JENS FRANKE

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HARMONIC ANALYSIS IN WEIGHTED L_2-SPACES

BY JENS FRANKE

ABSTRACT. – In this article, we prove a conjecture of Borel, which states that the inclusion of the space of automorphic forms into the space of all C_∞-functions induces an isomorphism on the cohomology with coefficients in a finite-dimensional representation. We also prove that every automorphic form is a sum of derivatives of Eisenstein series and apply these results to the generalisation of the Manin-Drinfeld theorem and to the trace formula for Hecke operators.

RÉSUMÉ. – Dans cet article, nous prouvons une conjecture de Borel affirmant que l'inclusion de l'espace des formes automorphes dans l'espace des fonctions C_∞ induit un isomorphisme en cohomologie à coefficients dans une représentation algébrique de dimension finie. Nous prouvons aussi que toute forme automorphe est somme de dérivées de séries d'Eisenstein et appliquons ces résultats à la généralisation de théorème de Manin-Drinfeld et à la formule des traces pour les opérateurs de Hecke.

Contents

Introduction

This preprint arose from the author's attempts to prove Zucker's conjecture on the L_2-cohomology of locally symmetric varieties. One way to try to prove the conjecture is to try to explicitly calculate the local L_2-cohomology at a point of the boundary of the Satake compactification. This does not seem to be the most efficient proof, since there are clever direct proofs by Loijenga, Saper and Stern, and Rapoport and Loijenga. However, the computation of the local L_2-cohomology still seems to be an interesting problem.

It is well-known that the local L_2-cohomology of a Satake compactification is isomorphic to a weighted L_2-cohomology of the Levi component of a parabolic subgroup. Thus, the essential problem is to compute the weighted L_2-cohomology for certain weight functions. This problem is also interesting for other purposes. For instance, the inductive limit of the weighted L_2-cohomology over all moderately decreasing weight functions is isomorphic to the cohomology of arithmetic subgroups. The conjecture that this cohomology can be computed in terms of automorphic forms is due to Borel, and will be proved in this paper. It was previously proved in the rank one case (after a partial result in that case had been obtained by Harder) in a joint work of Casselman and Speh. A description in the case of $SL_2(\mathbb{Z})$ appeared in [Cas84].

The most natural way to compute the weighted L_2-cohomology would be to prove a spectral decomposition of the weighted L_2-space. For rank one groups, this was done by W. Casselman [Cas84]. In this paper we use another approach. Instead of getting a spectral decomposition of the weighted L_2-space, we directly prove that its cohomology can be computed using the subspace of elements whose $\mathfrak{Z}(\mathfrak{g})$-image is finite-dimensional. This can be reduced to the vanishing of the higher derived functors $\mathfrak{Fin}^i_{\mathcal{J}}$ on the weighted L_2-space, where $\mathcal{J}$ is an ideal of finite codimension in $\mathfrak{Z}(\mathfrak{g})$ and $\mathfrak{Fin}_{\mathcal{J}}$ associates to a $(\mathfrak{g}, K)$-module its submodule of elements killed by a power of $\mathcal{J}$. For suitable weight function, this vanishing assertion is proved by an induction argument in section 7. In the cocompact case, the vanishing assertion is trivial. In the higher rank case, the unweighted L_2-space is $\mathfrak{Fin}_{\mathcal{J}}$-acyclic provided $\mathcal{J}$ was sufficiently generic. Consequently, it suffices to show that the quotient of the weighted L_2-space by the unweighted L_2-space is $\mathfrak{Fin}_{\mathcal{J}}$-acyclic. Using the fact that a moderately increasing C^∞-function is square integrable if and only if its constant terms are square integrable, we construct a resolution of this quotient by representations induced from parabolic subgroups. These representations will be $\mathfrak{Fin}_{\mathcal{J}}$-acyclic by the induction assumption. Thus it suffices to verify that the result of applying $\mathfrak{Fin}_{\mathcal{J}}$ to the resolution is exact. This is done using a filtration on the space of automorphic forms, which is investigated in section 6.

In the general case, the unweighted L_2-space will not be $\mathfrak{Fin}_{\mathcal{J}}$-acyclic. However, the logarithmically weighted L_2-space is always $\mathfrak{Fin}_{\mathcal{J}}$-acyclic, as we will see in section 5. It is the logarithmically weighted L_2-space which is used for the induction argument in section 7.

The results of the first four sections are probably known. Their proofs have been given for the reader's convenience, since the author could not quote them in the form in which they are needed, or since the methods we will use in their proof are reused later on.

In section 7, we prove Borel's conjecture and use the filtration on the space of automorphic forms to construct an Eisenstein spectral sequence converging to the cohomology of arithmetic subgroups. If the cohomology has coefficients in a representation with regular highest weights, then the Eisenstein spectral sequence collapses and one gets an explicit description of the cohomology in terms of cusp forms and cuspidal Eisenstein series. Otherwise, little seems to be known in general about the differentials of this spectral sequence. Among the applications that can be proved without knowing anything about the differentials is the rationality of the (analytically defined) $\{P\}$-decomposition for the cohomology of GL_n. This is closely related to the Drinfel'd-Manin Theorem for GL_2, and extends a theorem of Clozel about the rationality of the cuspidal summand. Also, we derive a trace formula for Hecke operators on full cohomology from Arthur's trace formula for L_2-cohomology.

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The Seminar which C. Mœglin, J.-L. Waldspurger, and L. Clozel organized about this paper produced many useful hints and corrections of smaller mistakes. W. Hoffman and C. Kaiser also pointed out some corrections. Finally, J. Schwermer persuaded me to eventually publish this paper.

1. General Notations

Let $\boldsymbol{Q}$, $\boldsymbol{R}$, and $\boldsymbol{C}$ be the fields of rational, real, and complex numbers. In this paper, we will denote:

- linear algebraic groups over $\boldsymbol{Q}$ by calligraphic letters, like $\mathcal{A}$ (usually the maximal $\boldsymbol{Q}$-split torus in a parabolic subgroup), $\mathcal{G}$ (usually reductive groups), $\mathcal{M}$ (usually the intersection of the kernels of the characters of the Levi component of a parabolic subgroup), $\mathcal{N}$ (usually the radical of a parabolic subgroup) and $\mathcal{P}$ (usually parabolic subgroups).

- their adelic groups by poor man's boldface, like $\mathbb{A}$, $\mathbb{G}$, $\mathbb{M}$, $\mathbb{N}$, $\mathbb{P}$. The groups of finite adeles get a subscript f, for instance, $\mathbb{G} = \mathcal{G}(\boldsymbol{R})\mathbb{G}_f$.
- their groups of rational points by $A = \mathcal{A}(\boldsymbol{Q}) \subset \mathbb{A}$, $G,\ldots,P$.
- their Lie-algebras, viewed as real vector spaces, by small gothics letters, like $\mathfrak{a} = Lie(\mathcal{A}/\boldsymbol{Q}) \otimes_{\boldsymbol{Q}} \boldsymbol{R}$, $\mathfrak{g}$, $\mathfrak{m}$, $\mathfrak{n}$, $\mathfrak{p}$.

The only exception to this scheme of notations are compact subgroups of adelic groups. For a reductive group $\mathcal{G}$ we will always assume that a good maximal compact subgroup $\mathbb{K} \subset G$ has been chosen. It is the product of maximal subgroups $K_v \subset \mathcal{G}(\boldsymbol{Q}_v)$ over all places v of $\boldsymbol{Q}$. We put

$$\mathbb{K}_f = \prod_{v\ finite} K_v$$

and assume that

$$K \subseteq K_\infty.$$

is an open subgroup.

Let $\mathcal{P}_o$ be a minimal parabolic subgroup of $\mathcal{G}$, which we assume to be fixed throughout the paper. We put

$$\check{\mathfrak{a}}_o = X^*(\mathcal{P}_o) \otimes \boldsymbol{R},$$

where X^* is the group of $\mathbb{Q}$- rational characters. Similar, if $\mathcal{P} \supset \mathcal{P}_o$ is a standard (with respect to $\mathcal{P}_o$) parabolic subgroup, then

$$\check{\mathfrak{a}}_{\mathcal{P}} = X^*(\mathcal{P}) \otimes \boldsymbol{R}.$$

Furthermore, let $\mathcal{M}_{\mathcal{P}}\mathcal{A}_{\mathcal{P}}\mathcal{N}_{\mathcal{P}}$ and $\mathcal{M}_o\mathcal{A}_o\mathcal{M}_o$ be Langlands decompositions for $\mathcal{P}$ and $\mathcal{P}_o$ such that $\mathcal{A}_{\mathcal{P}} \subset \mathcal{A}_o$ and $\mathcal{M}_{\mathcal{P}} \supset \mathcal{M}_o$. It will always be assumed that a Langlands decomposition for $\mathcal{P}_o$ has been fixed and that the Langlands decompositions of the standard parabolic subgroups are compatible in the above way to the Langlands decomposition of $\mathcal{P}_o$. In the special case $\mathcal{P} = \mathcal{G}$, $\mathcal{A}_{\mathcal{G}}$ is a maximal $\boldsymbol{Q}$-split torus in the center of $\mathcal{G}$, $\mathcal{M}_{\mathcal{G}}$ is the connected component of the intersection of the kernels of all $\boldsymbol{Q}$-rational characters of $\mathcal{G}$, and $\mathfrak{m}_{\mathcal{G}}$ is the Lie algebra of $\mathcal{M}_{\mathcal{G}}(\boldsymbol{R})$. For instance, $\mathcal{A}_{\mathcal{G}} = \{1\}$ and $\mathfrak{m}_{\mathcal{G}} = \mathfrak{g}$ if $\mathcal{G}$ is semisimple. We denote by $\mathcal{L}_{\mathcal{P}} = \mathcal{A}_{\mathcal{P}}\mathcal{M}_{\mathcal{P}}$ the Levi component of $\mathcal{P}$. If confusions are impossible, we will write $\mathcal{A}$, $\mathcal{L}$, $\mathcal{M}$, $\mathcal{N}$ for $\mathcal{A}_{\mathcal{P}}$, $\mathcal{L}_{\mathcal{P}}$, $\mathcal{M}_{\mathcal{P}}$ and $\mathcal{N}_{\mathcal{P}}$. Then

$$\mathfrak{a}_{\mathcal{P}} = X_*(\mathcal{A}) \otimes \boldsymbol{R},$$

where X_* is the group of $\boldsymbol{Q}$-rational cocharacters, and

$$\mathfrak{a}_o = X_*(\mathcal{A}_o) \otimes \boldsymbol{R},$$

are in the natural way in duality with $\check{\mathfrak{a}}_{\mathcal{P}}$ and $\check{\mathfrak{a}}_o$. In particular, $\mathfrak{a}_{\mathcal{P}}$ and $\mathfrak{a}_o$ are up to canonical isomorphism independent of the Langlands decomposition. The inclusion $\mathcal{A}_P \subset \mathcal{A}_o$ defines

$$\mathfrak{a}_{\mathcal{P}} \to \mathfrak{a}_o \tag{1}$$

and the restriction of characters of $\mathcal{P}$ to $\mathcal{P}_o$ defines

$$\check{\mathfrak{a}}_{\mathcal{P}} \to \check{\mathfrak{a}}_o$$

which is inverse to the dual of (1) and hence defines a direct sum decomposition

$$\mathfrak{a}_o = \mathfrak{a}_{\mathcal{P}} \oplus \mathfrak{a}_o^{\mathcal{P}}, \quad \check{\mathfrak{a}}_o = \check{\mathfrak{a}}_{\mathcal{P}} \oplus \check{\mathfrak{a}}_o^{\mathcal{P}}. \tag{2}$$

Let $\mathfrak{a}_{\mathcal{P}}^{\mathcal{Q}}$ be the intersection of $\mathfrak{a}_{\mathcal{P}}$ and $\mathfrak{a}_o^{\mathcal{Q}}$ in $\mathfrak{a}_o$. A similar notation will be used for $\check{\mathfrak{a}}$. Note that in general we will not consider $\mathcal{G}$ as a parabolic subgroup of itself. However, in the decomposition (2) we will allow the case $\mathcal{G} = \mathcal{P}$ as well. For instance, $\check{a}_{\mathcal{P}}^{\mathcal{G}}$ is the real vector space generated by the characters of $\mathcal{P}$ which vanish on the center of $\mathcal{G}$. The dimension of $\mathfrak{a}_{\mathcal{P}}^{\mathcal{G}}$ will be called the rank of $\mathcal{P}$ and denoted by $\mathrm{rank}(\mathcal{P})$.

For $p \in \mathbb{P}$ we define $H_{\mathcal{P}}(p) \in \mathfrak{a}_{\mathcal{P}}$ by

$$\prod_v \mid \chi(p) \mid_v = \mathrm{e}^{\langle \chi, H_{\mathcal{P}}(p) \rangle},$$

where χ is any character of $\mathcal{P}$ which on the right hand side of the equation is viewed as an element of $\check{\mathfrak{a}}_{\mathcal{P}}$. The pairing between $\mathfrak{a}_{\mathcal{P}}$ and $\check{\mathfrak{a}}_{\mathcal{P}}$ has been denoted by $\langle .,. \rangle$. If g is an element of $\mathbb{G}$, we choose an Iwasawa decomposition $g = pk$ with $k \in \mathbb{K}$ and $p \in \mathbb{P}$ and put

$$H_{\mathcal{P},\mathbf{K}}(g) = H_{\mathcal{P}}(p), \tag{3}$$

which is easily seen to be independent of the choice of the Iwasawa decomposition of g. If confusions are impossible, we will write $H_{\mathcal{P}}(g)$ for $H_{\mathcal{P},\mathbf{K}}(g)$ and $H(g)$ for $H_{\mathcal{P}_o}(g)$. One verifies $H(pg) = H(g)$ for $p \in P$.

We denote by $\Phi_o \subset X^*(\mathcal{A}_o) \subset \check{\mathfrak{a}}_o$ the set of roots of $\mathcal{A}_o$ in $\mathfrak{g}$. Then Φ_o is a (in general non-reduced) root system in the vector space $\check{\mathfrak{a}}_o$. We denote by $\check{\Phi}_o \subset \mathfrak{a}_o$ the dual root system and for every $\alpha \in \Phi_o$ by $\check{\alpha} \in \check{\Phi}_o$ the dual root. Let $\Delta_o \in \Phi_o$ be the set of simple positive roots, $\mathfrak{a}_o^{\mathcal{G}+} \subset \mathfrak{a}_o^{\mathcal{G}}$ and $\check{\mathfrak{a}}_o^{\mathcal{G}+} \subset \check{\mathfrak{a}}_o^{\mathcal{G}}$ the open positive Weyl chamber, and ${}^+\mathfrak{a}_o^{\mathcal{G}} \subset \mathfrak{a}_o^{\mathcal{G}}$ and ${}^+\check{\mathfrak{a}}_o^{\mathcal{G}} \subset \check{\mathfrak{a}}_o^{\mathcal{G}}$ the open positive cones dual to the positive Weyl chambers. For a real number D we put

$$\mathfrak{S}(D) = \left\{ g \in \mathbb{G} \mid \langle \alpha, H(g) \rangle > D \;\Big|\; \text{for all } \alpha \in \Delta_o \right\} \tag{4}$$

This set is obviously P_o-invariant. If D is small enough, then $G\mathfrak{S}(D) = \mathbb{G}$, we will often assume that such a small number D has been fixed and put $\mathfrak{S} = \mathfrak{S}(D)$. For every D, the number of $\gamma \in P_o \backslash \mathbb{G}$ with $\gamma g \in \mathfrak{S}(D)$ is known to be uniformly bounded for $g \in \mathbb{G}$.

Let For a parabolic subgroup $\mathcal{P}$, $\mathcal{A}_{\mathcal{P}}(\boldsymbol{R})^o$ is the connected component of $\mathcal{A}_{\mathcal{P}}(\boldsymbol{R})$. In particular, $\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o$ is the connected component of the group of real points of a maximal $\boldsymbol{Q}$-split torus in the center of $\mathcal{G}$.

For two standard parabolic subgroups $\mathcal{P} \subset \mathcal{R}$, let $\Delta_{\mathcal{P}}^{\mathcal{R}}$ be the set of those simple roots $\alpha \in \Delta_o$ which occur in the Lie algebra of the radical of $\mathcal{P}$ but not in the Lie algebra of the radical of $\mathcal{R}$. For $\alpha \in \Delta_{\mathcal{P}}^{\mathcal{R}}$, we denote by $\check{\alpha} \in \mathfrak{a}_{\mathcal{P}}^{\mathcal{R}}$ the corresponding coroot (or, more precisely, the projection to $\mathfrak{a}_{\mathcal{P}}^{\mathcal{R}}$ of the corresponding coroot in the co-root system of $\mathcal{P}_o$) and by $\{\omega_{\alpha}^{\mathcal{R}}\}_{\{\alpha\in\Delta_{\mathcal{P}}^{\mathcal{R}}\}}$ the base of $\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{R}}$ dual to the base of $\mathfrak{a}_{\mathcal{P}}^{\mathcal{R}}$ given by $\{\check{\alpha}\}_{\{\alpha\in\Delta_{\mathcal{P}}^{\mathcal{R}}\}}$. The definition of $\check{\omega}_{\alpha}^{\mathcal{R}} \in \mathfrak{a}_{\mathcal{P}}^{\mathcal{R}}$ is similar. The superscript $\mathcal{R}$ is necessary because for $\mathcal{P} \subset \mathcal{Q} \subset \mathcal{R}$ we have $\Delta_{\mathcal{P}}^{\mathcal{Q}} \subset \Delta_{\mathcal{P}}^{\mathcal{R}}$ but in general $\omega_{\alpha}^{\mathcal{Q}} \neq \omega_{\alpha}^{\mathcal{R}}$ for $\alpha \in \Delta_{\mathcal{P}}^{\mathcal{Q}}$. We will often write ω_{α} for $\omega_{\alpha}^{\mathcal{G}}$. Let $\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{R}+}$ and ${}^{+}\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{R}}$ be the open cones spanned by the $\omega_{\alpha}^{\mathcal{R}}$ and the α for $\alpha \in \Delta_{\mathcal{P}}^{\mathcal{R}}$. The Weyl chamber and the positive cone in $\mathfrak{a}_{\mathcal{P}}^{\mathcal{R}}$ are defined in a similar manner.

For a parabolic subgroup $\mathcal{P}$, let $\rho_{\mathcal{P}} \in \check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{G}}$ be one half the sum of the roots of $\mathcal{P}$ in $\mathcal{N}_{\mathcal{P}}$. If $\mathcal{Q} \subset \mathcal{P}$, then $\rho_{\mathcal{P}}$ is equal to the projection of $\rho_{\mathcal{Q}}$ to $\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{G}}$. We will write ρ_o for $\rho_{\mathcal{P}_o}$.

For complex numbers z, $\Re z$ and $\Im z$ denote the real and imaginary part of z. The same notation will be used for elements of the complexification of a real vector space.

Between two complete locally convex spaces of which at least one is nuclear, $\widehat{\otimes}$ refers to Grothendieck's completed tensor product. This notation is also used for complexes of locally convex spaces.

Let $\mathfrak{U}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$ and $\mathfrak{Z}(\mathfrak{g})$ its center. For $A \in \mathfrak{U}(\mathfrak{g})$, the action of A by right differentiation on a C^{∞}-function f will be denoted by Af or $R(A)f$.

2. Definition of the weighted L_2-cohomology

2.1. Basic definitions

We consider weight functions $\rho(g)$ on $G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^{o}\backslash\mathbb{G}/\mathbb{K}$ such that for every $D \in \mathfrak{U}(\mathfrak{g})$ there exists a constant c_D with

$$(1) \qquad |(D\rho)(g)| < c_D\rho(g).$$

For such a weight function, there exists a neighbourhood U of the identity in $\mathbb{G}$ such that

$$(2) \qquad \frac{1}{2}\rho(gh) < \rho(g) < 2\rho(gh)$$

holds for $g \in \mathbb{G}$ and $h \in U$. Conversely, if ρ is a weight function for which there exists neighbourhood U with (2), then we may choose a non-negative non-vanishing C^{∞}-function φ on $\mathbb{G}$ with support in U and replace ρ by the equivalent weight function $\rho * \varphi$ which always satisfies (1).

Let us assume that a small real number D has been chosen as explained after (1.4) and that a minimal parabolic subgroup $\mathcal{P}_o$ has been fixed. For a standard parabolic subgroup $\mathcal{P}$ and a real number $T > D$, we put

$$(3) \qquad \mathfrak{S}(\mathcal{P}, D, T) = \{g \in \mathfrak{S}(D) \text{ such that } \langle H(g), \alpha\rangle > T \text{ for all } \alpha \in \Delta_o - \Delta_o^{\mathcal{P}}.\}$$

Here $H(g)$ was defined after (1.3).

PROPOSITION. – *For every* $\lambda \in \check{\mathfrak{a}}_o^{\mathcal{G}}$ *there exists a weight function* ρ_λ *on the symmetric space* $G\mathcal{A}_G(\boldsymbol{R})^o\backslash\mathbb{G}/\mathbb{K}$ *which satisfies* (1) *and such that*

$$\rho_\lambda(g) \sim \exp(\langle\lambda, H(g)\rangle)$$

for $g \in \mathfrak{S}(D)$. *In addition,* ρ_λ *may be assumed to satisfy the following condition: If* D *has been fixed as above, then there exists a real number* T *such that*

$$\rho_\lambda(ng) = \rho_\lambda(g)$$

when $\mathcal{P} = \mathcal{MAN}$ *is a standard parabolic subgroup,* $n \in \mathbb{N}$, *and* $g \in \mathfrak{S}(\mathcal{P}, D, T)$.

Proof. – Let χ be a C^∞-function on $\boldsymbol{R}$ which is equal to zero on $(-\infty\,,\, D-1]$ and equal to one on $[D, \infty)$. We put

$$\rho_\lambda(g) = \sum_{\gamma\in P_o\backslash G} \mathrm{e}^{\langle\lambda, H(\gamma g)\rangle} \prod_{\alpha\in\Delta_o} \chi(\langle\alpha, H(\gamma g)\rangle). \tag{4}$$

By the fundamental property of the Siegel domain, the number of non-zero summands is uniformly bounded for $g \in \mathbb{G}$. To verify that the sum has the required properties, we quote the following results from reduction theory:

THEOREM 1.

(1) *There exists an* $\xi \in \mathfrak{a}_o^{\mathcal{G}}$ *(depending on D) such that*

$$H(\gamma g) - H(g) \in \xi - {}^+\mathfrak{a}_o^{\mathcal{G}}$$

for $\gamma \in G$ *and* $g \in \mathfrak{S}(D)$.

(2) *If* $g \in \mathfrak{S}(D)$ *and* $\gamma g \in \mathfrak{S}(D)$, *then*

$$H(\gamma g) - H(g)$$

lies in a certain compact subset of $\mathfrak{a}_o^{\mathcal{G}}$ *depending only on* D.

(3) *There exists a real number* T *(depending only on* D*) such that for a standard parabolic subgroup* $\mathcal{P}$ $g \in \mathfrak{S}(\mathcal{P}, D, T)$ *and* $\gamma g \in \mathfrak{S}(D)$ *implies* $\gamma \in P$.

By part 2. of this theorem, all members of the above sum are equivalent to $\mathrm{e}^{\langle\lambda, H(g)\rangle}$ if $g \in \mathfrak{S}(D)$. For the same reason, all derivatives of ρ_λ can be estimated from above by $\mathrm{e}^{\langle\lambda, H(g)\rangle}$ on the Siegel domain, and (1) is satisfied. By part 3. of the above theorem, if T has been chosen large enough (as specified by the theorem with D replaced by $D-1$), then for $g \in \mathfrak{S}(\mathcal{P}, D, T)$ and any summand in (4) which is different from zero, γ must be in P. But then $\gamma\mathbb{N} = \mathbb{N}\gamma$, and the corresponding term in the (4) is $\mathbb{N}$-invariant. The proof of the Lemma is complete. □

Proof of Theorem 1. – This fact seems to be well known, but for the reader's convenience a proof is added. The first assertion is an adelic reformulation of [HC68], §II.1, Corollary 3. Following is a short sketch of the arguments in [HC68].

Let $\xi \in \check{\mathfrak{a}}_o^{\mathcal{G}}$ such that there exists a finite-dimensional irreducible representation of $\mathcal{G}$ in a rational vector space X with a highest (with respect to $\mathcal{P}_o$) weight vector x_ξ. Let $\|\cdot\|_v$ be a collection of v-adic metrics on X such that for any non-zero $x \in X$ we have $\| x \|_v = 1$ for all but finitely many valuations v of $\mathbf{Q}$. It may be assumed that $\| \cdot \|_v$ is K_v-invariant.

