ANNALES SCIENTIFIQUES DE L'É.N.S.

ANNETTE HUBER
GUIDO KINGS

# Dirichlet motives via modular curves

*Annales scientifiques de l'É.N.S. 4e série*, tome 32, nº 3 (1999), p. 313-345
<http://www.numdam.org/item?id=ASENS_1999_4_32_3_313_0>

*Ann. scient. Éc. Norm. Sup.*,
4e série, t. 32, 1999, p. 313 à 345.

# DIRICHLET MOTIVES VIA MODULAR CURVES

BY ANNETTE HUBER AND GUIDO KINGS

ABSTRACT. – Generalizing ideas of Anderson, Harder has proposed a construction of extensions of Tate-motives (more precisely of Hodge structures and Galois modules, respectively) in terms of modular curves. The aim of this paper is to construct directly those elements of motivic cohomology of $\mathrm{Spec}\mathbb{Q}(\mu_N)$ (*i.e.* in $K_*(\mathrm{Spec}\mathbb{Q}(\mu_N)))$ which induce these extensions in absolute Hodge cohomology and continuous Galois cohomology. We give two such constructions and prove that they are equivalent. The key ingredient is Beilinson's Eisenstein symbol in motivic cohomology of powers of the universal elliptic curve over the modular curve. We also compute explicitly the Harder-Anderson element in absolute Hodge cohomology. It is given in terms of Dirichlet-$L$-functions. As a corollary, we get a new proof of Beilinson's conjecture for Dirichlet-$L$-functions. A second paper [HuK] treats the explicit computation in the $l$-adic case. © Elsevier, Paris

RÉSUMÉ. – Généralisant des idées de Anderson, Harder a proposé une construction d'extensions de motifs de Tate (plus précisément d'extensions de structure de Hodge et de modules galoisiens) par les courbes modulaires. Le but de cet article est de trouver des éléments dans la cohomologie motivique de $\mathrm{Spec}\mathbb{Q}(\mu_N)$ (c'est-à-dire dans $K_*(\mathrm{Spec}\mathbb{Q}(\mu_N)))$ qui induisent ces extensions dans la cohomologie de Hodge absolue et dans la cohomologie galoisienne continue. Nous donnerons deux constructions de tels éléments et montrerons qu'elles sont équivalentes. L'ingrédient clé est le symbole d'Eisenstein, introduit par Beilinson, qui donne des éléments dans la cohomologie motivique de puissances de la courbe elliptique universelle sur la courbe modulaire. Nous calculons aussi l'extension de Harder-Anderson dans la cohomologie de Hodge absolue. Elle est donnée explicitement en utilisant les fonctions $L$ de Dirichlet. Comme corollaire nous obtenons une nouvelle démonstration de la conjecture de Beilinson sur les fonctions $L$ de Dirichlet. Dans un second article [HuK], nous calculerons explicitement la variante $l$-adique. © Elsevier, Paris

## Introduction

Generalizing ideas of Anderson, Harder has proposed a construction of extensions of motives of Dirichlet-Tate type in sheaf theoretic terms (*cf.* [Ha] Section 4.2 or the review before Prop. 6.4). Let $\mathcal{M}_k(-1)$ be the smooth $l$-adic sheaf on $Y_1(p)$ given as $k$-th symmetric power of the Tate module of the universal elliptic curve over $Y_1(p)$. Decomposing a certain $l$-adic cohomology group with coefficients in $\mathcal{M}_k$ under the Hecke-algebra, Harder obtains (after extension of scalars to $\mathbb{Q}(\mu_{p-1})$), for each Dirichlet character $\eta$ mod $p$ with $\eta(-1) = (-1)^k$, an extension of Galois-modules

$$0 \longrightarrow \mathbb{Q}_l(k+1)(\eta) \longrightarrow E_{k,\eta} \longrightarrow \mathbb{Q}_l(0) \longrightarrow 0 \quad ,$$

*i.e.* elements in $\mathrm{H}^1_{\mathrm{cont}}(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\mu_p)), \mathbb{Q}_l(k+1)) \otimes \mathbb{Q}(\mu_{p-1})$. By the same method, he also constructs a family of extensions of Hodge structures, *i.e.* elements in $\mathrm{H}^1_{\mathcal{H}}(\mathrm{Spec}\mathbb{Q}(\mu_p)_{\mathbb{R}}, \mathbb{Q}(k+1)) \otimes \mathbb{Q}(\mu_{p-1})$ (absolute Hodge cohomology).

It is clear from the geometric nature of the construction that these extensions should correspond to an extension in the conjectural category of motivic sheaves, *i.e.* to elements in $\mathrm{Ext}^1_{\mathcal{MM}(\mathrm{Spec}\mathbb{Q}(\mu_p))}(\mathbb{Q}, \mathbb{Q}(k+1))$. Conjecturally, this group is isomorphic to $\mathrm{K}_{2k+1}(\mathrm{Spec}\mathbb{Q}(\mu_p))) \otimes \mathbb{Q}$. The main aim of this paper is to construct directly those elements in $\mathrm{K}_{2k+1}(\mathrm{Spec}\mathbb{Q}(\mu_p)) \otimes \mathbb{Q}$ which induce the Harder-Anderson extensions. The $k = 0$-case has already been treated by Brinkmann ([Br]).

In fact, we give in sections 3 and 4 two constructions on $X(N)$ $(N \geq 3)$ that produce elements in $\mathrm{K}_{2k+1}(\mathrm{Spec}\mathbb{Q}(\mu_N)) \otimes \mathbb{Q} = \mathrm{H}^1_{\mathcal{M}}(\mathrm{Spec}\mathbb{Q}(\mu_N), k+1)$. They are shown to agree (Prop. 5.2). We call these elements Harder-Anderson elements. We then identify their image under the regulator maps to absolute Hodge cohomology and $l$-adic cohomology and show that they induce Harder's elements. In the Hodge-theoretic setting, the extension class can be computed and is given by Dirichlet $L$-series (Cor. 8.3). (For $N = p$ this is also carried out by a different method in [Ha]). The Harder-Anderson elements span the whole motivic cohomology group. This gives a new proof of (the weak form of) Beilinson's conjecture for Dirichlet series using automorphic methods. The original proof is due to Beilinson (completed by Neukirch [Neu] and Esnault [Es]) in [Be2]. Deninger has shown this case using elliptic curves with complex multiplication ([De1]). There is also an alternative proof using ideas of Beilinson and Deligne that relies on the motivic polylogarithm in the classical case ([HuW]). The proof given here can be seen as a byproduct of the existence of the elliptic polylogarithm.

Elements in motivic cohomology are uniquely determined by their Hodge-theoretic regulator. It was a conjecture of Bloch and Kato ([BlK] Conjecture 6.2) that the Beilinson elements in $\mathrm{K}_{2k+1}(\mathrm{Spec}\mathbb{Q}(\mu_N)) \otimes \mathbb{Q}$ are mapped to the Soulé-Deligne elements in Galois cohomology $\mathrm{H}^1(\mathbb{Q}(\mu_N), \mathbb{Q}_l(k+1))$. This has been proved using the classical polylogarithm ([HuW] Corollary 9.7). As a corollary, we can identify the $l$-adic version of the Harder-Anderson extension. It is given precisely by those Soulé-Deligne elements (*cf.* Cor. 8.5). In a second paper ([HuK]), we will show this equality directly without recourse to the $K$-theoretic result. Harder has also announced that he can do this. Together with the results of the present paper this will conversely give an alternative proof of Bloch-Kato's compatibility conjecture 6.2 in [BlK].

We now sketch our motivic constructions. They both rely on the Eisenstein symbol in $\mathrm{H}^{k+1}_{\mathcal{M}}(\mathfrak{X}^k, k+1)$, where $\mathfrak{X}^k$ is the $k$-fold fibred product of the universal elliptic curve over the modular curve. It was defined by Beilinson [Be3]. The first construction closely imitates Harder's. We begin with an Eisenstein class that has residue 0 at the cusp $\infty$. It then extends to a class on $\widetilde{\mathfrak{X}}^k$, a partial compactification of $\mathfrak{X}^k$ that is smooth. Restriction to the fibre at $\infty$ gives a class in the cohomology of an $N$-gon which can be projected to an element in the cohomology of $\infty = \mathrm{Spec}\mathbb{Q}(\mu_N)$. This part of the paper can be seen as a sequel to Scholl's [Sch]. A construction similar to ours but in terms of $K$-theory with compact support was independently found by Scholl. One remark on the use of compactifications may be necessary: we cannot use the Néron-model because the restriction of our elements to the Néron-model vanishes. The next obvious choice would be powers of the standard compactification of $\mathfrak{X}$. However, this is singular and cannot be used for the $K$-theoretic computations like localization sequences. Hence we end up with desingularizations of such completions.

The second construction is quite natural in terms of absolute cohomology and does not need in fact such a precise understanding of the geometric situation. We begin with two Eisenstein classes. One has residue zero at the cusp $\infty$, the other has residue 1 there and residue zero at all other cusps. We take their cup-product and take residue at $\infty$ of the result. It lives in the same $\mathrm{H}^1_{\mathcal{M}}(\infty, k+1)$ as the result of the first construction and in fact we get the same element.

We want to give an overview of the article. We start by fixing the notation and reviewing the Eisenstein-symbol (sections 1 and 2). In sections 3 to 5 we give the Harder-Anderson and the cup-product constructions in terms of absolute cohomology and compare them. The underlying sheaf theoretic extension is identified in section 6. This allows to make the comparison with Harder's extensions explicit. Finally, we compute the Hodge-theoretic extension class in terms of the cup-product construction (Section 7). We round off by gathering the main results mentioned in this introduction in Section 8. There is an appendix on a motivic version of a splitting of the weight filtration for certain toric varieties. It is needed in order to imitate the Harder-Anderson construction in terms of absolute cohomology.

We thank D. Blasius, G. Harder, K. Künnemann, D. Ramakrishnan, A. Scholl and J. Wildeshaus for discussions on the subject and useful comments. The first author wants to thank particularly D. Blasius and D. Ramakrishnan whose questions first started her interest in Harder's construction.

## 1. Notations and conventions

Let $N \geq 3$ and $B = \mathrm{Spec}\mathbb{Q}(\mu_N)$. Fix a primitive root of unity $\zeta \in \mathbb{Q}(\mu_N)$. Let $\mathcal{M}$ be the modular curve of elliptic curves $E$ with level-$N$-structure $(\mathbb{Z}/N)^2 \xrightarrow{\cong} E[N]$. It is naturally a $B$-scheme which we also consider as a $\mathbb{Q}$-scheme via the natural map $B \to \mathrm{Spec}\mathbb{Q}$. Let $\overline{\mathcal{M}}$ be its compactification. Furthermore, let $\pi : \mathfrak{X} \to \mathcal{M}$ be the universal elliptic curve above $\mathcal{M}$ and $\overline{\pi} : \overline{\mathfrak{X}} \to \overline{\mathcal{M}}$ its compactification. This is a generalized elliptic curve. Let $\mathrm{Cusp} = \overline{\mathcal{M}} \setminus \mathcal{M}$ be the scheme of cusps. The standard Néron-$N$-gon has a level-$N$-structure $\mathbb{Z}/N \times \mathbb{Z}/N \subset \mathbb{G}_m \times \mathbb{Z}/N$ over $B$ where $1 \mapsto \zeta$ in the first component and of course $1 \mapsto 1$ in the second. This induces a section $\infty : B \to \mathrm{Cusp}$. Let $\overset{\circ}{\mathcal{M}} = \mathcal{M} \cup \infty$ as open subscheme of $\overline{\mathcal{M}}$. The notation can be read as "$\mathcal{M}$ plus one point". The inclusion into $\overline{\mathcal{M}}$ will be denoted $j$. We denote the fibre above $\infty$ by $\overline{\mathfrak{X}}_\infty$. It is a Néron-$N$-gon. We will also need the Néron model of $\mathfrak{X}$ over $\overline{\mathcal{M}}$, which we denote $\widehat{\mathfrak{X}}$. Its fibre at $\infty$ is isomorphic to $\mathbb{G}_{m,B} \times \mathbb{Z}/N$. Finally, we consider, for $k \geq 1$, the $k$-fold fibre products of these objects over $\overline{\mathcal{M}}$, namely $\mathfrak{X}^k = \mathfrak{X} \times_{\mathcal{M}} \mathfrak{X} \ldots \times_{\mathcal{M}} \mathfrak{X}$, $\overline{\mathfrak{X}}^k = \overline{\mathfrak{X}} \times_{\overline{\mathcal{M}}} \ldots \times_{\overline{\mathcal{M}}} \overline{\mathfrak{X}}$ and $\widehat{\mathfrak{X}}^k = \widehat{\mathfrak{X}} \times_{\overline{\mathcal{M}}} \widehat{\mathfrak{X}} \ldots \times_{\overline{\mathcal{M}}} \widehat{\mathfrak{X}}$. Whereas $\overline{\mathfrak{X}}$ itself is smooth over $\mathrm{Spec}\mathbb{Q}$, this is not true for the higher $\overline{\mathfrak{X}}^k$. Let $p : \widetilde{\mathfrak{X}}^k \to \overline{\mathfrak{X}}^k$ be the canonical desingularization, which is obtained by a sequence of blow-ups. For details see [Sch] section 2. $\widetilde{\mathfrak{X}}^k$ is always smooth. Let $\overline{\mathfrak{X}}^k_\infty$, $\widetilde{\mathfrak{X}}^k_\infty$ and $\widehat{\mathfrak{X}}_\infty$ be the fibres above $\infty$. These objects organize into the following cartesian diagram of $B$-schemes:

$$\begin{array}{ccccc} & & \widetilde{\mathfrak{X}} & \longleftarrow & \widetilde{\mathfrak{X}}^k_\infty \\ & \nearrow & \big\downarrow p & & \big\downarrow p_\infty \\ \mathfrak{X}^k & \longrightarrow & \overline{\mathfrak{X}}^k & \longleftarrow & \overline{\mathfrak{X}}^k_\infty \\ \pi^k \big\downarrow & & \overline{\pi}^k \big\downarrow & & \big\downarrow \\ \mathcal{M} & \longrightarrow & \overline{\mathcal{M}} & \longleftarrow & B \end{array}$$

Our object of central interest is the smooth variety $\widetilde{\mathfrak{X}}^k_{\overset{\circ}{\mathcal{M}}} = \widetilde{\mathfrak{X}}^k \times_{\overline{\mathcal{M}}} \overset{\circ}{\mathcal{M}}$. All these $B$-schemes have a canonical model over $\mathbb{Z}[1/N]$.

We are going to do computations in absolute cohomology by which we mean Ext-groups of Tate sheaves on the varieties. We will work in parallel in the following three settings:

1. Mixed algebraic Hodge modules over $\mathbb{R}$ (see [HuW] Definition A.2.4). They carry a weight filtration. The corresponding absolute cohomology theory is $\mathrm{H}^n_{\mathcal{H}}(V/\mathbb{R}, \mathbb{R}(m)) = \mathrm{Ext}^n(\mathbb{R}(0)_V, \mathbb{R}(0)_V)$, the absolute Hodge cohomology over $\mathbb{R}$ (*loc. cit.* Definition A.2.6 or [Be1] §7 which gives the same thing by [HuW] A.2.7). It agrees with Deligne cohomology over $\mathbb{R}$ for $n \leq m$ ($n \leq 2m$ if the variety is proper).
2. Mixed perverse $l$-adic sheaves on varieties over $\mathbb{Q}$ as in [Hu2] section 3. A weight filtration need not exist but we can speak of the weights of a given object. The absolute cohomology theory is $\mathrm{H}^n_{\mathrm{hor}}(V, \mathbb{Q}_l(m)) = \mathrm{Ext}^n(\mathbb{Q}_l(0), \mathbb{Q}(m))$, horizontal continuous $l$-adic cohomology (*loc. cit.* Definition 5.1). For a variety $V$ over $\mathbb{Q}$, it is given by the direct limit of the continuous $l$-adic cohomology groups of $\mathcal{V}$, where $\mathcal{V}$ runs through models of $V$ which are of finite type over $\mathbb{Z}$. It has a canonical map to continuous $l$-adic cohomology as defined by Jannsen ([J] sect. 3).
3. The conjectural theory of motivic sheaves. The corresponding absolute cohomology is motivic cohomology, $\mathrm{H}^n_{\mathcal{M}}(\,\cdot\,, m)$. If $\mathcal{V}$ is a scheme of finite type over $\mathbb{Z}$ whose irreducible components together with their iterated intersections are smooth over $\mathbb{Z}$, then let $\mathcal{V}_0$ be the disjoint union of the irreducible components of $\mathcal{V}$. Let

$$\mathcal{V}_* = \mathrm{cosq}_0(\mathcal{V}_0/\mathcal{V}).$$

This is a simplicial scheme, the Čech-nerve of the covering. As in [HuW] B.2.9 we can *define* motivic cohomology of $\mathcal{V}$ by $\mathrm{H}^n_{\mathcal{M}}(\mathcal{V}, m) = \mathrm{Gr}^m_\gamma K_{2m-n}(\mathcal{V}_*)_{\mathbb{Q}}$. By abuse of notation we will write $\mathrm{H}^n_{\mathcal{M}}(V, m)$ for the direct limit of the $\mathrm{H}^n_{\mathcal{M}}(\mathcal{V}, m)$ for all models of $V$.

There are higher Chern class maps

$$\begin{aligned} r_{\mathcal{H}} :& \mathrm{H}^n_{\mathcal{M}}(V, m) \to \mathrm{H}^n_{\mathcal{H}}(V \times_{\mathbb{Q}} \mathbb{R}/\mathbb{R}, \mathbb{R}(m)) \\ r_l :& \mathrm{H}^n_{\mathcal{M}}(V, m) \to \mathrm{H}^n_{\mathrm{hor}}(V, \mathbb{Q}_l(m)). \end{aligned}$$

For the existence of localization sequences and their compatibility with regulators, we again refer to [HuW], B.2.17, B.4.4, B.5.8 d).

For a morphism of varieties $f$, the functors $f_*$ *etc.* denote the functors on the derived category. Cohomology objects are perverse ones. Note that this leads to unfamiliar indexing.

