that ψ satisfies (2.8) and (2.10) and that ψ_{sc} is given by (4.13) and (4.14). We construct $\overline{u}_\tau : [0, +\infty) \to \mathscr{S}$ according to (4.15), (4.7), (4.8) and (4.9).*

Then there exist a locally absolutely continuous curve $u : [0, +\infty) \to \mathscr{S}$ and a subsequence $(\overline{u}_{\tau_k})_{k \in \mathbb{N}}$ such that

$$\overline{u}_{\tau_k}(t) \overset{\sigma}{\rightharpoonup} u(t) \text{ for all } t \geq 0 \tag{4.16}$$

and u satisfies the initial condition

$$u(0) = u^0 \tag{4.17}$$

and the energy inequality

$$\psi_{sc}(u(0)) - \psi_{sc}(u(t)) \geq \frac{1}{2} \int_0^t |\partial^- \psi_{sc}|^2(u(s)) \, \mathrm{d}s + \frac{1}{2} \int_0^t |u'|^2(s) \, \mathrm{d}s \tag{4.18}$$

for all $t \geq 0$.

In particular, if the relaxed slope of ψ_{sc} is a strong upper gradient, then

$$u \text{ is a curve of maximal slope for } \psi_{sc} \text{ with respect to } |\partial^- \psi_{sc}|. \tag{4.19}$$

Proof. We apply Theorem 3.4 with $\phi_\epsilon = \psi$ for all $\epsilon > 0$ and $\phi = \psi_{sc}$. The Assumptions 3.1 and 3.2 are clearly satisfied. It remains to be checked if Assumption 3.3 is fulfilled.

We can conclude from the fact that the conditions (2.8) and (2.10) hold for the functional ψ and from (4.13), (4.14) that Assumption 2.7 is satisfied for the functional ψ_{sc}. It can be proved [2] that if a functional satisfies Assumption 2.7 then it also satisfies (4.2). So for small $\tau > 0$ we have

$$J_\tau \psi_{sc}[w] \neq \emptyset \text{ for all } w \in \mathscr{S}.$$

Hence, Proposition 4.1 is applicable to ψ_{sc} and it holds that

$$\liminf_{\tau \to 0} \frac{\psi_{sc}(u_\tau) - \mathcal{Y}_\tau \psi_{sc}(u_\tau)}{\tau} \geq \frac{1}{2} |\partial^- \psi_{sc}|^2(u) \tag{4.20}$$

whenever $u, u_\tau \in \mathscr{S}$ ($\tau > 0$) satisfy

$$u_\tau \stackrel{\sigma}{\to} u, \ \sup_\tau \{\psi_{sc}(u_\tau), d(u_\tau, u)\} < +\infty.$$

Now, let $u, u_\tau \in \mathscr{S}$ ($\tau > 0$) with

$$u_\tau \stackrel{\sigma}{\to} u, \ \sup_\tau \{\psi(u_\tau), d(u_\tau, u)\} < +\infty$$

be given. For $\tau > 0$ fixed, we set $v_\tau \in J_\tau \psi_{sc}[u_\tau]$, we let $(v_\tau^k)_{k \in \mathbb{N}}$ be a recovery sequence for v_τ according to (4.14), i.e. $v_\tau^k \stackrel{d}{\to} v_\tau$, $\psi(v_\tau^k) \to \psi_{sc}(v_\tau)$ ($k \to \infty$), and we obtain

$$\frac{\psi(u_\tau) - \mathcal{Y}_\tau \psi(u_\tau)}{\tau} + \frac{\mathcal{Y}_\tau \psi_{sc}(u_\tau) - \psi_{sc}(u_\tau)}{\tau} \geq \frac{\mathcal{Y}_\tau \psi_{sc}(u_\tau) - \mathcal{Y}_\tau \psi(u_\tau)}{\tau}$$

$$\geq \frac{\psi_{sc}(v_\tau) - \psi(v_\tau^k)}{\tau} + \frac{1}{2\tau^2} \left[d^2(v_\tau, u_\tau) - d^2(v_\tau^k, u_\tau) \right] \geq -\tau.$$

The last inequality holds if we choose $k \in \mathbb{N}$ large enough for $\tau > 0$ fixed. Please note that we have $\psi_{sc}(w) \leq \psi(w)$ for all $w \in \mathscr{S}$.

All in all we obtain

$$\frac{1}{2} |\partial^- \psi_{sc}|^2(u) \leq \liminf_{\tau \to 0} \frac{\psi_{sc}(u_\tau) - \mathcal{Y}_\tau \psi_{sc}(u_\tau)}{\tau}$$

$$\leq \liminf_{\tau \to 0} \frac{\psi(u_\tau) - \mathcal{Y}_\tau \psi(u_\tau)}{\tau}.$$

The proof of Theorem 4.6 is complete. $\square$

We indirectly revealed that for small $\tau > 0$ we have $\mathcal{Y}_\tau \psi_{sc}(w) = \mathcal{Y}_\tau \psi(w)$ for all $w \in \mathscr{S}$. In fact, we could have proved Theorem 4.6 as a corollary of Theorem 4.3 since for small $\tau > 0$ any sequence u_τ^n constructed in accordance to (4.15), (4.7) is a solution to the relaxed minimizing movement scheme along the functional ψ_{sc}.

Restriction to a dense subset. Now, let a functional $\psi : \mathscr{S} \to (-\infty, +\infty]$ satisfying Assumption 2.7 with strong upper gradient $|\partial^- \psi|$ and a dense subset $\mathscr{V} \subset \mathscr{S}$ such that

$$\forall v \in D(\psi) \; \exists (v_k)_{k \in \mathbb{N}} \subset \mathscr{V} : \; v_k \overset{d}{\to} v, \; \psi(v_k) \to \psi(v)$$

be given. Our relaxation results make it possible to restrict the approximate minimization problems to this dense subset $\mathscr{V} \subset \mathscr{S}$, *i.e.* to replace (4.7) by

$$(\psi + \mathbb{I}_{\mathscr{V}})(u_\tau^n) + \frac{1}{2\tau} d^2(u_\tau^n, u_\tau^{n-1}) \; \leq \; \mathcal{Y}_\tau (\psi + \mathbb{I}_{\mathscr{V}})(u_\tau^{n-1}) + \gamma_\tau \tau \tag{4.21}$$

in which we set $\mathbb{I}_{\mathscr{V}} \equiv 0$ on $\mathscr{V}$, $\mathbb{I}_{\mathscr{V}} \equiv +\infty$ on $\mathscr{S} \setminus \mathscr{V}$. Thus, we construct approximations $\overline{u}_\tau : [0, +\infty) \to \mathscr{V}$ (with particular properties) to curves of maximal slope for ψ with respect to $|\partial^- \psi|$. In the following example, the desired property is higher regularity.