Let $\mathbb{X}$ be the adelic space of X. For $x \in \mathbb{X}$ we put

$$\| x \| = \prod_v \| x \|_v .$$

There exists a number $\varepsilon > 0$ with

$$\| x \| > \varepsilon$$

for $x \in X - \{0\}$. We may normalize the metrics by $\| v_\xi \| = 1$. Then we have

$$\langle H(g), \xi \rangle = -\log(\| g^{-1} v_\xi \|).$$

for any g.Then we have

$$\langle H(g), \xi \rangle = -\log(\| g^{-1} v_\xi \|).$$

for any g.

Let $g \in \mathfrak{S}(D)$ and $\gamma \in G$. At the price of changing g in its P_o-orbit, we may assume

$$g \in \Omega \mathbb{A}_o$$

for a fixed compact set $\Omega \subset \mathbb{G}$. Then there is a constant c (depending on D) with

$$\| g v_\xi \| \leq c e^{\langle H(g), \xi \rangle} \| v_\xi \| .$$

Consequently,

$$\begin{aligned} \varepsilon &\leq c \| \gamma^{-1} v_\xi \| \\ &\leq c \| g g^{-1} \gamma^{-1} v_\xi \| \\ &\leq c e^{\langle \xi, H(g) \rangle} \| g^{-1} \gamma^{-1} v_\xi \| \\ &= c' e^{\langle \xi, H(g) - H(\gamma g) \rangle}. \end{aligned}$$

Since the set of all ξ for which X and v_ξ exist spans $\check{\mathfrak{a}}_o^{\mathcal{G}+}$, this proves the first point.

In particular, the first part of the theorem implies that for every D there exists a compact subset Ω of $\mathfrak{a}_o$ such that $H(g) - H(\gamma g) \in \Omega$ if $g \in \mathfrak{S}(D)$ and $\gamma g \in \mathfrak{S}(D)$.

To prove the third point, we may by the Bruhat decomposition assume that $\gamma = w$ is in a finite set of representatives in the normalizer of $\mathcal{A}_o$ of the elements the Weyl group of Φ_o. We may also assume $g = p = ln \in \mathbb{P}$ with $l \in \mathbb{L}_o$ and $n \in \mathbb{N}_o$. Then

$$H(wp) - H(p) = H(wlw^{-1}) - H(l) + H(wn) = wH(l) - H(wn).$$

By the first part of the theorem, there exists a vector y with

$$H(wn) \in y - {}^+\mathfrak{a}_o^{\mathcal{G}}.$$

By the theory of root systems, for $w \notin \mathcal{P}$ there exists a vector $x \in \overline{{}^+\mathfrak{a}_o^{\mathcal{G}}} - \{0\}$ such that $l \in \mathfrak{S}(\mathcal{P}, D, T)$ implies

$$wH(l) - H(l) \in y' - Tx - {}^+\mathfrak{a}_o^{\mathcal{G}}.$$

Hence

$$H(wg) - H(g) \in y + y' - Tx - {}^+\mathfrak{a}_o^{\mathcal{G}},$$

and if T is large enough then the set on the right hand side does not intersect Ω.

The proof of Theorem 1 is complete. Q.E.D.

We consider a finite-dimensional algebraic representation $\boldsymbol{E}$ of $\mathcal{G}(\boldsymbol{R})$ in a complex vector space E. For every open subgroup $\mathbb{K}^f \subseteq \mathbb{K}_f$ which is small enough such that G acts without fixed points on $\mathbb{G}/\mathbb{K}^f K$, $\boldsymbol{E}$ defines a local system $\mathbb{E}$ on $G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o \backslash \mathbb{G}/\mathbb{K}^f K$ by descending the constant local system E on the symmetric space $\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o \backslash \mathbb{G}/\mathbb{K}^f K$ with the $G \subset \mathcal{G}(\boldsymbol{R})$-action given by the representation $\boldsymbol{E}$. Every K-invariant metric $| \, . \, |$ on E defines a metric on the constant local system E on the symmetric space $\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o \backslash \mathbb{G}/\mathbb{K}^f K$ by the rule

$$| e |_{p(g)} = | g_\infty^{-1} e |,$$

where $p : \mathbb{G} \longrightarrow \mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o \backslash \mathbb{G}/\mathbb{K}^f K$ is the projection and g satisfies the condition

$$\prod_v | \chi(g) |_v = 1 \tag{5}$$

for every rational character ξ of $\mathcal{G}$. This metric is G-invariant and hence descends to a metric on $\mathbb{E}$.

Similarly, every $Ad(K)$-invariant metric $| \, . \, |$ on $\mathfrak{g}/\mathfrak{k}$ defines a $\mathbb{G}$-invariant Riemannian metric on $\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o \backslash \mathbb{G}/\mathbb{K}^f K$ by

$$| \pi_*(\vec{v}(g)) |_{\pi(g)} = | v |,$$

where $g \in \mathbb{G}$, $\pi : \mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o \backslash \mathbb{G}/\mathbb{K}^f \longrightarrow \mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o \backslash \mathbb{G}/\mathbb{K}^f K$ is the projection, and v is a vector in $\mathfrak{g}$ which is orthogonal to the differentials of all rational characters of $\mathcal{G}$, giving rise to the left $\mathbb{G}$-invariant vector field $\vec{v}$ on $\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o \backslash \mathbb{G}/\mathbb{K}^f$. Such an invariant metric descends to a Riemannian metric on $G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o \backslash \mathbb{G}/\mathbb{K}^f K$ provided that $\mathbb{K}^f$ was small enough as indicated above. Let dg be the quotient of the Haar measure on $\mathbb{G}$, normalized by the condition that $\int_{G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o \backslash \mathbb{G}/\mathbb{K}} dg = 1$. This measure on $G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o \backslash \mathbb{G}/\mathbb{K}^f K$ is proportional to the measure defined by the Riemannian metric.

If ρ is weight function satisfying (1), we put

$$\begin{aligned} &L_{2,\rho}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o \backslash \mathbb{G}/\mathbb{K}^f K, \Lambda^i T^* \otimes \mathbb{E}) \\ &\quad = \Big\{ \mathbb{E}\text{-valued } i\text{-forms } \omega \text{ on } G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o \backslash \mathbb{G}/\mathbb{K}^f K \text{ with} \\ &\qquad \| f | L_{2,\rho} \|^2 = \int_{G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o \backslash \mathbb{G}/\mathbb{K}^f K} \rho(g)^2 (| \omega(g) |^2) dg < \infty \Big\} \end{aligned} \tag{6}$$

$$(7)\qquad L^i_\rho(G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G}/\mathbb{K}^fK,\mathbb{E}) = \Big\{\omega\in L_{2,\rho}(G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G}/\mathbb{K}^fK,\Lambda^iT^*\otimes\mathbb{E}) \text{ such that } d\omega\in L_{2,\rho}(G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G}/\mathbb{K}^fK,\Lambda^{i+1}T^*\otimes\mathbb{E})\Big\}$$

(As usual, this is to be interpreted in the way that the distribution $d\omega$ has to be representable by a function satisfying the above estimate. The absolute value of an $\mathbb{E}$-valued form is defined by the metrics on $\mathbb{E}$ and the Riemannian metric on $G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G}/\mathbb{K}^fK$). This is a complex with differential d (the differential of the local system $\mathbb{E}$). Its cohomology is the weighted L_2-cohomology of $\mathbb{E}$, it is denoted by $H^*_\rho(G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G}/\mathbb{K}^fK,\mathbb{E})$. It is usually more convenient to investigate the inductive limit

$$H^*_\rho(G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G}/K,\boldsymbol{E}) = \operatorname*{colim}_{\mathbb{K}^f} H^*_\rho(G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G}/\mathbb{K}^fK,\mathbb{E}),$$

where the limit is over all sufficiently small open subgroups $\mathbb{K}^f\subset\mathbb{K}_f$. This vector space is a $\mathbb{G}_f$-module. It is the cohomology of the complex of $\mathbb{G}_f$-modules

$$L^*_\rho(G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G}/K,\boldsymbol{E}) = \operatorname*{colim}_{\mathbb{K}^f} L^*_\rho(G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G}/\mathbb{K}^fK,\mathbb{E})$$

2.2. Cohomology of arithmetic subgroups

Perhaps the most interesting example is the inductive limit of the weighted L_2-cohomology over all admissible weight functions ρ. It turns out that this limit describes the cohomology of arithmetic subgroups. To be more precisely, we put

$$H^*(G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G}/K,\boldsymbol{E}) = \operatorname*{colim}_{\mathbb{K}^f} H^*(G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G}/\mathbb{K}^fK,\mathbb{E}),$$

where the limit is over all sufficiently small open compact subgroups $\mathbb{K}^f\subset\mathbb{K}_f$ and $H^*(G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G}/\mathbb{K}^fK,\mathbb{E})$ is the cohomology of the de Rham complex of $\mathbb{E}$-valued currents on $G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G}/\mathbb{K}^fK$. It is therefore clear that the problem of computing this cohomology is equivalent to the problem of computing the cohomology of arithmetic subgroups. It turns out that the currents can be replaced by moderately increasing forms. This fact is due to Borel.

THEOREM. – *The inclusion*

$$\operatorname*{colim}_{\lambda} L^i_{\rho_{-\lambda}}(G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G}/K,E)\subset\Big\{\mathbb{E}\textit{-valued currents on } G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G}/K\Big\}$$

induces an isomorphism on cohomology, where the inductive limit is over all $\lambda\in\check{\mathfrak{a}}^{\mathcal{G}+}_o$, ordered by the relation

$$\lambda\prec\tau \textit{ if and only if } \tau-\lambda\in{}^+\check{\mathfrak{a}}^{\mathcal{G}}_o.$$

Consequently,

$$H^*(G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G}/K,\boldsymbol{E})\cong\operatorname*{colim}_{\lambda} H^*_{\rho_{-\lambda}}(G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G}/K,\boldsymbol{E}).$$

Proof. – Sheafification of the left hand side of the inclusion gives us a complex of acyclic sheaves on the Borel-Serre compactification (this is so because for every open covering of the Borel-Serre compactification we have a C^∞-resolution of unity with moderately increasing (*not bounded*) differentials). The same is true for the de Rham complex of currents. Consequently, it suffices to verify that both complexes have the same local cohomology at any point of the Borel-Serre compactification. This is easily done by a direct computation. □

2.3. Regularization

The following regularization results could also be quoted from a paper of Borel [BOR83]. We give a self-contained explanation, which works for arbitrary complete Riemannian manifolds, using techniques which will be useful in later sections.

In this chapter, X may be any complete Riemannian manifold, and $\mathbb{E}$ any metrized local system on X. Weight functions ρ on X are called admissible if $D \log \rho$ is bounded whenever D is a finite superposition of d and d^*. In the special case $X = G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o \backslash \mathbb{G}/\mathbb{K}^f K$, this estimate is a consequence of (and probably equivalent to) (2.1.1). It will be necessary to assume that there exists a sequence of C_o^∞-functions ϕ_n on X such that for every compact subset Ω of X there exists an index n_o with $\phi_n |_\Omega = 1$ if $n > n_o$, and such that the differential forms and functions

$$(1) \qquad d\phi_n, \Delta\phi_n$$

are uniformly bounded on X and in n. Here $\Delta = dd^* + d^*d$ is the Laplacian on X, where d^* is the dual of d (the connection defined by $\mathbb{E}$) with respect to the metric on $\mathbb{E}$ and the Riemannian metric on X. In the case $X = G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o \backslash \mathbb{G}/\mathbb{K}^f K$, such functions can easily be constructed, *cf.* 5.1.5) below.

We define a Fréchet space of differential forms on X by

$$(2) \qquad S^i_\rho(X, \mathbb{E}) = \Big\{ \mathbb{E}\text{-valued i-forms } \omega \text{ on } X \text{ with } \int_X \rho(g)^2 \mid \Delta^k \omega(g) \mid^2 dg < \infty \quad \text{for every integer } k \geq 0 \Big\}.$$

As usual, (2) means that the distribution $\Delta^k \omega$ is representable by a function satisfying this estimate.

LEMMA 1. – *We have*

$$S^i_\rho(X, \mathbb{E}) = \Big\{ \mathbb{E}\text{-valued i-forms } \omega \text{ on } X \text{ with } \int_X \rho(g)^2 \mid D^k \omega(g) \mid^2 dg < \infty \quad \text{if } D \text{ is a finite superposition of } d \text{ and } d^* \Big\}.$$

Proof. – The embedding "$\supset$" is clear since Δ^k is a sum of superpositions of d and d^*. To prove the converse inclusion it suffices to prove that d and d^* are bounded operators in $S^i_\rho(X, \mathbb{E})$. Let A and B be the differential operators

$$A = \rho \Delta \rho^{-1}$$
$$B = \rho^{-1} \Delta \rho.$$

Let A_o and B_o be A and B, with the set of C_c^∞-forms as domain of definition. We consider these operators as unbounded operators in L_2. Let $B_{✠} = A^*$ and $A_{✠} = B^*$ be their adjoint operators. Then $A_{✠}$ and $B_{✠}$ are closed extensions of A and B. We want to verify that the domain of $A_{✠}$ is contained in

$$\Big\{ \phi \in L_2 \;\Big|\; d\phi \in L_2 \text{ and } d^*\phi \in L_2 \Big\}$$

and that

$$\|df|L_2\| + \|d^*f|L_2\| \le c\Big(\|f|L_2\| + \Big\|A_{\maltese}f|L_2\Big\|\Big). \tag{3}$$

This will imply that for every distribution f in $L_{2,\rho}$ such that $\Delta f \in L_{2,\rho}$, we have $df \in L_{2,\rho}$ and $d^*f \in L_{2,\rho}$ and

$$\|df|L_{2,\rho}\| + \|d^*f|L_{2,\rho}\| \le c(\|f|L_{2,\rho}\| + \|\Delta f|L_{2,\rho}\|).$$

Since Δ commutes with d and d^*, this will prove the lemma.

Let W^1 be the set of all distributions f such that

$$\|f|W^1\|^2 = \|f|L_2\|^2 + \|df|L_2\|^2 + \|d^*f|L_2\|^2 < \infty.$$

Then the space of C_c^∞-forms is dense in W^1. Indeed, by using local coordinates one derives from standard facts about Sobolev spaces on $\boldsymbol{R}^n$ that every distribution in W^1 with compact support can be approximated in W^1 by C_c^∞-forms. Now if f is any element in W^1, the forms $\phi_n f$ have compact support and converge to f in W^1, where ϕ_n is the same as in (1).

Let $A_{\clubsuit}$ and $B_{\clubsuit}$ be the restrictions of $A_{\maltese}$ and $B_{\maltese}$ to W^1. If $\rho = 1$ (such that $A = B$), then this is the Friedrichs extension of $A_{\grave{o}}$ (*cf.* [Yos74], Theorem XI.7.2). The aim of the next few remarks is to verify that some basic facts about this extension remain valid in our non-symmetric situation. We claim that for sufficiently large real numbers L

$$\Re\langle (A_{\clubsuit} + L)f, f\rangle_{L_2} = \Re\langle (B_{\clubsuit} + L)f, f\rangle_{L_2} \sim \|f|W^1\|^2 \tag{4}$$

and

$$\begin{aligned} \big|\Im\langle (A_{\clubsuit} + L)f, f\rangle_{L_2}\big| &\le c\|f|W^1\|^2 \\ \big|\Im\langle (B_{\clubsuit} + L)f, f\rangle_{L_2}\big| &\le c\|f|W^1\|^2 \end{aligned} \tag{5}$$

Indeed, let ϕ_n be the same as above. Then because $f \in W^1$, $A_{\clubsuit}\phi_n f$ converges to $A_{\clubsuit} f$ in L_2. By partial integration we get

$$\begin{aligned} \big\langle \rho\Delta\rho^{-1}\phi_n f, \phi_n f\big\rangle_{L_2} = {} & \langle d\phi_n f, d\phi_n f\rangle_{L_2} + \langle d^*\phi_n f, d^*\phi_n f\rangle_{L_2} \\ & - \langle (d\log\rho)\wedge\phi_n f, (d\log\rho)\wedge\phi_n f\rangle_{L_2} \\ & - \langle (d\log\rho)\vdash\phi_n f, (d\log\rho)\vdash\phi_n f\rangle_{L_2} \\ & + 2i\Im\langle \phi_n f, (d\log\rho)\wedge d^*\phi_n f\rangle - 2i\Im\langle (d\log\rho)\wedge\phi_n f, d(\phi_n f)\rangle. \end{aligned}$$

Here $\phi\vdash\cdot$ refers to contraction by ϕ, *i.e.*, to the adjoint operator with respect to the given Riemannian metric of $\phi\wedge\cdot$. We may pass to the limit $n\to\infty$ and obtain

$$\begin{aligned} \langle A_{\clubsuit}f, f\rangle_{L_2} = {} & \|df|L_2\|^2 + \|d^*f|L_2\|^2 \\ & - \|(d\log\rho)\wedge f|L_2\|^2 - \|(d\log\rho)\vdash f|L_2\|^2 \\ & + 2i\Im\langle f, (d\log\rho)\wedge d^*f\rangle - 2i\Im\langle (d\log\rho)\wedge f, df\rangle. \end{aligned} \tag{6}$$

Since $d\log\rho$ is bounded, this implies both (4) and (5).

Note that the norm in W^1 is a Hilbert space norm. Let $\langle \cdot, \cdot \rangle_{W^1}$ be the corresponding scalar product. For sufficiently big L it follows from (4), (5) and the Lax-Milgram theorem (*cf.* [Yos74], Theorem III.7) that there exists a bounded invertible operator S_L in W^1 with the property that

$$\langle S_L f, g \rangle_{W^1} = \langle (A_{\clubsuit} + L) f, g \rangle_{L_2}$$

for all f in the domain of definition of $A_{\clubsuit}$. We want to verify that for L sufficiently large, $B_{\clubsuit} + L$ is surjective. In fact, let $h \in L_2$, then by the Riesz representation theorem we find $g \in W^1$ such that

$$\langle f, h \rangle_{L_2} = \langle S_L f, g \rangle_{W^1}.$$

Then

$$\langle f, h \rangle_{L_2} = \langle S_L f, g \rangle_{W^1} = \langle (A_o + L) f, g \rangle_{L_2}$$

for all C_c^∞-forms f. It follows that g belongs to the domain of definition of $B_{\maltese}$, and that $(B_{\maltese} + L) g = h$. Since $g \in W^1$, it even belongs to the domain of definition of $B_{\clubsuit}$, whence our claim about the surjectivity of $B_{\clubsuit} + L$. Furthermore, if g is as above and if the ϕ_n are as above, then $(B_{\clubsuit} + L)(\phi_n g)$ converges to h in L_2. By well-known local results (*cf.* the facts about the relation between the Friedrichs mollifier J_ε and d and d^* mentioned in [Gaf51] (1)),the $\phi_n g$ can be approximated by C_o^∞-forms ψ_{nm} such that $(B_{\clubsuit} + L)\psi_{nm} \to (B_{\clubsuit} + L)(\phi_n g)$. It follows that $(B_o + L) C_o^\infty$ is dense in L_2 for sufficiently big L. This implies the injectivity of $A_{\maltese} + L$ for sufficiently big L. Of course, these facts all remain true if the roles of A and B are interchanged.

Now we want to prove $A_{\clubsuit} = A_{\maltese}$. It is clear that $A_{\clubsuit} \subseteq A_{\maltese}$. It remains to show that every f in the domain of definition of $A_{\maltese}$ belongs to the domain of definition of $A_{\clubsuit}$. Let L be sufficiently large, and determine $\widetilde{f}$ by $(A_{\clubsuit} + L)\widetilde{f} = (A_{\maltese} + L) f$. Then $(A_{\maltese} + L)(\widetilde{f} - f) = 0$, hence by the injectivity of $A_{\maltese} + L$ for big L we have $f = \widetilde{f}$, which belongs to the domain of definition of $A_{\clubsuit}$.

Now since $A_{\clubsuit} = A_{\maltese}$, (4) implies (3), and the proof is complete. □

THEOREM 3. – *If $L_\rho^*(\mathbb{E})$ is defined as in* (2.1.7) *but with $G\mathcal{A}_G(\boldsymbol{R})^\circ \backslash \mathbb{G}/\mathbb{K}^f K$ replaced by X, then the inclusion $S_\rho^*(\mathbb{E}) \subset L_\rho^*(\mathbb{E})$induces an isomorphism on cohomology.*

We need the following lemma:

LEMMA 2. – *There exists a constant K such that for every complex number λ with $\Re(\lambda) > K$ the closed operator $\Delta + \lambda$ has a continuous inverse in $L_{2,\rho}(X, \Lambda^i T^* \otimes \mathbb{E})$ (which is the weighted L_2-space defined analogous to* (2.16)*) which satisfies the estimate*

$$\left\| (\Delta + \lambda)^{-1} \middle| L_{2,\rho}(X, \Lambda^i T^* \otimes \mathbb{E}) \to L_{2,\rho}(X, \Lambda^i T^* \otimes \mathbb{E}) \right\| \leq \min((\Re(\lambda) - K)^{-1}, 2 \mid \lambda - K \mid^{-1}). \tag{7}$$

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(1) The point (which was only sketched by Gaffney) is roughly as follows: Let J_ε be the Friedrichs mollifier and let D be a first order differential operator. Since $\{J_\varepsilon | \varepsilon > 0\}$ is bounded in the algebra $S_{1,0}^0$ of pseudodifferential operators, $[J_\varepsilon, D]$ is also bounded in $S_{1,0}^0$ and therefore bounded in $L(L_2, L_2)$. Since $[J_\varepsilon, D]\psi \to 0$ if ψ is C_c^∞, $[J_\varepsilon, D] f \to 0$ in L_2 if f is in L_2, and hence $D J_\varepsilon f \to D f$ in L_2 if f and Df are in L_2.

Here Δ is the distributional Δ, viewed as a closed operator in $L_{2,\rho}(X, \Lambda^i T^* \otimes \mathbb{E})$ defined on the set of those currents ω with values in $\mathbb{E}$ such that both ω and $\Delta\omega$ belong to $L_{2,\rho}(X, \Lambda^i T^* \otimes \mathbb{E})$.

Proof. – Since the isomorphism $L_{2,\rho} \cong L_2$ given by multiplication by ρ identifies Δ with the closed operator $A_{\maltese}$ considered in the proof of the last lemma, we may consider $A_{\maltese}$ instead of Δ. Then the invertibility of $A_{\maltese} + \lambda$ for sufficiently big real numbers λ has already been verified in the proof of Lemma 1. The same argument also proves the invertibility of $A_{\maltese} + \lambda$ for $\Re\lambda \gg 0$. It remains to prove the necessary estimate for its inverse. Considering the real part of (6), we have

$$\begin{aligned}\Re\Big\langle (A_{\maltese} + \lambda) f, f \Big\rangle_{L_2} &= \|df|L_2\|^2 + \|d^* f|L_2\|^2 + \Re\lambda \|f|L_2\|^2 \\ &\quad - \|(d\log\rho) \wedge f|L_2\|^2 - \|(d\log\rho) \vdash f|L_2\|^2 \\ &\geq (\Re\lambda - K)\|f|L_2\|^2\end{aligned}$$

for sufficiently big K since $d\log\rho$ is bounded. This proves the first of the two estimates contained in (7). To prove the other estimate, we note that (6) implies

$$\begin{aligned}\mid \Big\langle (A_{\maltese} + \lambda) f, f \Big\rangle_{L_2} \mid &= \Big| \|df|L_2\|^2 + \|d^* f|L_2\|^2 + \lambda \|f|L_2\|^2 \Big| \\ &\quad + O(\|f|L_2\|(\|df|L_2\| + \|d^* f|L_2\| + \|f|L_2\|)).\end{aligned}$$

For $\Re\lambda \geq 0$, we get

$$\begin{aligned}&\frac{1}{\sqrt{2}}\Big(\|df|L_2\|^2 + \|d^* f|L_2\|^2 + \mid\lambda\mid \|f|L_2\|^2\Big) \\ &\leq \Big| \|df|L_2\|^2 + \|d^* f|L_2\|^2 + \lambda\|f|L_2\|^2 \Big| \\ &\leq \mid \Big\langle (A_{\maltese} + \lambda) f, f \Big\rangle_{L_2} \mid + C\|f|L_2\|(\|df|L_2\| + \|d^* f|L_2\| + \|f|L_2\|) \\ &\leq \mid \Big\langle (A_{\maltese} + \lambda) f, f \Big\rangle_{L_2} \mid + (C + \frac{C^2}{\sqrt{2}})\|f|L_2\|^2 + \frac{\|df|L_2\|^2 + \|d^* f|L_2\|^2}{\sqrt{2}}.\end{aligned}$$

We have used the inequality

$$\mid a \mid + \mid \lambda \mid \cdot \mid b \mid \leq \sqrt{2} \mid a + \lambda b \mid$$

for non-negative real numbers a and b and complex numbers λ with $\Re\lambda \geq 0$ in the first line, and the inequality

$$Cab \leq \frac{C^2 a^2}{2\sqrt{2}} + \frac{b^2}{\sqrt{2}}$$

for non-negative real numbers a, b, and C in the third line. If $L \geq \sqrt{2}C + C^2$, this implies

$$(\mid \lambda \mid - L)\|f|L_2\| \leq \sqrt{2}\Big\|(A_{\maltese} + \lambda) f|L_2\Big\|.$$

This implies the second estimate in (7) with $K = \sqrt{2}L$, since $\mid \lambda \mid - L \geq \frac{1}{\sqrt{2}} \mid \lambda - K \mid$ for $\Re(\lambda) \geq K$ by the first of the two inequalities for complex numbers mentioned above. □

Proof of Theorem 3. – By Lemma 2 and the Hille-Yosida theorem ([Yos74]), the Laplacian is the infinitesimal generator of a strongly continuous semi-group $\{e^{-t\Delta}\}_{t\geq 0}$ with

$$\left\| e^{-t\Delta} | L_{2,\rho}(X, \Lambda^i T^* \otimes \mathbb{E}) \to L_{2,\rho}(X, \Lambda^i T^* \otimes \mathbb{E}) \right\| \leq e^{Kt}.$$

Using the fact that d and d^* commute with Δ and that their domain of definition on $L_{2,\rho}$ is contained in that of Δ (*cf.* the proof of Lemma 1), on easily sees that they commute with the resolvents of Δ. By the construction used in the proof of [Yos74], Theorem IX.7. they also commute with $e^{-t\Delta}$ for $t \geq 0$. Using the second of the two arguments of min in (7), we see that the semi-group $e^{-(\Delta+K)t}$ satisfies the third of the three equivalent conditions for the definition of a holomorphic semigroup in [Yos74], IX.10. The first of these three conditions implies that $e^{-(\Delta+K)t} f$ is in the domain of definition of Δ^k for all positive k and t. Consequently, $e^{-t\Delta} f$ maps L^* into its subcomplex S^* for $t > 0$.