We use the unified notations

$$\mathrm{H}^n(V,m) = \begin{cases} \mathrm{H}^n_{\mathcal{H}}(V\times\mathbb{R}/\mathbb{R},\mathbb{R}(m)) & \text{Hodge context}\\ \mathrm{H}^n_{\mathrm{hor}}(V,\mathbb{Q}_l(m)) & l\text{-adic context}\\ \mathrm{H}^n_{\mathcal{M}}(V,m) & \text{motivic context}\end{cases}$$

$$F = \begin{cases} \mathbb{R} & \text{Hodge context}\\ \mathbb{Q}_l & l\text{-adic context}\end{cases}$$

If $v : V \longrightarrow \mathrm{Spec}\mathbb{Q}$ is a variety, let $F_V = v^*F$ be the object in the derived category $D(V)$ of Hodge modules respectively mixed perverse $l$-adic sheaves. If $V$ is smooth, then this will be a Hodge module respectively a perverse sheaf up to shift of complexes. With these conventions we put

$$\underline{\mathrm{H}}^n(V) = \mathrm{H}^n v_* v^* F_{\mathrm{Spec}\mathbb{Q}} = \begin{cases} \mathrm{H}^n_{\mathrm{sing}}(V,\mathbb{R}) \text{ as a mixed } \mathbb{R}\text{-Hodge structure over } \mathbb{R}\\ \mathrm{H}^n_{\mathrm{et}}(V_{\overline{\mathbb{Q}}},\mathbb{Q}_l) \text{ as mixed Galois module}\end{cases}$$

They carry a weight filtration. The absolute cohomology theories of 1. and 2. are part of a Bloch-Ogus cohomology theory. In particular there is also absolute homology

$$\mathrm{H}_n(V,m) = \mathrm{Hom}_{D(V)}(v^*F(m)[n], v^!F) = \mathrm{Hom}_{D(\mathrm{Spec}\mathbb{Q})}(v_!F_V(m)[n], F)$$

and a canonical isomorphism between homology and cohomology in the smooth case. Motivic cohomology is known to be a Bloch-Ogus cohomology theory at least when we restrict to smooth varieties.

For a morphism $f : X \longrightarrow Y$ and a sheaf $\mathcal{F}$ on $X$, we also use the notation

$$\underline{R\Gamma}(X/Y,\mathcal{F}) = f_*\mathcal{F}$$
$$\underline{R\Gamma}(X/Y) = f_*F_X \quad .$$

If $Y = B$, we drop the $Y$ from the notation. By adjointness we have the equalities

$$\mathrm{H}^n(V,m) = \mathrm{Hom}_{D(B)}(F_B, \underline{R\Gamma}(X/B)(m)[n])$$
$$\mathrm{H}_n(V,m) = \mathrm{Hom}_{D(B)}(F_B(m)[n], \mathbb{D}v_!F_V)$$

## 2. The Eisenstein symbol

We start by reviewing Beilinson's Eisenstein symbol [Be3]. There are several possible conventions for the action of the group $\mathrm{GL}_2(\mathbb{Z}/N)$ on $\mathcal{M}$ and the explicit parameterization of $\mathfrak{X}^k(\mathbb{C})$, which is important for the formulas for the Eisenstein symbol in Deligne cohomology. We follow, with our approach, the one in [De2] (see also [Den-Sch]), whose normalizations differ from the original ones in [Be3] and coincide with the ones in [Ki2,Ki3]. Although the Eisenstein symbol was first constructed in motivic cohomology, we work with all absolute cohomologies at the same time.

The Eisenstein symbol is a splitting of a certain residue map, which we now recall for the purpose of fixing some notations. Recall that $\widehat{\mathfrak{X}}/\overline{\mathcal{M}}$ denotes the Néron model of $\mathfrak{X}$ over $\overline{\mathcal{M}}$. Denote by $\widehat{\mathfrak{X}}_{\mathrm{Cusp}}$ the fibre over Cusp. It has $N$ connected components. Let $\widehat{\mathfrak{X}}^0_{\mathrm{Cusp}}$ be the component of the identity section. By our fixed choice of root of unity $\zeta$, the standard Néron-$N$-gon over $B$ has a canonical level-$N$-structure, which gives rise to a section $\infty : B \to \mathrm{Cusp}$. Denote by Isom the $\mu_{2,B} = \mathrm{Aut}_B(\mathbb{G}_m)$-torsor

$$\mathrm{Isom} := \mathrm{Isom}_B(\mathbb{G}_m, \widehat{\mathfrak{X}}^0_\infty)$$

so that we have a canonical isomorphism

$$\sigma : \widehat{\mathfrak{X}}_\infty \xrightarrow{\sim} ((\mathbb{G}_{m,B} \times \mathbb{Z}/N) \times_B \mathrm{Isom})/\mathrm{Aut}_B(\mathbb{G}_m) \, , \tag{1}$$

$\mathrm{Aut}_B(\mathbb{G}_m)$ acting on both factors.

Now $\mathrm{Cusp}/\mathbb{Q}$ is étale, hence completely described by the set

$$\mathrm{Cusp}(\overline{\mathbb{Q}}) = \pm U(\mathbb{Z}/N) \setminus \mathrm{GL}_2(\mathbb{Z}/N) \, ,$$

where $U = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2 \right\}$, with $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action given via its quotient $\mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q})$ as follows: fix the isomorphism $\mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q}) \cong (\mathbb{Z}/N)^*$, such that $\zeta \mapsto \zeta^{a^{-1}}$ for $a \in (\mathbb{Z}/N)^*$ and $\zeta \in \mu_N$. Then let $a \in (\mathbb{Z}/N)^*$ act via left multiplication by $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ on $\pm U(\mathbb{Z}/N) \setminus \mathrm{GL}_2(\mathbb{Z}/N)$. If we let $P := \left\{ \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2 \right\}$, then the cusp $\infty$ is given by

$$\infty(\overline{\mathbb{Q}}) = \pm U(\mathbb{Z}/N) \setminus \pm P(\mathbb{Z}/N)$$

and the set of closed points of Cusp is the set

$$\pm P(\mathbb{Z}/N) \setminus \mathrm{GL}_2(\mathbb{Z}/N) \, .$$

We get

$$\widehat{\mathfrak{X}}_{\mathrm{Cusp}} = \coprod_{g \in \pm P(\mathbb{Z}/N) \setminus \mathrm{GL}_2(\mathbb{Z}/N)} \widehat{\mathfrak{X}}_{g \cdot \infty}$$

so that

$$\mathrm{H}^\bullet(\widehat{\mathfrak{X}}_{\mathrm{Cusp}}, *) \cong \mathrm{Ind}_{\pm P(\mathbb{Z}/N)}^{\mathrm{GL}_2(\mathbb{Z}/N)} \mathrm{H}^\bullet(\widehat{\mathfrak{X}}_\infty, *) \, .$$

Composing with (1) we get:

$$\sigma^* : \mathrm{H}^\bullet(\widehat{\mathfrak{X}}_{\mathrm{Cusp}}, *) \xrightarrow{\cong} \mathrm{Ind}_{\pm P(\mathbb{Z}/N)}^{\mathrm{GL}_2(\mathbb{Z}/N)} \mathrm{H}^\bullet((\mathbb{Z}/N \times \mathbb{G}_{m,B}) \times \mathrm{Isom}, *)^{\mathrm{Aut}_B(\mathbb{G}_m) = \mathrm{id}}$$

or more generally for $k \geq 1$:

$$\sigma^* : \mathrm{H}^\bullet(\widehat{\mathfrak{X}}^k_{\mathrm{Cusp}}, *) \xrightarrow{\cong} \mathrm{Ind}_{\pm P(\mathbb{Z}/N)}^{\mathrm{GL}_2(\mathbb{Z}/N)} \mathrm{H}^\bullet((\mathbb{Z}/N \times \mathbb{G}_{m,B})^k \times \mathrm{Isom}, *)^{\mathrm{Aut}_B(\mathbb{G}_m) = \mathrm{id}} \, .$$

For the special case $\mathrm{H}^k(\mathbb{G}^k_{m,B} \times \mathrm{Isom}, k)$ we have the residue map

$$\mathrm{H}^k(\mathbb{G}^k_{m,B} \times \mathrm{Isom}, k) \xrightarrow{\mathrm{res}} \mathrm{H}^0(\mathrm{Isom}, 0)$$

induced by $\mathrm{H}^1(\mathbb{G}_{m,B}, 1) \to \mathrm{H}^0(B, 0)$ which in turn comes from the localization sequence for the pair $(\mathbb{A}^1, \mathbb{G}_m)$. This map becomes equivariant for the $\mathfrak{S}_k$-action on $\mathbb{G}^k_m$ if we let

$\tau \in \mathfrak{S}_k$ act via $\operatorname{sgn} \tau$ on $\mathrm{H}^0(\mathrm{Isom}, 0)$. The map res is not equivariant for the natural operation of $\mathrm{Aut}_B(\mathbb{G}_m)$, but induces rather

$$\mathrm{H}^k(\mathbb{G}^k_{m,B} \times \mathrm{Isom}, k)^{\mathrm{Aut}_B(\mathbb{G}_m)=\mathrm{id}} \xrightarrow{\mathrm{res}} \mathrm{H}^0(\mathrm{Isom}, 0)^{\mathrm{Aut}_B(\mathbb{G}_m)=(-1)^k} .$$

Let

$$\mathrm{H}^0(B,0)^{(k)} := \mathrm{H}^0(\mathrm{Isom}, 0)^{\mathrm{Aut}_B(\mathbb{G}_m)=(-1)^k} ,$$
$$\mathrm{H}^0(\mathrm{Cusp},0)^{(k)} := \mathrm{H}^0(\mathrm{Isom}_{\mathrm{Cusp}}(\mathbb{G}_m, \widehat{\mathfrak{X}}^0_{\mathrm{Cusp}}), 0)^{\mathrm{Aut}(\mathbb{G}_m)=(-1)^k} .$$

Observe that $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}/N)$ acts via $(-1)^k$ on $\mathrm{H}^0(B,0)^{(k)}$.

Composing the boundary map $\delta$ in the Gysin sequence for the pair $(\widehat{\mathfrak{X}}^k, \mathfrak{X}^k)$ with residue and projection to the cohomology of the identity component, one gets:

$$\mathrm{H}^{k+1}(\mathfrak{X}^k, k+1) \xrightarrow{\delta} \mathrm{H}^k(\widehat{\mathfrak{X}}^k_{\mathrm{Cusp}}, k) \xrightarrow{\mathrm{res}\circ\sigma^*} \mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Z}/N)}_{\pm P(\mathbb{Z}/N)} \bigoplus_{(\mathbb{Z}/N)^k} \mathrm{H}^0(B,0)^{(k)}$$
$$\to \mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Z}/N)}_{\pm P(\mathbb{Z}/N)} \mathrm{H}^0(B,0)^{(k)} .$$

Note that the same map can be obtained from the Gysin sequence for the pair $((\widehat{\mathfrak{X}}^0)^k, \mathfrak{X}^k)$. We put

$$\mathrm{Res} := \mathrm{res}\circ\sigma^*\circ\delta$$

which is a $\mathrm{GL}_2(\mathbb{Z}/N)$- and $\mathfrak{S}_k$-equivariant map

$$\mathrm{Res} : \mathrm{H}^{k+1}(\mathfrak{X}^k, k+1) \longrightarrow \mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Z}/N)}_{\pm P(\mathbb{Z}/N)} \mathrm{H}^0(B,0)^{(k)} .$$

The Eisenstein symbol for $k \geq 1$ is now a $\mathrm{GL}_2(\mathbb{Z}/N)$- and $\mathfrak{S}_k$-equivariant splitting of Res:

$$\mathcal{E}\mathrm{is} : \mathrm{H}^0(\mathrm{Cusp},0)^{(k)} = \mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Z}/N)}_{\pm P(\mathbb{Z}/N)} \mathrm{H}^0(B,0)^{(k)} \longrightarrow \mathrm{H}^{k+1}(\mathfrak{X}^k, k+1)$$

with

$$\mathrm{Res}\circ\mathcal{E}\mathrm{is} = \mathrm{id} ,$$

the motivic Eisenstein symbol constructed by Beilinson [Be3] section 3, respectively its image under the regulator maps. The Eisenstein symbol is the main ingredient in the constructions of this article.

Recall from [Sch] that the group

$$\Gamma_k := ((\mathbb{Z}/N)^2 \rtimes \mu_2)^k \rtimes \mathfrak{S}_k$$

acts on the $B$-scheme $\mathfrak{X}^k$. Here $(\mathbb{Z}/N)^2 \cong \mathfrak{X}[N]$ acts by translation, $\pm 1 \in \mu_2$ by multiplication on $\mathfrak{X}$ and $\mathfrak{S}_k$ acts by permuting the factors. Scholl introduced the character

$$\varepsilon : \Gamma_k \longrightarrow \mu_2$$

which is trivial on $(\mathbb{Z}/N)^{2k}$, the multiplication map on $\mu_2^k$ and the sign character on $\mathfrak{S}_k$. Denote by $V(\varepsilon)$ the $\varepsilon$-isotypical subspace of a vector space (over a field of characteristic 0) with $\Gamma_k$-action. We let $P_\varepsilon$ be the projector onto $V(\varepsilon)$. Res and Eis become $\Gamma_k$-equivariant if we let $\Gamma_k$ act via $\varepsilon$ on $\mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Z}/N)}_{\pm P(\mathbb{Z}/N)} \mathrm{H}^0(B,0)^{(k)}$ ([Be3] 3.1). This implies $P_\varepsilon\circ\mathcal{E}\mathrm{is} = \mathcal{E}\mathrm{is}$.

## 3. Harder-Anderson elements in $K$-theory

We want to show that, by restriction to a subspace of $\mathrm{Ind}_{\pm P(\mathbb{Z}/N)}^{\mathrm{GL}_2(\mathbb{Z}/N)}\mathrm{H}^0(B,0)^{(k)}$, we can actually get elements in $\mathrm{H}^{k+1}(\widetilde{\mathfrak{X}}^k_{\underset{\mathcal{M}}{\circ}}, k+1)$, where $\widetilde{\mathfrak{X}}^k_{\underset{\mathcal{M}}{\circ}}$ is the partial smooth compactification including $\infty$. Consider the partial Néron-model $\widehat{\mathfrak{X}}^k_{\underset{\mathcal{M}}{\circ}}$. As the computations of Scholl [Sch] which lead to the isomorphism

$$\mathrm{H}^\bullet(\widetilde{\mathfrak{X}}^k,*)(\varepsilon) \cong \mathrm{H}^\bullet(\widehat{\mathfrak{X}}^k,*)(\varepsilon)$$

(3.1.0 of *loc. cit.*) are completely local on Cusp, we get:

LEMMA 3.1. – *The inclusion* $\widehat{\mathfrak{X}}^k_{\underset{\mathcal{M}}{\circ}} \hookrightarrow \widetilde{\mathfrak{X}}^k_{\underset{\mathcal{M}}{\circ}}$ *induces an isomorphism*

$$\mathrm{H}^\bullet(\widetilde{\mathfrak{X}}^k_{\underset{\mathcal{M}}{\circ}},*)(\varepsilon) \cong \mathrm{H}^\bullet(\widehat{\mathfrak{X}}^k_{\underset{\mathcal{M}}{\circ}},*)(\varepsilon)\ .$$

The subscheme $\widehat{\mathfrak{X}}^k_{\underset{\mathcal{M}}{\circ}} \subset \widehat{\mathfrak{X}}^k$ is obviously stabilized by $\pm P(\mathbb{Z}/N) \subset \mathrm{GL}_2(\mathbb{Z}/N)$. As in section 2, we define a (only $\pm P(\mathbb{Z}/N)$-equivariant) map

$$\mathrm{H}^{k+1}(\widehat{\mathfrak{X}}^k_\infty, k+1) \xrightarrow{\mathrm{res}\circ\sigma^*} \mathrm{H}^0(B,0)^{(k)}\ ,$$

which is still $\Gamma_k$-equivariant, if $\Gamma_k$ acts via $\varepsilon$ on $\mathrm{H}^0(B,0)^{(k)}$. If we let again be $\mathrm{Res} := \mathrm{res}\circ\sigma^*\circ\delta$, we get a map

$$\mathrm{Res} : \mathrm{H}^{k+1}(\mathfrak{X}^k, k+1) \longrightarrow \mathrm{H}^0(B,0)^{(k)}\ .$$

For later use we denote by $\infty^*$ the $\pm P(\mathbb{Z}/N)$-morphism

$$\infty^* : \mathrm{Ind}_{\pm P(\mathbb{Z}/N)}^{\mathrm{GL}_2(\mathbb{Z}/N)}\mathrm{H}^0(B,0)^{(k)} \cong \mathrm{H}^0(\mathrm{Cusp},0)^{(k)} \xrightarrow{\infty^*} \mathrm{H}^0(B,0)^{(k)} \tag{2}$$

which is evaluation of $f \in \mathrm{Ind}_{\pm P(\mathbb{Z}/N)}^{\mathrm{GL}_2(\mathbb{Z}/N)}\mathrm{H}^0(B,0)^{(k)}$ at $\begin{pmatrix}1&0\\0&1\end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}/N)$.