Example 4.7. Let $\Omega \subset \mathbb{R}^n$ be bounded and open. We set

$$\mathscr{S} = L^p(\Omega) \; (1 \leq p < +\infty), \; \sigma = d, \; D(\psi) = W_0^{1,p}(\Omega),$$

$$\psi(u) = \int_\Omega L(x, \nabla u(x)) \, \mathrm{d}x + \int_\Omega j(x, u(x)) \, \mathrm{d}x, \; \text{if } u \in W_0^{1,p}(\Omega)$$

with convex, lower semicontinuous $L(x, \cdot) : \mathbb{R}^n \to \mathbb{R}$, $j(x, \cdot) : \mathbb{R} \to \mathbb{R}$. We suppose that $L(x, \cdot)$ satisfies a growth condition of order p, *i.e.* that there exist $c_1, c_2, c_3, c_4 > 0$ such that

$$c_1 |z|^p - c_2 \; \leq \; L(x, z) \; \leq \; c_3 |z|^p + c_4$$

for all $x \in \mathbb{R}^n$, $z \in \mathbb{R}^n$. Moreover, we suppose that j is bounded from below and that there exists $c > 0$ such that

$$j(x, z) \; \leq \; c(1 + |z|^p)$$

for all $x \in \mathbb{R}^n$, $z \in \mathbb{R}$.
We set

$$\mathscr{V} = C_0^\infty(\Omega).$$

One can easily check that all the required assumptions are satisfied in this example.

5. Convergence of gradient flows

Let us suppose that we have functionals $\phi_\epsilon : \mathscr{S} \to (-\infty, +\infty]$ $(\epsilon > 0)$, $\phi : \mathscr{S} \to (-\infty, +\infty]$ connected through a $\Gamma(\sigma)$-liminf inequality as in Assumption 3.2, with strong upper gradients $|\partial^- \phi_\epsilon|$, $|\partial^- \phi|$, and satisfying a particular property which we call COGF-Property (Convergence Of Gradient Flows):
COGF-Property. If $\vartheta_\epsilon : [0, +\infty) \to \mathscr{S}$ is a curve of maximal slope for ϕ_ϵ with respect to $|\partial^- \phi_\epsilon|$ $(\epsilon > 0)$ and we have

$$\vartheta_\epsilon(t) \overset{\sigma}{\to} \vartheta(t) \text{ for all } t \geq 0, \tag{5.1}$$

$$\phi_\epsilon(\vartheta_\epsilon(0)) \to \phi(\vartheta(0)), \tag{5.2}$$

for a curve $\vartheta : [0, +\infty) \to \mathscr{S}$, then ϑ is a curve of maximal slope for ϕ with respect to $|\partial^- \phi|$.

In addition, we suppose that there exist curves of maximal slope for ϕ_ϵ with respect to $|\partial^- \phi_\epsilon|$, for which (5.1), (5.2) indeed hold. For this to be guaranteed, a natural setting would include both some equi-coercivity and combined compactness property such as in Assumption 3.1 and some lower semicontinuity and compactness property of the single functionals ϕ_ϵ such as (2.9) and (2.10).

Our expectation. Intuitively, in this case, any choice $\epsilon = \epsilon(\tau)$ in (3.4)−(3.5) should be feasible in order to obtain the results of Theorem 3.4.

We specify this idea in the abstract situation that the COGF-Property is a consequence of the established considerations by Sandier and Serfaty [22, 23]. Indeed, our expectation is fulfilled:

Proposition 5.1. *Let $(\phi_\epsilon)_{\epsilon>0}$ satisfy the equi-coercivity condition (3.1) and let the lower semicontinuity and compactness conditions (2.9) and (2.10) hold for the single functionals ϕ_ϵ. If the Serfaty−Sandier condition*

$$u_\epsilon \overset{\sigma}{\rightharpoonup} u \;\Rightarrow\; \liminf_{\epsilon \to 0} |\partial^- \phi_\epsilon|(u_\epsilon) \geq |\partial^- \phi|(u) \tag{5.3}$$

holds, then Assumption 3.3 is satisfied for every choice $\epsilon = \epsilon(\tau) \to 0$ $(\tau \to 0)$.

Proof. Let $\epsilon(\tau) > 0$, $\epsilon(\tau) \to 0$ and $u, u_\tau \in \mathscr{S}$ $(\tau > 0)$ be given such that

$$u_\tau \overset{\sigma}{\rightharpoonup} u, \; \sup_\tau \left\{ \phi_{\epsilon(\tau)}(u_\tau), d(u_\tau, u) \right\} < +\infty.$$

For $\pi \in (0, 1)$ we set $v_{\pi \cdot \tau, \tau} \in J_{\pi \cdot \tau} \phi_{\epsilon(\tau)}[u_\tau]$. Then it holds that

$$\frac{1}{2}|\partial^- \phi|^2(u) \;=\; \frac{1}{2} \int_0^1 |\partial^- \phi|^2(u) \; \mathrm{d}\pi \;\leq\; \frac{1}{2} \int_0^1 \liminf_{\tau \to 0} |\partial \phi_{\epsilon(\tau)}|^2(v_{\pi \cdot \tau, \tau}) \; \mathrm{d}\pi$$

$$\leq \frac{1}{2} \int_0^1 \liminf_{\tau \to 0} \frac{d^2(v_{\pi \cdot \tau, \tau}, u_\tau)}{(\pi \cdot \tau)^2} \; \mathrm{d}\pi \;\leq\; \liminf_{\tau \to 0} \frac{1}{2} \int_0^1 \frac{d^2(v_{\pi \cdot \tau, \tau}, u_\tau)}{(\pi \cdot \tau)^2} \; \mathrm{d}\pi$$

$$= \liminf_{\tau \to 0} \frac{1}{\tau} \int_0^\tau \frac{d^2(v_{\varsigma, \tau}, u_\tau)}{2\varsigma^2} \; \mathrm{d}\varsigma \;=\; \liminf_{\tau \to 0} \frac{\phi_{\epsilon(\tau)}(u_\tau) - \mathcal{Y}_\tau \phi_{\epsilon(\tau)}(u_\tau)}{\tau}.$$

We applied (5.3) and we followed similar arguments as in the proof of Proposition 4.1. □

The λ-convex case. If ϕ, ϕ_ϵ $(\epsilon > 0)$ are λ-convex $(\lambda \in \mathbb{R})$, the σ-topology coincides with the topology induced by the distance d and $\phi_\epsilon \overset{\Gamma}{\to} \phi$, then condition (5.4) on the local slopes can be proved [20]

$$u_\epsilon \to u \;\Rightarrow\; \liminf_{\epsilon \to 0} |\partial \phi_\epsilon|(u_\epsilon) \geq |\partial \phi|(u). \tag{5.4}$$

The Γ-liminf condition (5.5) on the local slopes discussed in [20] can be viewed as discrete counterpart of the Serfaty−Sandier condition (5.3).