For sufficiently large $\Re(\lambda)$, the operator $T(\lambda) = (\Delta + \lambda)^{-1}(e^{-\Delta} - e^{\lambda})$ is well-defined by Lemma 2. It is clear that it commutes with d and d^*. By the explicit formula for the resolvent of the infinitesimal generator of a semigroup given in [Yos74], IX.4.(1), we have

$$T(\lambda) = \int_0^\infty e^{-\lambda t} e^{-t\Delta}(e^{-\Delta} - e^{\lambda})\, dt = -\int_0^1 e^{t\lambda} e^{(t-1)\Delta}\, dt$$

for those λ with $\Re(\lambda)$ large enough. However, the last expression makes sense for arbitrary complex numbers λ, defining an analytical continuation of $T(\lambda)$ which also commutes with d and d^*. By analytical continuation, we have

$$(\Delta + \lambda) T(\lambda) = e^{-\Delta} - e^{\lambda}$$

for arbitrary complex values of λ. Now let

$$\sigma(\omega) = d^* T(0)\omega, \tag{8}$$

this is an element of $L^i(X, \mathbb{E})$ if $\omega \in L^{i+1}(X, \mathbb{E})$. Then

$$d\sigma(\omega) + \sigma(d\omega) = (dd^* + d^*d)T(0)(\omega) = e^{-\Delta}\omega - \omega,$$

constructing a homotopy between $e^{-\Delta}$ and the identity. It is well known that the image of $e^{-\Delta}$ is contained in the domain of Δ^k for every positive integer k, We have seen that $e^{-\Delta}$ maps L^* into its subcomplex S^*. Since this subcomplex is also invariant under the homotopy (8), its embedding into L^* is a homotopy equivalence.

The proof of Theorem 3 is complete. Q.E.D.

In the case $X = G\mathcal{A}_G(\boldsymbol{R})^o \backslash \mathbb{G}/\mathbb{K}^f K$, this regularization result allows us to rewrite the weighted L_2-cohomology in terms of $(\mathfrak{m}_G, K)$-cohomology. Let

$$S_\rho(G\mathcal{A}_G(\boldsymbol{R})^o\backslash\mathbb{G}) = \Big\{ \mathbb{K}\text{-finite functions } \varphi \text{ on } G\backslash\mathbb{G} \text{ such that } \int_{G\mathcal{A}_G(\boldsymbol{R})^o\backslash\mathbb{G}} \rho(g)^2 \mid (D\varphi)(g) \mid^2 dg < \infty \text{ for every } D \in \mathfrak{U}(\mathfrak{g}). \Big\}. \tag{9}$$

More precisely, define the weighted Sobolev spaces as follows. For a base γ_i of $\mathfrak{g}$, let

$$(10)\quad W^k_\rho(G\mathcal{A}_G(\boldsymbol{R})^o\backslash\mathbb{G}) = \Big\{\mathbb{K}\text{-finite functions } \varphi \text{ on } G\backslash\mathbb{G} \text{ such that } \|\varphi|W^k_\rho\|^2 = \int_{G\mathcal{A}_G(\boldsymbol{R})^o\backslash\mathbb{G}} \sum_{|\alpha|\le k} \Big| \frac{\partial^\alpha\varphi}{\partial\gamma^\alpha}(g)\Big|^2\, dg < \infty\Big\},$$

where the sum is over all multi-indices $(\alpha_1,\dots,\alpha_{\dim\mathfrak{g}})$ with $|\alpha| = \sum\alpha_i \le k$ and $\frac{\partial^\alpha}{\partial\gamma^\alpha} = \frac{\partial^{\alpha_1}}{\partial\gamma_1^{\alpha_1}}\cdot\ldots\cdot\frac{\partial^{\alpha_{\dim\mathfrak{g}}}}{\partial\gamma_{\dim\mathfrak{g}}^{\alpha_{\dim\mathfrak{g}}}}$ with the order given by the choice of the base. It is clear that $S_\rho(G\mathcal{A}_G(\boldsymbol{R})^o\backslash\mathbb{G})$ is the intersection of $W^k_\rho(G\mathcal{A}_G(\boldsymbol{R})^o\backslash\mathbb{G})$ over $k\ge 0$.

For every finite set $\boldsymbol{S}$ of $\mathbb{K}$-types (= finite-dimensional $\mathbb{K}$-representations), the $\boldsymbol{S}$-isotypical subspace $W^{2m}_\rho(G\mathcal{A}_G(\boldsymbol{R})^o\backslash\mathbb{G})_{\boldsymbol{S}}$ is a Banach space. Similarly, $S_\rho(G\mathcal{A}_G(\boldsymbol{R})^o\backslash\mathbb{G})_{\boldsymbol{S}}$ is a Fréchet space. Thus, $S_\rho(G\mathcal{A}_G(\boldsymbol{R})^o\backslash\mathbb{G})$ is a locally convex $(\mathfrak{g},K)$-module if it is equipped with the topology of the inductive limit over all finite sets $\boldsymbol{S}$ of $\mathbb{K}$-types. Here the definition of a $(\mathfrak{g},K)$-module is as in [Vog81], §6.1. The group $\mathbb{G}_f$ also acts on this space by right translation, such that the stabilizer of each vector is open and the action is compatible with the $(\mathfrak{g},K)$-action. For the remaining part of this paper, we will refer to such a structure as a $(\mathfrak{g},K,\mathbb{G}_f)$-module.

PROPOSITION 1. – *For any finite set of* $\mathbb{K}$*-types* $\boldsymbol{S}$*, we have*

$$(11)\qquad W^{2m}_\rho(G\mathcal{A}_G(\boldsymbol{R})^o\backslash\mathbb{G})_{\boldsymbol{S}} = (\Lambda+\omega)^{-m}L_{2,\rho}(G\mathcal{A}_G(\boldsymbol{R})^o\backslash\mathbb{G})_{\boldsymbol{S}},$$

where ω *is the Casimir operator and* Λ *has been chosen so large that* $\Lambda+\omega$ *is invertible in* $L_{2,\rho}(G\mathcal{A}_G(\boldsymbol{R})^o\backslash\mathbb{G})_{\boldsymbol{S}}$.

Proof. – It suffices to prove this for $m=1$, since the general case will follow by induction. The fact that $\Lambda+\omega$ is invertible for sufficiently big Λ (depending on $\boldsymbol{S}$ and ρ) is proved in the same way as in the proof of Lemma 2. As in the proof of Lemma 1, one shows

$$(12)\qquad \|f|L_{2,\rho}\| + \sum_{i=1}^{\dim\mathfrak{g}} \|\gamma_i f|L_{2,\rho}\| \le c\|(\Lambda+\omega)f|L_{2,\rho}\|.$$

It remains to verify the estimate

$$(13)\qquad \|\gamma_i\gamma_j f|L_{2,\rho}\| \le c\|(\Lambda+\omega)f|L_{2,\rho}\|$$

for C^∞_o-functions f. Once this estimate is shown for C^∞_o-functions, it follows by an easy density and continuity argument that the bounded operator $(\Lambda+\omega)^{-1}$ in $L_{2,\rho}$ is in fact a continuous operator $L_{2,\rho}\to W^2_\rho$, and this proves the proposition. As

$$(\rho\omega\rho^{-1}-\omega)f = \sum_{i=1}^{\dim\mathfrak{g}} c_i\gamma_i f + c_0 f$$

with bounded coefficients c_i, it suffices to do this for $\rho = 1$. Let us denote the unweighted Sobolev space (W_ρ^1 with $\rho = 1$) by W^1. Let $\mathfrak{g} = \mathfrak{p} + \mathfrak{k}$ be the Cartan decomposition, and choose a base p_i of $\mathfrak{p}$ which is orthogonal with respect to the Killing form. Then

$$(14) \qquad \left| \sum_{i,j} \langle p_i^2 p_j^2 f, f \rangle_{L_2} - \|\omega f | L_2\|^2 \right| \leq c \|f | W^1\|^2 \leq c \|(\Lambda + \omega) f | L_2\|^2$$

as f is $\boldsymbol{S}$-isotypical, hence the L_2-norm of all derivatives in the $\mathfrak{k}$-direction of f can be majorized by the L_2-norm of f. But

$$\begin{aligned} \left| \langle p_i^2 p_j^2 f, f \rangle - \|p_j p_i f | L_2\|^2 \right| &= |\langle p_i [p_i, p_j] p_j f, f \rangle + \langle p_i p_j [p_i, p_j] f, f \rangle| \\ &= |\langle [p_i, p_j] p_j f, p_i f \rangle + \langle p_j [p_i, p_j] f, p_i f \rangle| \\ &\leq c (\|f | L_2\| + \|p_i f | L_2\|)(\|f | L_2\| + \|p_j f | L_2\|) \end{aligned}$$

since $[p_i, p_j] \in \mathfrak{k}$ and since $p_j f$ is $\widetilde{\boldsymbol{S}}$-isotypical for some finite $\widetilde{\boldsymbol{S}} \supset \boldsymbol{S}$. Combining this with (12) and (14), we find

$$\|p_i p_j f | L_2\| \leq c \|(\Lambda + \omega) f | L_2\|.$$

This completes the proof of (13) in the case $\rho = 1$. As we indicated above, this is sufficient to imply the general case. □

We need some facts about duality and interpolation of these spaces. The equation (11) can be used as a definition of Sobolev spaces of order $m < 0$. It follows from this description that the dual of $W_\rho^{2m}(G\mathcal{A}_G(\boldsymbol{R})^o \backslash \mathbb{G})_{\boldsymbol{S}}$ is the space $W_{\rho^{-1}}^{-2m}(G\mathcal{A}_G(\boldsymbol{R})^o \backslash \mathbb{G})_{\boldsymbol{S}}$. By the well-known result about the interpolation of weighted L_2-spaces, this description also proves

$$(15) \qquad [W_{\rho_1}^{2m}(G\mathcal{A}_G(\boldsymbol{R})^o \backslash \mathbb{G})_{\boldsymbol{S}}, W_{\rho_2}^{2m}(G\mathcal{A}_G(\boldsymbol{R})^o \backslash \mathbb{G})_{\boldsymbol{S}}]_\Theta = W_{\rho_1^\Theta \rho_2^{1-\Theta}}^{2m}(G\mathcal{A}_G(\boldsymbol{R})^o \backslash \mathbb{G})_{\boldsymbol{S}},$$

where $[.,.]_\Theta$ is the complex interpolation functor of Lions, Calderón and Krejn. (*cf.* [Tri78], Theorem 1.18.5).

To see that the weighted L_2-cohomology and the $(\mathfrak{m}_G, K)$-cohomology of the module $S_\rho(G\mathcal{A}_G(\boldsymbol{R})^o \backslash \mathbb{G})$ are isomorphic, we associate to every element

$$\phi \in \operatorname{Hom}_K\left(\Lambda^i(\mathfrak{m}_G/\mathfrak{k}), S_\rho(G\mathcal{A}_G(\boldsymbol{R})^o \backslash \mathbb{G}) \otimes \boldsymbol{E}\right)$$

the i-form

$$(16) \qquad \tau_\phi(\pi_*(\vec{v}_1(g) \wedge \ldots \wedge \vec{v}_i(g))) = \phi(v_1, \ldots, v_n)(g),$$

where $\vec{v}$ is the left-invariant vector field on $G\mathcal{A}_G(\boldsymbol{R})^o \backslash \mathbb{G}$ defined by $v \in \mathfrak{g}$ $\pi : G\mathcal{A}_G(\boldsymbol{R})^o \backslash \mathbb{G} \to G\mathcal{A}_G(\boldsymbol{R})^o \backslash \mathbb{G}/K$ is the projection, and $g \in \mathbb{G}$ satisfies the normalization condition (2.1.5). This defines an isomorphism between the de Rham complex and the standard complex [Vog81], Proposition 6.1.16 used for computing $(\mathfrak{m}_{\mathcal{G}}, K)$-cohomology. The result is an isomorphism

$$H^*_\rho\big(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G}/K, \boldsymbol{E}\big) \cong H^*_{(\mathfrak{m}_{\mathcal{G}},K)}\big(S_\rho\big(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G}\big) \otimes E\big).$$

However, this is not an isomorphism of $\mathbb{G}_f$-modules since the normalization condition (2.1.5) which is used in (16) is not invariant under right translation by $\mathbb{G}_f$. To get an isomorphism of $\mathbb{G}_f$-modules, it is convenient to assume that $\mathcal{A}_{\mathcal{G}}$ acts on E by a character ζ_E, which is automatically defined over the rationals (because it is an algebraic character and the torus $\mathcal{A}_{\mathcal{G}}$ is $\mathbb{Q}$-split). and defines an element of $\check{\mathfrak{a}}_{\mathcal{G}}$. For a $\mathbb{G}_f$-module M, we denote by $M(\zeta_E)$ the twist of M in which the action of g on M is multiplied by the factor $\mathrm{e}^{\langle H_{\mathcal{G}}(g), \zeta_E\rangle}$. Then it is easy to see that the following holds:

THEOREM 4. – *If $\mathcal{A}_{\mathcal{G}}$ acts on E by a character ζ_E, then* (16) *defines an isomorphism of $\mathbb{G}_f$-modules*

$$H^*_\rho\big(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G}/K, \boldsymbol{E}\big) \cong H^*_{(\mathfrak{m}_{\mathcal{G}},K)}\big(S_\rho\big(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G}\big) \otimes E\big)(\zeta_E), \tag{17}$$

where the twist (ζ_E) *was defined above.*

We will denote the union

$$\bigcup_\rho S_\rho(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G}) \tag{18}$$

over all admissible weight functions ρ by $S_\infty(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$. Instead of the weighted L_2-space, one often considers the space $C^\infty_{\mathrm{umg}}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$ consisting of all $\mathbb{K}$-finite C^∞-functions f such that there exists a $\lambda \in \check{\mathfrak{a}}^{\mathcal{G}}_o$ such that for any $D \in \mathfrak{U}(\mathfrak{g})$, Df can be bounded by a multiple of ρ_λ. Since any admissible weight function ρ can be majorized by ρ_λ for some λ and by the following Sobolev-type embedding theorem, we have $C^\infty_{\mathrm{umg}} = S_\infty$.

PROPOSITION 2. – *If* $2k > \dim(\mathfrak{g}/(\mathfrak{k} + \mathfrak{a}_{\mathcal{G}}))$ *and* $f \in W^k_\rho$, *then f is bounded by a multiple of $\rho^{-1}\rho_{\rho_o}$, where ρ_o is one half the sum of the positive roots of $\mathcal{A}_o$ and ρ_{ρ_o} is defined by* Proposition 2.1.1.

Proof. – It is clear from (10) that multiplication by ρ^{-1} defines an isomorphism of topological vector spaces between the unweighted Sobolev space W^k and W^k_ρ. Therefore, we may assume $\rho = 1$.

Fix a finite set $\boldsymbol{S}$ of $\mathbb{K}$-types, and let $\mathbb{K}^f \subset \mathbb{K}_f$ be the smallest open compact subgroup acting trivially on all the $\mathbb{K}_f$-representations in $\boldsymbol{S}$. Let $\Omega \subset \mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G}/\mathbb{K}^f$ be a compact domain with C^∞-boundary. We assume that Ω is so large that its image in $\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o P_o\backslash\mathbb{G}/\mathbb{K}^f$ contains a neighbourhood of the set

$$\big\{g \in \mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o P_o\backslash\mathbb{G}/\mathbb{K}^f \bigm| \langle\check{\alpha}, H(g)\rangle = D \text{ for } \alpha \in \Delta_o\big\},$$

where D is a sufficiently small real number. Then it suffices to prove that

$$|f(\exp(\lambda)g)| \le c_{\boldsymbol{S},\Omega}\big\|f\big|W^k\big\|\mathrm{e}^{\langle\lambda,\rho_o\rangle} \tag{19}$$

for $g \in \Omega$ and $\lambda \in \mathfrak{a}^+_o$, where the constant c depends only on $\boldsymbol{S}$ and the domain Ω.

For any $g \in \mathbb{G}/\mathbb{K}^f$ and $\lambda \in \mathfrak{a}_o^+$, we will prove that the cardinality of the intersection of the G-orbit of g with $\exp(\lambda)\Omega$ satisfies the following bound:

$$\#(\exp(\lambda)\Omega \cap Gg) \leq d_{\boldsymbol{S},\Omega} e^{\langle \lambda, 2\rho_o \rangle}. \tag{20}$$

Since we have fixed a finite set of $\mathbb{K}$-types, the restriction of f to Ω can be viewed as a C^∞-section of a finite-dimensional vector bundle over $\widetilde{\Omega} = \Omega/K_\infty$. Once it is proved, (20) will imply

$$\left\| f(\exp(\lambda)\cdot) \middle| W^k(\Omega) \right\| \leq e_{\boldsymbol{S},\Omega} e^{\langle \lambda, 2\rho_o \rangle} \left\| f \middle| W^k(G\mathcal{A}_G(\boldsymbol{R})^o \backslash \mathbb{G}/\mathbb{K}^f) \right\|,$$

which implies (20) by the Sobolev embedding theorem for the compact manifold with boundary $\widetilde{\Omega}$, provided of course that $2k > \dim(\mathfrak{g}/(\mathfrak{k} + \mathfrak{a}_G))$, the dimension of the manifold $\widetilde{\Omega}$.

To prove (20), let $\widetilde{\mathfrak{S}} \subset \mathcal{A}_G(\boldsymbol{R})^o \backslash \mathbb{G}/\mathbb{K}^f$ be a Siegel domain for $\mathcal{P}_o$ such that $G\mathfrak{S} = \mathcal{A}_G(\boldsymbol{R})^o \backslash \mathbb{G}/\mathbb{K}^f$ and $N_o\widetilde{\mathfrak{S}} \supset \exp \mathfrak{a}^+ \Omega$. Since $\widetilde{\mathfrak{S}}$ is a fundamental domain [Bor66], Theorem 1.10, the number of $\gamma \in G$ with $\gamma\widetilde{\mathfrak{S}} \cap \widetilde{\mathfrak{S}} \neq \emptyset$ is finite. Let $\{\gamma_1, \ldots, \gamma_N\}$ be the set of elements with this property. It is sufficient to prove (20) for all $g \in \widetilde{\mathfrak{S}}$. If $\gamma g \in \exp(\lambda)\Omega$ and $\lambda \in \mathfrak{a}_o^+$, then by our selection of $\widetilde{\mathfrak{S}}$ there exists $\nu \in N_o$ such that $\nu\gamma g \in \widetilde{\mathfrak{S}}$, hence $\nu\gamma \in \{\gamma_1, \ldots, \gamma_N\}$. Assume that $1 \leq i \leq N$ and

$$M_i = \{\gamma \in N_o\gamma_i | \gamma\widetilde{\mathfrak{S}} \cap \exp(\lambda)\Omega \neq \emptyset\}$$

is not empty. We will prove $\#M_i \leq \widetilde{c} e^{\langle 2\rho_o, \lambda \rangle}$, where $\widetilde{c}$ depends only on Ω. This proves (20). Let $\delta_i \in M_i$. If $\delta \in M_i$, then $\delta = \nu\delta_i$ for some $\nu \in N_o$, and $\delta g \in \exp(\lambda)\Omega \cap \nu\exp(\lambda)\Omega$. By Lemma 3 below, the number of all ν for which this is possible has a bound of the form described above. □

LEMMA 3. – *There exists a constant $\widetilde{c}$ such that for every $\lambda \in \mathfrak{a}_o^+$ the number of $\nu \in N_o$ with $\nu\exp(\lambda)\Omega \cap \exp(\lambda)\Omega \neq \emptyset$ is bounded by $\widetilde{c} e^{\langle 2\rho_o, \lambda \rangle}$.*

Proof. – By the compactness of Ω and $\mathbb{K}^f$, the finite adelic part of ν must belong to a compact subgroup which is independent of λ. This means that $\exp^{-1}(\nu)$ belongs to a fixed lattice in $\mathfrak{n}_o$. Also by the compactness of Ω, $\exp^{-1}(\nu)$ must belong to

$$\mathrm{Ad}(\exp(\lambda)) \cdot (\text{some ball in } \mathfrak{n}_o). \tag{21}$$

Since λ belongs to the positive Weyl chamber, all eigenvalues of $\mathrm{Ad}(\exp(\lambda))$ on $\mathfrak{n}_o$ are ≥ 1, hence the number of lattice points in (21) is bounded by a multiple of $e^{\langle \lambda, 2\rho_o \rangle}$, by the well-known argument of Gauß on counting lattice points in a Ball. □

3. Decomposition according to classes of associate parabolic subgroups

The aim of this chapter is to prove that the decomposition of $L_2(G\mathcal{A}_G(\boldsymbol{R})^o \backslash \mathbb{G})$ according to classes of associate parabolic subgroups (*cf.* [Lan76], Lemma 4.6.) remains valid with $L_2(G\mathcal{A}_G(\boldsymbol{R})^o \backslash \mathbb{G})$ replaced by $S_\rho(G\mathcal{A}_G(\boldsymbol{R})^o \backslash \mathbb{G})$. A similar decomposition for functions of moderate growth has also been proved by Langlands and written down, but never published, by Borel. The first published proof of Langlands' result seems to be in [Cas89], Theorem 4.6, which is dual to the things described by Borel. These proofs do not apply to the spaces $S_\rho(G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G})$.

We recall that two rational parabolic subgroups are called associate if their split components are $\mathcal{G}(\boldsymbol{Q})$-conjugate. A class of associate parabolic subgroups $\{P\}$ is an equivalence class for this equivalence relation. For formal reasons, the class $\{G\}$ consisting only of $\mathcal{G}$ itself will also be considered to be a class of associate parabolic subgroups. Our aim is to prove that $S_\rho(G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G})$ is the sum over all classes of associate parabolic subgroups $\{P\}$ of its subspaces $S_\rho(G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G})_{\{P\}}$. To define these subspaces, we first need a straightforward modification of a well-known inequality of Gel'fand. For a locally integrable function f on $G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G}$ and a rational parabolic subgroup $\mathcal{P} = \mathcal{MAN}$, the constant term of f along $\mathcal{N}$ is defined by

$$f_\mathcal{N}(g) = \int_{N\backslash\mathbb{N}} f(ng)\,dn, \tag{1}$$

where the Haar measure is normalized by

$$\int_{N\backslash\mathbb{N}} dn = 1.$$

Note that the formation of the constant term is compatible with the $(\mathfrak{g}, K, \mathbb{G}_f)$-module structure on the space of C^∞-functions. We need Gel'fand's inequality in the following form.