Our aim is to prove:

PROPOSITION 3.2. – *There is a commutative diagram of short exact sequences*

$$\begin{array}{ccccccc} 0 \to & \mathrm{H}^{k+1}(\widehat{\mathfrak{X}}^k_{\underset{\mathcal{M}}{\circ}}, k+1)(\varepsilon) & \to & \mathrm{H}^{k+1}(\mathfrak{X}^k, k+1)(\varepsilon) & & & \\ & \uparrow & & \| & \cdots & & \\ 0 \to & \mathrm{H}^{k+1}(\widehat{\mathfrak{X}}^k, k+1)(\varepsilon) & \to & \mathrm{H}^{k+1}(\mathfrak{X}^k, k+1)(\varepsilon) & & & \end{array}$$

$$\begin{array}{cccc} & \xrightarrow{\mathrm{Res}} & \mathrm{H}^0(B,0)^{(k)} & \to 0 \\ \cdots & & \uparrow \infty^* & \\ & \xrightarrow{\mathrm{Res}} & \mathrm{Ind}_{\pm P(\mathbb{Z}/N)}^{\mathrm{GL}_2(\mathbb{Z}/N)}\mathrm{H}^0(B,0)^{(k)} & \to 0 \end{array}$$

*where all maps are* $\pm P(\mathbb{Z}/N)$*-equivariant.*

*Proof.* – Consider the $\Gamma_k$-equivariant long exact localization sequence for the pair $(\widehat{\mathfrak{X}}^k_{\overset{\circ}{\mathcal{M}}}, \mathfrak{X}^k)$

$$\to \mathrm{H}^{k-1}(\widehat{\mathfrak{X}}^k_\infty, k) \to \mathrm{H}^{k+1}(\widehat{\mathfrak{X}}^k_{\overset{\circ}{\mathcal{M}}}, k+1) \to \mathrm{H}^{k+1}(\mathfrak{X}^k, k+1) \to \mathrm{H}^k(\widehat{\mathfrak{X}}^k_\infty, k) \to$$

and project to $\varepsilon$-eigenspaces. Because $\widehat{\mathfrak{X}}^k_\infty \cong (\mathbb{Z}/N \times \mathbb{G}_{m,B})^k$ we have

$$\mathrm{H}^{k-1}(\widehat{\mathfrak{X}}^k_\infty, k)(\varepsilon) = 0$$

and

$$\mathrm{H}^k(\widehat{\mathfrak{X}}^k_\infty, k)(\varepsilon) \cong \mathrm{H}^0(B, 0)^{(k)} .$$

The same localization sequence for $(\widehat{\mathfrak{X}}^k, \mathfrak{X}^k)$ is even right exact because Res has the splitting Eis. This, together with the functoriality of the localization sequence, implies the result. The $\pm P(\mathbb{Z}/N)$-equivariance follows from the construction of Res. □

We identify the kernel of $\infty^*$ with

$$\ker \infty^* \cong \mathrm{H}^0(\mathrm{Cusp} \smallsetminus \infty, 0)^{(k)} .$$

COROLLARY 3.3. – *The Eisenstein symbol induces a map*

$$\mathcal{E}\mathrm{is} : \mathrm{H}^0(\mathrm{Cusp} \smallsetminus \infty, 0)^{(k)} \longrightarrow \mathrm{H}^{k+1}(\widetilde{\mathfrak{X}}^k_{\overset{\circ}{\mathcal{M}}}, k+1)(\varepsilon) .$$

*Proof.* – This is (3.2) together with (3.1). □

In the appendix, we will establish the following: *If $V$ is $B$-variety whose irreducible components and their iterated intersections are smooth over $B$, then there is a simplicial variety $W_0V$, called the weight zero part of $V$ and a canonical morphism $V \to W_0V$. It induces maps on absolute and geometric cohomology by contravariant functoriality and indeed $\underline{\mathrm{H}}^i(W_0V) \cong W_0\underline{\mathrm{H}}^i(V)$ via the induced morphism.*

PROPOSITION 3.4. – *Let $V$ be a proper variety whose irreducible components and their iterated intersections are smooth toric varieties over $B$. In this case there exist projections*

$$\underline{R\Gamma}(V/B) \to \underline{R\Gamma}(W_0V/B),$$

*such that the induced map on cohomology is a projection to the weight zero part. In particular, $W_0\underline{\mathrm{H}}^i(V)$ is a direct summand. Moreover, there are projections*

$$\mathrm{H}^n(V, m) \to \mathrm{H}^n(W_0V, m) \qquad \textit{for all } n,\ m$$

*in all three absolute cohomology theories. These projections are compatible under realization functors and the Leray spectral sequence. They are uniquely determined by this property for $n = m = \dim(V) + 1$. In particular, they are invariant under endomorphisms of $V$.*

*Proof.* – See A.12 and A.14 □

Note that $\overline{\mathfrak{X}}^k_\infty$ and $\widetilde{\mathfrak{X}}^k_\infty$ are both of this special type (see the explicit description before Lemma A.5). Moreover:

PROPOSITION 3.5. – *The morphism* $\widetilde{\mathfrak{X}}^k_\infty \to \overline{\mathfrak{X}}^k_\infty$ *induces isomorphisms*

$$W_0\underline{\mathrm{H}}^n(\overline{\mathfrak{X}}^k_\infty) \cong W_0\underline{\mathrm{H}}^n(\widetilde{\mathfrak{X}}^k_\infty) = \bigoplus_{\substack{i_1+\ldots+i_k=n\\ i_j\in\{0,1\}}} \underline{\mathrm{H}}^0(B)$$

$$\mathrm{H}^{k+1}(W_0\overline{\mathfrak{X}}^k_\infty, k+1) \cong \mathrm{H}^{k+1}(W_0\widetilde{\mathfrak{X}}^k_\infty, k+1) \cong \mathrm{H}^1(B, k+1).$$

*Proof.* – This is A.6. Recall that by definition $\infty$ gives an isomorphism of $\overline{\mathfrak{X}}_\infty$ with the standard $N$-gon. □

We are now in a position to construct the Harder–Anderson elements.

DEFINITION 3.6. – *Composing* $W_0$ *with the map* $\mathcal{E}$is *from corollary* 3.3 *we get*

$$\mathcal{D}\mathrm{ir}^{\mathrm{HA}} : \mathrm{H}^0(\mathrm{Cusp} \setminus \infty, 0) \longrightarrow \mathrm{H}^1(B, k+1)\ .$$

*Elements in the image of*

$$r_? \mathcal{D}\mathrm{ir}^{\mathrm{HA}}_{\mathcal{M}} : \mathrm{H}^0_{\mathcal{M}}(\mathrm{Cusp} \setminus \infty, 0)^{(k)} \cong \mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Z}/N)}_{\pm P(\mathbb{Z}/N)} \mathrm{H}^0_{\mathcal{M}}(B, 0)^{(k)} \longrightarrow \mathrm{H}^1_?(B, k+1)$$

*in motivic, absolute Hodge or continuous l-adic cohomology are called* Harder-Anderson elements.

## 4. The cup-product construction

We give a second construction of elements in $\mathrm{H}^1(B, k+1)$ starting with the Eisenstein-symbol. We will show in section 5 that it is equivalent to the one given before.

To work with elements in $\mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Z}/N)}_{\pm P(\mathbb{Z}/N)} \mathrm{H}^0(B, 0)^{(k)}$ we want to give a slightly different description of this space. We have an identification

$$\mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Z}/N)}_{\pm P(\mathbb{Z}/N)} \mathrm{H}^0(B, 0)^{(k)} = \left(\mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Z}/N)}_{P(\mathbb{Z}/N)} \mathrm{H}^0(B, 0)\right)^{-E=(-1)^k}$$
$$=: \left(\mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Z}/N)}_{P(\mathbb{Z}/N)} \mathrm{H}^0(B, 0)\right)^{(k)}$$

with $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. This, together with the isomorphism $\mathrm{H}^0_{\mathcal{M}}(B, 0) = \mathbb{Q}$ allows us to write

$$\left(\mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Z}/N)}_{P(\mathbb{Z}/N)} \mathrm{H}^0_{\mathcal{M}}(B, 0)\right)^{(k)} = \left(\mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Z}/N)}_{P(\mathbb{Z}/N)} \mathbb{Q}\right)^{(k)}\ .$$

As in Section 3, (2) we consider the projection

$$\infty^* : \left(\mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Z}/N)}_{P(\mathbb{Z}/N)} \mathrm{H}^0(B, 0)\right)^{(k)} \to \mathrm{H}^0(B, 0)^{(k)}\ .$$

DEFINITION 4.1. – *With the above identification, let* $f_\infty \in \left(\mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Z}/N)}_{P(\mathbb{Z}/N)} \mathbb{Q}\right)^{(k)}$ *be the element such that*

$$f_\infty \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1\ ,$$
$$f_\infty \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = (-1)^k\ ,$$
$$f_\infty(g) = 0 \textit{ for } g \notin \pm P(\mathbb{Z}/N).$$

*We also denote by $f_\infty$ the image of the above element under the regulator in all realizations, so that we can write*

$$f_\infty \in \left(\mathrm{Ind}_{P(\mathbb{Z}/N)}^{\mathrm{GL}_2(\mathbb{Z}/N)} \mathrm{H}^0(B,0)\right)^{(k)} \cong \mathrm{H}^0(\mathrm{Cusp},0)^{(k)} .$$

Now consider the map

$$\mathcal{E}\mathrm{is}(\cdot) \cup \mathcal{E}\mathrm{is}(f_\infty) : \mathrm{H}^0(\mathrm{Cusp} \smallsetminus \infty, 0)^{(k)} \longrightarrow \mathrm{H}^{2k+2}(\mathfrak{X}^k, 2k+2).$$

We compose it with push-forward for the smooth and proper map $\pi^k : \mathfrak{X}^k \to \mathcal{M}$ and residue at infinity $\mathrm{Res} : \mathrm{H}^2(\mathcal{M}, k+2) \to \mathrm{H}^1(B, k+1)$.

DEFINITION 4.2. – *For $k \geq 1$ the composition $\mathrm{Res} \circ \pi^k_*(\mathcal{E}\mathrm{is}(\cdot) \cup \mathcal{E}\mathrm{is}(f_\infty))$ defines a map*

$$\mathcal{D}\mathrm{ir}^{\mathrm{Cup}} : \mathrm{H}^0(\mathrm{Cusp} \smallsetminus \infty, 0)^{(k)} \longrightarrow \mathrm{H}^1(B, k+1).$$

*We say that an element in $\mathrm{H}^1_?(B, k+1)$ is obtained by the* cup-product construction *if it is in the image of $\mathcal{D}\mathrm{ir}^{\mathrm{Cup}}$ on $\mathrm{H}^0_{\mathcal{M}}(\mathrm{Cusp} \smallsetminus \infty, 0)^{(k)}$.*

Note that this construction only involves smooth schemes.

*Remark.* – The same construction (without the restriction to $\ker \infty^*$) already appears in Beilinson's [Be3] 2.3.1. However, he states that the result is always zero. This is false as we shall show (*cf.* 8.4).

## 5. Cup-product vs. Harder-Anderson construction

We want to show that the cup-product construction is equivalent to the Harder-Anderson construction. We do this for absolute Hodge cohomology and $l$-adic cohomology. Because of the injectivity of the regulators, the final result will also hold in the motivic context.

We first use the fact that the Eisenstein symbol on $\mathrm{H}^0(\mathrm{Cusp} \smallsetminus \infty, 0)^{(k)}$ lifts to $\mathrm{H}^{k+1}(\widetilde{\mathfrak{X}}^k_{\overset{\circ}{\mathcal{M}}}, k+1)$. Applying the Poincaré duality isomorphism

$$\mathrm{H}^n(\mathfrak{X}^k, m) \longrightarrow \mathrm{H}_{2k+2-n}(\mathfrak{X}^k, k+1-m)$$

we can equivalently formulate the map in terms of cap-products rather than cup-products. This will have advantages in dealing with the singular varieties that occur. In these terms the cup-product construction is an application of the map

$$\mathrm{H}^{k+1}(\widetilde{\mathfrak{X}}^k_{\overset{\circ}{\mathcal{M}}}, k+1) \times \mathrm{H}_{k+1}(\mathfrak{X}^k, 0) \xrightarrow{\cap} \mathrm{H}_0(\mathfrak{X}^k, -k-1)$$
$$\xrightarrow{\pi^k_*} \mathrm{H}_0(\mathcal{M}, -k-1) \xrightarrow{\mathrm{res}} \mathrm{H}_{-1}(\infty, -k-1)$$

to $(\mathcal{E}\mathrm{is}(f), \mathcal{E}\mathrm{is}(f_\infty)^\vee)$ where $(\cdot)^\vee$ is the Poincaré dual and $f \in \mathrm{H}^0(\mathrm{Cusp} \smallsetminus \infty, 0)^{(k)}$. $\mathcal{E}\mathrm{is}(f) \cap \cdot$ is compatible with the long exact homology sequence for the triple $(\widetilde{\mathfrak{X}}^k_\infty, \widetilde{\mathfrak{X}}^k_{\overset{\circ}{\mathcal{M}}}, \mathfrak{X}^k)$. The connecting morphism and the functoriality of push-forward give the commutative diagram

$$\begin{array}{ccccc}
\mathrm{H}_{k+1}(\mathfrak{X}^k, 0) & \xrightarrow{\mathcal{E}\mathrm{is}(f)\cap} & \mathrm{H}_0(\mathfrak{X}^k, -k-1) & \xrightarrow{\pi^k_*} & \mathrm{H}_0(\mathcal{M}, -k-1) \\
\downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\mathrm{res}} \\
\mathrm{H}_k(\widetilde{\mathfrak{X}}^k_\infty, 0) & \xrightarrow{i^*(\mathcal{E}\mathrm{is}(f))\cap} & \mathrm{H}_{-1}(\widetilde{\mathfrak{X}}^k_\infty, -k-1) & \xrightarrow{\widetilde{\pi}^k_*} & \mathrm{H}_{-1}(B, -k-1)
\end{array}$$

We now have reformulated the cup-product construction in terms on $N$-gons.

*Remark.* – It is in applying the push-forward that we use the completeness of $\widetilde{\mathfrak{X}}^k$, *i.e.*, that the Néron-model would not do the job. This is not only a technical question – the very fact that we have something interesting in $\mathrm{H}_{-1}(\widetilde{\mathfrak{X}}^k_\infty, -k-1)$ relies on the vanishing of the corresponding element in the special fibre of the Néron model.

LEMMA 5.1. – *The map*

$$\mathrm{H}^{k+1}(\widetilde{\mathfrak{X}}^k_\infty, k+1) \xrightarrow{\cap \mathcal{E}\mathrm{is}(f_\infty)^\vee} \mathrm{H}_{-1}(\widetilde{\mathfrak{X}}^k_\infty, -k-1) \xrightarrow{\widetilde{\pi}^k_*} \mathrm{H}_{-1}(B, -k-1)$$

*factors through* $\mathrm{H}^{k+1}(W_0\widetilde{\mathfrak{X}}^k_\infty, k+1) \cong \mathrm{H}^1(\infty, k+1)$. *The induced map is, up to its sign, given by the Poincaré duality isomorphism*

$$\mathrm{H}^1(B, k+1) \longrightarrow \mathrm{H}_{-1}(B, -k-1).$$

*Proof.* – We abbreviate $\mathfrak{X}_\infty = \mathcal{N}$, $\widetilde{\mathfrak{X}}^k = \widetilde{\mathcal{N}}^k$. Our map

$$\mathrm{H}^{k+1}(\widetilde{\mathcal{N}}^k, k+1) \times \mathrm{H}_k(\widetilde{\mathcal{N}}^k, 0) \to \mathrm{H}_{-1}(\widetilde{\mathcal{N}}^k, -k-1) \to \mathrm{H}_{-1}(B, -k-1)$$

arises as the composition

$$\begin{aligned} F(-k-1)[-k-1] \otimes F[k] &\xrightarrow{i^*\mathcal{E}\mathrm{is}(f)\otimes\mathcal{E}\mathrm{is}(f_\infty)^\vee} \underline{R\Gamma}(\widetilde{\mathcal{N}}^k) \otimes \mathbb{D}\underline{R\Gamma}(\widetilde{\mathcal{N}}^k) \\ &\xrightarrow{\cap} \mathbb{D}\underline{R\Gamma}(\widetilde{\mathcal{N}}^k) \to \tau_{\geq 0}\mathbb{D}R(\widetilde{\mathcal{N}}^k) = \underline{\mathrm{H}}^0(B). \end{aligned}$$

Note first that any map $F[k] \to \mathbb{D}\underline{R\Gamma}(\widetilde{\mathcal{N}}^k)$ factors through $\tau_{\leq -k}\mathbb{D}\underline{R\Gamma}(\widetilde{\mathcal{N}}^k)$. Any map $F(-k-1)[-k-1] \to \underline{R\Gamma}(\widetilde{\mathcal{N}}^k)$ factors even through $\tau_{\leq k}\underline{R\Gamma}(\widetilde{\mathcal{N}}^k)$ because the weights of $\underline{\mathrm{H}}^{k+1}(\widetilde{\mathcal{N}}^k)$ are smaller than $k+1 \leq 2k+2$. Finally, a map $\underline{R\Gamma}(\widetilde{\mathcal{N}}^k)\otimes\mathbb{D}\underline{R\Gamma}(\widetilde{\mathcal{N}}^k) \to \underline{\mathrm{H}}^0(B)$ factors through $\tau_{\geq 0}(\underline{R\Gamma}(\widetilde{\mathcal{N}}^k) \otimes \mathbb{D}\underline{R\Gamma}(\widetilde{\mathcal{N}}^k))$. Putting this information together we see that our cap-product factors through the geometric cap-product

$$\underline{\mathrm{H}}^k(\widetilde{\mathcal{N}}^k) \otimes \underline{\mathrm{H}}_k(\widetilde{\mathcal{N}}^k) \longrightarrow \underline{\mathrm{H}}_0(\widetilde{\mathcal{N}}^k).$$

$\mathcal{E}\mathrm{is}(f_\infty)^\vee : F \to \underline{\mathrm{H}}_k(\widetilde{\mathcal{N}}^k)$ has to respect weights, *i.e.* it must factor through $\mathrm{Gr}^W_0\underline{\mathrm{H}}_k(\widetilde{\mathcal{N}}^k) = \underline{\mathrm{H}}_k(\mathcal{N}^k)(\varepsilon)$. Hence only the weight zero part of $\underline{\mathrm{H}}^k(\widetilde{\mathcal{N}}^k)$ contributes. The map of the proposition factors as stated through the projection to the weight zero part of $\widetilde{\mathcal{N}}^k$. $\mathcal{E}\mathrm{is}(f_\infty)^\vee$ gives an identification of $\mathrm{Gr}^W_0\underline{\mathrm{H}}_k(\widetilde{\mathcal{N}}^k)$ with $\underline{\mathrm{H}}_0(B)$. Using the dual identification for $W_0\underline{\mathrm{H}}^k(\widetilde{\mathcal{N}}^k)$ (as we did when we specified it in the remark after A.14) our map is identified with the cap-product for $B$. □

PROPOSITION 5.2. – *The maps*

$$\mathcal{D}\mathrm{ir}^{\mathrm{HA}}, \mathcal{D}\mathrm{ir}^{\mathrm{Cup}} : \mathrm{H}^0_?(\mathrm{Cusp} \smallsetminus \infty, 0)^{(k)} \to \mathrm{H}^1_?(B, k+1)$$

*are equal in motivic, absolute Hodge and continuous l-adic cohomology. We denote the map simply* $\mathcal{D}$ir.