Similarly as Proposition 5.1, we can prove

Proposition 5.2. *Let $(\phi_\epsilon)_{\epsilon>0}$ satisfy the equi-coercivity condition (3.1) and let the lower semicontinuity and compactness conditions (2.9) and (2.10) hold for the single functionals ϕ_ϵ. If the following condition on the local slopes holds*

$$u_\epsilon \overset{\sigma}{\rightharpoonup} u \;\Rightarrow\; \liminf_{\epsilon \to 0} |\partial \phi_\epsilon|(u_\epsilon) \geq |\partial \phi|(u), \tag{5.5}$$

then Assumption 3.3 is satisfied for every choice $\epsilon = \epsilon(\tau) \to 0$ $(\tau \to 0)$.

Again, we obtain that in our scheme (3.4)−(3.5) any choice $\epsilon = \epsilon(\tau)$ is feasible in order to apply Theorem 3.4.

In order to prove Proposition 5.1 and 5.2, it would be sufficient to impose the conditions (5.3) and (5.5) respectively on sequences $u_\epsilon \overset{\sigma}{\rightharpoonup} u$ with $\sup_\epsilon \{\phi_\epsilon(u_\epsilon), d(u_\epsilon, u)\} < +\infty$, and weaker conditions than (2.9) and (2.10).

Note: In the special case considered in Section 5, every choice $\epsilon = \epsilon(\tau)$ is possible. In general, the interrelation between the steepest descent movement along a Γ-converging sequence of functionals and the gradient flow motion of its limit functional is more involved.

Various choices of $(\epsilon(\tau))_{\tau>0}$ lead to different motions not coincident with the gradient flow motion of the limit functional. For illustrative purposes, the reader may have a look at (heuristical) comptutations of $J_\tau f_\epsilon[\cdot]$ $(\tau > 0, \epsilon > 0)$ for concrete examples of Γ-converging functionals f_ϵ in [5].

6. Existence of a suitable choice $\epsilon = \epsilon(\tau)$

There always exists a sequence $(\epsilon_\tau)_{\tau>0}$ ($\epsilon_\tau > 0$) such that the following holds: If we associate $\tau > 0$ with $\epsilon = \epsilon(\tau) \leq \epsilon_\tau$ in the relaxed minimizing movement scheme (3.4)−(3.5), our Theorem 3.4 with all its results is applicable.

Theorem 6.1. *We assume that $(\mathscr{S}, d)$ is a separable, complete metric space and the σ-topology coincides with the topology induced by the distance d.*

Let functionals $\phi, \phi_\epsilon : \mathscr{S} \to (-\infty, +\infty]$ ($\epsilon > 0$) be given. We suppose that $(\phi_\epsilon)_{\epsilon>0}$ satisfies Assumption 3.1 and $\phi_\epsilon \xrightarrow{\Gamma} \phi$, i.e.

$$\liminf_{\epsilon \to 0} \phi_\epsilon(u_\epsilon) \geq \phi(u) \text{ for all } u, u_\epsilon \in \mathscr{S}, \ u_\epsilon \xrightarrow{d} u,$$

$$\forall u \in \mathscr{S} \ \exists \tilde{u}_\epsilon \in \mathscr{S} : \ \tilde{u}_\epsilon \xrightarrow{d} u \text{ and } \phi_\epsilon(\tilde{u}_\epsilon) \to \phi(u).$$

Then there exists a sequence $(\epsilon_\tau)_{\tau>0}$ with $\epsilon_\tau > 0$ such that our main Assumption 3.3 is satisfied for all choices $(\epsilon(\tau))_{\tau>0}$ with $\epsilon(\tau) \leq \epsilon_\tau$ ($\epsilon(\tau) \to 0$). In particular, if we choose $\epsilon = \epsilon(\tau) \leq \epsilon_\tau$ for $\tau \to 0$ in (3.4)−(3.5), all the results of Theorem 3.4 hold.

Comment on Theorem 6.1. The special feature of Theorem 6.1 is that the sequence $(\epsilon_\tau)_{\tau>0}$ only depends on the velocity of Γ-convergence of the functionals ϕ_ϵ to the functional ϕ. We notice that the sequence $(\epsilon_\tau)_{\tau>0}$ in Theorem 6.1 is completely independent of initial values $u_\tau^0, u^0 \in \mathscr{S}$ and of (approximate) minimizers u_τ^n in the (relaxed) minimizing movement scheme. We establish a direct connection between the gradient flow motion along a Γ-converging sequence of functionals and the gradient flow motion of its limit functional.

Proof. We prove Theorem 6.1. Note that the functional ϕ satisfies assumption 2.7 since $(\phi_\epsilon)_{\epsilon>0}$ satisfies assumption 3.1 and ϕ_ϵ Γ-converges to ϕ.

On a separable metric space, Γ-convergence is metrizable ([9], Chap. 10) for a certain class of functionals.

Lemma 6.2 (see [9]).

Let us fix a dense subset $\{x_i : i \in \mathbb{N}\}$ of $\mathscr{S}$, a sequence $(\kappa_j)_{j \in \mathbb{N}}$ of positive real numbers converging to 0, and an increasing homeomorphism Φ between $[0, +\infty]$ and $[0, 1]$.

Then, for a sequence of functionals $f_n : \mathscr{S} \to [0, +\infty]$ ($n \in \mathbb{N}$) satisfying

$$(u_n)_{n \in \mathbb{N}} \subset \mathscr{S}, \ \sup_{n \in \mathbb{N}} f_n(u_n) < +\infty \ \Rightarrow \ \exists u \in \mathscr{S}, n_k \uparrow +\infty : \ u_{n_k} \xrightarrow{d} u, \tag{6.1}$$

it holds that $(f_n)_{n \in \mathbb{N}}$ Γ-converges to a functional $f : \mathscr{S} \to [0, +\infty]$ ($f_n \xrightarrow{\Gamma} f$) if and only if we have

$$\delta(f_n, f) \to 0 \ (n \to +\infty), \tag{6.2}$$

with

$$\delta(f_n, f) := \sum_{i,j=1}^{\infty} 2^{-i-j} |\Phi(\mathcal{Y}_{\kappa_j} f_n(x_i)) - \Phi(\mathcal{Y}_{\kappa_j} f(x_i))|. \tag{6.3}$$

We may assume that Φ is Lipschitz continuous.