THEOREM 5. – *Let ρ and ρ' be admissible (in the sense of* (2.1.1)*) weight functions on $G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G}$. Also, let us assume that D has been chosen small enough such that $G\mathfrak{S}(D) = \mathbb{G}$* (cf. (1.4)). *Let $T \geq D$ be a real number. Then if $f \in S_{\rho'}(G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G})$ and if*

$$\|f|\mathcal{P}, A, \rho\|^2 = \int_{P_o\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathfrak{S}(\mathcal{P},D,T)} \rho(g)^2 \mid (Af_\mathcal{N})(g) \mid^2\, dg$$

is finite for every $A \in \mathfrak{U}(\mathfrak{g})$, and every standard rank one parabolic subgroup $\mathcal{P} = \mathcal{MAN}$, then $f \in S_\rho(G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G})$. Moreover, this assertion holds uniformly in the sense that for every set of weight functions satisfying (2.11) *uniformly and for every finite set S of $\mathbb{K}$-types, there exist constants C_m and M such that*

$$\|f|W_\rho^{2m}\| \leq C_m\left(\left\|f\middle|W_{\rho'}^{2(m+M)}\right\| + \sum_\mathcal{P}\sum_{i=0}^{m} \|f|\mathcal{P}, \omega^i, \rho\|\right) \tag{2}$$

where the sum is over all rank one parabolic subgroups, holds whenever ρ' and ρ belong to this given set of weight functions and f is S-isotypical.

Remark. – If $\mathcal{G}$ is a reductive group over $\mathbb{Q}$, we denote by $\mathbb{G}^1 \subset \mathbb{G}$ the subgroup of all $g \in \mathbb{G}$ with $\prod_v |\chi(g)|_v = 1$ for all characters ξ of $\mathcal{G}$ defined over $\mathbb{Q}$, where the product is taken over all valuations of $\mathbb{Q}$. It is easy to see that we have a unique decomposition

$\mathbb{G} = \mathcal{A}_{\mathcal{G}}(\mathbb{R})^o\mathbb{G}^1$. Therefore, results about functions of $\mathbb{G}^1$ are equivalent to results about functions on $\mathcal{A}_{\mathcal{G}}(\mathbb{R})^o\backslash\mathbb{G}$. We will often use this tacitly when it is more convenient to apply Theorem 5 and Theorem 6 to functions on $\mathbb{G}^1$ than to use them as they are stated.

We defer the proof of this theorem until the end of this chapter. Recall that a function is called cuspidal if all its constant terms vanish (it is easy to see that it is sufficient to require this vanishing for rank one parabolic subgroups). In this case, Theorem 5 implies that f belongs to all the spaces $S_\rho(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$ (*i.e.*, f is rapidly decreasing) if it belongs to one of them. We define

$$S_\rho(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})_{\{P\}} \tag{3}$$

as the space of those $f \in S_\rho(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$ such that for every parabolic subgroup $\mathcal{P}$ which is not in $\{P\}$ and every $g \in \mathbb{A}_{\mathcal{P}}\mathbb{K}$, $f_{\mathcal{N}}(.g)$ is orthogonal to the space of rapidly decreasing (or, which amounts to an equivalent condition, moderately increasing) cuspidal functions on $L\backslash\mathbb{L}^1$. By Theorem 5, $S_\rho(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})_{\{G\}}$ is the space of rapidly decreasing (or moderately increasing) cuspidal functions on $G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G}$, it is independent of ρ.

The aim of this section is to prove

THEOREM 6. – *We have a decomposition*

$$S_\rho(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G}) = \bigoplus_{\{P\}} S_\rho(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})_{\{P\}}. \tag{4}$$

Moreover, this decomposition holds uniformly in the following sense: Let $\Pi_{\{P\}}$ be the projection to the $\{P\}$-component, then for every set of weight functions which satisfy (2.1.1) *uniformly and for every finite set S of $\mathbb{K}$-types, there exist constants M and C_m such that*

$$\left\|\Pi_{\{P\}}f\middle|W_\rho^{2m}\right\| \leq C_m\left\|f\middle|W_\rho^{2(m+M)}\right\|.$$

holds for all weight functions ρ in the given set and for all S-isotypical f.

Proof. – We proceed by induction on $r = \dim(\mathfrak{a}_o^{\mathcal{G}})$. If $r = 0$, the theorem becomes tautological because there is only one class of associate parabolic subgroups. Let us assume that the theorem is known for the Levi components of all parabolic subgroups of $\mathcal{G}$, we want to prove the assertion for $\mathcal{G}$. We proceed in several steps.

1 Since functions in S_ρ have uniformly moderate growth by Proposition 2.3.2, a well-known lemma of Langlands [Lan76], Lemma 3.7. (or [HC68], Theorem 4) implies that the intersection of any two of the summands in (4) is zero. It is therefore sufficient to prove that these summands actually generate the whole space, together with giving an estimate for the projection to each of the summands. Also, the analogue of (4) for the unweighted L_2-space is known (*cf.* [Lan76], Lemma 4.6.), consequently the assertion is true in the unweighted case $\rho = 1$. Next we treat the case $\rho \leq 1$.

2 Let $S_{\rho,k,D,T}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$ be the closed subspace of all functions $f \in S_\rho(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$ such that for every standard parabolic subgroup $\mathcal{P} = \mathcal{MAN}$ of rank k, the constant term $f_{\mathcal{N}}(g)$ vanishes for $g \in \mathfrak{S}(\mathcal{P}, D, T)$. Recall that the last domain has been defined by (2.1.3). Now let us assume $\rho \leq 1$. By Theorem 5, we have $S_{\rho,1,D,T}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G}) = S_{1,1,D,T}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$. Because, as we mentioned in Step 1, the theorem is true for $\rho = 1$ and because of $S_1(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G}) \subseteq S_\rho(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$, this implies that every $f \in S_{\rho,1,D,T}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$ decomposes as in (4), with the summands bounded by f. Therefore it suffices to prove, for every $k > 1$ and every real number T, the existence of a real number T' and of continuous operators

$$(\alpha) \qquad Q_k^{\{P\}}: S_{\rho,k,D,T}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G}) \to S_\rho(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})_{\{P\}}$$

such that

$$(\beta) \qquad f - \sum_{\{P\}} Q_k^{\{P\}}(f) \in S_{\rho,k-1,D,T'}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G}).$$

To construct such operators, we proceed in the following way. Let χ be a C^∞-function on $\boldsymbol{R}$ such that $\chi(t) = 0$ if $t < 1$ and $\chi(t) = 1$ if $t > 2$. Let $\mathcal{P} = \mathcal{MAN}$ be a standard parabolic subgroup and $\mathcal{L} = \mathcal{MA}$. For each $\alpha \in \Delta_o - \Delta_o^{\mathcal{P}}$, let $\rho_{\alpha|_{\mathfrak{a}_o^{\mathcal{P}}}}$ be the weight function on

$$\mathcal{A}_{\mathcal{P}}(\boldsymbol{R})^o L\backslash\mathbb{L}/\mathbb{K}_{\mathcal{L}}$$

($\mathbb{K}_{\mathcal{L}}$ = image of $\mathbb{K} \cap \mathbb{P}$ in $\mathbb{L}$) associated to $\alpha\,|_{\mathfrak{a}_o^{\mathcal{P}}}$. We put

$$(5) \qquad \varphi_{\mathcal{P}}(nlk) = \prod_{\alpha\in\Delta_o-\Delta_o^{\mathcal{P}}} \chi(v e^{\langle H_{\mathcal{P}}(l),\alpha\rangle} \rho_{\alpha|_{\mathfrak{a}_o^{\mathcal{P}}}}(l)).$$

For any given real number T'', v may be chosen small enough such that

$$\operatorname{supp}\varphi_{\mathcal{P}} \subset P\mathfrak{S}(\mathcal{P}, D, T'').$$

Also, it follows from the special property of ρ_α verified in Proposition 2.1.1 that for every standard parabolic subgroup $\mathcal{R} = \mathcal{M}_{\mathcal{R}}\mathcal{A}_{\mathcal{R}}\mathcal{N}_{\mathcal{R}}$ there exists a real number T' such that $\varphi_{\mathcal{P}}$ is $\mathbb{N}_{\mathcal{R}}$-invariant on $\mathfrak{S}(\mathcal{R}, D, T')$.

For a standard parabolic subgroup $\mathcal{P} = \mathcal{MAN}$ and a function ψ on $P\backslash\mathbb{G}$ we put formally

$$(6) \qquad E_{\mathcal{P}}^{\mathcal{G}}\psi(g) = \sum_{\gamma\in P\backslash G} \psi(\gamma g).$$

This is well-defined, for instance, if the support of ψ is contained in a Siegel domain, or compact.

We define

$$Q_k(f) = \sum_{\mathcal{P}} E_{\mathcal{P}}^{\mathcal{G}}\varphi_{\mathcal{P}} f_{\mathcal{N}} = \sum_{\mathcal{P}}\sum_{\gamma\in P\backslash G} \varphi_{\mathcal{P}}(\gamma g) f_{\mathcal{N}}(\gamma g).$$

for $f \in S_{\rho,k,D,T}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$, where the sum is over all parabolic subgroups $\mathcal{P}$ of rank $k-1$, and claim that for such f

$$(\gamma) \qquad f - Q_k(f) \in S_{\rho,k-1,D,T'}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$$

provided that v (in the definition of $\varphi_{\mathcal{P}}$) was chosen small enough and T' large enough.

For a parabolic subgroup $\mathcal{P} = \mathcal{M}\mathcal{A}\mathcal{N}$ of rank $k-1$ and a class of associate parabolic subgroups $\{^{\psi}P\}$ for $\mathcal{M}$, let

$$(\varphi_{\mathcal{P}} f_{\mathcal{N}})_{\{^{\psi}P\}}(g)$$

be the function on $P\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^{o}\backslash\mathbb{G}$ such that for every $g \in \mathbb{A}_{\mathcal{P}}\mathbb{K}$ the function

$$(\varphi_{\mathcal{P}} f_{\mathcal{N}})_{\{^{\psi}P\}}(mg) \quad m \in L\backslash\mathbb{L}^{1}$$

is the projection of $(\varphi_{\mathcal{P}} f_{\mathcal{N}})(mg)$ to the space of functions of type $\{^{\psi}P\}$ on $L\backslash\mathbb{L}^{1}$. (By Remark 1, it is easy to see that the induction assumption implies that this projection exists.) Our second claim is that the series

$$E^{\mathcal{G}}_{\mathcal{P}}\big((\varphi_{\mathcal{P}} f_{\mathcal{N}})_{\{^{\psi}P\}}\big) \tag{δ}$$

converges to an element of $S_{\rho}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^{o}\backslash\mathbb{G})_{\{P\}}$. If we take this for granted, we may put

$$Q^{\{P\}}_{k} f = \sum_{\mathcal{P}} \sum_{\{^{\psi}P\}} E^{\mathcal{G}}_{\mathcal{P}}\big((\varphi_{\mathcal{P}} f_{\mathcal{N}})_{\{^{\psi}P\}}\big).$$

The first sum is again over all standard parabolic subgroups $\mathcal{P}$ of rank $k-1$, and the second sum is over all classes of associate parabolic subgroups $\{^{\psi}P\}$ for $\mathcal{M}$ such that for $\mathcal{R} \in \{^{\psi}P\}$, the parabolic subgroup $\mathcal{R}\mathcal{A}\mathcal{N}$ belongs to $\{P\}$. It follows from (δ) that this is an operator as in (α), and by the induction assumption we have

$$\sum_{\{P\}} Q^{\{P\}}_{k} = Q_{k} \tag{ε}$$

such that (β) is a consequence of (γ) provided that υ (in the definition of $\varphi_{\mathcal{P}}$) was chosen small enough. To complete the induction step for $\rho \leq 1$, it remains to prove our claims concerning (γ) and (δ).

3 To prove (γ) for υ small enough, we consider a parabolic subgroup $\mathcal{P}' = \mathcal{M}'\mathcal{A}'\mathcal{N}'$ of rank $k-1$ and assume that $g \in \mathfrak{S}(\mathcal{P}', D, T')$, where T' has been chosen large enough (depending on υ!) such that $\varphi_{\mathcal{P}'} = 1$ on $\mathfrak{S}(\mathcal{P}', D, T')$. We have

$$Q_{k}(f)_{\mathcal{N}'} = \sum_{\mathrm{rank}(\mathcal{P})=k-1} \sum_{\gamma \in P\backslash G} \int_{N'\backslash\mathbb{N}'} \varphi_{\mathcal{P}}(\gamma n' g) f_{\mathcal{N}}(\gamma n' g)\, dn'.$$

By Theorem 1, there exists a real number $T'' > T$ such that $g \in \mathfrak{S}(\mathcal{P}, D, T'')$ and $\gamma g \in \mathfrak{S}(D)$ implies $\gamma \in P$ for any standard parabolic subgroup $\mathcal{P}$. We assume that υ was chosen so small that $\operatorname{supp}\varphi_{\mathcal{P}} \subset P\mathfrak{S}(\mathcal{P}, D, T'')$. Then if γ gives a contribution to the second sum in the above equation, we have $\gamma \in P$ because $\gamma n' g \in P\mathfrak{S}(\mathcal{P}, D, T'')$ and $\gamma^{-1}\gamma n' g = n' g \in \mathfrak{S}(D)$. Consequently, the expression simplifies to

$$\int_{N'\backslash\mathbb{N}'} \varphi_{\mathcal{P}}(n' g) f_{\mathcal{N}}(n' g)\, dn' = \varphi_{\mathcal{P}}(g) \int_{N'\backslash\mathbb{N}'} f_{\mathcal{N}}(n' g)\, dn'.$$

The equation holds for sufficiently large T' because then the special property of the weight functions constructed by Proposition 2.1.1 makes sure that $\varphi_{\mathcal{P}}$ is $\mathbb{N}'$-invariant on

$\mathfrak{S}(\mathcal{P}', D, T')$. If $\mathcal{P} = \mathcal{P}'$, then the first factor on the right hand side is one by the choice of T' and we recover the constant term of f along $\mathcal{N}'$. Otherwise, the second factor of the right hand side is the constant term of f along the radical of the parabolic subgroup $\mathcal{P} \cap \mathcal{P}'$, which vanishes at $g \in \mathfrak{S}(\mathcal{P}', D, T') \cap \mathfrak{S}(\mathcal{P}, D, T'') \subset \mathfrak{S}(\mathcal{P} \cap \mathcal{P}', D, T)$. The proof of (γ) is complete.

4 To prove the convergence of (δ), we construct weight functions Θ_S on $P\mathbb{N}\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G}/\mathbb{K}$ as follows: Let

$$\Theta'_S(g) = \max(\mathrm{e}^{-SD(g)}, 1),$$

where $D(g)$ is the largest real number such that $g \in P\mathfrak{S}(D)$. This weight function on $P\mathbb{N}\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G}/\mathbb{K}$ satisfies an obvious analogue of (2.1.2), and consequently it can be modified to an equivalent weight function Θ_S which satisfies the obvious analogue of (2.1.1) as well. We define a locally convex $(\mathfrak{g}, K, \mathbb{G}_f)$-module $S_{\rho,S}(\mathcal{P})$ consisting of all $\mathbb{K}$-finite C^∞-functions f on $P\mathbb{N}\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G}$ such that

$$\int_{P\mathbb{N}\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G}} \mid \rho^{(\mathcal{N})}(g)\Theta_S(g)(Af)(g) \mid^2 dg < \infty,$$

where $\rho^{(\mathcal{N})}(g) = \max_{n\in\mathbb{N}}\rho(ng)$, for every $A \in \mathfrak{Z}(\mathfrak{g})$. Since the weight functions on $\mathbb{L}^1$ $\rho^{(\mathcal{N})}(lg)\Theta_S(lg)$ satisfy (2.1.1) on $\mathbb{L}^1$ uniformly for $g \in \mathbb{A}_\mathcal{P}\mathbb{K}$, we may apply the induction assumption to get a direct sum decomposition over all classes of associate parabolic subgroups $\{{}^\psi P\}$ of $\mathcal{M}$

$$S_{\rho,S}(\mathcal{P}) = \bigoplus_{\{{}^\psi P\}} S_{\rho,S}(\mathcal{P})_{\{{}^\psi P\}}$$

into subspaces of functions f such that $f(mg)$ is a function of type $\{{}^\psi P\}$ on $L\backslash\mathbb{L}^1$ for every $g \in \mathbb{A}_\mathcal{P}\mathbb{K}$. Since $(\varphi_\mathcal{P} f_\mathcal{N})$ lies in $S_{\rho,S}(\mathcal{P})$ for every real number S (its support is contained in the P-orbit of a Siegel domain where Θ_S is bounded), this implies that the functions $(\varphi_\mathcal{P} f_\mathcal{N})_{\{{}^\psi P\}}$ also belong to $S_{\rho,S}(\mathcal{P})$. Consequently, it suffices to establish the convergence of $E_\mathcal{P}^\mathcal{Q} h$ for $h \in S_{\rho,S}(\mathcal{P})$. The following lemma of Langlands ([Lan76], Lemma 2.11) easily establishes the absolute convergence of the series with respect to each of the semi-norms

$$\int_{\mathfrak{S}(D)} \mid \rho(g)(A\phi)(g) \mid^2 dg,$$

which is sufficient to establish (δ) and (ε). Consequently, the induction argument in the case of weight functions which are bounded from above is finished.

LEMMA 1. – *There exists a constant S such that for each $g \in \mathbb{G}$, the number of $\gamma \in P\backslash G$ with $\gamma g \in \mathfrak{S}(D)$ is bounded by a multiple of* $\mathrm{e}^{SD(g)}$.

This proves the convergence of (δ). It remains to prove that the function defined by (δ) belongs to $S_\rho(G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G})_{\{P\}}$. It suffices to do this for $E_\mathcal{P}^\mathcal{Q}(\psi)$, where ψ is an arbitrary element of $S_{\rho,S}(\mathcal{P})_{\{{}^\psi P\}}$. We may also assume that ψ is rapidly decreasing (in the sense that $\psi \in S_\tau(L_\mathcal{P}\backslash\mathbb{L}_\mathcal{P})$ for every weight function τ on $L_\mathcal{P}\backslash\mathcal{L}_\mathcal{P}$ satisfying (2.1.2)). Indeed, it follows from the induction assumption and the density of rapidly decreasing functions in $S_{\rho,S}(\mathcal{P})$ that the subspace of rapidly decreasing functions is dense in $S_{\rho,S}(\mathcal{P})_{\{\psi P\}}$. We now have to prove that, for rapidly decreasing functions ψ of type $\{^\psi P\}$ on $L_\mathcal{P}\backslash\mathbb{L}_\mathcal{P}$, $E^\mathcal{G}_\mathcal{P}(\psi)$ is of type ψ.

This can be done in the same way as the proof of the similar assertion about cuspidal Eisenstein series. Indeed, the same considerations as in [MW95], II.1.7.(4) lead to

$$(7)\qquad (E^\mathcal{G}_\mathcal{P}\psi)_{N_\mathcal{Q}}(lak) = \sum_{w\in W^\bullet_{\mathcal{L}_\mathcal{P},\mathcal{L},\mathcal{Q}}} \sum_{\mu\in L_\mathcal{Q}\cap wPw^{-1}\backslash L_\mathcal{Q}} \int_{N_\mathcal{Q}\cap wPw^{-1}\backslash\mathbb{N}_\mathcal{Q}} \psi(w^{-1}\mu nlak)\,dn$$

for $l \in \mathbb{L}^1$, $a \in \mathbb{A}_\mathcal{Q}$, and $k \in \mathbb{K}$ and where $W^\bullet_{\mathcal{L}_\mathcal{P},\mathcal{L}_\mathcal{Q}}$ is a certain subset of a set of representatives for the Weyl group of $\mathcal{A}_o$ in $\mathcal{G}(\mathbb{Q})$. In the same way as in [MW95], II.1.7, one sees that the summand belonging to w in (7) vanishes unless $w\mathcal{L}_\mathcal{P}w^{-1} \subseteq \mathcal{L}_\mathcal{Q}$, in which case $\widetilde{\mathcal{P}} = \mathcal{L}_\mathcal{Q} \cap w\mathcal{P}w^{-1}$ is a parabolic subgroup of $\mathcal{L}_\mathcal{Q}$ and the summand belonging to w in (7) equals

$$(8)\qquad \big(E^{\mathcal{L}_\mathcal{Q}}_{\widetilde{\mathcal{P}}}\widetilde{\psi}\big)(l),$$

where

$$\widetilde{\psi}(l) = \int_{NQ\cap wPw^{-1}\backslash\mathbb{N}_Q} \psi(w^{-1}nlak)\,dn.$$

But functions of type (8) are easily seen to be orthogonal to the space of cusp forms on $L\backslash\mathbb{L}^1$ unless $\widetilde{\mathcal{P}} = \mathcal{L}_\mathcal{Q}$, in which case $w\mathcal{L}_\mathcal{P}w^{-1} = \mathcal{L}_\mathcal{Q}$, so that $\mathcal{P}$ and $\mathcal{Q}$ are associate. This proves our claim $E^\mathcal{G}_\mathcal{P}(\psi) \in S_\rho(GA_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G})_{\{P\}}$.

5 The case of weight functions which are bounded from below is dual to the previous case. To be more precisely we consider a weight function ρ which is bounded from above and note that what we have proved in the previous step is equivalent to the existence of a constant k such that the projection to functions of type $\{P\}$ is continuous from $W^{2(m+k)}_\rho$ to W^{2m}_ρ. By (2.3.11), for every finite set X of $\mathbb{K}$-types there exists an isomorphism from $(W^{2m}_\rho)_X$ to $(W^{2(m+l)}_\rho)_X$ given by a certain l-th order polynomial in the Casimir operator. Since this polynomial commutes with the projection to the space of functions of type $\{P\}$, the integer k is independent of m. By duality as discussed after the proof of (2.3.11), we conclude that the projection to the space of functions of type $\{P\}$ is also continuous from $W^{2(m+k)}_{\rho^{-1}}$ to $W^{2m}_{\rho^{-1}}$. This completes the induction argument in the case of weight functions which are bounded fro below.

To treat the case of weight functions ρ which are not bounded from any side (like the weight function $\rho\Theta_S$ we used in step 4), we use interpolation. If ρ is any weight function satisfying (2.1.1), then $\min(\rho^2,1)$ and $\max(\rho^2,1)$ satisfy (2.1.2), hence they could be replaced by equivalent weight functions satisfying (2.1.1) and our previous considerations imply the existence of a k such that the projection to the space of functions of type $\{P\}$ is continuous from $W^{2(m+k)}_\tau$ to W^{2m}_τ if τ is one of these two functions. But (2.3.15) implies

$$[W^{2l}_{\min(\rho^2,1)}, W^{2l}_{\max(\rho^2,1)}]_{1/2} = W^{2l}_\rho,$$

hence the projection to the space of functions of type $\{P\}$ is also continuous from $W_\rho^{2(m+k)}$ to W_ρ^{2m}. This proves the induction argument in the general case, and we are through. $\square$

It remains to prove our version of Gel'fand's estimate. The argument for doing this is standard.

Proof of Theorem 5. – For a rank one parabolic subgroup $\mathcal{P} = \mathcal{MAN}$, let $C_{\mathcal{P}}$ be the set of those $\xi \in \mathfrak{a}_o^{\mathcal{G}}$ such that $\langle \check{\beta}, \xi \rangle \geq D$ for all $\beta \in \Delta_o$ and such that

$$\langle \check{\alpha}, \xi \rangle = \max_{\beta \in \Delta_o} \langle \check{\beta}, \xi \rangle,$$

where α is the unique element of $\Delta_o - \Delta_o^{\mathcal{P}}$. We put

$$\mathfrak{S}(D)_{\mathcal{P}} = \{ g \in \mathfrak{S}(D) \mid H(g) \in C_{\mathcal{P}} \}.$$

Then $\mathfrak{S}(D)$ is the union of the domains $\mathfrak{S}(D)_{\mathcal{P}}$.