*Proof.* – As the regulator maps are injective on $\mathrm{H}^1_{\mathcal{M}}(B, k+1)$, it is enough to prove equality in the realizations. We have shown that

$$\begin{aligned} \mathcal{D}\mathrm{ir}^{\mathrm{Cup}}(f)^\vee &= i^*(\mathcal{E}\mathrm{is}(f) \cap \delta\mathcal{E}\mathrm{is}(f_\infty)^\vee \\ &= W_0(i^*(\mathcal{E}\mathrm{is}(f)) \cap \delta\mathcal{E}\mathrm{is}(f_\infty) \\ &= \mathcal{D}\mathrm{ir}^{\mathrm{HA}}(f) \cap \delta\mathcal{E}\mathrm{is}(f_\infty)^\vee \quad . \end{aligned} \tag{5.1}$$

By the last lemma this implies the equality. □

## 6. The sheaf theoretic interpretation

We are going to represent the Harder-Anderson extensions by explicit Yoneda-extensions. This is done by reinterpreting all objects sheaf-theoretically. We start by collecting the main ingredients. Let $j : \mathcal{M} \to \mathring{\mathcal{M}}$ be the inclusion. Let $\varepsilon$ be the character defined in section 3.

LEMMA 6.1. – *The following isomorphisms of sheaves on $B$ hold:*

$$\begin{aligned}
\underline{\mathrm{H}}^{k+1}(\mathring{\mathcal{M}}, j_!\pi^k_* F_{\mathfrak{X}^k})(\varepsilon) &= \underline{\mathrm{H}}^1(\mathring{\mathcal{M}}, j_!\mathrm{Sym}^k R^1\pi_* F_{\mathfrak{X}}),\\
\underline{\mathrm{H}}^{k}(\mathring{\mathcal{M}}, j_!\pi^k_* F_{\mathfrak{X}^k})(\varepsilon) &= 0,\\
\underline{\mathrm{H}}^k(\overline{\mathfrak{X}}^k_\infty)(\varepsilon) &\cong \underline{\mathrm{H}}^0(B),\\
\underline{\mathrm{H}}^k(\widetilde{\mathfrak{X}}^k_{\mathring{\mathcal{M}}})(\varepsilon) &= 0.
\end{aligned}$$

*Proof.* – Analyzing the operation of $\Gamma_k$ on the different cohomology groups of an elliptic, curve respectively on Néron-$N$-gons, we get for (perverse!) cohomology on $\mathring{\mathcal{M}}$:

$$\begin{aligned}
H^2\pi_* F_{\mathfrak{X}}(\varepsilon) &= H^2\pi_* F_{\mathfrak{X}}\\
H^1\overline{\pi}_* F_{\overline{\mathfrak{X}}_\infty}(\varepsilon) &= H^1\overline{\pi}_* F_{\overline{\mathfrak{X}}_\infty} = F_\infty,
\end{aligned}$$

whereas the $\varepsilon$-eigenspaces of the other cohomology objects vanish. ($H^2\pi_* F_{\mathfrak{X}}$ corresponds to $R^1\pi_* F_{\mathfrak{X}}[1]$ in usual cohomology.) The sign-eigenspace of the symmetric group translates into the symmetric power because of the anti-commutativity of the cup-product. Hence:

$$\begin{aligned}
H^{k+1}\pi^k_* F_{\mathfrak{X}^k}(\varepsilon) &= \mathrm{Sym}^k H^2\pi_* F_{\mathfrak{X}},\\
H^k\overline{\pi}^k_* F_{\overline{\mathfrak{X}}^k_\infty}(\varepsilon) &= \mathrm{Sym}^k H^1\overline{\pi}_* F_{\overline{\mathfrak{X}}_\infty} = F_\infty,
\end{aligned}$$

and again the $\varepsilon$-eigenspaces of the other cohomology objects vanish. By the Leray spectral sequence this implies the first three equalities. By [Sch] 1.2.1, the group $\underline{\mathrm{H}}^k(\widetilde{\mathfrak{X}}^k)(\varepsilon)$ vanishes. Using the localization sequence we see easily that this implies the same vanishing for $\widetilde{\mathfrak{X}}^k_{\mathring{\mathcal{M}}}$. □

Recall the map $p : \widetilde{\mathfrak{X}}^k \to \overline{\mathfrak{X}}^k$. We also need its base change with $\infty$ or $\mathring{\mathcal{M}}$. The lemma and Proposition A.5 imply that we have the commutative diagram

$$\begin{array}{ccc}
W_0\underline{\mathrm{H}}^k(\widetilde{\mathfrak{X}}^k_\infty)(\varepsilon) & \stackrel{=}{\longleftarrow} & W_0\underline{\mathrm{H}}^k(\overline{\mathfrak{X}}^k_\infty)(\varepsilon)\\
\uparrow & & \uparrow\cong\\
\underline{\mathrm{H}}^k(\widetilde{\mathfrak{X}}^k_\infty)(\varepsilon) & \stackrel{p_\infty}{\longleftarrow} & \underline{\mathrm{H}}^k(\overline{\mathfrak{X}}^k_\infty)(\varepsilon)
\end{array}$$

In particular $p_\infty$ is injective and can be identified with the inclusion of the weight zero part. We now recommend to the reader to write out the big diagram containing the above square, the $\varepsilon$-eigenspaces of the long exact sequences for the triples $(j_!\pi^k_* F_{\mathfrak{X}^k} \to \widetilde{\pi}^k_* F_{\widetilde{\mathfrak{X}}^k} \to i_*\widetilde{\pi}^k_* F_{\widetilde{\mathfrak{X}}^k_\infty})$ and $(j_!\pi^k_* F_{\mathfrak{X}^k} \to \overline{\pi}^k_* F_{\overline{\mathfrak{X}}^k} \to i_*\overline{\pi}^k_* F_{\overline{\mathfrak{X}}^k_\infty})$ as well as the other maps considered below. In particular, we have

$$\begin{array}{ccccccc}
\underline{\mathrm{H}}^k(\widetilde{\mathfrak{X}}^k_\infty)^\varepsilon & \hookrightarrow & \underline{\mathrm{H}}^1(\mathring{\mathcal{M}}, j_!\mathrm{Sym}^k R^1\pi_* F_{\mathfrak{X}})^\varepsilon & \stackrel{\tilde{d}^\varepsilon}{\to} & \underline{\mathrm{H}}^{k+1}(\widetilde{\mathfrak{X}}^k_{\mathring{\mathcal{M}}})^\varepsilon & \to & \underline{\mathrm{H}}^{k+1}(\widetilde{\mathfrak{X}}^k_\infty)^\varepsilon\\
{\scriptstyle p_\infty}\uparrow & & \uparrow = & & \uparrow{\scriptstyle p_{\mathring{\mathcal{M}}}} & & \uparrow\\
\underline{\mathrm{H}}^k(\overline{\mathfrak{X}}^k_\infty)^\varepsilon & \hookrightarrow & \underline{\mathrm{H}}^1(\mathring{\mathcal{M}}, j_!\mathrm{Sym}^k R^1\pi_* F_{\mathfrak{X}})^\varepsilon & \stackrel{\bar{d}^\varepsilon}{\to} & \underline{\mathrm{H}}^{k+1}(\overline{\mathfrak{X}}^k_{\mathring{\mathcal{M}}})^\varepsilon & \to & \underline{\mathrm{H}}^{k+1}(\overline{\mathfrak{X}}^k_{\mathring{\mathcal{M}}})^\varepsilon
\end{array}$$

$p_\infty$ has a section (the projection to the weight zero part), hence there is also a projection inverse to $p_{\overset{\circ}{\mathcal{M}}}$ on $\mathrm{Im}(\tilde{d})^\varepsilon$. Recall that for each $f \in \mathrm{H}^0(\mathrm{Cusp} \setminus \infty, 0)^{(k)}$ the Eisenstein symbol $\mathcal{E}\mathrm{is}(f)$ defines a map $F(-k-1) \to \underline{\mathrm{H}}^{k+1}(\widetilde{\mathfrak{X}}^k_{\overset{\circ}{\mathcal{M}}})(\varepsilon)$. Its image in $\underline{\mathrm{H}}^{k+1}(\widetilde{\mathfrak{X}}^k_\infty)(\varepsilon)$ vanishes for weight reasons. Hence it factors through $\mathrm{Im}(\tilde{d})(\varepsilon)$ and using the projection it factors even through $\mathrm{Im}(\bar{d})(\varepsilon)$. Pull-back via this map induces a Yoneda-extension

$$0 \longrightarrow \underline{\mathrm{H}}^0(B) \longrightarrow E_f \longrightarrow F(-k-1) \longrightarrow 0 \qquad (*_f)$$

where we fix an isomorphism $\underline{\mathrm{H}}^0(\overline{\mathfrak{X}}^k_\infty)(\varepsilon)$ with $\underline{\mathrm{H}}^0(B)$.

Proposition 6.2. – *The Harder-Anderson element $\mathcal{D}\mathrm{ir}^{\mathrm{HA}}(f)$ for $f \in \mathrm{H}^0(\mathrm{Cusp} \setminus \infty, 0)$ as in* 3.6 *is given by the above Yoneda-extension $(*_f)$.*

*Proof.* – A priori, the Harder-Anderson extension is given by a composition of morphisms in the derived category of sheaves on $\mathrm{Spec}\mathbb{Q}$:

$$F(-k-1)[-k-1] \to \underline{R\Gamma}(\widetilde{\mathfrak{X}}^k_{\overset{\circ}{\mathcal{M}}}) \to \underline{R\Gamma}(\widetilde{\mathfrak{X}}^k_\infty) \to \tau_{\geq k} W_0 \underline{R\Gamma}(\widetilde{\mathfrak{X}}^k_\infty) = W_0 \underline{\mathrm{H}}^k(\widetilde{\mathfrak{X}}^k_\infty)[-k]$$

The additional truncation $\tau_{\geq k}$ does not change the composition. From the above considerations, we see that the compositions

$$F(-k-1) \to \underline{\mathrm{H}}^{k+1}(\widetilde{\mathfrak{X}}^k_{\overset{\circ}{\mathcal{M}}}) \to \underline{\mathrm{H}}^{k+1}(\widetilde{\mathfrak{X}}^k_\infty),$$
$$\underline{\mathrm{H}}^k(\widetilde{\mathfrak{X}}^k_{\overset{\circ}{\mathcal{M}}}) \to \underline{\mathrm{H}}^{k+1}(\widetilde{\mathfrak{X}}^k_\infty) \to W_0 \underline{\mathrm{H}}^k(\widetilde{\mathfrak{X}}^k_\infty) = W_0 \underline{\mathrm{H}}^k(\widetilde{\mathfrak{X}}^k_\infty)(\varepsilon)$$

vanish. Hence the assumptions of the small lemma below are valid and we see that we find the Harder-Anderson extension as pull-back and push-out in the diagram

$$\begin{array}{ccccc}
W_0 \underline{\mathrm{H}}^k(\widetilde{\mathfrak{X}}^k_\infty) & & & & \\
\uparrow & & & & \\
\underline{\mathrm{H}}^k(\widetilde{\mathfrak{X}}^k_\infty) & \longrightarrow & \underline{\mathrm{H}}^{k+1}(\overset{\circ}{\mathcal{M}}, j_! \pi^k_* F_{\mathfrak{X}^k}) & \longrightarrow & \underline{\mathrm{H}}^{k+1}(\widetilde{\mathfrak{X}}^k_{\overset{\circ}{\mathcal{M}}}) \\
 & & & & \uparrow \\
 & & & & F(-k-1)
\end{array}$$

Everything factors through $\varepsilon$-eigenspaces. From there it is an easy exercise that the push-out can be identified as we did. □

Lemma 6.3. – *Let $D$ be a triangulated category with a t-structure. Let $(A \to B \to C)$ be a distinguished triangle. Let $F$ and $W$ be objects in the heart of the t-structure. Suppose we are given morphisms $f : F \to C$ and $w : A \to W$ in $D$ such that the induced compositions $F \to H^0(C) \to H^1(A)$ and $H^{-1}(C) \to H^0(A) \to W$ vanish. Then the 1-extension defined by the composition $F \to C \to A[1] \to W[1]$ is represented by the push-out via $w$ and pull-back via $f$ of the exact sequence*

$$H^0(A) \to H^0(B) \to H^0(C).$$

*Proof.* – Apply the truncation functor $\tau_{\geq 0}$ to $B$ and $C$. Extend to a morphism of triangles $(A, B, C) \to (A', \tau_{\geq 0}B, \tau_{\geq 0})$. Obviously $H^0(A') \cong H^0(A)/\mathrm{Im}H^{-1}(C)$. By our assumption on $w$ it factors through $A'$. We apply the dual manipulation using $\tau_{\leq 0}$. Hence we can assume without loss of generality that the $A, B, C$ are in the heart. The equivalence of Yondeda-extensions with morphisms in the derived category identifies $C \to A[1]$ with the short exact sequence. The lemma follows by functoriality. $\square$

Now we want to show how the construction given here relates to Harder's original one, *e.g.* in [Ha] 4.2. He uses $Y_1(p)$ where $p$ is a prime. The scheme of cusps is decomposed as $\Sigma_\infty \cup \Sigma_0$. He starts with the canonical module $M_k$ on $Y_1(p)$, *i.e.*, with $\mathrm{Sym}^k R^1\pi_* F_{\widetilde{\mathfrak{X}}}$. (In the identification in [Ha] 4.2.1 we have to take the antisymmetric part rather than the symmetric part because cup-product is anti-commutative.) He forms $M_k^\#$ on $\overline{Y_1(p)}$ by extending via $j_*$ at the cusp $\Sigma_\infty$ and via $j_!'$ at $\Sigma_0$. Harder then considers the mixed object $\underline{\mathrm{H}}^{k+1}(\overline{Y_1(p)}, M_k^\#)$. It is mixed of weights 0 (corresponding to the image of $\underline{\mathrm{H}}(\overline{\mathfrak{X}}^k_{\Sigma_0})(\varepsilon)$), $k+1$ (the cuspidal part of cohomology) and $2k+2$ (preimage of $\underline{\mathrm{H}}(\overline{\mathfrak{X}}^k_\infty)$). Analysis of the operation of the Hecke-algebra first splits off the cuspidal part as a direct summand. We are left with an extension

$$0 \to \underline{\mathrm{H}}^0(\Sigma_0, 0) \to E \to \underline{\mathrm{H}}^0(\Sigma_\infty, -k-1) \to 0$$

which can be decomposed into the eigen-components under the Hecke-algebra after extension of scalars to $\mathbb{Q}(\mu_{p-1})$. This yields extensions

$$0 \to \mathbb{Q} \to E_B(\chi) \to \underline{\mathrm{H}}^0(\mathrm{Spec}\mathbb{Q}(\mu_p), -k-1)(\chi) \to 0$$

for Dirichlet characters $\chi$ satisfying $\chi(-1) = (-1)^k$.

PROPOSITION 6.4. – *Let* $H = \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \subset \mathrm{GL}_2(\mathbb{Z}/p)$ *be the covering group of* $\mathcal{M} = Y(p)$ *over* $Y_1(p)$. *For* $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *let* $f_w$ *be the function on* $\mathrm{GL}_2(\mathbb{Z}/p)$ *defined by* $f_w(g) = f_\infty(gw)$ *for* $f_\infty$ *as in* 4.1. *The dual of the Harder-Anderson element* $\mathcal{D}\mathrm{ir}^{\mathrm{HA}}(f_w)$ *induces the extension classes constructed by Harder* ([Ha]) 4.2 *in the Hodge or l-adic realization by projecting to H-invariants and decomposing under the Galois group.*

*Proof.* – We first have to understand the cusps of $Y(p)$ relative to those of $Y_1(p)$. There is one $H$-orbit above $\Sigma_\infty$. It is isomorphic to $\coprod_{h\in\pm U(\mathbb{Z}/p)\backslash H} B$. There are $(p-1)/2$ more orbits, each of them mapping to one copy of $\mathrm{Spec}\mathbb{Q} \subset \Sigma_0$. The group $P(\mathbb{Z}/p) = \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$ operates from the right on Cusp. It respects the cusps in the $H$-orbit of $\infty$. Note that the operation here is the same as the operation of the Galois group which was by $\begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix}$ from the left. On the other hand $P(\mathbb{Z}/p)$ operates simply transitively on the $H$-orbits above $\Sigma_0$.

On $Y_1(p)$ we consider the continuation of $M_k$ via $j_!$ at $\Sigma_\infty$ and via $j_*$ at $\Sigma_0$. Note that we have exchanged the roles of 0 and $\infty$ with regard to Harder's construction. But this is (up to a twist) the dual of the original one because the canonical module is dual to $M_k(k)$ whereas $j_!$ and $j_*$ are exchanged. We can repeat the construction for $Y(p)$, *i.e.*, we consider the canonical module on the modular curve and extend by $j_!$ at all cusps above $\Sigma_\infty$ and by $j_*$ at all others. By taking $H$-invariants we can pass from $Y(p)$ to $Y_1(p)$. Cohomology of the mixed sheaf can be identified with an explicit extension as in the beginning of this section. After splitting off the cuspidal cohomology, we are now dealing with an extension

$$0 \to \underline{\mathrm{H}}^0(\infty \cdot H, 0) \to E \to \underline{\mathrm{H}}^0(\mathrm{Cusp} \smallsetminus \infty \cdot H, -k-1) \to 0. \qquad (*)$$

The construction of $\mathcal{D}\mathrm{ir}^{\mathrm{HA}}(f_w)$ given in this article means push-out via the projection to $\infty$, *i.e.*, $\underline{\mathrm{H}}^0(\infty \cdot H, 0) \to \underline{\mathrm{H}}^0(B, 0)$ and pull-back under $f_w : \underline{\mathrm{H}}^0(B) \to \underline{\mathrm{H}}^0(\mathrm{Cusp} \smallsetminus \infty \cdot H, 0)$. Moreover, we project to $H$-invariants in order to get something coming from $Y_1(p)$. Taking $H$-invariants turns the first map into an isomorphism. Note that the decomposition of $\underline{\mathrm{H}}^0(B, 0)$ under $P(\mathbb{Z}/p)$ is the same as the decomposition under the operation of the Galois group. On the other hand the $H$-invariants of $\underline{\mathrm{H}}^0(\mathrm{Cusp} \smallsetminus \infty \cdot H, -k-1)$ are given by $\underline{\mathrm{H}}^0(\Sigma_0, -k-1)$. We apply the projection to the $P(\mathbb{Z}/p)$-eigencomponents in $\underline{\mathrm{H}}^0(\mathrm{Cusp} \smallsetminus \infty \cdot H, -k-1)$ to the $H$-invariant map associated to $f_w$. The result is non-zero and hence an isomorphism. Note finally that in Harder's construction the decomposition under $P(\mathbb{Z}/p)/H$ is the same as under the Hecke-algebra on $Y_1(p)$. These considerations give the identification we claimed. □

## 7. The absolute Hodge-realization of the cup-product construction

In this section we prepare the computation of $\mathcal{D}\mathrm{ir}_{\mathcal{H}}^{\mathrm{Cup}}(f)$. After Proposition 5.2, stating that $\mathcal{D}\mathrm{ir}_{\mathcal{H}}^{\mathrm{Cup}}(f) = \mathcal{D}\mathrm{ir}_{\mathcal{H}}^{\mathrm{HA}}(f)$, and after Harder's computation [Ha] 4.3, it should be no surprise that $\mathcal{D}\mathrm{ir}_{\mathcal{H}}^{\mathrm{Cup}}(f)$ can be expressed in terms of special values of the Dirichlet L-series. Note that Harder does only a very special case and that our method of construction is completely different. We think that the following computation sheds some additional light on the nature of our construction.