Now, let $(\alpha_k)_{k \in \mathbb{N}}$ with $\alpha_k > 0$ ($k \in \mathbb{N}$) and $\alpha_k \to 0$ ($k \to +\infty$) be given and for $\epsilon, \tau > 0$, $k \in \mathbb{N}$, define $F_{\epsilon,\tau,\alpha_k}, F_{\tau,\alpha_k} : \mathscr{S} \to [0, +\infty]$ by

$$F_{\epsilon,\tau,\alpha_k} := \frac{\phi_\epsilon - \mathcal{Y}_\tau \phi_\epsilon}{\tau} + \alpha_k(\phi_\epsilon + A + Bd^2(\cdot, u_\star)) + d(\cdot, u_\star),$$

$$F_{\tau,\alpha_k} := \frac{\phi - \mathcal{Y}_\tau \phi}{\tau} + \alpha_k(\phi + A + Bd^2(\cdot, u_\star)) + d(\cdot, u_\star).$$

Then, for $\tau \in \left(0, \frac{1}{2B}\right)$ and $\alpha_k > 0$ fixed, it holds that

$$F_{\epsilon,\tau,\alpha_k} \xrightarrow{\Gamma} F_{\tau,\alpha_k} \ (\epsilon \to 0).$$

We used the fact that $\mathcal{Y}_\tau \phi_\epsilon$ converges locally uniformly to $\mathcal{Y}_\tau \phi$ as $\epsilon \to 0$, which follows from the Fundamental Theorem of Γ-convergence on the convergence of minimum problems (see [4,9]). The Fundamental Theorem is applicable due to Assumption 3.1 on $(\phi_\epsilon)_{\epsilon>0}$.

At the end, we will be interested in the limit $\tau \to 0$, so in the following, if we consider $\tau > 0$, we tacitly suppose that $\tau \in \left(0, \frac{1}{2B}\right)$.

Let $(\theta_\tau)_{\tau>0}$ with $\theta_\tau > 0$ for $\tau > 0$ and $\theta_\tau \to 0 \ (\tau \to 0)$ be given.

Since $(\phi_\epsilon)_{\epsilon>0}$ satisfies Assumption 3.1, condition (6.1) holds for $(F_{\epsilon,\tau,\alpha_k})_{\epsilon>0}$. We apply Lemma 6.2 and deduce that

$$\delta(F_{\epsilon,\tau,\alpha_k}, F_{\tau,\alpha_k}) \to 0 \ (\epsilon \to 0)$$

for all $\tau, \alpha_k > 0$. Thus, for all $\tau, \alpha_k > 0$ there exists $\epsilon_{\tau,\alpha_k} > 0$ such that

$$\delta(F_{\eta,\tau,\alpha_k}, F_{\tau,\alpha_k}) \leq \theta_\tau \text{ for all } 0 < \eta \leq \epsilon_{\tau,\alpha_k}. \tag{6.4}$$

Next, we show that we can choose $\epsilon_\tau = \epsilon_{\tau,\alpha_k}$ independent of $\alpha_k \ (k \in \mathbb{N})$ in (6.4). For fixed $\tau > 0$ and for $\epsilon_k \to 0 \ (\epsilon_k > 0)$ we have

$$F_{\epsilon_k,\tau,\alpha_k} \xrightarrow{\Gamma} \frac{\phi - \mathcal{Y}_\tau \phi}{\tau} + d(\cdot, u_\star) \ (k \to +\infty),$$

$$F_{\tau,\alpha_k} \xrightarrow{\Gamma} \frac{\phi - \mathcal{Y}_\tau \phi}{\tau} + d(\cdot, u_\star) \ (k \to +\infty).$$

This means that the two families of functionals $\{F_{\epsilon,\tau,\alpha_k} : \epsilon > 0, \ k \in \mathbb{N}\}$ and $\{F_{\tau,\alpha_k} : k \in \mathbb{N}\}$ are uniformly Γ-equivalent. The notion of Γ-equivalence was introduced by Braides and Truskinovsky [6].

Moreover, both $(F_{\epsilon_k,\tau,\alpha_k})_{k\in\mathbb{N}}$ and $(F_{\tau,\alpha_k})_{k\in\mathbb{N}}$ satisfy condition (6.1). It easily follows (see [6] for the corresponding statement) that

$$\sup_{k\in\mathbb{N}} |\mathcal{Y}_\kappa F_{\epsilon,\tau,\alpha_k}(x) - \mathcal{Y}_\kappa F_{\tau,\alpha_k}(x)| \to 0 \ (\epsilon \to 0) \tag{6.5}$$

for all $\kappa > 0, x \in \mathscr{S}$.

In view of definition (6.3) of $\delta(\cdot, \cdot)$ with Φ Lipschitz continuous, we can conclude from (6.5) with basic straightforward arguments that for all $\tau > 0$ there exists $\epsilon_\tau > 0$ such that

$$\delta(F_{\eta,\tau,\alpha_k}, F_{\tau,\alpha_k}) \leq \theta_\tau \text{ for all } 0 < \eta \leq \epsilon_\tau, \ \alpha_k > 0. \tag{6.6}$$

We want to pass on to the limit $\tau \to 0$.

Let us consider $(F_{\tau,\alpha_k})_{\tau>0}$ for $\alpha_k > 0$ fixed.

On a separable metric space, Γ-convergence is compact (see [4,9]). Thus, every subsequence of $(F_{\tau,\alpha_k})_{\tau>0}$ admits a Γ-converging (sub)subsequence.

If we have

$$F_{\tau_n,\alpha_k} \xrightarrow{\Gamma} \Sigma \ (n \to +\infty)$$

for some functional $\Sigma : \mathscr{S} \to [0,+\infty]$ and a subsequence $\tau_n \to 0$, then the following facts hold.

As $(F_{\tau,\alpha_k})_{\tau>0}$ satisfies condition (6.1), we can apply Lemma 6.2 to deduce that

$$\delta(F_{\tau_n,\alpha_k}, \Sigma) \to 0 \ (n \to +\infty). \tag{6.7}$$

It can be proved [2] that if a functional satisfies Assumption 2.7, then it also satisfies (4.2). Hence, Proposition 4.1 is applicable to ϕ. By the definition of Γ-convergence it holds that

$$\Sigma(u) = \inf \left\{ \liminf_{n\to+\infty} F_{\tau_n,\alpha_k}(u_{\tau_n}) : \ u_{\tau_n} \xrightarrow{d} u \right\}.$$

We can conclude that

$$\Sigma(u) \geq \frac{1}{2}|\partial^- \phi|^2(u) + \alpha_k(\phi(u) + A + Bd^2(u, u_\star)) + d(u, u_\star) \tag{6.8}$$

for $u \in \mathscr{S}$.

Now, we can put together all the building blocks of our proof.

Let $(\epsilon(\tau))_{\tau>0}$ with $\epsilon(\tau) \leq \epsilon_\tau$ and $\epsilon(\tau) \to 0$ $(\tau \to 0)$ be given with $(\epsilon_\tau)_{\tau>0}$ as in (6.6).

We consider $(F_{\epsilon(\tau),\tau,\alpha_k})_{\tau>0}$ for an arbitrary but fixed $\alpha_k > 0$.

Let a subsequence $\tau_n \to 0$ and a functional $\Sigma : \mathscr{S} \to [0, +\infty]$ be given with

$$F_{\tau_n,\alpha_k} \xrightarrow{\Gamma} \Sigma \ (n \to +\infty),$$

i.e. as in the preceding considerations on $(F_{\tau,\alpha_k})_{\tau>0}$.