Let $\nu_1, \ldots, \nu_k$ be a base of $\mathfrak{n}$, and assume that ν_i is an eigenvector for the positive root $\alpha_i \in \Phi_o$. By the Sobolev inequalities, for every open subgroup $U \subset \mathbb{N}_f$ there exists a constant c_U with

$$\int_{N\backslash\mathbb{N}} | h(n) |^2 \, dn \leq c_U \left(\left| \int_{N\backslash\mathbb{N}} h(n) \, dn \right|^2 + \sum_{i=1}^{k} \int_{N\backslash\mathbb{N}} | (\nu_i h)(n) |^2 \, dn \right)$$

for all h which are right U-invariant. Iterating this inequality and using the fact that $\int_{N\backslash\mathbb{N}} (\nu_i h)(n) \, dn = 0$, we get

$$\int_{N\backslash\mathbb{N}} | h(n) |^2 \, dn \leq c_{k,U} \left(\left| \int_{N\backslash\mathbb{N}} h(n) \, dn \right|^2 + \sum_{i_1,\ldots,i_l} \int_{N\backslash\mathbb{N}} | (\nu_{i_1} \ldots \nu_{i_l} h)(n) |^2 \, dn \right).$$

Let $\Omega \subset \mathbb{L}_o\mathbb{K}$ be a compact subset such that $\mathfrak{S}(D)_{\mathcal{P}} \subset P_o \mathbb{N} \exp(C_{\mathcal{P}})\Omega$. For every finite set S of $\mathbb{K}$-types there exists an open subgroup $U \subset \mathbb{N}_f$ such that for every S-isotypical f, every $\xi \in C_{\mathcal{P}}$, and every $\omega \in \Omega$ the function $h(n) = f(n\mathrm{e}^{\xi}\omega)$ is U-invariant. We get

(9)
$$\begin{aligned}
&\int_{\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^\circ P_o \backslash \mathfrak{S}(D)_{\mathcal{P}}} \rho(g)^2 \, | f(g) |^2 \, dg \leq \\
&\leq c \int_{N\backslash\mathbb{N}} dn \int_{C_{\mathcal{P}}} d\xi \int_{\Omega} d\omega \mathrm{e}^{-2\langle \rho_o, \xi \rangle} \rho(\mathrm{e}^{\xi}\omega)^2 \, | f(n\mathrm{e}^{\xi}\omega) |^2 \\
&\leq c_{l,U} c \int_{C_{\mathcal{P}}} d\xi \int_{\Omega} d\omega \left(\left| \int_{N\backslash\mathbb{N}} f(n\mathrm{e}^{\xi}\omega) \, dn \right|^2 + \right. \\
&\left. + \int_{N\backslash\mathbb{N}} \sum_{i_1,\ldots,i_l} \left| (\nu_{i_1} \ldots \nu_{i_l} f(\cdot \mathrm{e}^{\xi}\omega))(n) \right|^2 dn \right) \mathrm{e}^{-2\langle \rho_o, \xi \rangle} \rho(\mathrm{e}^{\xi}\omega).
\end{aligned}$$

The second summand is equal to

(10)
$$\mathrm{e}^{-2\langle \rho_o, \xi \rangle} \rho(\mathrm{e}^{\xi}\omega) \sum_{i_1,\ldots,i_l} \mathrm{e}^{-2\sum_{j=1}^{l} \langle \alpha_{i_j}, \xi \rangle} \int_{N\backslash\mathbb{N}} | (Ad(\omega^{-1})\nu_{i_1} \ldots Ad(\omega^{-1})\nu_{i_l} f)(n\mathrm{e}^{\xi}\omega) |^2 \, dn.$$

Since

$$\langle \alpha_i, \xi \rangle \geq \langle \alpha, \xi \rangle + C,$$

where α is the unique element of $\Delta_o - \Delta_o^{\mathcal{P}}$, and since for sufficiently large l $(\rho\, {\rho'}^{-1})(\mathrm{e}^{\xi}\omega)$ is bounded by a multiple of $\mathrm{e}^{l\langle \alpha, \xi \rangle}$, for sufficiently large l (10) can be bounded by a multiple of

$$\sum_{i_1, \dots, i_l} \mathrm{e}^{-2\langle \rho_o, \xi \rangle} \rho'(\mathrm{e}^{\xi}\omega) \int_{N \backslash \mathbb{N}} \mid (Ad(\omega^{-1})\nu_{i_1} \dots Ad(\omega^{-1})\nu_{i_l} f)(n\mathrm{e}^{\xi}\omega) \mid^2 \, dn.$$

We conclude that (9) is bounded by a multiple of

$$\int_{\mathfrak{S}(D)_{\mathcal{P}}} (\rho(g) \mid f_{\mathcal{N}}(g) \mid)^2 + \sum_i (\rho'(g)(A_i f)(g))^2 \, dg,$$

where the A_i form a base of the elements of degree $\leq l$ in $\mathfrak{U}(\mathfrak{g})$. Taking the sum of these inequalities over all rank one parabolic subgroups, and using the fact that $\mathfrak{S}(D)_{\mathcal{P}} - \mathfrak{S}(P, D, T)$ is compact, we arrive at (2) with $m = 0$. To get the general case of (2) from this special case, one uses (2.3.11). □

The proof of Theorem 15 is complete. Q.E.D.

4. The functors $\mathfrak{Fin}^i_{\mathcal{J}}$

The aim of this section is to study certain derived functors which reduce the study of Borel's conjecture to a vanishing assertion. This allows an inductive approach to the conjecture in which part of the analysis is replaced by algebra. The functors are given by cohomology with support in a finite set of the spectrum of the center of the universal enveloping algebra with coefficients in the quasi-coherent sheaf given by a $(\mathfrak{g}, K)$-module. They have independently been studied by W. Casselman, and the spectral sequence (4) is also contained in the section "An algebraic result" of his unpublished notes [Cas].

Let $\mathfrak{g}$ be a real reductive Lie algebra and K a compact Lie group acting on $\mathfrak{g}$ such that the assumptions made in [Vog81], Definition 6.1.1 are satisfied. Let $\mathcal{J}$ be an ideal of finite codimension in $\mathfrak{Z}(\mathfrak{g})$ (the center of the complexification of the universal enveloping algebra). To every $(\mathfrak{g}, K)$-module V we associate

(1) $$\mathfrak{Fin}_{\mathcal{J}} V = \left\{ v \in V \;\middle|\; \text{such that } \mathcal{J}^n v = \{0\} \text{ for some n} \right\}.$$

It is evident that $\mathfrak{Fin}_{\mathcal{J}}$ is a left-exact functor on the category of $(\mathfrak{g}, K)$-modules. Let $\mathfrak{Fin}^i_{\mathcal{J}}$ be its derived functors.

We want to study the relations of this functor to $(\mathfrak{g}, K)$-cohomology (the derived functor of the functor of $(\mathfrak{g}, K)$-invariants, *cf.* [Vog81], Definition 6.1.13) and to parabolic induction of $(\mathfrak{g}, K)$-modules. Here parabolic induction will be defined in the following way: Let $\mathfrak{p} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n} \subset \mathfrak{g}$ be a parabolic subalgebra and $K_{\mathfrak{p}}$ its normalizer in K.

For a $(\mathfrak{m}+\mathfrak{a}, K_{\mathfrak{p}})$-module W (note that $K_{\mathfrak{p}}$ acts on $\mathfrak{m}+\mathfrak{a}$ because it acts on $\mathfrak{p}$ and leaves $\mathfrak{n}$ invariant) we put

$$\operatorname{Ind}_{(\mathfrak{p},K_{\mathfrak{p}})}^{(\mathfrak{g},K)} W = \Gamma_{(\mathfrak{g},K_{\mathfrak{p}})}^{(\mathfrak{g},K)} \, pro_{(\mathfrak{p},K_{\mathfrak{p}})}^{(\mathfrak{g},K_{\mathfrak{p}})} W, \tag{2}$$

where W is viewed as a $(\mathfrak{p}, K_{\mathfrak{p}})$-module with trivial $\mathfrak{n}$-action, *pro* is the functor defined in [Vog81], Definition 6.1.21, and Γ is the Zuckerman functor defined in [Vog81], Definition 6.2.9. Note that (2) is an abuse of language because Vogan also has a functor *ind* defined in a different way than *pro*. We think, however, that the notation used for the left side of (2) is more suggestive. We also mention that while *pro* is an exact functor preserving injectives ([Vog81], Lemma 6.2.22), Γ is only left exact but preserves injective objects ([Vog81], Lemma 6.2.10). However, since we have restricted our attention to a situation where $\mathfrak{p}$ is defined over the real numbers, the derived functors of the composition (2) vanish ([Vog81], Proposition 6.3.5.a)), and (2) is exact and preserves injectives. Actually, [Vog81], Proposition 6.3.5.a) is only formulated for Harish-Chandra-modules, but the proof of the crucial assertion

$$\left(\operatorname{Ind}_{(\mathfrak{p},K_{\mathfrak{p}})}^{(\mathfrak{g},K)}\right) W|_K = \operatorname{Ind}_{K_{\mathfrak{p}}}^{K} \left(W|_{K_{\mathfrak{p}}}\right)$$

(where Ind is the same as in (2)) holds in full generality. The functor $\operatorname{Ind}_{K_{\mathfrak{p}}}^{K}$ can be described as

$$\operatorname{Ind}_{K_{\mathfrak{p}}}^{K} = \left\{ f : K \to X \mid f \text{ is } K\text{-finite and fulfils } f(lk) = lf(k) \text{ for } l \in K_{\mathfrak{p}} \right\}.$$

It follows from this description that it is exact and commutes with direct limits. Since the restriction functors from $(\mathfrak{g}, K)$ to K and from $(\mathfrak{p}, K_{\mathfrak{p}})$ to $K_{\mathfrak{p}}$ are exact and faithful and commute with direct limits, this implies that $\operatorname{Ind}_{K_{\mathfrak{p}}}^{K}$ is exact and commutes with direct limits.

THEOREM 7.

(1) *If V is an injective $(\mathfrak{g}, K)$-module, then so is $\mathfrak{Fin}_{\mathcal{J}} V$.*

(2) *We have*

$$\mathfrak{Fin}_{\mathcal{J}}^{i} V = \operatorname*{colim}_{n} \operatorname{Ext}_{\mathfrak{Z}(\mathfrak{g})}^{i}(\mathfrak{Z}(\mathfrak{g})/\mathcal{J}^{n}, V). \tag{3}$$

More generally if the $\mathfrak{Z}(\mathfrak{g})$-module structure of V extends to the structure of an R-module for a flat $\mathfrak{Z}(\mathfrak{g})$-algebra R, then

$$\mathfrak{Fin}_{\mathcal{J}}^{i} V = \operatorname*{colim}_{n} \operatorname{Ext}_{R}^{i}(R/(R\mathcal{J})^{n}, V).$$

(3) *If E is a finite-dimensional $(\mathfrak{g}, K)$-module and $\mathcal{J}$ the annihilator of $\check{E}$, then there is a spectral sequence*

$$E_2^{p,q} = H_{(\mathfrak{g},K)}^{p}(\mathfrak{Fin}_{\mathcal{J}}^{q}(V) \otimes E) \Rightarrow H_{(\mathfrak{g},K)}^{p+q}(V \otimes E). \tag{4}$$

(4) *Let $\mathfrak{p} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n}$ be a parabolic subalgebra of $\mathfrak{g}$. We extend $\mathfrak{a}$ to a Cartan subalgebra and denote by*

$$\xi \colon \mathfrak{Z}(\mathfrak{g}) \to S(\mathfrak{h})_{\boldsymbol{C}}^{W}$$
$$\xi' \colon \mathfrak{Z}(\mathfrak{m}+\mathfrak{a}) \to S(\mathfrak{h})_{\boldsymbol{C}}^{W'}$$

the Harish-Chandra isomorphisms, where W and W' are the Weyl groups of $(\mathfrak{h}, \mathfrak{g})$ and $(\mathfrak{m} \cap \mathfrak{h}, \mathfrak{m})$ (cf. [Wal88], Theorem 3.2.3.)*. There exists a finite W-invariant subset $X \subset \check{h}_{\boldsymbol{C}}$ such that the radical of $\xi(\mathcal{J})$ consists of all W-invariant polynomials vanishing in X. Let*

(5) $$\mathcal{J}' = \{D \in \mathfrak{Z}(\mathfrak{m}+\mathfrak{a}) \mid \text{such that } \xi'(D) \text{ vanishes on } X + \rho\},$$

where ρ is one half the sum of the roots of $\mathfrak{h}$ in $\mathfrak{n}$. Then we have a canonical isomorphism

$$\mathfrak{Fin}^{i}_{\mathcal{J}}(\mathrm{Ind}^{(\mathfrak{g},K)}_{(\mathfrak{p},K_{\mathfrak{p}})})W \cong \mathrm{Ind}^{(\mathfrak{g},K)}_{(\mathfrak{p},K_{\mathfrak{p}})}\mathfrak{Fin}^{i}_{\mathcal{J}'}W$$

for $(\mathfrak{m}+\mathfrak{a}, K_{\mathfrak{p}})$-modules W.

(5) *Let $\mathfrak{g} = \mathfrak{m}+\mathfrak{a}$, where $\mathfrak{a}$ is contained in the center of $\mathfrak{g}$, $\mathfrak{k} \subseteq \mathfrak{m}$ and K normalizes $\mathfrak{m}$. If*

$$\mathcal{J} = \mathfrak{Z}(\mathfrak{g})(\mathcal{J}_{\mathfrak{a}} + \mathcal{J}_{\mathfrak{m}}),$$

where $\mathcal{J}_{\mathfrak{a}} \subset \mathfrak{Z}(\mathfrak{a})$ and $\mathcal{J}_{\mathfrak{m}} \subset \mathfrak{Z}(\mathfrak{m})$ are ideals of finite codimension, then we have a spectral sequence

(6) $$E_2^{p,q} = \mathfrak{Fin}^{p}_{\mathcal{J}_{\mathfrak{a}}}\mathfrak{Fin}^{q}_{\mathcal{J}_{\mathfrak{m}}}V \Rightarrow \mathfrak{Fin}^{p}+q_{\mathcal{J}}V$$

for all $(\mathfrak{g}, K)$-modules V, where the first $\mathfrak{Fin}$ on the left hand side is computed in the category of $\mathfrak{a}$-modules and the second $\mathfrak{Fin}$ is computed in the category of $(\mathfrak{m}, K)$-modules. In both cases, the result inherits the structure of a $(\mathfrak{g}, K)$-module from V.

(6) *If $\mathcal{I} = \mathcal{J}\mathcal{K}$, where $\mathcal{J}$ and $\mathcal{K}$ are ideals of finite codimension which generate $\mathfrak{Z}(\mathfrak{g})$, then*

$$\mathfrak{Fin}^{p}_{\mathcal{I}} \cong \mathfrak{Fin}^{p}_{\mathcal{J}} \oplus \mathfrak{Fin}^{p}_{\mathcal{K}}.$$

Proof.

Proof of 1. – Let V be an injective $(\mathfrak{g}, K)$-module, $X \subset Y$, and $X \xrightarrow{\phi} \mathfrak{Fin}_{\mathcal{J}}V$. We have to extend ϕ to all of Y. Since $\mathfrak{U}(\mathfrak{g})$ is a Noetherian ring ([Wal88], Lemma 0.6.1), we may assume that X and Y are finitely generated. Indeed, if Y is arbitrary, then by Zorn's lemma there is a maximal extension $\widetilde{X} \xrightarrow{\psi} \mathfrak{Fin}_{\mathcal{J}}V$ of ϕ to some $(\mathfrak{g}, K)$-submodule $\widetilde{X}$ of Y containing X. If $\widetilde{X} \neq Y$, there would be a finite-dimensional K-invariant subspace $Z \subseteq Y$ which is not contained in $\widetilde{X}$. Let $Y_1 = \mathfrak{U}(\mathfrak{g})Z \subset Y$ and $X_1 = Y_1 \cap \widetilde{X}$. As $\mathfrak{U}(\mathfrak{g})$ is Noetherian, X_1 is finitely generated. Since we are assuming that the assertion to be proved holds in the case of finitely generated $(\mathfrak{g}, K)$-modules, there is an extension of $\psi|_{X_1}$ to a morphism $Y_1 \xrightarrow{\widetilde{\psi}} \mathfrak{Fin}_{\mathcal{J}}V$, which can be patched with ψ to a morphism $\widetilde{X} + Y_1 \to \mathfrak{Fin}_{\mathcal{J}}V$. This contradicts our assumption about the maximality of ψ, and proves $\widetilde{X} = Y$.

From now on we assume that X and Y are finitely generated. Then there exists a $k \geq 0$ such that $\phi(\mathcal{J}^k X) = 0$.

Let $Y_l = \mathcal{J}^{k+l}Y + \mathcal{J}^k X$. Then $(Y_l \cap X) \,/\, \mathcal{J}^k X \subseteq X \,/\, \mathcal{J}^k X$. The last $(\mathfrak{g}, K)$-module is obviously finitely generated, consequently it is admissible by [Wal88], Corollary 3.4.7. By [Wal88], Theorem 4.2.1, this implies that it has finite length. Consequently, the sequence of the $Y_l \cap X$ stabilizes for big l. We want to show that $Y_l \cap X = \mathcal{J}^k X$ for big l, and we have seen that to do this it suffices to show that for every irreducible representation σ of K, we have

$$\mathcal{J}^{k+l}Y_\sigma \cap X_\sigma \subset \mathcal{J}^k X_\sigma$$

for big l, where the subscript σ refers to σ-isotypical components. But Y_σ and X_σ are finitely generated $\mathfrak{Z}(\mathfrak{g})$-modules by [Wal88], Theorem 3.4.1, hence the last assertion follows from the Artin-Rees theorem (*cf.* [Ser66], II.5, Théorème 1, p. II-9 or [Wal88], 0.6.3).

Let l be so big that $Y_l \cap X = \mathcal{J}^k X$. Then the morphism $\phi: X \,/\, \mathcal{J}^k X \to V$ can be extended to $\hat{\phi}: Y \,/\, Y_l \to V$, which clearly has its image in $\mathfrak{Fin}_{\mathcal{J}} V$. This provides the desired extension of ϕ.

PROOF OF 2. – Let $\widetilde{\mathfrak{Fin}}^i_{\mathcal{J}}$ be the functor on the right hand side of (3). It suffices to show that every injective $(\mathfrak{g}, K)$-module V is $\widetilde{\mathfrak{Fin}}_{\mathcal{J}}$-acyclic.

We first want to verify that $\widetilde{\mathfrak{Fin}}^i_{\mathcal{J}} X = 0$ for positive i if X is annihilated by a power of $\mathcal{J}$. Without loosing generality we may assume that $\mathcal{J}$ is the product of finitely many maximal ideals. Since $\mathfrak{Z}(\mathfrak{g})$ is a regular ring (*cf.* [Bou68], Ch. V, §5.3., Théorème 3), there exist r ($r = \boldsymbol{C}$-rank of $\mathcal{G}$) elements $\boldsymbol{D} = (D_1, ..., D_r)$ of $\mathfrak{Z}(\mathfrak{g})$ which generate $\mathcal{J}$. By a well-known result of commutative algebra (*cf.* the discussion following the proof of Proposition 3 in [Ser65], Chapter IV.A.2), the i-th cohomology group of the Koszul complex

$$C^*(\boldsymbol{D}, X) = \ldots \to X \otimes \Lambda^k(\boldsymbol{C}^r) \xrightarrow{\wedge(D_1, ..., D_r)} X \otimes \Lambda^{k+1}(\boldsymbol{C}^r) \to \ldots$$

is $\mathrm{Ext}^i_{\mathfrak{Z}(\mathfrak{g})}(\mathfrak{Z}(\mathfrak{g})/\mathcal{J}, X)$. It follows that for any $(\mathfrak{g}, K)$-module X there is a natural isomorphism

$$\widetilde{\mathfrak{Fin}}^*_{\mathcal{J}} X \cong \operatorname*{colim}_n H^*(C^*(\boldsymbol{D}^n, X),) \tag{7}$$

where $\boldsymbol{D}^n = (D_1^n, \ldots, D_r^n)$ and the transition map for the inductive limit of the Koszul complexes maps $x \otimes (e_{i_1} \wedge \ldots \wedge e_{i_r})$ to $(D_{i_1} \ldots D_{i_r}) x \otimes (e_{i_1} \wedge \ldots \wedge e_{i_r})$. If X is annihilated by $\mathcal{J}^k$, then the composition of k transition homomorphisms vanishes if $i > 0$, consequently the inductive limit vanishes for positive i.

If $X = \mathfrak{Fin}_{\mathcal{J}} X$, then X is the inductive limit of $(\mathfrak{g}, K)$-modules which are annihilated by a finite power of $\mathcal{J}$, consequently $\widetilde{\mathfrak{Fin}}^i_{\mathcal{J}} X$ vanishes for positive i.

Let V be an injective $(\mathfrak{g}, K)$-module. We have already verified that $\mathfrak{Fin}_{\mathcal{J}} V$ is also injective. Consequently, $\mathfrak{Fin}_{\mathcal{J}} V$ is a direct summand of V. Consider the resulting embedding

$$V \hookrightarrow \mathfrak{Fin}_{\mathcal{J}} V \oplus \bigoplus_{k=1}^{r} D_k^{-1} V.$$

We claim that the right hand side is $\widetilde{\mathfrak{Fin}}_{\mathcal{J}}$-acyclic. Then V itself must be $\widetilde{\mathfrak{Fin}}_{\mathcal{J}}$-acyclic because being injective it is a direct summand of the right hand side. The first summand on the right hand side is $\widetilde{\mathfrak{Fin}}_{\mathcal{J}}$-acyclic by the previous remarks. The localization $D_k^{-1}V$ is $\widetilde{\mathfrak{Fin}}_{\mathcal{J}}$-acyclic because D_k induces an automorphism of that module, while each element of $\widetilde{\mathfrak{Fin}}^i_{\mathcal{J}}(D_k^{-1}V)$ must be annihilated by a power of D_k.

The proof of the first assertion is complete, and the second assertion is an immediate consequence of the first one.

PROOF OF 3. – This is an immediate consequence of the first part of the theorem we are proving.

PROOF OF 4. – Let

$$\gamma\colon \mathfrak{U}(\mathfrak{g})/(\mathfrak{n}\mathfrak{U}(\mathfrak{g}) + \mathfrak{U}(\mathfrak{g})\mathfrak{n}^{op}) \to \mathfrak{U}(\mathfrak{m}+\mathfrak{a}) \quad (\mathfrak{n}^{op} = \text{opposite of } \mathfrak{n})$$

be the natural projection. It defines an embedding of $\mathfrak{Z}(\mathfrak{g})$ into $\mathfrak{Z}(\mathfrak{m}+\mathfrak{a})$, and we have $\xi = (\xi'\gamma)(.+\rho)$ by the very definition of Harish-Chandra's isomorphism. Consequently, if $D_1,\ldots,D_r$ are elements of $\mathcal{J}$ such that $(D_1,\ldots,D_r)_{\mathfrak{Z}(\mathfrak{g})}$ contains a power of $\mathcal{J}$, then $\gamma(D_1),\ldots,\gamma(D_r)$ satisfy the same property with $\mathcal{J}$ replaced by $\mathcal{J}'$. It follows from the proof of 2. that for any $(\mathfrak{g},K)$-module B there is a natural isomorphism

$$\mathfrak{Fin}^*_{\mathcal{J}}B \cong \operatorname*{colim}_n H^*(C^*(\boldsymbol{D}^n, B),)$$

where $\boldsymbol{D}^n = (D_1^n,\ldots,D_r^n)$ and the transition map for the inductive limit of the Koszul complexes maps $b\otimes(e_{i_1}\wedge\ldots\wedge e_{i_r})$ to $(D_{i_1}\ldots D_{i_r})b\otimes(e_{i_1}\wedge\ldots\wedge e_{i_r})$. Similarly, $\mathfrak{Fin}^*_{\mathcal{J}'}W$ is isomorphic to the inductive limit of the cohomology of $C^*(\gamma(\boldsymbol{D})^n, W)$, where $\gamma(\boldsymbol{D}) = (\gamma(D_1),\ldots,\gamma(D_r))$. One checks easily that

$$\mathrm{Ind}_{(\mathfrak{p},K_\mathfrak{p})}^{(\mathfrak{g},K)}(C^*(\gamma(\boldsymbol{D})^n, W)) \cong C^*(\boldsymbol{D}^n, \mathrm{Ind}_{(\mathfrak{p},K_\mathfrak{p})}^{(\mathfrak{g},K)}W),$$

which proves our assertion since as we mentioned before formulating this theorem we are in a situation where the induction functor is exact and commutes with direct limits.

PROOF OF 5. – This follows from 2. and a spectral sequence for the Ext-groups.

PROOF OF 6. – This is trivial. □

Remark 1. – In part 5. of the theorem, the functors $\mathfrak{Fin}^p_{\mathcal{J}_\mathfrak{a}}$ and $\mathfrak{Fin}^q_{\mathcal{J}_\mathfrak{m}}$ could also be computed in the category of $(\mathfrak{g},K)$-modules with the same result. Indeed, the restriction to $(\mathfrak{m},K)$ of an injective $(\mathfrak{g},K)$-module is injective because this restriction has an exact left adjoint functor (*cf.* [Vog81], Lemma 6.1.5). Also, an injective $(\mathfrak{g},K)$-module is injective as a $\mathfrak{Z}(\mathfrak{a}) = \mathfrak{U}(\mathfrak{a})$-module because for every finitely generated $\mathfrak{a}$-module M and every $(\mathfrak{g},K)$-module V we have

$$\mathrm{Hom}_\mathfrak{a}(M,V) \cong \sum_\sigma \mathrm{Hom}_{(\mathfrak{g},K)}\Big(\mathfrak{U}(\mathfrak{g}) \bigotimes_{\mathfrak{U}(\mathfrak{a})\otimes\mathfrak{U}(\mathfrak{k})} (M\otimes E_\sigma), V\Big),$$

where the tensor product in the second Hom is given the structure of a $(\mathfrak{g},K)$-module in the same way as in [Vog81], Definition 6.1.5. The coproduct is over all finite-dimensional irreducible representations σ of K, and E_σ is the endomorphism ring of the vector space σ, which is an $\mathfrak{U}(\mathfrak{k})$-algebra in the obvious way. The inner tensor product is over $\boldsymbol{C}$, and is given the structure of a $\mathfrak{U}(\mathfrak{a}\oplus\mathfrak{k})$-module in the obvious way, where $\mathfrak{a}$ acts trivially on the second factor and $\mathfrak{m}$ on the first. The construction of the isomorphism is obvious. As $\mathfrak{U}(\mathfrak{g})$ is a flat $\mathfrak{U}(\mathfrak{a})$-module, the right hand side is exact in M, hence so is the left hand side, provided of course that M is finitely generated. As $\mathfrak{U}(\mathfrak{a})$ is Noetherian, it is well-known that this suffices to prove the injectivity of V as an $\mathfrak{U}(\mathfrak{a})$-module.