### Realization of the Eisenstein symbol

Let us start by recalling the realization of the Eisenstein symbol in absolute Hodge cohomology. As in section 2 we follow [De2].

Let $\mathfrak{H} := \{\tau \in \mathbb{C} \,|\, \mathrm{Im}\tau > 0\}$ be the upper half plane with the usual action of $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) : \tau \mapsto \frac{a\tau+b}{c\tau+d}$. Consider the action of $\left(\binom{m}{n}, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \in \mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{C} \times \mathfrak{H}$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z, \tau) := \left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right),$$
$$\begin{pmatrix} m \\ n \end{pmatrix}(z, \tau) := (z + m - n\tau, \tau)\,.$$

Then we have:

$$\mathfrak{X}^k(\mathbb{C}) = \mathbb{Z}^{2k} \rtimes \mathrm{SL}_2(\mathbb{Z}) \setminus (\mathbb{C}^k \times \mathfrak{H}) \times ((\mathbb{Z}/N)^{2k} \rtimes \mathrm{GL}_2(\mathbb{Z}/N))\,.$$

As in [De2] (and [Be3]) consider ($\pm U = \pm\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$):

$$\mathcal{Y}^k(\mathbb{C}) = \mathbb{Z}^{2k} \rtimes \pm U(\mathbb{Z}) \setminus (\mathbb{C}^k \times \mathfrak{H}) \times ((\mathbb{Z}/N)^{2k} \rtimes \mathrm{GL}_2(\mathbb{Z}/N))$$

together with its projection $p_k : \mathcal{Y}^k(\mathbb{C}) \to \mathfrak{X}^k(\mathbb{C})$. Let $q := \exp(2\pi i\tau)$ and $z_1, \ldots, z_k$ be the coordinates on $\mathbb{C}^k$. We recall the identification from section 4:

$$\mathrm{Ind}_{\pm P(\mathbb{Z}/N)}^{\mathrm{GL}_2(\mathbb{Z}/N)} \mathrm{H}^0_{\mathcal{M}}(B, 0)^{(k)} \cong \left(\mathrm{Ind}_{P(\mathbb{Z}/N)}^{\mathrm{GL}_2(\mathbb{Z}/N)} \mathbb{Q}\right)^{(k)}\,.$$

For any function $f \in \left(\mathrm{Ind}_{P(\mathbb{Z}/N)}^{\mathrm{GL}_2(\mathbb{Z}/N)}\mathbb{Q}\right)^{(k)}$ consider the holomorphic form on $\mathcal{Y}^k(\mathbb{C})$

$$(q, z_1, \ldots, g) \mapsto \mathrm{Eis}^k(f)(g) := p_{k*}\left(\frac{(2\pi i)^k}{N} f(g)\frac{\mathrm{d}q}{q} \wedge \mathrm{d}z_1 \wedge \ldots \wedge \mathrm{d}z_k\right).$$

As in [De2] 3.7. this is:

$$\mathrm{Eis}^k(f)(g) = \frac{(2\pi i)^k}{N} \sum_{\gamma \in \pm U(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{Z})} \frac{f(\gamma g)}{(c\tau + d)^{k+2}} \frac{\mathrm{d}q}{q} \wedge \mathrm{d}z_1 \wedge \ldots \wedge \mathrm{d}z_k ,$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. This is a holomorphic $k+1$-form on $\mathfrak{X}^k(\mathbb{C})$. Let $\mathbb{R}(k) := (2\pi i)^k\mathbb{R}$ and

$$\pi_k : \mathbb{C} \to \mathbb{R}(k), \quad \alpha \mapsto \frac{1}{2}(\alpha + (-1)^k\overline{\alpha}).$$

On $\mathbb{C}$-valued differential forms this induces

$$\pi_k : \omega \mapsto \frac{1}{2}(\omega + (-1)^k\overline{\omega})$$

Recall that any element in $\mathrm{H}_{\mathcal{H}}^{k+1}(\mathfrak{X}^k, k+1)$ is of the form $\eta$ where $\eta$ is an $\mathbb{R}$-valued $\mathcal{C}^\infty$-$k$-form on $\mathfrak{X}^k(\mathbb{C})$ with $\mathrm{d}\eta = \pi_k\omega$, where $\omega$ is a holomorphic $k+1$-form with logarithmic poles along $\widetilde{\mathfrak{X}}^k(\mathbb{C}) \setminus \mathfrak{X}^k(\mathbb{C})$ (see [Den-Sch] for this description).

LEMMA 7.1. – *The class of $\mathcal{E}\mathrm{is}_{\mathcal{H}}^k(f) \in \mathrm{H}_{\mathcal{H}}^{k+1}(\mathfrak{X}^k, k+1)$ is represented by a real analytic, $\mathbb{R}(k)$-valued differential k-form, also called $\mathcal{E}\mathrm{is}_{\mathcal{H}}^k(f)$ on $\mathfrak{X}^k(\mathbb{C})$, such that*

$$\mathrm{d}\mathcal{E}\mathrm{is}_{\mathcal{H}}^k(f) = \pi_k \mathrm{Eis}_{\mathcal{H}}^k(f).$$

*Proof.* – This is [Be3] theorem 3.1.7 and 2.2. □

*Remark.* – Note that formula (3.16) in [De2] is not correct. There is a factor $\frac{1}{N}$ missing which is due to the fact that the residue of $\frac{\mathrm{d}q}{q}$ in loc. cit. is $N$. The map res in [Be3] 2.1.2 differs from Deninger's by this factor $\frac{1}{N}$ due to ramification at the cusp $\infty$.

This form can be constructed as follows: define $(z_0 := \tau)$,

$$\kappa_{\mathcal{H}}^k := -(2\pi i)^k \cdot \sum_{r,s=0}^{k} (-1)^s 2\pi C_r \cdot y_s \eta_s^r$$

where $y_s := \mathrm{Im}(z_s), C_r := \left((k+1)\binom{k}{r}\right)^{-1}$ and

$$\eta_s^r := \sum (\mathrm{d}\overset{(-)}{z_0} \wedge \ldots \wedge \widehat{\mathrm{d}\overset{(-)}{z_s}} \wedge \ldots \wedge \mathrm{d}\overset{(-)}{z_k})^{(r,k-r)}$$

where the sum is over the exterior products of $r$ factors $\mathrm{d}z$ and $k-r$ factors $\mathrm{d}\overline{z}$, the $\widehat{\mathrm{d}\,z_s{}^{(-)}}$ being omitted. Then we get

$$\mathrm{d}\kappa_{\mathcal{H}}^k = \pi_k\left((2\pi i)^k \frac{\mathrm{d}q}{q} \wedge \mathrm{d}z_1 \wedge \ldots \wedge \mathrm{d}z_k\right).$$

LEMMA 7.2. – *The differential form $\mathcal{E}\mathrm{is}^k_{\mathcal{H}}(f)$ is given explicitly by the series*

$$\frac{1}{N}\sum_{\gamma\in\pm U(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{Z})} f(\gamma g)\cdot\gamma^*(\kappa^k_{\mathcal{H}}).$$

*Proof.* – This is [Be3] 2.2. □

For later use we need a formula for the projection of $\mathcal{E}\mathrm{is}^k_{\mathcal{H}}(f)$ into the relative differential forms for $\mathfrak{X}(\mathbb{C})\to\mathcal{M}(\mathbb{C})$. A straightforward calculation gives, with $\eta_0^{k-j}$ as above (*cf.* [De2] 3.12),

$$\mathcal{E}\mathrm{is}^k_{\mathcal{H}}(f)(g)\equiv\frac{(2\pi i)^{k+1}\pi_1(\tau)}{N(k+1)}\sum_{j=0}^{k}\binom{k}{j}^{-1}\eta_0^{k-j}$$

$$\left(\sum_{\gamma\in\pm U(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{Z})} f(\gamma g)\frac{(c\tau+d)^j(c\bar\tau+d)^{k-j}}{|c\tau+d|^{2(k+1)}}\right)\quad \bmod \mathrm{d}\tau,\mathrm{d}\bar\tau. \quad (3)$$

**Cup-products**

The next thing we need to recall is the explicit formula for the cup-product of two Deligne cohomology classes from [Den-Sch], 2.5. For cohomology classes $\eta_i$, let $\omega_i$ be the unique holomorphic solution of $\mathrm{d}\eta_i=\pi_k(\omega_i)$. The cup-product $\eta_1\cup\eta_2\in \mathrm{H}^{2k+2}_{\mathcal{H}}(\mathfrak{X}^k,2k+2)$ is then given by

$$\eta_1\wedge\pi_{k+1}\omega_2+(-1)^{k+1}\pi_{k+1}\omega_1\wedge\eta_2$$

(*cf.* [Den-Sch]).

If we apply this to the Eisenstein symbols $\mathcal{E}\mathrm{is}^k_{\mathcal{H}}(f_i)$ (with $f_1,f_2$ being in $\mathrm{Ind}^{\mathrm{GL}_2(\mathbb{Z}/N)}_{\pm P(\mathbb{Z}/N)}\mathbb{Q}^{(k)}$), we get

$$\mathcal{E}\mathrm{is}^k_{\mathcal{H}}(f_1)\wedge\pi_{k+1}\mathrm{Eis}^k(f_2)+(-1)^{k+1}\pi_{k+1}\mathrm{Eis}^k(f_1)\wedge\mathcal{E}\mathrm{is}^k_{\mathcal{H}}(f_2)\ .$$

**Push-forward to $\mathcal{M}$**

Denote by $\mathcal{E}\mathrm{is}^{(k,0)}_{\mathcal{H}}(f_i)$ (resp. $\mathcal{E}\mathrm{is}^{(0,k)}_{\mathcal{H}}(f_i)$), $i=1,2$, the $(k,0)$ resp. the $(0,k)$-component of $\mathcal{E}\mathrm{is}^k_{\mathcal{H}}(f_i)$ along the fibres of $\pi^k:\mathfrak{X}^k(\mathbb{C})\to\mathcal{M}(\mathbb{C})$. Then one has

$$\begin{aligned}&\pi^k_*(\mathcal{E}\mathrm{is}^k_{\mathcal{H}}(f_1)\cup\mathcal{E}\mathrm{is}^k_{\mathcal{H}}(f_2))=\\&\quad\frac{1}{2}\pi^k_*\Big[\mathcal{E}\mathrm{is}^{(0,k)}_{\mathcal{H}}(f_1)\wedge\mathrm{Eis}^k(f_2)+(-1)^{k+1}\mathcal{E}\mathrm{is}^{(k,0)}_{\mathcal{H}}(f_1)\wedge\overline{\mathrm{Eis}^k(f_2)}\\&\qquad+(-1)^{k+1}\mathrm{Eis}^k(f_1)\wedge\mathcal{E}\mathrm{is}^{(0,k)}_{\mathcal{H}}(f_2)+\overline{\mathrm{Eis}^k(f_1)}\wedge\mathcal{E}\mathrm{is}^{(k,0)}_{\mathcal{H}}(f_2)\Big]\ .\end{aligned}\quad (4)$$

The formula for $\mathcal{E}\mathrm{is}^k_{\mathcal{H}}(f_i)$ gives, $y_0:=\mathrm{Im}\tau$:

$$\mathcal{E}\mathrm{is}^{(k,0)}(f_i)(g)=(2\pi i)^k\frac{-2\pi y_0}{N(k+1)}\sum_{\gamma\in\pm U(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{Z})} f_i(\gamma g)\frac{(c\bar\tau+d)^k\mathrm{d}z_1\wedge\ldots\wedge\mathrm{d}z_k}{|c\tau+d|^{2(k+1)}}\quad (5)$$

and

$$\mathcal{E}\mathrm{is}_{\mathcal{H}}^{(0,k)}(f_i) = (2\pi i)^k \frac{-2\pi y_0}{N(k+1)} \sum_{\gamma \in \pm U(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{Z})} \frac{f_i(\gamma g)(c\tau+d)^k \mathrm{d}\overline{z}_1 \wedge \ldots \wedge \mathrm{d}\overline{z}_k}{|c\tau+d|^{2(k+1)}} .$$

In order to compute $\pi_*^k$ of the summands in the formula for $\mathcal{E}\mathrm{is}_{\mathcal{H}}^k(f_1) \cup \mathcal{E}\mathrm{is}_{\mathcal{H}}^k(f_2)$, we need:

LEMMA 7.3.

$$\pi_*^k(\mathrm{d}z_1 \wedge \mathrm{d}\overline{z}_1 \wedge \ldots \wedge \mathrm{d}z_k \wedge \mathrm{d}\overline{z}_k) = \frac{y_0^k}{\pi^k} .$$

*Proof.* – Let $(\tau, g) \in \mathfrak{H} \times \mathrm{GL}_2(\mathbb{Z}/N)$. Then the fiber of $\pi^1$ over $(\tau, g)$ is isomorphic to $\mathbb{C}/1\mathbb{Z} + (-\tau)\mathbb{Z}$. Because $\mathrm{d}z \wedge \mathrm{d}\overline{z} = -2i\mathrm{d}x \wedge \mathrm{d}y$ and $\pi_*^1(\mathrm{d}x \wedge \mathrm{d}y) = (2\pi i)^{-1}$ and $\det\begin{pmatrix} -1 & \mathrm{Re}\tau \\ 0 & \mathrm{Im}\tau \end{pmatrix} = -\mathrm{Im}\tau = -y_0$ we get

$$\pi_*^1(\mathrm{d}z \wedge \mathrm{d}\overline{z}) = \frac{y_0}{\pi} .$$

This implies the result. □

Write:

$$\mathcal{E}^k(f_i, \overset{(-)}{\tau}, g) := \frac{1}{N} \sum_{\gamma \in \pm U(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{Z})} \frac{f_i(\gamma g)(c\overset{(-)}{\tau} + d)^k}{|c\tau+d|^{2k+2}}$$

and

$$E^k(f_i, \tau, g) := \frac{1}{N} \sum_{\gamma \in \pm U(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{Z})} \frac{f_i(\gamma g)}{(c\tau+d)^{k+2}}$$

This, together with lemma 7.3 gives:

$$\pi_*^k(\mathcal{E}\mathrm{is}_{\mathcal{H}}^k(f_1) \wedge \pi_{k+1}\mathrm{Eis}^k(f_2)) = \frac{(-1)^{k+1}(2\pi i)^{2k}\pi^{1-k}y_0^{k+1}}{(k+1)} \cdot \left[\mathcal{E}^k(f_1, \tau, g)E^k(f_2, \tau, g)\tfrac{\mathrm{d}q}{q} - \mathcal{E}^k(f_1, \overline{\tau}, g)E^k(\overline{f_2}, \overline{\tau}, g)\frac{\mathrm{d}\overline{q}}{\overline{q}}\right]$$

and

$$\pi_*^k((-1)^{k+1}\pi_{k+1}\mathrm{Eis}^k(f_1) \wedge \mathcal{E}\mathrm{is}_{\mathcal{H}}^k(f_2)) = \frac{(2\pi i)^{2k}\pi^{1-k}y_0^{k+1}}{(k+1)} \left[E^k(f_1, \tau, g)\mathcal{E}^k(f_2, \tau, g)\tfrac{\mathrm{d}q}{q} - E^k(\overline{f_1}, \overline{\tau}, g)\mathcal{E}^k(f_2, \overline{\tau}, g)\frac{\mathrm{d}\overline{q}}{\overline{q}}\right]$$

**Residue computation**

The next thing we are going to do is to give a formula for the residue in terms of the Fourier coefficients of the Eisenstein series involved. To compute the residue of

$$\pi_*^k(\mathcal{E}\mathrm{is}_{\mathcal{H}}^k(f_1) \cup \mathcal{E}\mathrm{is}_{\mathcal{H}}^k(f_2)) \in \mathrm{H}_{\mathcal{H}}^2(\mathcal{M}_{\mathbb{R}}/\mathbb{R}, \mathbb{R}(k+2))$$

first observe that for $k \geq 0$

$$\mathrm{H}_{\mathcal{H}}^2(\mathcal{M}_{\mathbb{R}}/\mathbb{R}, \mathbb{R}(k+2)) \cong \mathrm{H}_B^1(\mathcal{M}(\mathbb{C}), \mathbb{R}(k+1))^{\overline{F}_\infty}$$

and

$$\mathrm{H}^1_{\mathcal{H}}(B_{\mathbb{R}}/\mathbb{R}, \mathbb{R}(k+1)) \cong \mathrm{H}^0_B(B(\mathbb{C}), \mathbb{R}(k))^{\overline{F}_\infty} = \left( \bigoplus_{\sigma \in B(\mathbb{C})} \mathbb{R}(k) \right)^{\overline{F}_\infty} .$$

Here, $\mathrm{H}_B(\ldots)$ denotes singular cohomology, $F_\infty$ is the complex conjugation on $\mathcal{M}(\mathbb{C})$ and $B(\mathbb{C})$ and $\overline{(\ldots)}$ denotes the complex conjugation on $\mathbb{R}(k+1) = (2\pi i)^{k+1}\mathbb{R}$ and $\mathbb{R}(k) = (2\pi i)^k\mathbb{R}$. The residue is given via this identification by the zeroth Fourier coefficient

$$\begin{aligned} &\mathrm{res}(\pi^k_*(\mathcal{E}\mathrm{is}^k_{\mathcal{H}}(f_1) \cup \mathcal{E}\mathrm{is}^k_{\mathcal{H}}(f_2))(\tau, g)) \\ &\qquad = \frac{1}{2\pi i N} \int_{\mathbb{R}/N\cdot\mathbb{Z}} \pi^k_*(\mathcal{E}\mathrm{is}^k_{\mathcal{H}}(f_1) \cup \mathcal{E}\mathrm{is}^k_{\mathcal{H}}(f_2))(x_0, y_0, g) \end{aligned}$$

where $\tau = x_0 + iy_0$ and $y_0 \gg 0$ is fixed. The result is then independent of $y_0$.