In this case we also have

$$F_{\epsilon(\tau_n),\tau_n,\alpha_k} \xrightarrow{\Gamma} \Sigma \ (n \to +\infty).$$

In fact, this follows from (6.6), (6.7), the triangle inequality

$$\delta(F_{\epsilon(\tau_n),\tau_n,\alpha_k}, \Sigma) \leq \delta(F_{\epsilon(\tau_n),\tau_n,\alpha_k}, F_{\tau_n,\alpha_k}) + \delta(F_{\tau_n,\alpha_k}, \Sigma).$$

and Lemma 6.2 (note that condition (6.1) is satisfied for $(F_{\epsilon(\tau_n),\tau_n,\alpha_k})_{n\in\mathbb{N}}$).

All in all, remembering the compactness of Γ-convergence on a separable metric space, we obtain

$$\liminf_{\tau \to 0} F_{\epsilon(\tau),\tau,\alpha_k}(u_\tau) \geq \frac{1}{2}|\partial^- \phi|^2(u) + \alpha_k(\phi(u) + A + Bd^2(u, u_\star)) + d(u, u_\star)$$

for all $u, u_\tau \in \mathscr{S}$ $(\tau > 0)$ with $u_\tau \xrightarrow{d} u$.

Now, we can directly deduce that Assumption 3.3 holds.

Let $u, u_\tau \in \mathscr{S}$ $(\tau > 0)$ be given with

$$u_\tau \xrightarrow{d} u, \ \sup_\tau \phi_{\epsilon(\tau)}(u_\tau) < +\infty,$$

and set $C := \sup_\tau \left(\phi_{\epsilon(\tau)}(u_\tau) + A + Bd^2(u_\tau, u_\star) \right) < +\infty$. It follows from the preceding steps that

$$\frac{1}{2}|\partial^- \phi|^2(u) \ \leq \ \liminf_{\tau \to 0} \frac{\phi_{\epsilon(\tau)}(u_\tau) - \mathcal{Y}_\tau \phi_{\epsilon(\tau)}(u_\tau)}{\tau} + \alpha_k C \tag{6.9}$$

for all $\alpha_k > 0$.

The proof of Assumption 3.3, and thus, of Theorem 6.1 is complete. $\qquad\square$

7. TWO FINITE DIMENSIONAL EXAMPLES

Let a C^1-function $\phi : \mathbb{R} \to \mathbb{R}$ be given, with

$$\phi(\cdot) \geq -A - Bd^2(\cdot, u_\star)$$

for some $A, B > 0$, $u_\star \in \mathbb{R}$. (We write $d(x, y) := |x - y|$.)

Then Assumption 2.7 is satisfied and the relaxed slope $|\partial^- \phi| = |\phi'|$ is a strong upper gradient for ϕ. It can be proved [2] that if a functional satisfies Assumption 2.7, then it also satisfies (4.2). We consider two examples of perturbations $\phi_\epsilon : \mathbb{R} \to \mathbb{R}$ of ϕ with $\phi_\epsilon \xrightarrow{\Gamma} \phi$.

Example 7.1. Let an arbitrary function $a : \mathbb{R} \to [0, +\infty)$ be given and define

$$\phi_\epsilon(x) = \phi(x) + a(x) \cos^2 \left(\frac{x}{\epsilon} \right).$$

If we choose $\epsilon = \epsilon(\tau)$ with $\frac{\epsilon(\tau)}{\tau} \to 0$ ($\tau \to 0$) in our scheme (3.4)–(3.5), then we can apply Theorem 3.4 and any curve u constructed in accordance with (3.9) solves

$$u'(t) = -\phi'(u(t)) \text{ for all } t \in (0, +\infty), \ u(0) = u^0.$$

Proof. We only have to check Assumption 3.3. Let $u, u_\tau \in \mathbb{R}$ ($\tau > 0$) be given such that

$$u_\tau \to u, \ \sup_\tau \phi_{\epsilon(\tau)}(u_\tau) < +\infty.$$

By Proposition 4.1, it is sufficient to show

$$\liminf_{\tau \to 0} \frac{\phi_{\epsilon(\tau)}(u_\tau) - \mathcal{Y}_\tau \phi_{\epsilon(\tau)}(u_\tau)}{\tau} \geq \liminf_{\tau \to 0} \frac{\phi(u_\tau) - \mathcal{Y}_\tau \phi(u_\tau)}{\tau}. \tag{7.1}$$

We set $v_\tau \in J_\tau \phi[u_\tau]$ and $\tilde{v}_\tau \in \mathbb{R}$ such that

$$\cos \left(\frac{\tilde{v}_\tau}{\epsilon(\tau)} \right) = 0 \text{ and } d(\tilde{v}_\tau, v_\tau) \leq \pi \epsilon(\tau).$$

As in the proof of Proposition 4.1, we have $v_\tau \to u$ ($\tau \to 0$).

The C^1-function ϕ is locally Lipschitz continuous and a Lipschitz constant for a neighbourhood of u is denoted by $C(u) > 0$. We use the fact that

$$\frac{1}{2\tau} d^2(u_\tau, v_\tau) \leq \phi(u_\tau) - \phi(v_\tau) \leq C(u) d(u_\tau, v_\tau).$$

All in all, we obtain

$$\frac{\phi_{\epsilon(\tau)}(u_\tau) - \mathcal{Y}_\tau \phi_{\epsilon(\tau)}(u_\tau)}{\tau} + \frac{\mathcal{Y}_\tau \phi(u_\tau) - \phi(u_\tau)}{\tau} \geq \frac{\mathcal{Y}_\tau \phi(u_\tau) - \mathcal{Y}_\tau \phi_{\epsilon(\tau)}(u_\tau)}{\tau}$$

$$\geq \frac{\phi(v_\tau) - \phi(\tilde{v}_\tau)}{\tau} + \frac{1}{2\tau^2} \left[d^2(u_\tau, v_\tau) - d^2(u_\tau, \tilde{v}_\tau) \right]$$

$$\geq -C(u) \frac{d(v_\tau, \tilde{v}_\tau)}{\tau} + \frac{1}{2\tau^2} \left[-d^2(v_\tau, \tilde{v}_\tau) - 2d(u_\tau, v_\tau) d(v_\tau, \tilde{v}_\tau) \right]$$

$$\geq -C(u) \pi \frac{\epsilon(\tau)}{\tau} - \frac{(\pi \epsilon(\tau))^2}{2\tau^2} - 2C(u) \frac{\pi \epsilon(\tau)}{\tau}.$$