We conclude this section with some remarks about the action of the induction functor on spaces of functions. Let us fix a parabolic subgroup $\mathcal{P} = \mathcal{MAN}$ of $\mathcal{G}$. Let $C^\infty(P\mathbb{N}\backslash\mathbb{G})$ be the space of $\mathbb{K}$-finite C^∞-functions on $P\mathbb{N}\backslash\mathbb{G}$, and let $C^\infty(P\mathbb{N}\backslash\mathbb{P})$ be the space of $\mathbb{K}\cap\mathbb{P}$-finite C^∞-functions on $P\mathbb{N}\backslash\mathbb{P}$. We want to verify

$$\mathrm{Ind}_{\mathbb{P}_f}^{\mathbb{G}_f}\mathrm{Ind}_{(\mathfrak{p},K\cap\mathcal{P}(\boldsymbol{R}))}^{(\mathfrak{g},K)} C^\infty(P\mathbb{N}\backslash\mathbb{P}) \cong C^\infty(P\mathbb{N}\backslash\mathbb{G}). \tag{8}$$

We first note that for every $(\mathfrak{p},K\cap\mathcal{P}(\boldsymbol{R}),\mathbb{P}_f)$-module V, there is a homomorphism of $(\mathfrak{p},K\cap\mathcal{P}(\boldsymbol{R}),\mathbb{P}_f)$-modules

$$\mathrm{Ind}_{\mathbb{P}_f}^{\mathbb{G}_f}\mathrm{Ind}_{(\mathfrak{p},K\cap\mathcal{P}(\boldsymbol{R}))}^{(\mathfrak{g},K)} V \to V \tag{9}$$

such that every homomorphism $W\to V$ of $(\mathfrak{p},K\cap\mathcal{P}(\boldsymbol{R}),\mathbb{P}_f)$-modules, where W is a $(\mathfrak{g},K,\mathbb{G}_f)$-module, factorises in a unique manner over (9) and a homomorphism

$$W \to \mathrm{Ind}_{\mathbb{P}_f}^{\mathbb{G}_f}\mathrm{Ind}_{(\mathfrak{p},K\cap\mathcal{P}(\boldsymbol{R}))}^{(\mathfrak{g},K)} V$$

of $(\mathfrak{g},K,\mathbb{G}_f)$-modules. For the induction at the real place, this universality property follows from [Vog81], (6.1.23) and Lemma 6.2.10. For the induction at the finite places it is easily verified.

Now let W be a $(\mathfrak{g},K,\mathbb{G}_f)$-module, and let

$$\phi: W \to C^\infty(P\mathbb{N}\backslash\mathbb{P}) \tag{10}$$

be a homomorphism of $(\mathfrak{p},K\cap\mathcal{P}(\boldsymbol{R}),\mathbb{P}_f)$-modules. We put for $w\in W$

$$\Phi(w)(g) = \phi(k^{-1}w)(p), \tag{11}$$

where $g = pk$, $p\in\mathbb{P}$ and $k\in\mathbb{K}$. It is easy to see that (11) is independent of the Iwasawa decomposition of g, that it defines a homomorphism

$$W \to C^\infty(P\mathbb{N}\backslash\mathbb{G})$$

of $(\mathfrak{g},K,\mathbb{G}_f)$-modules, and that this is the unique homomorphism over which (10) factorises. Consequently, we get an isomorphism (8). If in the following sections the induced module from a module of functions is silently identified with another module of functions, it is assumed that the identification is made in a similar way.

Let write

$$\mathrm{Ind}_{\mathcal{P}}^{\mathcal{G}} = \mathrm{Ind}_{\mathbb{P}_f}^{\mathbb{G}_f}\mathrm{Ind}_{(\mathfrak{p},K\cap\mathcal{P}(\boldsymbol{R}))}^{(\mathfrak{g},K)}.$$

5. $\mathfrak{Fin}$-acyclicity of $S_{\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$

Let

$$w_1(g) = \log \rho_\lambda(g),$$

where λ is in the interior of the positive Weyl chamber. Up to equivalence, this weight function is independent of λ. We put

$$w_n = w_1^n$$

for an integer n and define

$$S_{\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G}) = \bigcup_0^\infty S_{w_{-n}}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G}).$$

We are also interested in

$$S_{-\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G}) = \bigcap_0^\infty S_{w_n}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G}).$$

These are locally convex (with the topology of the inductive limit or of the intersection) $(\mathfrak{g}, K)$-modules. The main goal of this section is to show that they are $\mathfrak{Fin}$-acyclic, *i.e.*, that

$$\mathfrak{Fin}^i_{\mathcal{J}} S_{\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G}) = 0$$

for any $i > 0$ and any ideal $\mathcal{J}$ of finite codimension in $\mathfrak{Z}(\mathfrak{m}_{\mathcal{G}})$. We proceed in several steps. 5.1 reduces the proof to the consideration of the sub-$(\mathfrak{g}, K)$-module of functions belonging to a bounded part of the spectral decomposition of the Laplace operator. For this part of $S_{\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$ there is a spectral decomposition (which will also be used in the next chapter) identifying it essentially as the Fourier-Eisenstein transform of the space of distributions with bounded support. For $S_{-\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$, the procedure is similar. However, $\mathfrak{Fin}^j_{\mathcal{J}} S_{-\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$ does not vanish for $j > 0$, but can be explicitly computed.

It is useful to know the inequality

$$| (R(\gamma)w_n)(g) | \leq C_n w_{n-1} \tag{1}$$

for $\gamma \in \mathfrak{g}$.

5.1. Reduction to the bounded spectrum

We first want to show that for every function ϕ of at most polynomial growth, the function $\phi(\omega)$ of the Casimir operator ω, which is *a priori* defined in the space $S_1(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$, has a continuous extension to $S_{\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$. This is easily reduced to the case of the exponential function, which is treated by a modification of the Hille-Yosida theorem.

Let us fix a finite set of $\mathbb{K}$-types X. Let γ_m be a base for $\mathfrak{g}$ such that there exist real numbers λ_m with

$$\omega = \sum_m \lambda_m \gamma_m^2.$$

This becomes a closed operator in $L_{2,X}$ (the X-isotypical component of L_2) if its domain of definition is taken to be the set of all X-isotypical distributions f such that f and ωf are representable by square integrable functions. For every distribution f with this property, the distributions $\gamma_m f$ are also representable by square integrable functions.

PROPOSITION 1. – *If $n \geq 0$ and $\varepsilon \neq 0$, then the bounded linear operator $(\omega + i\varepsilon)^{-1}$ in $L_2(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o \backslash \mathbb{G})$ restricts to a bounded linear operator in*

$$A_n = \bigcap_{i=0}^{n} W_{w_i}^{(n-i)}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o \backslash \mathbb{G})_X.$$

Furthermore,

$$\left\|(\omega + i\varepsilon)^{-k} | A_n \to A_n\right\| \leq C_n \mid \varepsilon \mid^{-k} \left(1 + \frac{k}{\mid \varepsilon \mid}\right)^n. \tag{1}$$

Proof. – It is sufficient to prove by induction on n (the case $n = 0$ being clear) that for every $f \in L_2$ with $(\omega + i\varepsilon)f \in A_n$ we have $f \in A_n$ and

$$\|f | L_{2,w_n}\| \leq \mid \varepsilon \mid^{-1} \left(\|(\omega + i\varepsilon)f | L_{2,w_n}\| + d_n \left\|f \middle| W_{w_{n-1}}^1\right\|\right). \tag{2}$$

Let us define the Sobolev space norms inductively by

$$\left\|f \middle| W_{w_n}^0\right\| = \|f | L_{2,w_n}\|$$

and

$$\left\|f \middle| W_{w_n}^{k+1}\right\| = \left\|f \middle| W_{w_n}^k\right\| + \sum_m \left\|\gamma_m f \middle| W_{w_n}^k\right\|.$$

If this convention is made, one derives the estimate

$$\left\|(\omega + i\varepsilon)^{-k} f | L_{2,w_n}\right\| \leq \mid \varepsilon \mid^{-k} \sum_{j=0}^{n} \left(\prod_{l=j+1}^{n} d_l\right) \left(\frac{k}{\mid \varepsilon \mid}\right)^{n-j} \left\|f \middle| W_{w_j}^{n-j}\right\| \tag{3}$$

from (2) by induction on n and k. It is clear that (3) implies (1).

To prove (2), we consider $f \in L_{2,X}$ such that $(\omega + i\varepsilon)f \in A_n$. By the induction assumption, we already know that $f \in A_{n-1}$. Furthermore, $\gamma_m f \in L_2$ since this is true for all distributions in the domain of definition of ω in L_2 and $(\omega + i\varepsilon)\gamma_m f = \gamma_m(\omega + i\varepsilon)f \in A_{n-1}$ by the definition of A_n. Hence the induction assumption also implies $\gamma_m f \in A_{n-1}$. To prove $f \in A_n$, it remains to prove $f \in L_{2,w_n}$.

Let φ_k be a uniformly bounded sequence of C_c^∞-functions on $G\mathcal{A}_G(\boldsymbol{R})^o\backslash\mathbb{G}$ such that

$$|(\gamma_m\varphi_k)(g)| \leq cw_{-1}(g) \tag{4}$$

and such for every compact set Ω there exists a k_Ω such that $\varphi_k(g) = 1$ if $g \in \Omega$ and $k > k_\Omega$. For instance, the sequence

$$\varphi_k(g) = \phi\big(2^{-k}w_1(g)\big) \tag{5}$$

will do this job, where ϕ is any C_c^∞-function on $\boldsymbol{R}$ which is equal to one in some neighbourhood of zero.

Consider the imaginary part of the scalar product

$$\begin{aligned}&\langle\varphi_k^2 w_{2n}(\omega + i\varepsilon)f, f\rangle_{L_2}\\ &\quad= i\varepsilon\|\varphi_k f|L_{2,w_n}\|^2 + \sum_m \lambda_m\Big(-\big\langle\big(\gamma_m(\varphi_k^2 w_{2n})\big)f, \gamma_m f\big\rangle_{L_2} - \big\langle w_{2n}\varphi_k^2\gamma_m f, \gamma_m f\big\rangle_{L_2}\Big).\end{aligned}$$

Using (5.1) and (4), we get

$$\begin{aligned}&|\,\varepsilon\,|\,\|\varphi_k f|L_{2,w_n}\|^2\\ &\quad\leq \|\varphi_k(\omega + i\varepsilon)f|L_{2,w_n}\|\,\|\varphi_k f|L_{2,w_n}\| + c_n\sum_m \|\varphi_k f|L_{2,w_n}\|\,\big\|\gamma_m f\big|L_{2,w_{n-1}}\big\|,\end{aligned}$$

hence

$$|\,\varepsilon\,|\quad \|\varphi_k f|L_{2,w_n}\| \leq \|\varphi_k(\omega + i\varepsilon)f|L_{2,w_n}\| + c_n\sum_m \big\|\gamma_m f\big|L_{2,w_{n-1}}\big\|.$$

Since we already know that $f \in A_{n-1} \subset W^k_{w_{n-1}}$, the right hand side of the last inequality remains bounded as $k \to \infty$. We conclude that $f \in L_{2,w_n}$. In the limit $k \to \infty$, we obtain (2) $\square$

The estimate proved in this proposition is suitable for an application of the following modification of the Hille-Yosida theorem.

THEOREM 8. – *Let B be a closed operator in a Banach space A and n a positive integer. Then the following two assertions are equivalent:*

- *B defines a strongly continuous semigroup $\big(\mathrm{e}^{-tB}\big)_{t\geq 0}$, and there exists a constant C with*

$$\big\|\mathrm{e}^{-tB}|A \to A\big\| \leq C(1 + t)^n.$$

- *For $\varepsilon > 0$, the operator $B + \varepsilon$ has a bounded inverse in A and there exists a constant D with*

$$\big\|(B + \varepsilon)^{-k}|A \to A\big\| \leq D\varepsilon^{-k}\left(1 + \frac{k}{\varepsilon}\right)^n$$

for all positive integers k.

The proof is omitted since it is a straightforward modification of the proof of the Hille-Yosida theorem given in [Yos74].

COROLLARY 1. – *Let X be a finite set of $\mathbb{K}$-types, n an integer, and φ a function on* $\boldsymbol{R}$ *such that*

$$(1+t^2)^{\frac{m}{2}}\varphi(t) \in W_2^{|n|+1}(\boldsymbol{R})$$

for some integer $m \geq \frac{|n|}{2}$. If $n < 0$, then the bounded operator $\varphi(\omega)$ in $L_2(G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G})$ extends to a bounded operator in $L_{2,w_n}(G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G})_X$. If $n > 0$, then the bounded operator $\varphi(\omega)$ in $L_2(G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G})$ restricts to a bounded operator in $L_{2,w_n}(G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G})_X$.

Proof. – We may assume $n > 0$ since the case $n < 0$ is dual to that case. By (2.3.11), there is a constant λ such that $(\omega+\lambda)^m$ defines an isomorphism of $W_\rho^{2m}(G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G})_X$ onto $L_{2,\rho}(G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G})_X$ for $\rho = 1$ and $\rho = w_n$. Let λ be chosen in this way, and let

$$\psi(t) = \varphi(t)(t+\lambda)^m.$$

Then

$$\psi \in W_2^{n+1}(\boldsymbol{R}). \tag{6}$$

By Proposition 1 and Theorem 8, we have a continuous group of operators

$$\left(\mathrm{e}^{it\omega}\right)_{-\infty<t<\infty}$$

in A_n such that

$$\left\|\mathrm{e}^{it\omega}|A_n \to A_n\right\| \leq C(1+|\,t\,|)^n.$$

Consequently, by (6), the integral

$$\psi(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-it\omega}\hat{\psi}(t)\,dt$$

converges in the space of endomorphism of A_n, where

$$\hat{\psi}(t) = \int_{-\infty}^{\infty} \mathrm{e}^{ist}\psi(s)\,ds$$

is the Fourier transform of ψ. Thus we see that the continuous operator in L_2 $\psi(\omega)$ restricts to a continuous operator in A_n. Since the Banach space A_n contains $W_{w_n}^{2m}(G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G})$ (by our assumption $m \geq \frac{n}{2}$) and is contained in $L_{2,w_n}(G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G})$, it also restricts to a continuous operator from $W_{w_n}^{2m}(G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G})$ to $L_{2,w_n}(G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G})$. Consequently, $\varphi(\omega) = \psi(\omega)(\omega+\lambda)^{-m}$ restricts to a continuous operator in $L_{2,w_n}(G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G})_X$. □

THEOREM 9. – *Let ϕ be a C^∞-function on $\boldsymbol{R}$ satisfying estimates*

$$\left|\frac{d^k\phi}{dt^k}(t)\right| \leq C_k(1+\mid t\mid)^{N_k}$$

(we will refer to such functions as polynomially increasing). Then the operator $\phi(\omega)$ (which originally is defined in $S_1(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$ by spectral theory, extends to a continuous operator in $S_{\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$ and restricts to a continuous operator in $S_{-\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$.

Proof. – This is an easy consequence of the previous corollary. □

Let $S_{\log,b}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$ be the dense subspace of $S_{\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$ consisting of functions f such that there exists an R with

$$\phi(\omega)(f) = f$$

if ϕ is a C^∞-function on $\boldsymbol{R}$ which is polynomially increasing and equal to one on the interval $[-R, R]$. As an easy consequence of Theorem 9, we have

THEOREM 10. – *The inclusion $S_{\log,b}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G}) \to S_{\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$ induces an isomorphism on $\mathfrak{Fin}^*_{\mathcal{J}}$ for every ideal $\mathcal{J}$ of finite codimension in $\mathfrak{Z}(\mathfrak{m}_{\mathcal{G}})$. The same holds for $S_{-\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$.*

Proof. – Let $P(\omega)$ be a non-vanishing polynomial in the Casimir operator which belongs to $\mathcal{J}$. We claim that $P(\omega)$ defines an automorphism of the $(\mathfrak{g}, K)$-module

$$S_{\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})/S_{\log,b}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G}).$$

This implies that the endomorphism of $\mathfrak{Fin}^*_{\mathcal{J}}(S_{\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})/S_{\log,b}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G}))$ induced by $P(\omega)$, which must be locally nilpotent by the definition of $\mathfrak{Fin}^*_{\mathcal{J}}$, is an automorphism. This implies the vanishing of

$$\mathfrak{Fin}^*_{\mathcal{J}}(S_{\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})/S_{\log,b}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G}))$$

and proves the theorem.

Let χ be a C^∞-function on $\boldsymbol{R}$ which vanishes in a neighbourhood of the zeros of P but is equal to one outside a compact subset of $\boldsymbol{R}$. Then the desired inverse of $P(\omega)$ is given by $(\chi P^{-1})(\omega)$. □

5.2. Review of Eisenstein Series

The aim of this subsection is to reformulate some of the results of [Lan76], §7 in a language in which they will be used in this and in the next section. Some hints about where these facts are proved in the various expositions of Langlands' theory will be given at the end of this subsection.

Let $\{P\}$ be a class of associate parabolic subgroups. If k is a non-negative integer such that $k \leq \operatorname{rank}(\{P\})$, we denote by $U^k_{\{P\}}$ be the set of triples $u = (\mathcal{R}, \Lambda, \chi)$ with the following properties:

- $\mathcal{R} = \mathcal{M}_{\mathcal{R}}\mathcal{A}_{\mathcal{R}}\mathcal{N}_{\mathcal{R}}$ is a standard parabolic subgroup containing an element of $\{P\}$, and $\operatorname{rank}(\{P\}) = \operatorname{rank}(\mathcal{R}) + k$.
- $\Lambda: \mathbb{A}_{\mathcal{R}}/A_{\mathcal{R}}\mathcal{A}_{\mathcal{R}}(\boldsymbol{R})^{o} \to \mathbf{U}(1)$ is a continuous character.
- The last entry is a unitary character $\chi: \mathfrak{Z}(\mathfrak{m}) \to \mathbf{C}$ of the center of the universal enveloping algebra of $\mathfrak{m}$.

For such u, let $V(u)$ be the space of square integrable $\mathbb{K} \cap \mathbb{R}$-finite functions f on

$$R\mathbb{N}_{\mathcal{R}}\mathcal{A}_{\mathcal{R}}(\boldsymbol{R})^{o}\backslash\mathbb{R}$$

with the following properties:

- For every standard parabolic subgroup $\mathcal{Q} \subseteq \mathcal{R}$ which does not belong to $\{P\}$, the constant term of f along $\mathbb{N}_{\mathcal{R}}$ is orthogonal to the space of cusp forms on $M_{\mathcal{Q}}\backslash\mathbb{M}_{\mathcal{Q}}$.
- $f(ag) = \Lambda(a)f(g)$ for $a \in \mathbb{A}_{\mathcal{Q}}$.
- f is a χ-eigenvector for $\mathfrak{Z}(\mathfrak{m}_{\mathcal{R}})$.

Let $W(u)$ be the space of all $\mathbb{K}$-finite functions f on $R\mathbb{N}_{\mathcal{R}}\mathcal{A}_{\mathcal{R}}(\boldsymbol{R})^{o}\backslash\mathbb{G}$ such that for every every $g \in \mathbb{K}$, the function $f(rg)$ of $r \in R\mathbb{N}_{\mathcal{R}}\mathcal{A}_{\mathcal{R}}(\boldsymbol{R})^{o}\backslash\mathbb{R}$ belongs to $V(u)$.

For $f \in W(u)$ we define formally for $\lambda \in \check{a}^{\mathcal{G}}_{\mathcal{R}\boldsymbol{C}}$

$$E(f,\lambda) = \sum_{\delta \in R\backslash G} \mathrm{e}^{\langle \lambda + \rho_{\mathcal{R}}, H(\delta g)\rangle} f(\delta g). \tag{1}$$

This Eisenstein series converges to a real analytic function on $G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^{o}\backslash\mathbb{G}$ if $\Re(\lambda)$ is far inside $\check{\mathfrak{a}}^{\mathcal{G}+}_{\mathcal{R}}$. One of the main results of [Lan], §7 is the fact that it has a meromorphic continuation to all of $(\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{R}})_{\boldsymbol{C}}$.

Let $D'(i\check{a}^{\mathcal{G}}_{\mathcal{R}})$ be the space of all distributions on $i\check{a}^{\mathcal{G}}_{\mathcal{R}}$. Then $W(u) \otimes D'(i\check{a}^{\mathcal{G}}_{\mathcal{R}})$ can be given the structure of an $(\mathfrak{g}, K, \mathbb{G}_f)$-module by the following rules for $g \in \mathbb{G}$ and $x \in i\check{a}^{\mathcal{G}}_{\mathcal{R}}$:

- For $\gamma \in \mathfrak{g}$,

$$(\gamma f)(g,x) = \gamma f(g,x) + f(g,x)\langle x + \rho_{\mathcal{R}}, (\gamma H)(g)\rangle. \tag{2}$$

- For $k \in \mathbb{K}$, $(kf)(g,x) = f(gk,x)$.
- For $h \in \mathbb{G}_f$, $(hf)(g,x) = \mathrm{e}^{\langle x+\rho_{\mathcal{R}}, H(gh)-H(g)\rangle} f(gh,x)$.

Let $D'(i\check{a}^{\mathcal{G}}_{\mathcal{R}})$ be equipped with the structure of a module over the commutative Lie algebra $a_{\mathcal{R}}$ defined by the rule

$$\xi \in \mathfrak{a}_{\mathcal{R}}: f(\eta) \Rightarrow \langle \xi, \eta + \rho_{\mathcal{R}}\rangle f(\eta)\ (\eta \in i\check{\mathfrak{a}}_{\mathcal{R}}).$$

We extend this structure to an $\mathfrak{r}$-action by letting $\mathfrak{m}_{\mathcal{R}}$ and $\mathfrak{n}_{\mathcal{R}}$ act trivially and give $D'(i\check{a}^{\mathcal{G}}_{\mathcal{R}})$ an $\mathbb{R}_f$-action by

$$r \in \mathbb{R}_f: f(\eta) \Rightarrow \mathrm{e}^{\langle H_{\mathcal{R}}(r), \eta+\rho_{\mathcal{R}}\rangle} f(\eta)\ (\eta \in i\check{\mathfrak{a}}_{\mathcal{R}}).$$

There is an obvious isomorphism of $(\mathfrak{g}, K, \mathbb{G}_f)$-modules

$$W(u) \otimes D'(i\check{a}^{\mathcal{G}}_{\mathcal{R}}) \cong \operatorname{Ind}^{\mathcal{G}}_{\mathcal{R}}\big(V(u) \otimes D'(i\check{a}^{\mathcal{G}}_{\mathcal{R}})\big). \tag{3}$$

A similar isomorphism exists for various subspaces of $D'(i\check{a}_{\mathcal{R}}^{\mathcal{G}})$, like $C_c^\infty(i\check{a}_{\mathcal{R}}^{\mathcal{G}})$, $L_{2,c}(i\check{a}_{\mathcal{R}}^{\mathcal{G}})$, and $D'_c(i\check{a}_{\mathcal{R}}^{\mathcal{G}})$ (the subspaces of C^∞-functions, square integrable functions, and distributions with compact support). Let us fix an Euclidean measure on $i\check{a}_{\mathcal{R}}^{\mathcal{G}}$. Then

$$(4) \qquad E(\phi)(g) = \int_{i\check{a}_{\mathcal{R}}^{\mathcal{G}}} E(\phi(\lambda),\lambda)\,d\lambda; \quad \phi \in W(u) \otimes D'_c(i\check{a}_{\mathcal{R}}^{\mathcal{G}})$$

is an homomorphism of $(\mathfrak{g}, K, \mathbb{G}_f)$-modules.