LEMMA 7.4. – *If we denote by* $a_0(\ldots) = \frac{1}{N}\int_{\mathbb{R}/N\mathbb{Z}}(\ldots)\mathrm{d}x_0$ *the* 0*-th Fourier coefficient of a function, then the formula*

$$\begin{aligned} \mathrm{res}\circ\pi^k_*(\mathcal{E}\mathrm{is}^k_{\mathcal{H}}(f_1) \cup \mathcal{E}\mathrm{is}^k_{\mathcal{H}}(f_2)) = \frac{(2\pi i)^{2k}\pi^{1-k}y_0^{k+1}}{k+1} \\ \Big[(-1)^{k+1}\Big(a_0(\mathcal{E}^k(f_1,\tau,g))a_0(E^k(f_2,\tau,g)) + a_0(\mathcal{E}^k(f_1,\overline{\tau},g))a_0(E^k(\overline{f}_2,\overline{\tau},g))\Big) \\ + a_0(E^k(f_1,\tau,g))a_0(\mathcal{E}^k(f_2,\tau,g)) + a_0(E^k(\overline{f}_1,\overline{\tau},g))a_0(\mathcal{E}^k(f_2,\overline{\tau},g))\Big] \end{aligned}$$

*holds.*

*Proof.* – The residue of the products $\mathcal{E}^k(f_1,\tau,g)E^k(f_2,\tau,g)\frac{dq}{q}$ etc. is the 0-th Fourier coefficient of this product of two absolutely and uniformly convergent series on compact sets. Hence it is given by

$$\sum_n a_n(\mathcal{E}^k(f_1,\tau,g))a_{-n}(E^k(f_2,\tau,g)) .$$

As the residue is independent of $y_0 \gg 0$ and the $n$-th Fourier coefficients of $\mathcal{E}^k(f_1,\tau,g)$ and $E^k(f_1,\tau,g)$ grow like $\frac{c}{y_0^{k+1}}e^{-|y_0\cdot n|}$ for $y_0 \to \infty$ (see [Fr] III.4.4), only the 0-th Fourier coefficients matter for the computation of the residue. □

Before we can write down the Fourier coefficients of our Eisenstein series, we need some more notation.

Let $\varphi : (\mathbb{Z}/N)^2 \to \mathbb{C}$ be any function. Then for $g \in \mathrm{GL}_2(\mathbb{Z}/N)$ we put

$$(g\varphi)(x) = \varphi(g^{-1}x)$$

and for $\mathrm{Re}s > 1$

$$L(\varphi, s) := \sum_{d \geq 1} \frac{\varphi(0,d)}{d^s} .$$

We introduce the partial Fourier transforms (here $\zeta$ is the $N$-root of unity chosen as above)

$$P_1\varphi(a,b) = \sum_{d\in\mathbb{Z}/N} \varphi(d,b)\zeta^{ad}$$
$$P_2\varphi(a,b) = \sum_{d\in\mathbb{Z}/N} \varphi(a,d)\zeta^{bd}$$

and the symplectic Fourier transform

$$\widehat{\varphi}(c,d) = \frac{1}{N}\sum_{(a,b)\in(\mathbb{Z}/N)^2} \varphi(a,b)\zeta^{\det\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)} .$$

Let $\varphi^t(a,b) := \varphi(b,a)$, then $P_1\widehat{\varphi} = (P_1\varphi)^t$ and $P_1\varphi = P_2(\widehat{\varphi}^t)$. Note also that the functional equation for $L(\varphi,s)$ implies that if $\varphi(-c,-d) = (-1)^k\varphi(c,d)$, then

$$L(P_2\varphi, 1-k) = \frac{(-1)^k 2N^k(k-1)!}{(2\pi i)^k} L(\varphi,k) .$$

Let

$$\mathbb{Q}^{(k)}\left[(\mathbb{Z}/N)^2\right] := \left\{\varphi : (\mathbb{Z}/N)^2 \to \mathbb{Q} \mid \varphi(-c,-d) = (-1)^k\varphi(c,d)\right\} .$$

DEFINITION 7.5. – *The* horospherical map *is defined as*

$$\begin{aligned} \varrho : \mathbb{Q}^{(k)}\left[(\mathbb{Z}/N)^2\right] &\longrightarrow \left(\mathrm{Ind}_{P(\mathbb{Z}/N)}^{\mathrm{GL}_2(\mathbb{Z}/N)}\mathbb{Q}\right)^{(k)} \\ \varphi &\longmapsto \left\{ g \mapsto \frac{-1}{Nk!} L(P_2(\widehat{g\varphi}^t), -k-1)\right\} . \end{aligned}$$

Note that

$$\frac{-1}{Nk!}L(P_2(\widehat{g\varphi})^t, -k-1) = \frac{N^k}{k!}\sum_{(t_1,t_2)\in(\mathbb{Z}/N)^2} \varphi(g^{-1}t)\frac{B_{k+2}(< \frac{t_2}{N} >)}{k+2} \in \mathbb{Q}.$$

Here $B_{k+2}(x)$ is the $k+2$-th Bernoulli polynomial and $< x > := x - [x]$ is the difference between $x$ and the largest integer smaller than $x$. Following the arguments in [Be3] 3.1 or [Sch-Sch] 7.5 one sees that this map is actually surjective.

We want to apply this to the function $f_\infty \in \left(\mathrm{Ind}_{P(\mathbb{Z}/N)}^{\mathrm{GL}_2(\mathbb{Z}/N)}\mathbb{Q}\right)^{(k)}$ defined in 4.1.

LEMMA 7.6. – (*cf.* [Sch-Sch] 7.5) *For any Dirichlet character* $\chi : (\mathbb{Z}/N)^* \to \mathbb{C}^*$ *considered as a function on* $\mathbb{Z}/N$ *extended by zero, consider*

$$\varphi_\chi(a,b) := \chi(b) .$$

a) *Then* $P_2\varphi_\chi$ *has degree* 0.
b) *The element*

$$\varphi_\infty(a,b) := \frac{2}{\sharp(\mathbb{Z}/N)^*}\sum_{\substack{\chi \\ \chi(-1)=(-1)^k}} \frac{-k!P_2\varphi_\chi(a,b)}{L(P_2\varphi_\chi, -k-1)}$$

*is in* $\mathbb{Q}^{(k)}[(\mathbb{Z}/N)^2]$ *and has degree* 0.

c) $\varrho(\varphi_\infty) = f_\infty$.

*Proof.* – For a) we check

$$\sum_{a,b} P_2\varphi_\chi(a,b) = \sum_{x,a,b} \chi(x)\zeta^{xb} = N^2 \cdot \chi(0) = 0\,.$$

A straightforward computation shows that for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\varrho(P_2\phi_\chi)(g) = \begin{cases} \dfrac{-\chi(d)}{k!} L(P_2\varphi_\chi, -k-1) & c = 0 \\ 0 & c \neq 0\,. \end{cases}$$

Observe that for $\chi(-1) = (-1)^k$

$$L(P_2\varphi_\chi, -k-1) = \frac{(-1)^k 2N^{k+2}(k+1)!}{(2\pi i)^{k+2}} L(\chi, k+2)$$

so that $L(P_2\varphi_\chi, -k-1) \neq 0$. It follows that

$$\varrho(\varphi_\infty)(g) = \begin{cases} \dfrac{2}{\sharp(\mathbb{Z}/N)^*} \displaystyle\sum_{\substack{\chi \\ \chi(-1)=(-1)^k}} \chi(d) = f_\infty(g) & \text{for } c = 0, \\ 0 = f_\infty(g) & \text{for } c \neq 0. \end{cases}$$

As

$$L(P_2\varphi_\chi, -k-1) = -N^{k+1} \sum_{(t_1,t_2)\in(\mathbb{Z}/N)^2} \chi(t_1)\zeta^{t_1t_2} \frac{B_{k+2}(< \frac{t_2}{N} >)}{k+2}$$

it is easy to see that for $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$

$$\sigma(\varphi_\infty)(a,b) = \varphi_\infty(a,b)\,.$$

□

We now want to apply lemma 7.4 to compute the residue of

$$\pi_*^k(\mathcal{E}\mathrm{is}^k_{\mathcal{H}}(f) \cup \mathcal{E}\mathrm{is}^k_{\mathcal{H}}(f_\infty)).$$

For this we must write the series $\mathcal{E}^k(f_i, \overset{(-)}{\tau}, g)$ and $E^k(f_i, \tau, g)$ in a different way with the help of the horospherical map $\varrho$. Choose $\varphi \in \mathbb{Q}^{(k)}[(\mathbb{Z}/N)^2]$, such that

$$\varrho(\varphi) = f \in \mathrm{H}^0_{\mathcal{M}}(\mathrm{Cusp} \setminus \infty, 0)^{(k)}.$$

Then we have

$$\mathcal{E}^k(f, \overset{(-)}{\tau}, g) = \frac{(-1)^{k+1}N^k(k+1)}{(2\pi i)^{k+2}} \sum_{(c,d)\in\mathbb{Z}^2\setminus(0,0)} \frac{(\widehat{g\varphi})(d,c)(c\overset{(-)}{\tau}+d)^k}{|c\tau+d|^{2k+2}}$$

and

$$E^k(f, \tau, g) = \frac{(-1)^{k+1}N^k(k+1)}{(2\pi i)^{k+2}} \sum_{(c,d)\in\mathbb{Z}^2\setminus(0,0)} \frac{(\widehat{g\varphi})(d,c)}{(c\tau+d)^{k+2}}.$$

LEMMA 7.7. – *For* $g \in \pm P(\mathbb{Z}/N)$ *and* $\varphi \in \mathbb{Q}^{(k)}[(\mathbb{Z}/N)^2]$ *with* $\varrho(\varphi) = f$ *and* $f \in \mathrm{H}^0(\mathrm{Cusp} \setminus \infty, 0)^{(k)}$

$$
\begin{aligned}
a_0(\mathcal{E}^k(f,\tau,g)) &= \frac{(-1)^{k+1} i^{k-1} N^{k-1}(k+1)}{2^k y_0^{k+1}(2\pi i)^{k+1}} L(P_1(\widehat{g\varphi}), k+1)\\
a_0(\mathcal{E}^k(f,\overline{\tau},g)) &= \frac{-i^{k-1} N^{k-1}(k+1)}{2^k y_0^{k+1}(2\pi i)^{k+1}} L(P_1(\widehat{g\varphi}), k+1)\\
a_0(E^k(f_\infty,\tau,g)) &= \begin{cases} 1 & \text{for } g \in P(\mathbb{Z}/N)\\ (-1)^k & \text{for } g \in -P(\mathbb{Z}/N)\end{cases}\\
a_0(E^k(\overline{f}_\infty,\overline{\tau},g)) &= (-1)^k a_0(E^k(f_\infty,\tau,g))\\
a_0(E^k(f,\tau,g)) &= a_0(E^k(\overline{f},\overline{\tau},g)) = 0\,.
\end{aligned}
$$

*Proof.* – We do the computation, which is standard, only in the case of $\mathcal{E}^k(f,\tau,g)$:

$$
\sum_{(c,d)\neq(0,0)} \frac{\widehat{g\varphi}(d,c)(c\tau+d)^k}{|c\tau+d|^{2k+2}} = 2\cdot L(\widehat{g\varphi}^t, k+2) + 2\sum_{c\geq 1}\sum_{d\in\mathbb{Z}} \frac{\widehat{g\varphi}(d,c)(c\tau+d)^k}{|c\tau+d|^{2k+2}}
$$

and the 0-th Fourier coefficient of the second summand is

$$
\begin{aligned}
\frac{2}{N}\int_{\mathbb{R}/N\cdot\mathbb{Z}} &\sum_{c\geq 1}\sum_{d\in\mathbb{Z}} \frac{\widehat{g\varphi}(d,c)(c\tau+d)^k}{|c\tau+d|^{2k+2}}\,\mathrm{d}x_0\\
&= \frac{2}{N}\sum_{c\geq 1}\sum_{d\in\mathbb{Z}/N\cdot c} \frac{\widehat{g\varphi}(d,c)}{c^{k+2}} \int_{\mathbb{R}} \frac{(x_0+iy_0)^k}{|x_0+iy_0|^{2k+2}}\,\mathrm{d}x_0\\
&= \frac{2i^k}{N y_0^{k+1}} \sum_{c\geq 1} \frac{P_1(\widehat{g\varphi})(0,c)}{c^{k+1}} \int_{\mathbb{R}} \frac{(1-it)^k}{|1-it|^{2k+2}}\,\mathrm{d}t\\
&= \frac{2^{1-k} i^k \pi}{N y_0^{k+1}} L(P_1(\widehat{g\varphi}), k+1)\,.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&a_0(\mathcal{E}^k(f,\tau,g))\\
&= \frac{(-1)^{k+1} 2 N^k (k+1)}{(2\pi i)^{k+2}} L(\widehat{g\varphi}^t, k+2) + \frac{(-1)^{k+1} N^{k-1}(k+1) i^{k-1}}{2^k (2\pi i)^{k+1} y_0^{k+1}} L(P_1(\widehat{g\varphi}), k+1)\\
&= \frac{-1}{N^2 k!} L(P_2(\widehat{g\varphi}^t), -k-1) + \frac{(-1)^{k+1} N^{k-1}(k+1) i^{k-1}}{2^k (2\pi i)^{k+1} y_0^{k+1}} L(P_1(\widehat{g\varphi}), k+1)\,.
\end{aligned}
$$

But $\varrho(\varphi) \in \mathrm{H}^0(\mathrm{Cusp} \setminus \infty, 0)^{(k)}$, so that

$$
g \longmapsto \frac{-1}{N^2 k!} L(P_2(\widehat{g\varphi}^t), -k-1) \text{ is zero on } \pm P(\mathbb{Z}/N)\,.
$$

For the other series, observe that

$$
\overline{\varphi_\infty} = (-1)^k \varphi_\infty\,.
$$

□

## 8. The main theorems

Identify the $\mathbb{C}$-valued points of the cusp $\infty = B$ with

$$B(\mathbb{C}) = \pm U(\mathbb{Z}/N) \setminus \pm P(\mathbb{Z}/N) = U(\mathbb{Z}/N) \setminus P(\mathbb{Z}/N) .$$

Then we can write

$$\mathrm{H}^1_{\mathcal{H}}(B_{\mathbb{R}}/\mathbb{R}, \mathbb{R}(k+1)) \cong \left( \bigoplus_{U(\mathbb{Z}/N)\setminus P(\mathbb{Z}/N)} \mathbb{R}(k) \right)^{\overline{F}_\infty}$$

and accordingly we consider elements in $\mathrm{H}^1_{\mathcal{H}}(B_{\mathbb{R}}/\mathbb{R}, \mathbb{R}(k+1))$ as $\mathbb{R}(k)$-valued functions on $U(\mathbb{Z}/N) \setminus P(\mathbb{Z}/N)$ invariant under $\overline{F}_\infty$.

Lemmas 7.4–7.7 together give the following theorem:

THEOREM 8.1. – *Let* $f \in \mathrm{H}^0_{\mathcal{M}}(\mathrm{Cusp} \setminus \infty, 0)^{(k)}$ *and* $\varphi \in \mathbb{Q}^{(k)}[(\mathbb{Z}/N)^2]$ *be such that* $\varrho(\varphi) = f$ *under the horospherical map* (*cf.* definition 7.5). *Then, with the above identification, for each* $g \in P(\mathbb{Z}/N)$

$$r_{\mathcal{H}}\mathcal{D}\mathrm{ir}(\varrho(\phi))(g) = r_{\mathcal{H}}\mathcal{D}\mathrm{ir}(f)(g) = (-1)^{k+1}N^{k-1}L(P_1(\widehat{g\varphi}), k+1)$$

*in* $\mathrm{H}^1_{\mathcal{H}}(B_{\mathbb{R}}/\mathbb{R}, \mathbb{R}(k+1))$.

The following lemma allows us to make the $L$-value in the above theorem more explicit. Recall that with our notations the value of the $(k+1)$-th polylogarithm function at the root of unity $\zeta^{ux}$ for $u \in \mathbb{Z}/N$ is

$$L(\zeta^u, k+1),$$

if $\zeta^u$ denotes denotes also the function $\zeta^u(x,y) = \zeta^{uy}$ (*cf.* [Neu] p. 221).