The proof is finished. □

Example 7.2. Let functions $\zeta_\epsilon : \mathbb{R} \to \mathbb{R}$ be given with

$$\zeta_\epsilon(\cdot) \geq -\tilde{A} - \tilde{B} d^2(\cdot, \tilde{u}_\star)$$

for some $\tilde{A}, \tilde{B} > 0$, $\tilde{u}_\star \in \mathbb{R}$ and such that $(\zeta_\epsilon)_{\epsilon > 0}$ converges locally uniformly to a continuous function $\zeta : \mathbb{R} \to \mathbb{R}$. We define

$$\phi_\epsilon(x) = \phi(x) + \epsilon \zeta_\epsilon(x).$$

If we choose $\epsilon = \epsilon(\tau)$ with $\frac{\epsilon(\tau)}{\tau} \leq C$ for some $C > 0$ in (3.4)–(3.5), then we can apply Theorem 3.4 and any curve u constructed in accordance with (3.9) solves

$$u'(t) = -\phi'(u(t)) \text{ for all } t \in (0, +\infty), \ u(0) = u^0.$$

Proof. We only have to check Assumption 3.3. Let $u, u_\tau \in \mathbb{R}$ ($\tau > 0$) be given such that

$$u_\tau \to u, \ \sup_\tau \phi_{\epsilon(\tau)}(u_\tau) < +\infty.$$

Again, it suffices to show (7.1). We set $v_\tau \in J_\tau \phi[u_\tau]$ and as in Example 7.1 it holds that $v_\tau \to u$ ($\tau \to 0$). We have

$$\frac{\phi_{\epsilon(\tau)}(u_\tau) - \mathcal{Y}_\tau \phi_{\epsilon(\tau)}(u_\tau)}{\tau} + \frac{\mathcal{Y}_\tau \phi(u_\tau) - \phi(u_\tau)}{\tau}$$
$$\geq \frac{\phi_{\epsilon(\tau)}(u_\tau) - \phi_{\epsilon(\tau)}(v_\tau)}{\tau} + \frac{\phi(v_\tau) - \phi(u_\tau)}{\tau} = \frac{\epsilon(\tau)}{\tau}\left(\zeta_{\epsilon(\tau)}(u_\tau) - \zeta_{\epsilon(\tau)}(v_\tau)\right),$$

and we can conclude. $\qquad\square$

8. Perturbations, time-space discretizations

We return to the general case of a complete metric space $(\mathscr{S}, d)$ with a (compatible) topology σ on it, in accordance with the topological assumptions of Section 2.2, and study some stimulating aspects.

8.1. Perturbations

Let a functional $\phi : \mathscr{S} \to (-\infty, +\infty]$ satisfying Assumption 2.7 and functionals $\mathcal{P}_\epsilon : \mathscr{S} \to (-\infty, +\infty]$ ($\epsilon > 0$) be given with

$$\mathcal{P}_\epsilon(\cdot) \geq -\tilde{A} - \tilde{B}d^2(\cdot, \tilde{u}_\star) \tag{8.1}$$

for some $\tilde{A}, \tilde{B} > 0$, $\tilde{u}_\star \in \mathscr{S}$. We suppose that for $\epsilon_n \to 0$ ($\epsilon_n > 0$) it holds that

$$\sup_{n,m} d(u_n, u_m) < +\infty, \ u_n \xrightarrow{\sigma} u \ \Rightarrow \ \sup_{n \in \mathbb{N}} |\mathcal{P}_{\epsilon_n}(u_n)| < +\infty. \tag{8.2}$$

We define

$$\phi_\epsilon(u) = \phi(u) + \epsilon \mathcal{P}_\epsilon(u). \tag{8.3}$$

The functionals $\phi_\epsilon : \mathscr{S} \to (-\infty, +\infty]$ ($\epsilon > 0$) satisfy Assumption 3.1, and moreover, Assumption 3.2 holds. In order to apply Theorem 3.4 it remains to check Assumption 3.3:

Proposition 8.1. *Let ϕ and $\phi_\epsilon = \phi + \epsilon \mathcal{P}_\epsilon$ be given as above. If we choose $\epsilon = \epsilon(\tau)$ with $\frac{\epsilon(\tau)}{\tau} \to 0$ ($\tau \to 0$), then Assumption 3.3 is satisfied with this choice $\epsilon = \epsilon(\tau)$ ($\tau > 0$). In particular, all the results of Theorem 3.4 hold.*

Proof. We prove Assumption 3.3 with similar arguments as in Example 7.2. Let $u, u_\tau \in \mathscr{S}$ ($\tau > 0$) be given with

$$u_\tau \xrightarrow{\sigma} u, \ \sup_\tau \left\{\phi_{\epsilon(\tau)}(u_\tau), d(u_\tau, u)\right\} < +\infty.$$

We have already remarked several times that if a functional satisfies Assumption 2.7, Proposition 4.1 is applicable to it. Hence, in order to prove Assumption 3.3, it suffices to show

$$\liminf_{\tau \to 0} \frac{\phi_{\epsilon(\tau)}(u_\tau) - \mathcal{Y}_\tau \phi_{\epsilon(\tau)}(u_\tau)}{\tau} \geq \liminf_{\tau \to 0} \frac{\phi(u_\tau) - \mathcal{Y}_\tau \phi(u_\tau)}{\tau}.$$

We set $v_\tau \in J_\tau \phi[u_\tau]$. The sequence $\left(\frac{d(u_\tau, v_\tau)}{\sqrt{\tau}}\right)_{\tau > 0}$ is bounded (see the beginning of the proof of Prop. 4.1). We have

$$\frac{\phi_{\epsilon(\tau)}(u_\tau) - \mathcal{Y}_\tau \phi_{\epsilon(\tau)}(u_\tau)}{\tau} + \frac{\mathcal{Y}_\tau \phi(u_\tau) - \phi(u_\tau)}{\tau}$$
$$\geq \frac{\phi_{\epsilon(\tau)}(u_\tau) - \phi_{\epsilon(\tau)}(v_\tau)}{\tau} + \frac{\phi(v_\tau) - \phi(u_\tau)}{\tau} = \frac{\epsilon(\tau)}{\tau}\left(\mathcal{P}_{\epsilon(\tau)}(u_\tau) - \mathcal{P}_{\epsilon(\tau)}(v_\tau)\right).$$

Applying (8.2) we conclude. $\qquad\square$

In particular, if $|\partial^- \phi|$ is a strong upper gradient for ϕ, the relaxed minimizing movement scheme $(3.4)-(3.5)$ along $(\phi + \epsilon \mathcal{P}_\epsilon)_{\epsilon>0}$ leads to curves of maximal slope for ϕ with respect to $|\partial^- \phi|$.