Let $\mathcal{R}'$ be another standard parabolic subgroup which is associate to $\mathcal{R}$. Let $\Omega(\mathfrak{a}_{\mathcal{R}}, \mathfrak{a}_{\mathcal{R}'})$ be the set of all linear transformations from $\mathfrak{a}_{\mathcal{R}}$ to $\mathfrak{a}_{\mathcal{R}'}$ obtained by restricting elements of the Weyl group of Φ_o (the root system for $\mathfrak{a}_o$) to $\mathfrak{a}_{\mathcal{R}}$. The contragredient of an element of this Weyl set maps $\check{\mathfrak{a}}_{\mathcal{R}}$ to $\check{\mathfrak{a}}_{\mathcal{R}'}$ and will be denoted by the same letter. For $w \in \Omega(\mathfrak{a}_{\mathcal{R}}, \mathfrak{a}_{\mathcal{R}'})$, we may choose a representative $\tilde{w}$ for w and put $u' = \tilde{w}u \in U_{\{P\}}^k$. Then for $\lambda \in (\check{\mathfrak{a}}_{\mathcal{R}}^{\mathcal{G}})_{\boldsymbol{C}}$, we may formally define

$$(5) \qquad M(w,\lambda): W(u) \to W(u')$$

$$(M(w,\lambda)f)(g) = \int_{\mathbb{N}_{\mathcal{R}'} \cap \tilde{w}\mathbb{N}_{\mathcal{R}}\tilde{w}^{-1} \backslash \mathbb{N}_{\mathcal{R}'}} f(\tilde{w}^{-1}nx) e^{\langle \lambda + \rho_{\mathcal{R}}, H(\tilde{w}^{-1}nx) - w^{-1}H(x)\rangle} dn.$$

The integral converges absolutely if $\Re\lambda$ is far enough inside the positive Weyl chamber, but has a meromorphic continuation to all of $(\check{a}_{\mathcal{R}}^{\mathcal{G}})_{\boldsymbol{C}}$. Then we have the following functional equations of the Eisenstein series:

FUN 1.

$$E(w\lambda, M(w,\lambda)\phi) = E(\lambda,\phi)$$

FUN 2.

$$M(vw,\lambda) = M(v,w\lambda)M(w,\lambda),$$

where in the second functional equation $v \in \Omega(\mathfrak{a}_{\mathcal{R}'}, \mathfrak{a}_{\mathcal{R}''})$ for yet another standard parabolic subgroup $\mathcal{R}''$. Furthermore,

FUN 3.

$$M(s_\alpha, x) \text{ depends only on } \langle \check{\alpha}, x\rangle$$

if s_α is the generalised symmetry (in the sense of [MW95], I.1.7, I.1.8) belonging to the simple root α.

Now we can turn $U_{\{P\}}^k$ into a groupoid. Let $\mathcal{U}_{\{P\}}^k$ be the following groupoid:

- Objects are elements $t = (R, \Lambda, \chi) \in U_{\{P\}}^k$.
- $\mathrm{Hom}_{\mathcal{U}_{\{P\}}^k}(t,t')$ consists of those elements w of the Weyl set $\Omega(\mathfrak{a}_{\mathcal{R}}, \mathfrak{a}_{R'})$ which map χ to χ' and Λ to Λ'.

Note that $M(w,\lambda)$ is holomorphic at $\lambda \in i\check{\mathfrak{a}}_{\mathcal{R}}^{\mathcal{G}}$. Consequently, the association $u \to W(u) \otimes D'_c(i\check{\mathfrak{a}}_{\mathcal{R}}^{\mathcal{G}})$ can be turned into a functor from $\mathcal{U}_{\{P\}}^k$ to the category of $(\mathfrak{g}, K, \mathbb{G}_f)$-modules by

$$(6) \qquad (wf)(g,\lambda) = \big(M(w,\lambda)f(.,w^{-1}\lambda)\big)(g)$$

for $f \in W(u) \otimes D'_c(i\check{\mathfrak{a}}_{\mathcal{R}}^{\mathcal{G}})$. By **Fun 1**, $E(wf) = E(f)$. Consequently, (4) defines a homomorphism

$$(7) \qquad E : \operatorname*{colim}_{\mathcal{U}_{\{P\}}^k} W(u) \otimes D'_c(i\check{\mathfrak{a}}_{\mathcal{R}}^{\mathcal{G}}) \to \mathbb{K}\text{-finite } C^\infty\text{-functions on } G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G}$$

of $(\mathfrak{g}, K, \mathbb{G}_f)$-modules, where colim is the colimit (*cf.* [Mac], §III.3) in the category of $(\mathfrak{g}, K, \mathbb{G}_f)$-modules.

Let $S_{1,b}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$ be defined in the same way as we defined $S_{\log,b}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$ and $S_{-\log,b}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$ in the last section. Then [Lan76], Theorem 7.1 implies

THEOREM 11. – *The restriction of* (7) *to the space of square integrable functions defines an isomorphism of* $(\mathfrak{g}, K, \mathbb{G}_f)$-*modules*

$$\bigoplus_{\{P\}} \bigoplus_{k=o}^{\operatorname{rank}(\{P\})} \operatorname*{colim}_{\mathcal{U}_{\{P\}}^k} W(u) \otimes L_{2,c}(i\check{\mathfrak{a}}_{\mathcal{R}}^{\mathcal{G}}) \to S_{1,b}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G}),$$

where the outer sum is about all classes of associate parabolic subgroups.

The fact that in this theorem the usual algebraic colimit (rather than some construction involving direct sums of Hilbert spaces) occurs is due to the fact that every finite number R and every finite set X of $\mathbb{K}$-types, only a finite number of $u \in U_{\{P\}}^k$ can give a contribution to the part of the spectrum bounded by R. This follows for instance from [Mül89], Theorem 0.1.

It is the aim of the next subsection to prove a similar spectral decomposition for $S_{\log,b}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$ and $S_{-\log,b}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$, replacing $L_{2,c}(i\check{\mathfrak{a}}_{\mathcal{R}}^{\mathcal{G}})$ by $D'_c(i\check{\mathfrak{a}}_{\mathcal{R}}^{\mathcal{G}})$ in the first case and by $C_c^\infty(i\check{\mathfrak{a}}_{\mathcal{R}}^{\mathcal{G}})$ in the second case. We have recalled all the results of Langlands' book which are necessary to do this. The remainder of this paragraph is to formulate other consequences of Langlands' §7, which will be used in the next section to study the space of automorphic forms.

To facilitate induction arguments, we will also formulate the next results for partial Eisenstein series. For $\phi \in W(u)$ and for a parabolic subgroup $\mathcal{Q} \supseteq \mathcal{R}$, let the partial Eisenstein series be defined by

$$(8) \qquad E_{\mathcal{R}}^{\mathcal{Q}}(\phi,\lambda) = \sum_{\delta \in R\backslash Q} \mathrm{e}^{\langle \lambda + \rho_{\mathcal{R}}, H(\delta g)\rangle} f(\delta g).$$

If $\tilde{\mathcal{R}}$ is yet another parabolic subgroup contained in $\mathcal{Q}$ and if $w \in \Omega(\mathfrak{a}_{\mathcal{R}}, \mathfrak{a}_{\tilde{\mathcal{R}}})$ identically acts on $\mathfrak{a}_{\mathcal{Q}}$, then **Fun 1** still holds with E replaced by $E_{\mathcal{R}}^{\mathcal{Q}}$.

We will need a description of the constant term of such Eisenstein series along elements $\mathcal{P}$ of $\{P\}$. It is given by certain meromorphic functions $N(\omega, \lambda)$. The first argument is an affine function

$$\omega\colon \check{\mathfrak{a}}_{\mathcal{R}} \rightarrow \check{\mathfrak{a}}_{\mathcal{P}}$$

which varies in a finite set $\Omega(u, \check{\mathfrak{a}}_{\mathcal{P}})$ of such transformations. For all $\omega \in \Omega(u, \check{\mathfrak{a}}_{\mathcal{P}})$, the linear part $\hat{\omega}$ of ω can be expressed as the composition of the inclusion $\check{\mathfrak{a}}_{\mathcal{R}} \subseteq \check{\mathfrak{a}}_o$ and an element of $\Omega(\mathfrak{a}_o, \mathfrak{a}_o)$. The second argument to $N(\omega, \lambda)$ is $\lambda \in \left(\check{\mathfrak{a}}_{\mathcal{R}}^{\mathcal{G}}\right)_{\boldsymbol{C}}$. The value of N is a linear transformation from $W(u)$ to the space of polynomials on $\mathfrak{a}_{\mathcal{P}}$ with values in the space of cusp forms on $\mathbb{N}_{\mathcal{P}}\mathcal{A}(\boldsymbol{R})^o P\backslash\mathbb{G}$. Thus, for $\phi \in W(u)$, $x \in \mathfrak{a}_{\mathcal{P}}$, $\omega \in \Omega(u, \check{\mathfrak{a}}_{\mathcal{P}})$, and $g \in \mathbb{N}_{\mathcal{P}}\mathcal{A}(\boldsymbol{R})^o P\backslash\mathbb{G}$,

$$(N(\omega, \lambda)\phi)(x, g)$$

is a complex number.

The formula for the constant term of a full or partial Eisenstein series is a sum over all $\omega \in \Omega(u, \check{\mathfrak{a}}_{\mathcal{P}})$ for which $\hat{\omega}$ identically acts on $\check{\mathfrak{a}}_{\mathcal{Q}}$.

FUN 4.
$$\left(E_{\mathcal{R}}^{\mathcal{Q}}(\phi, \lambda)\right)_{\mathcal{N}_{\mathcal{P}}}(g) = \sum_{\substack{\omega \in \Omega(u, \check{\mathfrak{a}}_{\mathcal{P}}) \\ hat\omega|_{\check{\mathfrak{a}}_{\mathcal{Q}}} = \mathrm{Id}}} \mathrm{e}^{\langle H_{\mathcal{P}}(g), \rho_{\mathcal{P}} + \omega\lambda\rangle} (N(\omega, \lambda)\phi)(H_{\mathcal{P}}(g), g)$$

Of course, it is assumed that $\mathcal{P}$ is contained in $\mathcal{Q}$.

Since **Fun 4** is part of the structure defining an Eisenstein system, the image of $N(\omega, \lambda)$ consists of polynomials given by elements of the symmetric algebra of the orthogonal complement of the image of $\hat{\omega}$. However, we will not need this fact.

In the case of cuspidal Eisenstein series, the functions $N(\omega, \lambda)$ and $M(\omega, \lambda)$ agree. In general, they are related by the functional equation

FUN 5.
$$N(\omega w, \lambda) = N(\omega, w\lambda)M(w, \lambda).$$

We will also need the following geometrical property of the elements of $\Omega(u, \mathcal{P})$: For each $\omega \in \Omega(u, \mathcal{P})$, the affine space $\sigma = -\omega\check{\mathfrak{a}}_{\mathcal{R}}$ is an affine space along which one of the Eisenstein systems of [Lan76], Theorem 7.1 is defined, and $X(\sigma) = -\omega 0$ is the point of minimal norm in that affine space. Let $\mathcal{Q}$ be the smallest parabolic subgroup containing $\mathcal{P}$ such that $\check{\mathfrak{a}}_{\mathcal{Q}}$ is contained in $\hat{\omega}\check{\mathfrak{a}}_{\mathcal{R}}$. Then Langlands has shown (2) that $X(\sigma) \in {}^{+}\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{Q}}$. Consequently,

FUN 6.
$$\omega 0 \in -{}^{+}\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{Q}}.$$

We will also need the following lemma:

LEMMA 1. – *Let $u = (\mathcal{R}, \Lambda, \chi) \in U_{\{P\}}^k$, $B \subset W(u)$ a bounded subset, and $\boldsymbol{K} \subset (\check{\mathfrak{a}}_{\mathcal{R}})_{\boldsymbol{C}}$ be a compact subset which does not meet any singular hyperplane of $E(f, \cdot)$ for $f \in B$. Let D be a differential operator with constant coefficients on $\check{\mathfrak{a}}_{\mathcal{R}}$.*

Then $\left\{DE(f, \lambda) \mid f \in B, \lambda \in K\right\}$ is bounded in $S_\infty(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$.

(2) In Langlands' book, this is part of the assumptions of Lemma 7.5, which are guaranteed by Theorem 7.1.

Proof. – In the case where $\boldsymbol{K}$ does not meet a singular hyperplane of any of the functions $N(\omega,\cdot)$ for $\mathcal{P}\in\{P\}$ and $\omega\in\Omega(u,\check{\mathfrak{a}}_{\mathcal{P}})$, the assertion follows from [Lan76], Lemma 5.2.

In general the functions $N(\omega,\cdot)$ have singular hyperplanes which are not necessarily singular hyperplanes of $E(\cdot)$. The singular hyperplanes of N are parallel to root hyperplanes. It follows that for a generic vector $\xi\in\check{\mathfrak{a}}_{\mathcal{R}}^{\mathcal{G}}$, for every $\lambda\in(\check{\mathfrak{a}}_{\mathcal{R}})_{\boldsymbol{C}}$ the intersection of $\lambda+\boldsymbol{C}\cdot\xi$ with the union $\mathcal{S}$ of the singular sets of $N(\omega,\cdot)$ over $\mathcal{P}\in\{P\}$ and $\omega\in\Omega(u,\check{\mathfrak{a}}_{\mathcal{P}})$ is discrete. Let ξ be a normed vector with this property.

Since $\boldsymbol{K}$ can be covered by finitely many balls which still satisfy the assumptions of the theorem, we may assume $\boldsymbol{K}$ to be convex. Then there exists a compact convex neighbourhood $\tilde{\boldsymbol{K}}$ of $\boldsymbol{K}$ which still satisfies the assumptions of the theorem. For every $x\in\boldsymbol{K}$, there exists a closed neighbourhood U_x of x and a number $\varepsilon_x>0$ so that for $\boldsymbol{M}_x=\{y+z\xi\mid y\in U_x, z\in\boldsymbol{C}, |z|=\varepsilon\}$ is contained in $\tilde{\boldsymbol{K}}$ and does not intersect $\mathcal{S}$. There exist finitely many $x_i\in\boldsymbol{K}$ such that the U_{x_i} cover $\boldsymbol{K}$. Let $\boldsymbol{M}$ be the union of the $\boldsymbol{M}_{x_i}$. We know that the assertion of the lemma is true if $\boldsymbol{K}$ is replaced by $\boldsymbol{M}$. But if $x\in\boldsymbol{K}$, $y\in U_x$ and $f\in B$, $DE(f,y)$ is the mean value of $DE(f,y+\lambda\xi)$ over all comple xnumbers λ of absolute value ε_x. Since these $y+\lambda\xi$ are all contained in $\boldsymbol{M}$, the assertion follows. □

COROLLARY 1. – *If in addition to the assumptions of the lemma we have* $\omega\Re\lambda\in-\overline{{}^{+}\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{G}}}$ *for every* $\lambda\in\boldsymbol{K}$, $\mathcal{P}\in\{P\}$ *and* $\omega\in\Omega(u,\check{\mathfrak{a}}_{\mathcal{P}})$, *then* $\{DE(f,\lambda)\mid f\in B,\lambda\in K\}$ *is also bounded in* $S_{\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^{o}\backslash\mathbb{G})$.

Proof. – We will need the fact that lemma I.4.1 of [MW95] holds uniformly for a set of automorphic forms which is bounded in $S_{\infty}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^{o}\backslash\mathbb{G})$ and whose constant terms uniformly satisfy the assumption of that lemma. This is easy to see from the proof given [MW95]. The corollary now easily follows from this fact, **Fun 6**, and the lemma. □

The meromorphic continuation of the residual Eisenstein series and the spectral decomposition Theorem 11 are the main result of [Lan76] (*cf.* also [MW95] and [OW81]). The functional equations **Fun 1** and **Fun2** can in principle be derived from the results of Langlands' §7. A detailed proof can be found in the book of Mœglin and Waldspurger. **Fun 3** is easily checked in the domain of absolute convergence of the integral in (5), and holds in general by meromorphic continuation. **Fun 4** follows from the analogous fact about cuspidal Eisenstein series since a residual Eisenstein series is a sum of iterated residues of cuspidal Eisenstein series. In the books of Langlands and Osborne/Warner, it is part of the definition of an Eisenstein system. **Fun 5** follows from **Fun 1** since a description of the constant term by **Fun 4** is unique on an open dense subset of the parameter space. It was already explained how to derive **Fun 6** from two facts proved in Langlands' book. In the book of Osborne and Warner, these two facts are conditions **Geom III** on p. 157 and **E-S I** on p. 159. In the book of Mœglin and Waldspurger, they are Corollary V.3.16 and Proposition VI.1.6(c).

5.3. Spectral decomposition of $S_{\log,b}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^{o}\backslash\mathbb{G})$

The aim of this subsection is to prove the following spectral decomposition of $S_{\log,b}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^{o}\backslash\mathbb{G})$.

THEOREM 12.
- *We have an isomorphism induced by* (5.2.7)

$$(1)\qquad S_{\log,b}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G}) \cong \bigoplus_{\{P\}} \bigoplus_{k=0}^{\operatorname{rank}\{P\}} \operatorname*{colim}_{\mathcal{U}^k_{\{P\}}} W(u)\otimes D'_c(i\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{R}}) \cong \bigoplus_{\{P\}} \bigoplus_{k=0}^{\operatorname{rank}\{P\}} \operatorname*{colim}_{\mathcal{U}^k_{\{P\}}} \operatorname{Ind}^{\mathcal{G}}_{\mathcal{R}} V(u)\otimes D'_c(i\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{R}}),$$

where the second isomorphism is given by (5.2.3).
- *Similarly,*

$$(2)\qquad S_{-\log,b}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G}) \cong \bigoplus_{\{P\}} \bigoplus_{k=0}^{\operatorname{rank}\{P\}} \operatorname*{colim}_{\mathcal{U}^k_{\{P\}}} W(u)\otimes C^\infty_c(i\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{R}}) \cong \bigoplus_{\{P\}} \bigoplus_{k=0}^{\operatorname{rank}\{P\}} \operatorname*{colim}_{\mathcal{U}^k_{\{P\}}} \operatorname{Ind}^{\mathcal{G}}_{\mathcal{R}} V(u)\otimes C^\infty_c(i\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{R}}).$$

Remark 1. – This result can also be written as

$$S_{\log,b}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G}) \cong \bigoplus_{\{R\}} \operatorname*{colim}_{\mathrm{Weyl}(\{R\})} \operatorname{Ind}^{\mathcal{G}}_{\mathcal{R}}\big(L_{2,\mathrm{disc}}(\mathcal{A}_{\mathcal{L}}(\mathbb{R})^o L_{\mathcal{R}}\backslash\mathbb{L}_{\mathcal{R}})\otimes D'_c(i\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{R}})\big)$$
$$S_{-\log,b}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G}) \cong \bigoplus_{\{R\}} \operatorname*{colim}_{\mathrm{Weyl}(\{R\})} \operatorname{Ind}^{\mathcal{G}}_{\mathcal{R}}\big(L_{2,\mathrm{disc}}(\mathcal{A}_{\mathcal{L}}(\mathbb{R})^o L_{\mathcal{R}}\backslash\mathbb{L}_{\mathcal{R}})\otimes C^\infty_c(i\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{R}})\big).$$

Here Weyl$(\{R\})$ is the Weyl groupoid of $\{R\}$: Objects are elements $\mathcal{R}$ of $\{R\}$, and a morphism from $\mathcal{R}$ to $\tilde{\mathcal{R}}$ is an element of $\Omega(\mathfrak{a}_{\mathcal{R}},\mathfrak{a}_{\tilde{\mathcal{R}}})$.

For the reasons explained after the formulation of Theorem 11, for every finite number R and every finite set X of $\mathbb{K}$-types, only a finite number of $u\in U^k_{\{P\}}$ can give a contribution to the part of the spectrum bounded by R, so that we have the usual algebraic colimit of $(\mathfrak{g},K,\mathbb{G}_f)$-modules in Theorem 12.

The proof will require two propositions which use Arthur's truncation operator as a technical tool. Let $\{\omega_\alpha\}_{\alpha\in\Delta_o}$ be the base of $\check{\mathfrak{a}}_o$ dual to the base $\{\check{\alpha}\}$ of $\mathfrak{a}_o$. For a standard parabolic subgroup $\mathcal{P}$ and $x\in\mathfrak{a}_o$, we put

$$\hat{\tau}_{\mathcal{P}}(x)=\begin{cases}1 & \text{if } \langle\omega_\alpha,x\rangle>0 \text{ for all } \alpha\in\Delta_{\mathcal{P}}\\ 0 & \text{otherwise.}\end{cases}$$

This function depends only on the projection of x to $\mathfrak{a}_{\mathcal{P}}$. Consequently, the following expression formally makes sense:

$$(\lambda^t f)(g)=\sum_{\mathcal{P}}(-1)^{\operatorname{rank}\mathcal{P}}\sum_{\gamma\in P\backslash G}\hat{\tau}_{\mathcal{P}}(H(\gamma g)-t\check{\rho}_o)f_{\mathcal{N}}(\gamma g),$$

where $\check{\rho}_o$ is one half the sum of the positive coroots. In Arthur's notation, $\lambda^t = \Lambda^{t\check{\rho}_o}$. For functions with compact support, this expression converges. It is known that this is an orthogonal projection in L_2 if t is large enough. Also for sufficiently large t, functions which together with their derivatives are moderately increasing become rapidly decreasing after truncation, and the sums in the definition of λ^t converge. Thus, the use of λ^t in the following proposition makes sense for sufficiently large t. Throughout this subsection, we will also use the abbreviations $\hat{\tau}_1$ for $\hat{\tau}_{\mathcal{P}_1}$ and Δ_1^2 for $\Delta_{\mathcal{P}_1}^{\mathcal{P}_2}$ as they are used in Arthur's papers.

PROPOSITION 1. – *Let k be a positive integer. Then there exists a semi-norm c on the space $S_{w_{-k}}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$ such that for every sufficiently large t we have*

$$\left\|\lambda^t f|L_2(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})\right\| \leq c(f)t^k. \tag{3}$$

Proof. – Let χ_R be the characteristic function of the set of all $g \in G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G}$ such that $w_1(g) < R$. We first claim the existence of a $d > 0$ such that

$$\chi_R\lambda^t f = \chi_R\lambda^t\chi_{dR}f \tag{4}$$

for every $t > 0$. It suffices to chose d such that whenever $g \in \mathbb{G}$, $\mathcal{P}$ is a standard parabolic subgroup, $\delta \in P\backslash G$, and $u \in \mathbb{N}_{\mathcal{P}}$ such that $\hat{\tau}_{\mathcal{P}}(u\delta g) \neq 0$, we have

$$w_1(u\delta g) \leq dw_1(g) \tag{5}$$

Let $\mathfrak{F} \subseteq \omega\exp(\check{\mathfrak{a}}_o^{\mathcal{G}+})\mathbb{K} \subset \mathbb{G}^1 = \bigcap_{\xi\in X^*(\mathcal{G})}\ker(|\chi|)$, where $\omega \subset \mathbb{P}_o$ is compact, be a fundamental domain for the action of G on $\mathbb{G}^1$. By varying g in its left G-orbit and changing δ accordingly, we may assume $g \in \mathfrak{F}$. Also, we may assume that u belongs to a fixed compact subset of $\mathbb{N}_{\mathcal{P}}$. Let $\|\,.\,\|$ be a norm function of $\mathbb{G}$ as in [Art78], §1, p. 919. Then

$$w_1(g) < L\log(\|g\|) \tag{6}$$

for $g \in \mathbb{G}$ and there exists a constant L' such that

$$\log(\|g\|) < L'\,w_1(g) \tag{7}$$

for $g \in \mathfrak{F}$.

The considerations made in the proof of [Art78], Lemma 5.1 show that δ can be varied in its left P-orbit such that its norm is bounded by a multiple of a power of the norm of g. By changing u in its left $N_{\mathcal{P}}$-orbit, we may also assume that u varies only in a compact subset, where its norm is bounded. Consequently, the norm of $u\delta g$ can be assumed to be bounded by a multiple of a power of the norm of g. Together with (6) and (7), this proves (5), and the proof of (4) is complete.