LEMMA 8.2. – *Let* $\delta_{(u,v)}$ *be the delta function at* $(u,v) \in (\mathbb{Z}/N)^2$. *Then for* $g = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$

$$P_1(g\widehat{\delta_{(u,v)}})(0,x) = \begin{cases} \zeta^{aux} & v = 0 \\ 0 & v \neq 0 \end{cases}$$

*Hence*

$$L(P_1(g\widehat{\delta_{u,v}}), k+1) = \begin{cases} L(\zeta^{au}, k+1) & v = 0 \\ 0 & v \neq 0 \end{cases}$$

*Proof.* – This is a straightforward computation. □

COROLLARY 8.3. – *For* $u \neq 0$, *let*

$$\Psi_u := \frac{(-1)^{k+1}}{N^{k-1}}(u,0) - \frac{(-1)^{k+1}N^2}{1-N^{k+1}} \sum_{v\neq 0}(u,v) ,$$

$$\widetilde{\Psi}_u := \frac{1}{2}(\Psi_u + (-1)^k \Psi_{-u}) \in \mathbb{Q}^{(k)}[(\mathbb{Z}/N)^2]$$

*and for* $g = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in P(\mathbb{Z}/N)$

$$r_{\mathcal{H}}\mathcal{D}\mathrm{ir}(\widetilde{\Psi}_u)(g) = \frac{1}{2}\left[L(\zeta^{au}, k+1) + (-1)^k L(\zeta^{-au}, k+1)\right] .$$

*Proof.* – $\varrho(\Psi_u)(\mathrm{id}) = 0$ by the distribution relation for Bernoulli polynomials. Using the last lemma we get the explicit value. □

*Remark.* – The parametrization of $\mathrm{H}^1_{\mathcal{H}}(B_{\mathbb{R}}/\mathbb{R}, \mathbb{R}(k+1))$ used in [HuW] is translated to ours by projecting $z \in \mathbb{C}$ to $-\frac{1}{2}(z + (-1)^k \overline{z}) \in \mathbb{R}(k)$. Hence the corollary is the same as the result in [HuW] corollary 9.6 a).

We are now in a position to prove that our elements span the vector space $\mathrm{H}^1_{\mathcal{M}}(B, k+1)$.

PROPOSITION 8.4. – *The image of the map*

$$\mathbb{Q}^{(k)}\left[(\mathbb{Z}/N)^2\right] \xrightarrow{\mathcal{D}\mathrm{ir}\circ\varrho} \mathrm{H}^1_{\mathcal{M}}(B, \mathbb{Q}(k+1)) \xrightarrow{r_{\mathcal{H}}} \mathrm{H}^1_{\mathcal{H}}(B_{\mathbb{R}}/\mathbb{R}, \mathbb{R}(k+1)),$$

*spans the* $\mathbb{R}$ *vector space* $\mathrm{H}^1_{\mathcal{H}}(B_{\mathbb{R}}/\mathbb{R}, \mathbb{R}(k+1))$. *In particular*

$$\begin{aligned} \mathrm{H}^0_{\mathcal{M}}(\mathrm{Cusp} \smallsetminus \infty, 0)^{(k)} &\longrightarrow \mathrm{H}^1_{\mathcal{M}}(B, \mathbb{Q}(k+1)) \\ f &\longmapsto \mathcal{D}\mathrm{ir}(f) \end{aligned}$$

*is surjective.*

*Proof.* – We prove this after extension of scalars to $\overline{\mathbb{Q}}$ and show that we get a non trivial element in each $\chi$ eigenspace, for $\chi(-1) = (-1)^k$. Consider the function

$$\Psi_\chi := \sum_{u \in \mathbb{Z}/N} \widetilde{\Psi}_u \otimes \chi(u),$$

for $\chi(-1) = (-1)^k$. Then for $g = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in P(\mathbb{Z}/N)$ we get

$$\mathcal{D}\mathrm{ir}_{\mathcal{H}}(\varrho(\Psi_\chi))(g) = \sum_u L(\zeta^{au}, k+1) \otimes \chi(u) \in \left(\bigoplus_{U(\mathbb{Z}/N)\backslash P(\mathbb{Z}/N)} \mathbb{R}(k)\right)^{\overline{F}_\infty} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} ,$$

because

$$\sum_u L(\zeta^{-au}, k+1) \otimes \chi(u) = (-1)^k \sum_u L(\zeta^{au}, k+1) \otimes \chi(u).$$

But that this is non zero is shown in [Neu] p. 221. The second statement of the proposition follows from Borel's theorem, which states that

$$\mathrm{H}^1_{\mathcal{M}}(B, \mathbb{Q}(k+1)) \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{r_{\mathcal{H}}} \mathrm{H}^0_B(B(\mathbb{C}), \mathbb{R}(k))^{\overline{F}_\infty}$$

is an isomorphism. □

Another application of corollary 8.3 above is that we can identify the $l$-adic realization of our elements with the help of [HuW]. To state the result, recall that we fixed a primitive

$N$-th root of unity $\zeta \in \mathbb{Q}(\mu_N)$ and recall that Kummer theory allows us to identify $\mathrm{H}^1_{\mathrm{hor}}(B, \mathbb{Q}_l(k+1))$ with a $\mathbb{Q}_l$ subspace of

$$\left(\left(\varprojlim_{r\geq 1} \mathbb{Q}(\mu_{l^\infty}, \zeta)^*/(\mathbb{Q}(\mu_{l^\infty}, \zeta)^*)^{l^r} \otimes \mu_{l^r}^{\otimes k}\right) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l\right)^{\mathrm{Gal}(\mathbb{Q}(\mu_{l^\infty},\zeta)/\mathbb{Q}(\zeta))}.$$

PROPOSITION 8.5. – *With $\tilde{\Psi}_u$ as in corollary 8.3 we have*

$$r_l \mathcal{D}\mathrm{ir}(\varrho(\widetilde{\Psi}_u)) = \frac{1}{N^k k!}\left(\sum_{\alpha^{l^r}=\zeta^u} [1-\alpha] \otimes (\alpha^N)^{\otimes k}\right)_r.$$

*Proof.* – This is corollary 9.7 in [HuW], once we have shown that $\mathcal{D}\mathrm{ir}(\varrho(\widetilde{\Psi}_u))$ agrees with $\varepsilon_{k+1}(T^u)$ (notation of *loc. cit.*) in $\mathrm{H}^1_{\mathcal{M}}(B, k+1)$. The injectivity of the regulator implies that this can be checked in the absolute Hodge cohomology. There the result follows from corollary 8.3 above and corollary 9.6 a) of *loc. cit.*. □

*Remark.* – According to proposition 8.4 the $\mathcal{D}\mathrm{ir}(\widetilde{\Psi}_u)$ generate the image of $\mathcal{D}\mathrm{ir}$. Hence this proposition computes the $l$-adic regulators of all Harder-Anderson elements. In a second paper we will give a direct computation of the $l$-adic realization ([HuK]). Harder has also announced that he can do this. Conversely this will give an alternative proof of [HuW] 9.7, *i.e.*, of [BlK] Conjecture 6.2.

The final application concerns the Beilinson conjecture for Dirichlet characters. Recall that the Beilinson $\mathbb{Q}$-structure on

$$\mathrm{H}^1_{\mathcal{H}}(B_{\mathbb{R}}/\mathbb{R}, \mathbb{R}(k+1)) \cong \mathrm{H}^0_B(B(\mathbb{C}), \mathbb{R}(k))^{\overline{F}_\infty}$$

is given by the $\mathbb{Q}$-subspace

$$\mathrm{H}^0_B(B(\mathbb{C}), \mathbb{Q}(k))^{\overline{F}_\infty} \subset \mathrm{H}^0_B(B(\mathbb{C}), \mathbb{R}(k))^{\overline{F}_\infty}.$$

This induces a $\overline{\mathbb{Q}}$-structure in each $\chi$-eigenspace in $\mathrm{H}^0_B(B(\mathbb{C}), \mathbb{R}(k))^{\overline{F}_\infty} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$. The Beilinson conjecture compares this $\overline{\mathbb{Q}}$-structure with the one coming from $K$-theory. Let $f$ be the conductor of the Dirichlet character $\chi$. We define

$$a(\chi) := \frac{k! N^k}{2} \prod_{\substack{p\mid N\\ p\nmid f}} \frac{1-\chi^{-1}(p)p^{-k-1}}{1-\chi(p)p^k}.$$

PROPOSITION 8.6. – *For $\Psi_\chi = \sum_u \widetilde{\Psi}_u \otimes \chi(u)$ and $\chi(-1) = (-1)^k$ the formula*

$$r_{\mathcal{H}} \mathcal{D}\mathrm{ir}(a(\chi)\varrho\Psi_\chi)(\mathrm{id}) = (2\pi i)^k L'(\chi, -k)$$

*holds, where $L'(\chi, -k)$ is the derivative of the L-function $L(\chi, s)$ at $s = -k$. In particular Beilinson's conjecture is true for $\chi$.*

*Proof.* – This is theorem 4.4 in [Neu] together with the formula for $\mathcal{D}\mathrm{ir}_{\mathcal{H}}(\varrho\Psi_\chi)$ in the proof of proposition 8.4. □

*Remark.* – The above corollary has first been obtained by Beilinson [Be2] (with some subsequent work by Neukirch and Esnault) by completely different methods. See [Neu] and [Be2] for this approach. Another approach, following ideas of Deligne and Beilinson, was worked out in [HuW]. A third version, due to Deninger [De1], starts from the Eisenstein symbol for elliptic curves with complex multiplication.

## A. The weight-zero part of cohomology

In the main part, we construct and we identify elements in $\mathrm{H}^{k+1}(\widetilde{\mathfrak{X}}^k_\infty, k+1)$ which we want to project to $\mathrm{H}^1(B, k+1)$. This corresponds to a projection from the motive $h^k(\widetilde{\mathfrak{X}}^k_\infty)$ to $h^0(B)$. In order to do so we define and compute here in the appendix the "weight-zero-part of cohomology" and afterwards construct a projection to it.

All schemes in this section are $B$-schemes where for the purpose of the appendix $B$ can be the spectrum of any number field.

DEFINITION A.1. – *Let $V$ be a $B$-scheme whose irreducible components $C_1, \ldots, C_K$ and their iterated intersections are smooth and geometrically irreducible over $B$. Let $V_0$ be the disjoint union of these irreducible components. Let $V_.$ be the simplicial scheme given by the Čech-nerve of the cover $V_0 \to V$, i.e.,*

$$\begin{aligned} V_s =& V_0 \times_V \ldots \times_V V_0 \qquad s+1 \textit{ factors} \\ =& \coprod_{t_0,\ldots,t_s} C_{t_0} \cap \ldots \cap C_{t_s} \ . \end{aligned}$$

*Let $W_0V_.$ be the simplicial scheme given by*

$$W_0V_s = \coprod_{t_0,\ldots,t_s} B^{\#\{\text{connected components of } C_{t_0} \cap \ldots \cap C_{t_s}\}} \ .$$

*There is a natural map $V_. \to W_0V_.$ given by the structural map of each connected component. We call $W_0V_.$ the weight-zero-part of $V$.*

LEMMA A.2. – *Let $v : V \to B$ be as in the definition. Then $\underline{\mathrm{H}}^n(W_0V_.) \to \underline{\mathrm{H}}^n(V)$ induces an isomorphism $\underline{\mathrm{H}}^n(W_0V_.) \to W_0\underline{\mathrm{H}}^n(V)$*

*Proof.* – Note that $\underline{\mathrm{H}}^n(V_.) = \underline{\mathrm{H}}^n(V)$ because of cohomological descent for proper maps (Mayer-Vietoris for closed covers in our case). The morphism is induced by the canonical projection $V_. \to W_0V_.$. We consider the skeletal spectral sequence for $\underline{\mathrm{H}}^*$ attached to $V_.$

$$E_1^{p,q} = \underline{\mathrm{H}}^p(V_q) \Rightarrow \underline{\mathrm{H}}^{p+q}(V_.) \ .$$

All $V_s$ are smooth. Hence

$$W_0\underline{\mathrm{H}}^n(V_s) = \begin{cases} \underline{\mathrm{H}}^0(B)^{\#\{\text{connected components of } V_s\}} & \text{if } n = 0 \\ 0 & \text{else.} \end{cases}$$

This is precisely what appears in the $E_1$-term of the same spectral sequence for $W_0V_.$. The differentials of the spectral sequence are strictly compatible with the weight filtration. The isomorphism for the weight-zero part on the $E_1$-term induces one on the limit terms. □

*Remark.* – Note that the skeletal spectral sequence for $W_0V_.$ degenerates at $E_2$, *i.e.*,

$$W_0\underline{\mathrm{H}}^n(V_.) = H^n\big(\underline{\mathrm{H}}^0(W_0V_s)_{s\in\mathbb{N}_0}\big)$$

where $\underline{\mathrm{H}}^0(W_0V_s)_{s\in\mathbb{N}_0}$ is the cohomological complex attached to the cosimplicial object.

DEFINITION A.3. – *Let $V$ and $W_0V_.$ be as above. Then we call* $\mathrm{H}^n(W_0V_., m)$ *the weight-zero-part of the absolute cohomology of* $V$.

*Remark.* – There is a natural map $\mathrm{H}^n(W_0V_., m) \to \mathrm{H}^n(V_., m)$. It is *not* injective in general.

PROPOSITION. – *Let $V$ and $V'$ be proper varieties that satisfy the condition of A.1. Let us be given a morphism $p : V' \to V$ such that*

1. *$p$ is an isomorphism on an open subscheme $U$ of $V$;*
2. *$Z = V \setminus U$ is smooth;*
3. *$p|_Z$ is Zariski-locally a fibration with fibre $X$, i.e., it is locally isomorphic to $X \times Z$;*
4. $W_0\underline{\mathrm{H}}^n(X) = \underline{\mathrm{H}}^n(B) = \begin{cases} \underline{\mathrm{H}}^0(B) & \text{if } n = 0, \\ 0 & \text{else.} \end{cases}$

*Then $W_0\underline{\mathrm{H}}^n(V) \cong W_0\underline{\mathrm{H}}^n(V')$ and the weight zero part of* $\mathrm{H}^n(V, m)$ *agrees with the weight zero part of* $\mathrm{H}^n(V', m)$ *for all* $n, m \in \mathbb{Z}$.

*Proof.* – First we want to reduce the question on motivic cohomology to the one in absolute Hodge cohomology. For each $m$ we apply the functor $\mathrm{H}^*_{\mathcal{M}}(\,\cdot\,, m)$ to $W_0V'_s \to W_0V_s$. The only one non-trivial contributions are for $* = 0$ in the case $m = 0$ respectively $* = 1$ in the case $m > 0$. Hence the skeletal spectral sequence for $\mathrm{H}^*_{\mathcal{M}}(W_0V_., m)$ degenerates again at $E_2$ and we have

$$\mathrm{H}^n_{\mathcal{M}}(W_0V_., m) = \mathrm{H}^{n-e}(\mathrm{H}^e_{\mathcal{M}}(W_0V_*, m)) \ .$$

Here $e = 0, 1$ accordingly. Computing these cohomology groups is purely a combinatorial question. We have the same combinatorics in absolute Hodge cohomology, hence it suffices to consider the question there.

Instead of attacking the combinatorics of the situation, we now use sheaf theoretic methods. By the Leray spectral sequence and Lemma A.2 it suffices to show that all $W_0\underline{\mathrm{H}}^i(p_*)$ are isomorphisms. For the triple $i : Z \xrightarrow{i} V \xleftarrow{j} U$ we have the exact triangle $j_!j^*F_U \to F_V \to i_*F_Z$ which induces a long exact sequence

$$\ldots \to \underline{\mathrm{H}}^n_c(U) \to \underline{\mathrm{H}}^n(V) \to \underline{\mathrm{H}}^n(Z) \to \underline{\mathrm{H}}^{n+1}_c(U) \to \cdots \ .$$

The corresponding sequence for $V'$ reads

$$\ldots \to \underline{\mathrm{H}}^n_c(U) \to \underline{\mathrm{H}}^n(V') \to \underline{\mathrm{H}}^n(p^{-1}(Z)) \to \underline{\mathrm{H}}^{n+1}_c(U) \to \cdots \ .$$

To prove equality of the weight zero part for $V$ and $V'$ it suffices to do so for $Z$ and $Z' = p^{-1}(Z)$. In this case we can use the Leray spectral sequence for $p_*$. The higher direct images of $F_{Z'}$ are variations of Hodge structures (respectively smooth $l$-adic sheaves). Their fibre in a point of $Z$ is $\underline{\mathrm{H}}^i(X)$. If $\mathcal{F}$ is a (perverse) sheaf on $Z$ whose weights are $> 0$, then the same is true for its $\underline{\mathrm{H}}^*(Z, \mathcal{F})$ because $Z$ was assumed smooth. Hence we only have to consider the weight zero part of the $\mathrm{H}^i(p_*F_{Z'})$. By assumption on the cohomology of the fibre it agrees with $\mathrm{H}^i(F_Z)$. This proves the claim. □

*Remark.* – The weight zero part of motivic and absolute Hodge cohomology is compatible under the regulator maps which are even isomorphisms if $m \neq 1$. Note that the argument in the proof does *not* use the isomorphy.

Let $\mathcal{N}$ be the Néron-$N$-gon over $B$ and $\mathcal{N}^k$ its $k$-th power. In the notation of the main part of the article this is $\overline{\mathfrak{X}}^k_\infty$. It is given Zariski-locally by the equation $x_1y_1 = 0$ in $E_k = B[x_1, y_1, \ldots, x_k, y_k]/(x_1y_1 = \ldots = x_ky_k)$. Let $\widetilde{\mathcal{N}}^k = \widetilde{\mathfrak{X}}^k_\infty$ be the preimage of $\mathcal{N}^k$ in the desingularization of $\overline{\mathfrak{X}}^k$ ([Sch] section 2 and 3). We apply our concept to the cohomology of $\widetilde{\mathcal{N}}^k$ and $\mathcal{N}^k$.