Remark on the choice $\epsilon = \epsilon(\tau)$. Note that other choices $\epsilon = \epsilon(\tau)$ (*e.g.* of order $\mathcal{O}(\tau)$) might be possible besides, depending on the special features of the functionals $\mathcal{P}_\epsilon$. Indeed, in order to prove Assumption 3.3, the following condition is sufficient:

Whenever $(u_\tau)_{\tau>0}, (v_\tau)_{\tau>0} \subset \mathscr{S}$ satisfy

$$u_\tau, v_\tau \stackrel{\sigma}{\to} u, \ \sup_\tau \left\{ \frac{d(u_\tau, v_\tau)}{\sqrt{\tau}}, d(u_\tau, u) \right\} < +\infty,$$

then it holds that

$$\lim_{\tau \to 0} \frac{\epsilon(\tau)}{\tau} \left(\mathcal{P}_{\epsilon(\tau)}(u_\tau) - \mathcal{P}_{\epsilon(\tau)}(v_\tau) \right) = 0.$$

Remark on condition (8.2). If the σ-topology coincides with the topology induced by the distance d, an equivalent formulation of (8.2) is given by

$$\limsup_{n \to +\infty} \sup_{v:d(v,u)\leq\rho_n} |\mathcal{P}_{\epsilon_n}(v)| \ < \ +\infty \tag{8.4}$$

for every $u \in \mathscr{S}$, $\rho_n \to 0$ $(\rho_n > 0)$.

8.2. Restriction to bounded subsets

Step by step, we have gained the impression that the deciding factor in our theory is the **local** behaviour of the sequence of functionals under consideration, whereas its global behaviour appears to be irrelevant.

Indeed, let functionals $\phi, \phi_\epsilon : \mathscr{S} \to (-\infty, +\infty]$ $(\epsilon > 0)$ be given satisfying the Assumptions 3.1, 3.2 and Assumption 3.3 with the choice $\epsilon = \epsilon(\tau)$, and let $(r(\epsilon))_{\epsilon>0}$ be an arbitrary sequence of positive real numbers $r(\epsilon)$ with $r(\epsilon) \uparrow +\infty$ $(\epsilon \to 0)$. For $u_{\star\star} \in \mathscr{S}$ arbitrary but fixed, we define

$$\tilde{\phi}_\epsilon = \phi_\epsilon + \mathbb{I}_{\{d(u_{\star\star}, \cdot) \leq r(\epsilon)\}}, \tag{8.5}$$

in which we set $\mathbb{I}_{\mathscr{V}_\epsilon} \equiv 0$ on $\mathscr{V}_\epsilon$, $\mathbb{I}_{\mathscr{V}_\epsilon} \equiv +\infty$ on $\mathscr{S} \setminus \mathscr{V}_\epsilon$, for $\mathscr{V}_\epsilon = \{d(u_{\star\star}, \cdot) \leq r(\epsilon)\}$.

Then the functionals $\phi, \tilde{\phi}_\epsilon : \mathscr{S} \to (-\infty, +\infty]$ $(\epsilon > 0)$ satisfy the Assumptions 3.1, 3.2 (this is obviously true), and moreover, we can show that Assumption 3.3 holds for $\phi, (\tilde{\phi}_\epsilon)_{\epsilon>0}$ as well, with the same choice $\epsilon = \epsilon(\tau)$.

Proof. Let $u, u_\tau \in \mathscr{S}$ $(\tau > 0)$ be given with

$$u_\tau \stackrel{\sigma}{\to} u, \ \sup_\tau \left\{ \tilde{\phi}_{\epsilon(\tau)}(u_\tau), d(u_\tau, u) \right\} < +\infty.$$

In order to prove Assumption 3.3, it suffices to show

$$\liminf_{\tau \to 0} \frac{\tilde{\phi}_{\epsilon(\tau)}(u_\tau) - \mathcal{Y}_\tau \tilde{\phi}_{\epsilon(\tau)}(u_\tau)}{\tau} \ \geq \ \liminf_{\tau \to 0} \frac{\phi_{\epsilon(\tau)}(u_\tau) - \mathcal{Y}_\tau \phi_{\epsilon(\tau)}(u_\tau)}{\tau}.$$

We set $v_\tau \in \mathscr{S}$ with

$$\phi_{\epsilon(\tau)}(v_\tau) + \frac{1}{2\tau} d^2(v_\tau, u_\tau) \leq \mathcal{Y}_\tau \phi_{\epsilon(\tau)}(u_\tau) + \tau^2.$$

It holds that $d(u_\tau, v_\tau) \to 0$ $(\tau \to 0)$. This can be proved with similar arguments as at the beginning of the proof of Proposition 4.1. Thus, there exists $\eta > 0$ such that

$$v_\tau \in \{d(u_{\star\star}, \cdot) \leq r(\epsilon(\tau))\}$$

for all $0 < \tau < \eta$. So we have

$$
\frac{\tilde{\phi}_{\epsilon(\tau)}(u_\tau) - \mathcal{Y}_\tau \tilde{\phi}_{\epsilon(\tau)}(u_\tau)}{\tau} + \frac{\mathcal{Y}_\tau \phi_{\epsilon(\tau)}(u_\tau) - \phi_{\epsilon(\tau)}(u_\tau)}{\tau}
$$
$$
= \frac{\mathcal{Y}_\tau \phi_{\epsilon(\tau)}(u_\tau) - \mathcal{Y}_\tau \tilde{\phi}_{\epsilon(\tau)}(u_\tau)}{\tau} \geq \frac{\phi_{\epsilon(\tau)}(v_\tau) - \tilde{\phi}_{\epsilon(\tau)}(v_\tau)}{\tau} - \tau = -\tau
$$

for all $\tau \in (0, \eta)$. The proof is finished. $\qquad\square$

The preceding proof also reveals that whenever the convergence condition (3.15) is satisfied for $\phi, (\phi_\epsilon)_{\epsilon>0}$, then it is satisfied for $\phi, (\tilde{\phi}_\epsilon)_{\epsilon>0}$ too.

All in all, we obtain that we can always replace (3.5) by

$$
(\phi_{\epsilon(\tau)} + \mathbb{I}_{\mathcal{V}_{\epsilon(\tau)}})(u_\tau^n) + \frac{1}{2\tau}d^2(u_\tau^n, u_\tau^{n-1}) \leq \mathcal{Y}_\tau(\phi_{\epsilon(\tau)} + \mathbb{I}_{\mathcal{V}_{\epsilon(\tau)}})(u_\tau^{n-1}) + \gamma_\tau \tau \tag{8.6}
$$

with $\mathcal{V}_{\epsilon(\tau)} = \{d(u_{\star\star}, \cdot) \leq r(\epsilon(\tau))\}$, and still, all the results of Section 3 hold.

8.3. Time-space discretizations along a single functional

Let a functional $\phi : \mathscr{S} \to (-\infty, +\infty]$ satisfying Assumption 2.7 and subsets $\mathscr{W}_\epsilon \subset \mathscr{S}$ ($\epsilon > 0$) be given. We define

$$
\phi_\epsilon = \phi + \mathbb{I}_{\mathscr{W}_\epsilon}, \tag{8.7}
$$

in which we set $\mathbb{I}_{\mathscr{W}_\epsilon} \equiv 0$ on $\mathscr{W}_\epsilon$, $\mathbb{I}_{\mathscr{W}_\epsilon} \equiv +\infty$ on $\mathscr{S} \setminus \mathscr{W}_\epsilon$.