PROPOSITION A.5. – *The morphism $\widetilde{\mathcal{N}}^k \to \mathcal{N}^k$ induces an isomorphism on the weight zero part of cohomology.*

*Proof.* – We consider the situation Zariski-locally. $\widetilde{\mathcal{N}}^k$ is obtained as a sequence of blow-ups in smooth subvarieties ([Sch] 3.0 and 2.1.1). The exceptional fibres of these blow-ups are projective cones

$$F_i = \{[s_1 : t_1 : \ldots : s_i : t_i] \mid s_1t_1 = \ldots s_it_i\} \subset \mathbb{P}^{2i-1} \quad i = k, k-1, \ldots, 2$$

By the previous proposition it suffices to show that $W_0\underline{\mathrm{H}}^n(F_i) = \underline{\mathrm{H}}^n(B)$.
We prove this by induction on $i$. Note that $F_i$ is Zariski-locally isomorphic to $E_{i-1}$. For $i = 1$ it is smooth and there is nothing to prove. If $i \geq 2$ we can desingularize it by a sequence of blow-ups. The exceptional fibres are $F_{i'}$'s for $i' < i$. By inductive hypothesis and the previous proposition the weight zero part is unchanged. Hence the weight zero part of $F_i$ agrees with the weight zero part of a smooth variety. This gives the desired computation. □

LEMMA A.6.

$$W_0\underline{\mathrm{H}}^n(\widetilde{\mathcal{N}}^k) = W_0\underline{\mathrm{H}}^n(\mathcal{N}^k) \cong \bigoplus_{\substack{i_1+\ldots+i_k=n \\ i_j\in\{0,1\}}} \underline{\mathrm{H}}^0(B) \quad .$$

*Proof.* – In the $k = 1$ case we see easily by closed Mayer-Vietoris that

$$\underline{\mathrm{H}}^0(\mathcal{N}) = \underline{\mathrm{H}}^1(\mathcal{N}) \cong \underline{\mathrm{H}}^0(B) \text{ and } \underline{\mathrm{H}}^2(\mathcal{N}) \cong \underline{\mathrm{H}}^0(B)(-1)^N \; .$$

$\mathcal{N}^k$ is a $k$-fold product over $B$. By the Künneth formula the general case follows. □

*Remark.* – The isomorphism in the lemma depends on choices which can be made uniformly for all cohomology theories.

Now we construct the promised projection to the weight zero part of cohomology. It will correspond to a splitting of the weight filtration. Such a splitting does of course not exist for general varieties. If $V$ is smooth and proper and geometrically irreducible above $B$, then the choice of a $B$-point in $V$ induces such a splitting. In general this splitting depends on the choice of a point. However, if $V$ is rational, then the splitting becomes canonical because two points are homotopic. Or expressed differently, we have a splitting of Chow motives. We will give an ad-hoc version of the same idea for our singular varieties (cf. A.11).

DEFINITION A.7. – *By* path *we mean a variety $L$ whose irreducible components $L_1, \ldots L_n$ are isomorphic to $\mathbb{A}^1_B$ and whose singular locus consist of $L_1 \cap L_2$, …, $L_{n-1} \cap L_n$ where each of these intersections is isomorphic to $B$. This means that $L(\mathbb{C})$ is contractible. A* graph *is a variety which consist of finitely many paths which are glued together in simple double points. A graph $G$ is called* simply connected *if $G \times_B \mathbb{C}(\mathbb{C})$ is in the classical topology.*

*If $V$ is a variety, we say that two points $P_1$, $P_2$ can be joined by a path if there is a path $L$ and a map $L \to V$ such that $P_1$ and $P_2$ are in the image of $L$. We say that $V$ is* pathwise connected *if any two points can be joined by a path.*

*Remark.* – A simply connected graph $G$ has the cohomological type of the point $B$ in all cohomology theories in section 1. In absolute Hodge and $l$-adic cohomology this is a consequence of Mayer-Vietoris for closed covers. In the motivic case it was built into our definition.

LEMMA A.8. – *Smooth complete toric varieties over $B$ are pathwise connected.*

*Proof.* – Recall that such a variety has a cover by finitely many $\mathbb{A}^d$ where $d$ is the dimension. All these $\mathbb{A}^d$ intersect in one and the same torus $\mathbb{G}_m^d$. Now consider two points $P_1$ and $P_2$ in $V$. Pick a point $S$ in the torus. Join $P_1$ to $S$ by the $\mathbb{A}^1$ defined by them in the copy of $\mathbb{A}^d$ containing $P_1$. Do the same thing for $P_2$. This gives the required path joining $P_1$ and $P_2$. □

LEMMA A.9. – *Let $V$ be a pathwise connected variety over $B$. Suppose we are given finitely many simply connected graphs $G_1, \dots, G_n$ mapping to $V$. Then there is simply connected graph $G$ containing the $G_i$ as disjoint subvarieties such that there is a map $G \to V$ extending the ones on the $G_i$'s.*

*Proof.* – Pick two points in each of the $G_i$. Let $P_i$ and $Q_i$ be their images in $V$. Join $P_i$ to $Q_{i+1}$ by a path. Clearly this solves the question. □

PROPOSITION A.10. – *Let $V$ be a variety such that all irreducible components and all connected components of their iterated intersections are pathwise connected and smooth. Then there is simplicial variety $G_.$ and a map $g : G_. \to V_.$ such that for each connected component $D$ of $W_0 V_s$, the inverse image $g^{-1}D$ in $G_s$ is a simply connected graph.*

*Proof.* – Denote the irreducible components by $C_i$ for $1 \leq i \leq n$. For simplicity let us assume that the iterated intersections $\bigcap_{i_0,\dots,i_s} C_{i_j}$ are connected. The same method works in general. We want to define simply connected graphs $G_{i_0,\dots,i_s}$. They will only depend on the set $\{i_0, \dots, i_s\}$. Hence it suffices to construct by (descending) induction the graphs $G_{i_0<\dots<i_s}$ corresponding to the $\bigcap_{i_0<\dots<i_s} C_{i_j}$. We start with $s = n-1$ (the maximal possible set of indices). If the intersection is non-empty, chose two arbitrary points and join them by a path. For general $s$ take the disjoint union of the graphs $G_{k_0<\dots<k_{s+1}}$ where the tuples $k_0 < \dots < k_{s+1}$ are mapped to $i_0 < \dots < i_s$ by any of the boundary maps of the simplicial object. If there is more than one component extend this to one big, simply connected graph. If the union of graphs is empty but $\bigcap_{i_0<\dots<i_s} C_{i_j}$ is not, pick a path. It is easy to see that this construction serves the purpose. □

*Remark.* – Each $G_s$ is a graph by construction but not simply connected. It reflects precisely the incidence relations between the irreducible components of $V$.

PROPOSITION A.11. – *If $V$ is as in the previous proposition, there is then a projection from the cohomology of $V$ to its weight zero part.*

*Proof.* – We first show that $G_. \xrightarrow{g} V_. \to W_0 V_.$ induces an isomorphism in all cohomology theories. In order to see this, consider the skeletal spectral sequences for $W_0 V_.$ and $G_.$. Recall that the connected components of $W_0 V_s$ have the same cohomological type as those of $G_s$. Hence $g$ induces the projection. □

*Remark.* – A priori this projection will depend on the simplicial graph that was chosen. However, on geometric cohomology, this gives a splitting of the weight filtration which is necessarily unique.

We can apply these results to $\mathcal{N}^k$ and $\widetilde{\mathcal{N}}^k$ because all their irreducible components and their iterated intersections are smooth complete toric varieties. We get:

THEOREM A.12. – *For $\mathcal{N}^k$ and $\widetilde{\mathcal{N}}^k$ there is a projection on the weight zero part of absolute cohomology compatible with a splitting of the weight filtration on geometric cohomology. It is also compatible under regulators.*

*It is uniquely determined by these properties on $\mathrm{H}^{k+1}(\mathcal{N}^k, k+1)$ respectively $\mathrm{H}^{k+1}(\widetilde{\mathcal{N}}^k, k+1)$.*

*Proof.* – We can apply the previous proposition together with Lemma A.8 in order to construct the splitting. For uniqueness we first consider the Hodge case. As absolute Hodge cohomology has cohomological dimension one, we get a canonical short exact sequence

$$0 \to \mathrm{Ext}^1(\mathbb{R}(-k-1), \underline{\mathrm{H}}^k_{\mathcal{H}}(\mathcal{N}^k)) \to \mathrm{H}^{k+1}_{\mathcal{H}}(\mathcal{N}^k, k+1) \to \mathrm{Hom}(\mathbb{R}(-k-1), \underline{\mathrm{H}}^{k+1}_{\mathcal{H}}(\mathcal{N}^k)) \to 0\ .$$

The last group vanishes for weight reasons: as $\mathcal{N}^k$ is only $k$-dimensional, the weights of $\underline{\mathrm{H}}^{k+1}_{\mathcal{H}}(\mathcal{N}^k))$ are at most $2k$ whereas $\mathbb{R}(-k-1)$ has weight $2k+2$. A splitting of $\mathrm{H}^{k+1}_{\mathcal{H}}(\mathcal{N}^k, k+1)$ which is compatible with the splitting on geometric cohomology is uniquely determined by the unique splitting of $\underline{\mathrm{H}}^k_{\mathcal{H}}(\mathcal{N}^k)$.

In the $l$-adic situation we can apply the same argument. A priori there might be a contribution of $\mathrm{Ext}^2(\mathbb{Q}_l(-k-1), \underline{\mathrm{H}}^{k-1}_l(\mathcal{N}^k))$, but $\mathbb{Q}_l(-k)$ does not occur in $\underline{\mathrm{H}}^{k-1}_l(\mathcal{N}^k)$, hence we have again

$$\mathrm{H}^{k+1}_{\mathrm{hor}}(\mathcal{N}^k, k+1) = \mathrm{Ext}^1(\mathbb{Q}_l(-k-1), \underline{\mathrm{H}}_l(\mathcal{N}^k))\ .$$

For the motivic case we show that the map $\mathrm{H}^i_{\mathcal{M}}(\mathcal{N}^k, k+1) \to \mathrm{H}^i_{\mathrm{hor}}(\mathcal{N}^k, k+1)$ is injective. We use the simplicial version

$$\mathcal{N}^k_s = \coprod_{t_0,\ldots,t_s} C_{t_0} \cap \ldots \cap C_{t_s}$$

where $C_i$ are the irreducible components of $\mathcal{N}^k$. Recall that all components of $\mathcal{N}^k_s$ are smooth complete toric varieties. In order to compute the cohomology, by [HuW] B.6.2, it is enough to consider only part of $\mathcal{N}^k$,

$$\mathrm{str}(\mathcal{N}^k_s) = \coprod_{t_0<\ldots<t_s} C_{t_0} \cap \ldots \cap C_{t_s}\ .$$

The connected components of $\mathrm{str}(\mathcal{N}^k_s)$ have dimension $\leq k-s$. We argue in the skeletal spectral sequence

$$E_1^{s,t} = \mathrm{H}^t_{\mathcal{M}}(\mathrm{str}(\mathcal{N}^k_s), k+1) \Rightarrow \mathrm{H}^{s+t}_{\mathcal{M}}(\mathcal{N}^k, k+1)$$

By Lemma A.13 below the $l$-adic regulator tensored with $\mathbb{Q}_l$ is an isomorphism on the initial terms with the two exceptions

$$E_1^{0,2k+1} = \mathrm{H}^{2k+1}_{\mathcal{M}}(\mathcal{N}^k_0, k+1),$$
$$E_1^{0,2k+2} = \mathrm{H}^{2k+2}_{\mathcal{M}}(\mathcal{N}^k_0, k+1).$$

All differentials vanish on these terms, hence we have the same behaviour for all $E_r^{p,q}$. The arguments for $\widetilde{\mathcal{N}}^k$ are the same. □

We now give the lemma used in the last proof.

LEMMA A.13. – *Let $V$ be a smooth and complete toric variety of dimension $d \leq k$ over $B$. Then:*

1. *The absolute cohomology groups $\mathrm{H}^*(V, k+1)$ vanish for $* > 2k+2$ if $d = k$ and for $* > 2d+1$ if $d < k$.*
2. *The Beilinson regulator on $\mathrm{H}^*_{\mathcal{M}}(V, k+1)$ tensored with $\mathbb{R}$ is an isomorphism with the exception of $\mathrm{H}^{2k+1}_{\mathcal{M}}(V, k+1)$ if $d = k$.*
3. *The $l$-adic regulator tensored with $\mathbb{Q}_l$ is an isomorphism with exception of $\mathrm{H}^{2k+1}_{\mathcal{M}}(V, k+1)$ and $\mathrm{H}^{2k+2}_{\mathcal{M}}(V, k+1)$ for $k = d$. It is injective in these exceptional cases.*

*Proof.* – We can decompose $V$ as a Chow-motive ([Kk] Appendix). We get

$$h(V) = \bigoplus_{0 \leq i \leq d} \mathbb{Q}(-i)_B^{v_i}$$

for appropriate $v_i \geq 0$. Hence

$$\begin{aligned}\mathrm{H}^n_{\mathcal{M}}(V, k+1) &= \bigoplus_{0 \leq i \leq d} \mathrm{H}^{n-2i}_{\mathcal{M}}(B, k+1-i)^{v_i} \\ &= \begin{cases} \mathrm{H}^1_{\mathcal{M}}(B, k+1-\frac{n-1}{2})^{v_i} & \text{if } n-1 \text{ even, } n \leq 2d+1, \\ 0 & \text{else.} \end{cases}\end{aligned}$$

By the results for the Beilinson regulator for number fields we get an isomorphism with the only exception for $k+1-\frac{n-1}{2} = 1$, hence $i = k = d$.

The same computation works in the $l$-adic context; the extra term $\mathrm{H}^2_{\mathrm{hor}}(B, 1)$ appears as $i = k = d$, $n = 2k+2$. □

All in all we get the desired map.

PROPOSITION A.14. – *There is a projection from $\mathrm{H}^*(\widetilde{\mathcal{N}}^k, m)$* (resp. $\mathrm{H}^*(\mathcal{N}^k, m)$) *to the weight zero part which in both cases is isomorphic to*

$$\mathrm{H}^{n+e}(W_0\mathcal{N}^k, m) = \mathrm{H}^{n+e}(W_0\widetilde{\mathcal{N}}^k, m) \cong \bigoplus_{\substack{i_1+\ldots+i_k=n \\ i_j \in \{0,1\}}} \mathrm{H}^e(B, m)$$

*where $e = 0$ for $m = 0$, $e = 1$ for $m \geq 1$ (respectively $m > 1$ in the $l$-adic case).*

*Proof.* – The projection was constructed in proj. The explicit computation is a consequence of A.6 for $l$-adic and absolute Hodge cohomology. By comparing the combinatorics in the Hodge case to the motivic case the latter follows. Note that the $m = 1$-case in $l$-adic cohomology can be treated but is more complicated to write out. □

*Remark.* – We will have to know the explicit identification of $\mathrm{H}^{k+1}(W_0\mathcal{N}^k, k+1)$ with $\mathrm{H}^1(B, k+1)$. We use the one that is induced on geometric cohomology by the $k$-th power of

$$\underline{\mathrm{H}}^0(B) \xrightarrow{\mathrm{res}} \underline{\mathrm{H}}^1_c(\mathbb{G}_{m,B}) \to \underline{\mathrm{H}}^1(\mathcal{N})$$

where the first map is (dual to) residue at zero.

## REFERENCES

[Be1] A. A. BEILINSON, , Notes on absolute Hodge cohomology, in Applications of Algebraic $K$–theory to Algebraic Geometry and Number Theory, Proceedings of a Summer Research Conference held June 12–18, 1983, in Boulder, Colorado, Contemp. Math., vol. 55, Part I, AMS, Providence, pp. 35–68.

[Be2] A. BEILINSON, Higher regulators and values of $L$–functions, *Jour. Soviet Math.* **30**, 1985, pp. 2036–2070.

[Be3] A. BEILINSON, Higher regulators of modular curves,. Contemporary Math. Vol. 55, I (1986).

[BlK] S. BLOCH and K. KATO, $L$–functions and Tamagawa Numbers of Motives, in P. Cartier et al. (eds.), in The Grothendieck Festschrift, Volume I, Birkhäuser 1990, pp. 333–400.

[Br] C. BRINKMANN, Die Andersonextension und 1-Motive, *Bonner Mathematische Schriften.* **223**. Bonn: Univ. Bonn, Math.-Naturwiss. Fak. (1991).

[De1] C. DENINGER, Higher regulators and Hecke L-series of imaginary quadratic fields. II, *Ann. Math.*, II. Ser. 132, No. 1, 131-158 (1990).

[De2] C. DENINGER, Extensions of motives associated to symmetric powers of elliptic curves and to Hecke characters of imaginary quadratic fields, in: F. Catanese (ed.): Proc. Arithmetic Geometry, Cortona, 1994, Cambridge University Press, 1997, pp. 99–137.

[Den-Sch] C. DENINGER and A. J. SCHOLL, The Beilinson conjecture. in: J. Coates, M.J. Taylor (eds.): $L$–functions and Arithmetic. London Math. Soc. LNS 153 (1991)

[Es] H. ESNAULT, On the Loday Symbol in the Deligne–Beilinson Cohomology, *$K$–Theory* **3**, 1989, pp. 1–28.

[Fr] E. FREITAG, Hilbert modular forms, Springer (1990).

[Ha] G. HARDER, Eisensteinkohomologie und die Konstruktion gemischter Motive, LNM 1562, Springer 1993.

[Hu1] A. HUBER, Mixed Motives and their Realization in Derived Categories. Springer LNM 1604 (1995).

[Hu2] A. HUBER, Mixed perverse sheaves on schemes over number fields, *Comp. Math.* **108**, 1997, pp. 107–121.

[HuK] A. HUBER and G. KINGS, Degeneration of $l$-adic Eisenstein classes and of the elliptic polylog, *Invent. Math.* **135**, 1999, pp. 545–594.

[HuW] A. HUBER and J. WILDESHAUS, Classical motivic polylogarithm according to Beilinson and Deligne, *Doc. Math. J. DMV* **3**,1998, pp. 27–133.

[J] U. JANNSEN, Continuous étale cohomology, *Math. Ann.* **280**, 1988, pp. 207–245.

[Ki1] G. KINGS, Extensions of motives of modular forms, *Math. Ann.* **309**, 1997, pp. 375–399.

[Ki2] G. KINGS, Extensions of motives associated to $K$-theory elements for Hilbert modular forms, *J. reine angew. Math.* **503**, 1998, pp. 109–128.,

[Kk] B. KÖCK, Chow Motif and Higher Chow Theory of $G/P$, *manuscripta mathematica* **70** 91, 363–372.

[Neu] J. NEUKIRCH, The Beilinson Conjecture for algebraic number fields, in: M. Rapoport et al.: Beilinson's conjectures on special values of $L$-functions, Academic press 1988

[Sch-Sch] N. SCHAPPACHER and A. J. SCHOLL, The boundary of the Eisenstein symbol, *Math. Ann.*, **290**, 1991, pp. 303-321.

[Sch] A. J. SCHOLL, Motives for modular forms, *Invent. Math.* **100**, No. 2, 1990, pp. 419-430.

(Manuscript received January 15, 1998.)

A. HUBER and G. KINGS
Math. Institut, Einsteinstr. 62,
48 149 Münster, Germany