The Assumptions 3.1, 3.2 are clearly satisfied in this case. We want to derive conditions on $(\mathscr{W}_\epsilon)_{\epsilon>0}$ and $(\epsilon(\tau))_{\tau>0}$ such that Assumption 3.3 holds:

Let $u, u_\tau \in \mathscr{S}$ ($\tau > 0$) be given with

$$
u_\tau \xrightarrow{\sigma} u, \ \sup_\tau \left\{\phi_{\epsilon(\tau)}(u_\tau), d(u_\tau, u)\right\} < +\infty.
$$

In view of Proposition 4.1, it would suffice to show

$$
\liminf_{\tau \to 0} \frac{\phi_{\epsilon(\tau)}(u_\tau) - \mathcal{Y}_\tau \phi_{\epsilon(\tau)}(u_\tau)}{\tau} \geq \liminf_{\tau \to 0} \frac{\phi(u_\tau) - \mathcal{Y}_\tau \phi(u_\tau)}{\tau}.
$$

We set $v_\tau \in J_\tau \phi[u_\tau]$ and we have

$$
\frac{\phi_{\epsilon(\tau)}(u_\tau) - \mathcal{Y}_\tau \phi_{\epsilon(\tau)}(u_\tau)}{\tau} + \frac{\mathcal{Y}_\tau \phi(u_\tau) - \phi(u_\tau)}{\tau} = \frac{\mathcal{Y}_\tau \phi(u_\tau) - \mathcal{Y}_\tau \phi_{\epsilon(\tau)}(u_\tau)}{\tau}
$$
$$
\geq \frac{\phi(v_\tau) - \phi(w_\tau)}{\tau} + \frac{1}{2\tau^2}\left[d^2(u_\tau, v_\tau) - d^2(u_\tau, w_\tau)\right]
$$
$$
\geq \frac{\phi(v_\tau) - \phi(w_\tau)}{\tau} + \frac{1}{2\tau^2}\left[-2d(u_\tau, v_\tau)d(v_\tau, w_\tau) - d^2(v_\tau, w_\tau)\right]
$$

for all $w_\tau \in \mathscr{W}_{\epsilon(\tau)}$.

If there existed $(w_\tau)_{\tau>0} \subset \mathscr{S}$, $w_\tau \in \mathscr{W}_{\epsilon(\tau)}$ ($\tau > 0$), such that

$$
\liminf_{\tau \to 0} \left(\frac{\phi(v_\tau) - \phi(w_\tau)}{\tau} - \frac{d(u_\tau, v_\tau)d(v_\tau, w_\tau)}{\tau^2} - \frac{d^2(v_\tau, w_\tau)}{2\tau^2}\right) \geq 0, \tag{8.8}
$$

Assumption 3.3 would have been proved for this choice $\epsilon = \epsilon(\tau)$.

The sequence $(\frac{d(u_\tau, v_\tau)}{\sqrt{\tau}})_{\tau>0}$ is bounded (see the beginning of the proof of Prop. 4.1). So, a sufficient condition in order to show (8.8) would be

$$\liminf_{\tau \to 0} \frac{\phi(v_\tau) - \phi(w_\tau)}{\tau} \geq 0, \tag{8.9}$$

$$\frac{d(v_\tau, w_\tau)}{\sqrt{\tau^3}} \to 0 \ (\tau \to 0). \tag{8.10}$$

Example 8.2. We set $\mathscr{S} = L^2(\Omega)$ with $\Omega \subset \mathbb{R}^N$ and define the functional $\phi : L^2(\Omega) \to (-\infty, +\infty]$, $D(\phi) = H^1(\Omega)$,

$$\phi(u) = \frac{1}{2} \int_\Omega |\nabla u(x)|^2 \ \mathrm{d}x + \int_\Omega f(x, u(x)) \ \mathrm{d}x, \ \text{if } u \in H^1(\Omega).$$

Let us suppose that there exist $c_1, c_2 > 0$ such that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies

$$|f(x, w) - f(x, z)| \leq c_1(1 + |z| + |w|)|z - w|,$$
$$f(x, z) \geq -c_2(1 + |z|^2)$$

for all $x \in \Omega, w, z \in \mathbb{R}$.

Then the conditions (8.9), (8.10) hold for $w_\tau \in \mathscr{W}_{\epsilon(\tau)}$ with

$$\|\nabla w_\tau\|_{L^2(\Omega)} \leq \|\nabla v_\tau\|_{L^2(\Omega)}, \tag{8.11}$$

$$\frac{\|v_\tau - w_\tau\|_{L^2(\Omega)}}{\sqrt{\tau^3}} \to 0 \ (\tau \to 0). \tag{8.12}$$

The sequence $\left(\|\nabla v_\tau\|_{L^2(\Omega)}\right)_{\tau>0}$ is bounded. Now, estimates like (8.11), (8.12) are well-known in the finite element theory (see *e.g.* [7]).

9. SOME FINAL REMARKS

We would like to mention possible generalizations of our theory which we have not considered so far for the sake of clear presentation.

The theory can be easily extended to p-curves of maximal slope (see [2] for the definition of p-curves of maximal slope) with $p \in (1, +\infty)$, as well as to time discretizations with non-equidistant time steps.

Note that we do not use the special structure of the relaxed slope in the proof of Theorem 3.4. Section 3 and the ideas of Section 5 remain valid if we replace $|\partial^- \phi|$ by some upper gradient $g : \mathscr{S} \to [0, +\infty]$.

One might consider in addition to the distance d a sequence $(d_\epsilon)_{\epsilon>0}$ of distances with

$$u_n \overset{\sigma}{\to} u, \ v_n \overset{\sigma}{\to} v \ \Rightarrow \ \liminf_{n \to \infty} d_{\epsilon_n}(u_n, v_n) \geq d(u, v), \tag{9.1}$$

replace (3.5) by

$$\phi_{\epsilon(\tau)}(u_\tau^n) + \frac{1}{2\tau} d_{\epsilon(\tau)}^2(u_\tau^n, u_\tau^{n-1}) \ \leq \ \overline{\mathscr{Y}}_\tau \phi_{\epsilon(\tau)}(u_\tau^{n-1}) + \gamma_\tau \tau, \tag{9.2}$$

in which

$$\overline{\mathscr{Y}}_\tau \phi_{\epsilon(\tau)}(u) := \inf_{v \in \mathscr{S}} \left\{ \phi_{\epsilon(\tau)}(v) + \frac{1}{2\tau} d_{\epsilon(\tau)}^2(v, u) \right\}, \ u \in \mathscr{S},$$

and appropriately adapt the assumptions of Section 3. Then the proof of Theorem 3.4 can be adapted as well in this case, with a yet refined version of Ascoli−Arzelà theorem. The ideas of Section 5 remain valid too.

However, we do not want to expound the possible generalizations of our theory with regard to the topology, just give an impetus in case different topological assumptions are of interest.

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