As a consequence of (4) and because λ^t is known to be an orthogonal projection in L_2 if t is sufficiently large, we have

$$\left\|\chi_R\lambda^t f|L_2(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})\right\| \leq \|\chi_{dR}f|L_2(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})\| \leq c_1(f)\,(1+R)^k. \tag{8}$$

It remains to estimate $(1-\chi_R)\lambda^t f$. We have Arthur's estimate [Art80], (1.3)

$$\begin{aligned} &|\lambda^t f(g)| \\ &\leq \sum_{\mathcal{P}_o\subseteq\mathcal{P}_1\subseteq\mathcal{P}_2}\sum_{\delta\in P_1\backslash G}\sum_I\sum_i \chi^{12}(t,\delta g)\cdot\int_{N_I\backslash\mathbb{N}_I}|R(X_i)f(u\delta g)|\,du\,\mathrm{e}^{-n\beta_I(H(\delta g))},\end{aligned}$$

where

$$\chi^{12}(t,\delta g)=F^1(\delta g,t\check{\rho}_o)\sigma_1^2(H(\delta g)-t\check{\rho}_o).$$

For all unfamiliar notations, we refer to the considerations leading to [Art80], (1.3). Note that the measure space S present in [Art80], (1.3) has been chosen to consist of a single point with positive measure. Let $N_{12}(t,g)$ be the (finite, by [Art78], Lemma 5.1) number of classes δ such that $\chi^{12}(t,\delta g)\neq 0$. This number is left G-invariant. Also, let $f_i=R(X_i)f$. We have then

$$\begin{aligned} &|\lambda^t f(g)|^2 \\ &\leq\left(\sum_{\mathcal{P}_o\subseteq\mathcal{P}_1\subseteq\mathcal{P}_2}\sum_{\delta\in P_1\backslash G}\sum_I\sum_i\cdot\chi^{12}(t,\delta g)\int_{N_I\backslash\mathbb{N}_I}|f_i(u\delta g)|\,du\,\mathrm{e}^{-n\beta_I(H(\delta g))}\right)^2 \\ &\leq c\sum_{\mathcal{P}_o\subseteq\mathcal{P}_1\subseteq\mathcal{P}_2}\sum_{\delta\in P_1\backslash G}\sum_I\sum_i\chi^{12}(t,\delta g)N_{12}(t,g)\left(\int_{N_I\backslash\mathbb{N}_I}|f_i(u\delta g)|\,du\,\mathrm{e}^{-n\beta_I(H(\delta g))}\right)^2 \\ &\leq c\sum_{\mathcal{P}_o\subseteq\mathcal{P}_1\subseteq\mathcal{P}_2}\sum_{\delta\in P_1\backslash G}\sum_I\sum_i\chi^{12}(t,\delta g)N_{12}(t,g)\int_{N_I\backslash\mathbb{N}_I}|f_i(u\delta g)|^2\,du\,\mathrm{e}^{-2n\beta_I(H(\delta g))}\end{aligned}$$

and consequently

$$\begin{aligned} (9)\qquad &\int_{G\mathcal{A}_G(\boldsymbol{R})^o\backslash\mathbb{G}}(1-\chi_R(g))\,|\lambda^t f(g)|^2\,dg \\ &\quad\leq c\sum_{\mathcal{P}_o\subseteq\mathcal{P}_1\subseteq\mathcal{P}_2}\sum_I\sum_i\int_{P_1\mathcal{A}_G(\boldsymbol{R})^o\backslash\mathbb{G}}(1-\chi_R(g))\chi^{12}(t,g)N_{12}(t,g) \\ &\quad\times\int_{N_I\backslash\mathbb{N}_I}|f_i(ug)|^2\,du\,\mathrm{e}^{-2n\beta_I(H(g))}\end{aligned}$$

Our first task is to bound $N_{12}(t,g)$. It is obviously bounded by

$$\sum_{P_1\backslash G}\hat{\tau}_1(H(\delta g)-t\check{\rho}_o)\leq\sum_{P_1\backslash G}\hat{\tau}_1(H(\delta g))$$

if $t>0$. Arthur has shown (in [Art78], Lemma 5.1) that the last sum is bounded by a multiple of a power of $\|g\|$. Consequently,

$$(10)\qquad N_{12}(t,g)\leq c\|g\|^l.$$

Next, we want to estimate $\beta_I(H(g))$ from below. We may choose a representative $\hat{g}$ for g which has the form cak, where $a\in\mathbb{A}_o$ and c belongs to a fixed compact subset of $\mathbb{P}_o$. Since

β_I is a positive linear combination of roots in Δ_1^2, it follows from the proof of [Art78], Corollary 6.2. that $|H(a)|$ is bounded by a multiple of $t+\beta_I(H(g))$, hence

$$\| \hat{g} \| \leq c\, e^{D(t+\beta_I(H(g)))} \tag{11}$$

whenever $\chi^{12}(t,g) \neq 0$. A combination of (10) and (11) gives

$$N_{12}(t,g) \leq c\, e^{lD(t+\beta_I(H(g)))}. \tag{12}$$

As the special case $\delta = 1$ in (5), we have

$$1 - \chi_R(g) \leq 1 - \chi_{d^{-1}R}(ug) \tag{13}$$

whenever u and g contribute to (9). It follows from (12) and (13) that (9) is bounded by a multiple of

$$\begin{aligned} (14)\quad & e^{lDt} \sum_{\mathcal{P}_o \subseteq \mathcal{P}_1 \subseteq \mathcal{P}_2} \sum_I \sum_i \int_{P_1 \mathcal{A}_G(\boldsymbol{R})^\circ \backslash \mathbb{G}} \chi^{12}(t,g) e^{(lD-2n)\beta_I(H(g))} \\ & \cdot \int_{N_I \backslash \mathbb{N}_I} (1 - \chi_{d^{-1}R}(ug)) \mid f_i(ug) \mid^2 \, du\, dg \\ & = e^{lDt} \sum_{\mathcal{P}_o \subseteq \mathcal{P}_1 \subseteq \mathcal{P}_2} \sum_I \sum_i \int_{P_1 \mathcal{A}_G(\boldsymbol{R})^\circ \backslash \mathbb{G}} \chi^{12}(t,g) e^{(lD-2n)\beta_I(H(g))} \\ & \times (1 - \chi_{d^{-1}R}(g)) \mid f_i(g) \mid^2 \, dg, \end{aligned}$$

and the equality holds because the factor between the two integrals on the left hand side is $\mathbb{N}_I$-invariant. We may assume that n has been chosen large enough such that $lD - 2n < 0$–its choice influences only the i,X_i, and the constant by which (14) has to be multiplied. Note that by (11) and (6), we have

$$e^{-\beta_I(H(g))} \leq c e^{D't - \varepsilon w_1(g)}.$$

Consequently, (14) is bounded by a multiple of

$$\begin{aligned} (15)\quad & e^{D''t} \sum_{\mathcal{P}_o \subseteq \mathcal{P}_1 \subseteq \mathcal{P}_2} \sum_i \int_{P_1 \mathcal{A}_G(\boldsymbol{R})^\circ \backslash \mathbb{G}} \chi^{12}(t,g) e^{\varepsilon(lD-2n)w_1(g)} \big(1 - \chi_{d^{-1}R}(g)\big) \mid f_i(g) \mid^2 \, dg \\ & = e^{D''t} \sum_{\mathcal{P}_o \subseteq \mathcal{P}_1 \subseteq \mathcal{P}_2} \sum_i \int_{\mathfrak{F}} e^{\varepsilon(lD-2n)w_1(g)} \big(1 - \chi_{d^{-1}R}(g)\big) N_{12}(t,g) \mid f_i(g) \mid^2 \, dg, \end{aligned}$$

where D'' depends on n and $\mathfrak{F}$ was the fundamental domain we had chosen in the proof of (5). By (10) and (7), $N_{12}(t,g)$ is bounded by a multiple of a power of $e^{w_1(g)}$. Consequently, if n was chosen large enough we can estimate the last expression by a multiple of

$$e^{D''t} \sum_i \int_{\mathfrak{F}} (1 - \chi_{d^{-1}R}(g)) e^{-\delta w_1(g)} \mid f_i(g) \mid^2 \, dg,$$

where δ is positive. Now $w_1(g)^{2k}$ is bounded by a multiple of $e^{\delta w_1(g)/2}$, and the last expression can be bounded by

$$e^{D''t-\frac{\delta R}{2}} \sum_i \int_{\mathfrak{F}} w_1(g)^{-2k} \mid f_i(g) \mid^2 \, dg \le e^{D''t-\frac{\delta R}{2}} c_2(f).$$

If we put $R = \frac{2D''t}{\delta}$ and add this estimate to (8), we arrive at (3). The proof of Proposition 1 is complete. □

PROPOSITION 2. – *Let $\{P\}$ be a class of associate parabolic subgroups, k a non-negative integer, $u = (\mathcal{R}, \Lambda, \xi) \in U^k_{\{P\}}$, and φ a $W(u)$-valued distribution with compact support on $\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{R}}$.*

(1) *We have*

$$\int_{\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{R}}} E(\varphi(x), ix)\, dx \in S_{\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o \backslash \mathbb{G}). \tag{16}$$

(2) *We have*

$$\int_{\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{R}}} E(\varphi(x), ix)\, dx \in S_{-\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o \backslash \mathbb{G})$$

if and only if $\psi(\lambda) = \sum_{w \in \Omega(\mathfrak{a}_{\mathcal{R}}, \mathfrak{a}_{\mathcal{R}})} M(w^{-1}, i\lambda)\varphi(w\lambda)$ is in fact a C^∞_c-function.

Proof. – The first assertion is a consequence of Corollary 5.2.1.

The 'only if'-part of the second assertion follows from the first assertion by duality since

$$\langle \psi(\lambda), \chi \rangle = c \langle E_\varphi, E(\chi, i\lambda) \rangle_{L_2}$$

for some constant c, where E_φ is the left hand side of (16) and χ is an arbitrary element of $W(u)$. The right hand side of this equation is C^∞ because $\lambda \to E(\chi, \lambda)$ is a $S_{\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o \backslash \mathbb{G})$-valued C^∞-function by Corollary 5.2.1. To prove the 'if'-part, we first state the following lemma:

LEMMA 1. – *Let $f \in L_2(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o \backslash \mathbb{G})_b$ be a $\mathbb{K}$-finite function, where the subscript 'b' stands for bounded spectrum as in* 5.1. *Then $f \in S_{-\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o \backslash \mathbb{G})$ if for every N there is a C_N such that*

$$\left\| \lambda^t f - f | L_2(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o \backslash \mathbb{G}) \right\| \le C_N t^{-N}$$

for all sufficiently large t.

Thus, we have to show that

$$\mid i(\infty) - i(t) \mid \le C_N t^{-N}$$

with

$$i(t) = \left\| \int_{\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{R}}} \lambda^t E(\varphi(x), ix)\, dx | L_2 \right\|$$

and

$$i(\infty) = \lim_{t\to\infty} i(t).$$

By the result of [Art82], we have

$$| i(t) - j(t) | \leq C_N t^{-N},$$

where

$$j(t) = \int_{\check{\mathfrak{a}}_{\mathcal{R}}\times\check{\mathfrak{a}}_{\mathcal{R}}} \sum_{\tilde{\mathcal{R}}} \sum_{u,v\in\Omega(\mathfrak{a}_{\mathcal{R}},\mathfrak{a}_{\tilde{\mathcal{R}}})} \frac{\langle M(u,ix)\varphi(x), M(v,iy)\varphi(y)\rangle \exp(it\langle\check{\rho}_o, ux-vy\rangle)}{\prod_{\alpha\in\Delta_{\tilde{\mathcal{R}}}}\langle\check{\alpha}, ux-vy\rangle}\, dx\, dy.$$

The first sum is over all parabolic subgroups $\tilde{\mathcal{R}}$ which are associate to $\mathcal{R}$. The following lemma implies that the integrand is in fact a C_c^∞-function and that $j(t)$ has the asymptotic behaviour we need.

LEMMA 2. – *Let $\mathcal{A}$ be the space of collections of complex-valued Schwartz functions $\varphi_{\tilde{\mathcal{R}},u,v}$ on $\check{\mathfrak{a}}_{\mathcal{R}}\times\check{\mathfrak{a}}_{\mathcal{R}}$ where $\tilde{\mathcal{R}}$ is a parabolic subgroup associate to $\mathcal{R}$ and s and t are in $\Omega(\mathfrak{a}_{\mathcal{R}},\mathfrak{a}_{\tilde{\mathcal{R}}})$, with the following property: If $\langle\check{\alpha}, ux-vy\rangle = 0$ for $\alpha\in\Delta_{\tilde{\mathcal{R}}}$ and if $s_\alpha\in\Omega(\mathfrak{a}_{\tilde{\mathcal{R}}},\mathfrak{a}_{\hat{\mathcal{R}}})$ is the generalised symmetry (in the sense of*[MW95], I.1.7,I.1.8*) belonging to α, then $\varphi_{\tilde{\mathcal{R}},u,v}(x,y) = \varphi_{\hat{\mathcal{R}},s_\alpha u,s_\alpha v}(x,y)$. Then for all elements of $\mathcal{A}$, the integrand*

$$(17)\qquad j(t) = \int_{\check{\mathfrak{a}}_{\mathcal{R}}\times\check{\mathfrak{a}}_{\mathcal{R}}} \sum_{\tilde{\mathcal{R}}} \sum_{u,v\in\Omega(\mathfrak{a}_{\mathcal{R}},\mathfrak{a}_{\tilde{\mathcal{R}}})} \frac{\varphi_{\tilde{\mathcal{R}},u,v}(x,y) \exp(it\langle\check{\rho}_o, ux-vy\rangle)}{\prod_{\alpha\in\Delta_{\tilde{\mathcal{R}}}}\langle\check{\alpha}, ux-vy\rangle}\, dx\, dy.$$

is a C_c^∞-function and we have

$$| j(t) - j(\infty) | \leq C_N t^{-N},$$

where $j(\infty) = \lim_{t\to\infty} j(t)$.

To finish the proof of Proposition 2, it suffices to put

$$\varphi_{\tilde{\mathcal{R}},u,v} = \langle M(u,ix)\varphi(x), M(v,iy)\varphi(y)\rangle$$

in this lemma. It is a consequence of the functional equation **Fun 3** in 5.2 and of the unitarity of the operators $M(w,iz)$ for $z\in\check{\mathfrak{a}}_{\mathcal{R}}$ that this collection is in $\mathcal{A}$. □

Proof of Lemma 1. – As f is concentrated in the bounded part of the spectrum of the Casimir operator, there exists a function φ with compact support such that $f = \varphi(\omega)f$. Since by Corollary 5.1.1 $\omega^k\varphi(\omega)$ is a bounded operator in $(L_{2,w_l})_X$ for all integers l and $k\geq 0$, it suffices to prove $f\in L_{2,w_l}$ for all l. By duality, it suffices to prove that

$$\langle f, g\rangle_{L_2}$$

is finite for all $\mathbb{K}$-finite functions g which belong to L_{2,w_l} for some integer l. But then $\varphi(\omega)g$ is in $S_{\log}(G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G})$, such that we may as well assume that $g \in S_{\log}(G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G})$. By Proposition 1, we know that $\|\lambda^t g|L_2\|$ increases only polynomially. This implies that

$$\langle f, g\rangle_{L_2} = \langle f, \lambda^{t_o} g\rangle_{L_2} + \sum_{k=0}^{\infty}\langle(\lambda^{2^{k+1}t_o} - \lambda^{2^k t_o})f, \lambda^{2^{k+1}t_o} g\rangle_{L_2}$$

is finite, where t_o is a sufficiently large number.

The proof of Lemma 1 is complete. Q.E.D.

Proof of Lemma 2. – The sum inside the integral defines a C_c^∞-function because the singularity of the summand containing $\varphi_{\tilde{\mathcal{R}},u,v}$ along the hyperplane $\langle\check{\alpha}, ux - vy\rangle = 0$ is cancelled by the singularity of the summand containing $\varphi_{\hat{\mathcal{R}},s_\alpha u,s_\alpha v}$ along the same hyperplane. In particular, if the Fourier transforms of the $\varphi_{\tilde{\mathcal{R}},u,v}$ have compact support (such that the $\varphi_{\tilde{\mathcal{R}},u,v}$ are holomorphic by the Paley-Wiener-Schwartz-theorem), then the integrand is again holomorphic. To estimate $j(t) - j(\infty)$, we first claim that

$$(18) \qquad j(t) = \lim_{\varepsilon\to+0}\int_{\check{\mathfrak{a}}_\mathcal{R}\times\check{\mathfrak{a}}_\mathcal{R}}\sum_{\tilde{\mathcal{R}}}\sum_{u,v\in\Omega(\mathfrak{a}_\mathcal{R},\mathfrak{a}_{\tilde{\mathcal{R}}})}\frac{\varphi_{\tilde{\mathcal{R}},u,v}(x,y)\exp(it\langle\check{\rho}_o, ux - vy\rangle)}{\prod_{\alpha\in\Delta_{\tilde{\mathcal{R}}}}\langle\check{\alpha}, u(x + i\varepsilon\rho_\mathcal{R}) - vy\rangle}\,dx\,dy.$$

By Cauchy's integration formula, this is true if the Fourier transforms of the $\varphi_{\tilde{\mathcal{R}},u,v}$ have compact support. Since the space of functions with that property is easily seen to be dense in $\mathcal{A}$ and since both (17) and (18) are continuous functionals on $\mathcal{A}$, the same holds in general. Let $\hat{\varphi}_{\tilde{\mathcal{R}},u,v}$ be the Fourier transform of $\varphi_{\tilde{\mathcal{R}},u,v}$. This is a Schwartz function on $\mathfrak{a}_\mathcal{R}\times\mathfrak{a}_\mathcal{R}$. Then the summand in (18) belonging to $(\tilde{\mathcal{R}}, u, v)$ is proportional to

$$(19) \qquad \int_{C_{\tilde{\mathcal{R}},u,v}}\hat{\varphi}_{\tilde{\mathcal{R}},u,v}(u^{-1}\xi + tu^{-1}\check{\rho}_o, -v^{-1}\xi - tv^{-1}\check{\rho}_o)\,d\xi,$$

where $C_{\tilde{\mathcal{R}},u,v}$ is the set of all linear combinations

$$\xi = \sum_{\alpha\in\Delta_{\tilde{\mathcal{R}}}} t_\alpha\check{\alpha}$$

with

$$t_\alpha\begin{cases} > 0 & \text{if } \langle\check{\alpha}, u\rho_{\tilde{\mathcal{R}}}\rangle < 0 \\ < 0 & \text{if} \langle\check{\alpha}, u\rho_{\tilde{\mathcal{R}}}\rangle > 0\end{cases}.$$

Since $\hat{\varphi}_{\tilde{\mathcal{R}},s,t}$ is a Schwartz function, the difference between (19) and its limit as $t\to\infty$ is a rapidly decreasing function of t.

The proof of Lemma 2 is complete. Q.E.D.

Proof of Theorem 12. – It follows from Proposition 2 that the right hand sides of (1) and of (2) map continuously into the corresponding left hand sides. By the second part of Proposition 2 and the well-known spectral decomposition of L_2, (2) is an isomorphism. To derive the first assertion from this, note that $S_{\log,b}(G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G})_X$ is the bounded part (in a sense similar to 5.1) of the dual of $S_{-\log}(G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G})_X$. A similar duality holds for the right hand sides of (1) and (2). By the unitarity of the Fourier-Eisenstein transform, this means that (1) follows from (2) by duality.

The proof of Theorem 12 is complete. Q.E.D.

5.4. Proof of Acyclicity

We fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}_{\boldsymbol{C}}$ containing $(\mathfrak{a}_o)_{\boldsymbol{C}}$. Thus, $\mathfrak{h}$ is a complex vector space. Let $u = (\mathcal{R}, \Lambda, \chi) \in U^k_{\{P\}})$ and $\mathcal{R} = \mathcal{M}_{\mathcal{R}} A_{\mathcal{R}} N_{\mathcal{R}}$ be the Langlands decomposition for $\mathcal{R}$ compatible with the chosen Langlands decomposition for $\mathcal{P}_o$. Then there are Harish-Chandra isomorphisms

$$\xi : \mathfrak{Z}(\mathfrak{m}_{\mathcal{G}}) \to \boldsymbol{S}(\mathfrak{h} \cap \mathfrak{m}_{\mathcal{G}})^{W(\mathfrak{h},\mathfrak{g})}$$

and

$$\xi_{\mathcal{R}} : \mathfrak{Z}(\mathfrak{m}_{\mathcal{R}}) \to \boldsymbol{S}(\mathfrak{h} \cap \mathfrak{m}_{\mathcal{R}})^{W(\mathfrak{h}\cap\mathfrak{m}_{\mathcal{R}},\mathfrak{m})}$$

(*cf.* [Wal88], Theorem 3.2.3.). Under $\xi_{\mathcal{R}}$, χ corresponds to a Weyl group orbit in $\mathfrak{h} \cap \mathfrak{m}_{\mathcal{R}}$. We will fix a representative of this orbit and denote it by the same letter χ. For an ideal $\mathcal{J}$ of finite codimension in $\mathfrak{Z}(\mathfrak{m}_{\mathcal{G}})$, let

$$D'(i\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{R}})_{\mathrm{supp}_u \mathcal{J}} \subset D'(i\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{R}})$$

be the vector space of distributions supported in the intersection of $i\mathfrak{a}^{\mathcal{G}}_{\mathcal{R}}$ with the finite set of points

$$\mathrm{supp}_u \mathcal{J} = \{x \in (\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{R}})_{\boldsymbol{C}} \mid \text{such that } \{A \in \mathfrak{Z}(\mathfrak{m}_{\mathcal{G}}) \mid \xi(A)(x+\chi) = 0\} \text{ contains } \mathcal{J}.\}$$

Let

$$H^n_{\mathrm{supp}_u \mathcal{J} \cap i\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{R}}}(i\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{R}}) = \sum_{x \in \mathrm{supp}_u \mathcal{J} \cap i\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{R}}} \det\left(\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{R}}\right) \otimes \boldsymbol{S}(\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{R}}).$$

This space can be canonically identified with the cohomology with support in $i\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{R}} \cap \mathrm{supp}_u \mathcal{J}$ of the affine scheme defined by $\left(\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{R}}\right)_{\boldsymbol{C}}$. It is a module over the Abelian Lie algebra $\mathfrak{a}_{\mathcal{R}}$ if $\xi \in \mathfrak{a}_{\mathcal{R}}$ acts on the first factor (the highest exterior power of $\left(\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{R}}\right)_{\boldsymbol{C}}$) by multiplication by $\langle x, \xi \rangle$ and on the second factor (the symmetric algebra of $\left(\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{R}}\right)_{\boldsymbol{C}}$) by contraction.

We are now ready to formulate and prove the main theorem of this chapter.

THEOREM. 13

- *We have*

$$\mathfrak{Fin}^j_{\mathcal{J}} S_{\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o \backslash \mathbb{G}) = \{0\} \quad \text{if } j > 0$$

and

$$(1)\quad \mathfrak{Fin}_{\mathcal{J}} S_{\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o \backslash \mathbb{G}) \cong \bigoplus_{\{P\}} \bigoplus_{k=0}^{\mathrm{rank}(\{P\})} \operatorname*{colim}_{\mathcal{U}^k_{\{P\}}} W(u) \otimes D'(i\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{R}})_{\mathrm{supp}_u \mathcal{J}}$$

$$\cong \bigoplus_{\{P\}} \bigoplus_{k=0}^{\mathrm{rank}(\{P\})} \operatorname*{colim}_{\mathcal{U}^k_{\{P\}}} \mathrm{Ind}^{\mathcal{G}}_{\mathcal{R}} V(u) \otimes D'(i\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{R}})_{\mathrm{supp}_u \mathcal{J}}.$$

- *We have*

$$\mathfrak{Fin}^j_{\mathcal{J}} S_{-\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o \backslash \mathbb{G}) \cong \bigoplus_{\mathrm{rank}(\{P\}) \geq j} \operatorname*{colim}_{\mathcal{U}^j_{\{P\}}} \mathrm{Ind}^{\mathcal{G}}_{\mathcal{R}} V(u) \otimes H^n_{\mathrm{supp}_u \mathcal{J} \cap i\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{R}}}(\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{R}})$$

The spectral sequence (4.4) *degenerates, and furthermore its limit filtration splits canonically.*

Proof. – By 10, we may substitute $S_{\log,b}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$ for $S_{\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$ and $S_{-\log,b}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$ for $S_{-\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$ in the assertion we want to prove. In Theorem 7.2 we may put $R = S(\mathfrak{h}\cap\mathfrak{m}_{\mathcal{G}})$ to compute $\mathfrak{Fin}^j_{\mathcal{J}}$ because R is free over $\mathfrak{Z}(\mathfrak{m}_{\mathcal{G}})$ (*cf.* [Bou68], Ch. V, §5.2., Théorème 1). Then the assertion about $S_{\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$ reduces to the following lemma:

LEMMA 1. – *Let $\boldsymbol{x}^k = \left(x_1^k, \ldots, x_n^k\right)$ be the collection of the k-th powers of the coordinate functions on $\boldsymbol{R}^n$ acting by pointwise multiplication on $D'_c(\boldsymbol{R}^m)$, the space of distributions on the subspace characterised by the vanishing of the last $n-m$ coordinate functions. Then*

$$\operatorname*{colim}_k H^j(C^*(\boldsymbol{x}^k, D'_c(\boldsymbol{R}^m))) = \begin{cases} D'_{\{0\}}(\boldsymbol{R}^m), \text{ the space of distributions supported at } 0 & \text{if } j=0 \\ 0 & \text{if } j>0 \end{cases}$$

where the inductive limit is taken in the same way as in the proof of Theorem 7.2

The assertion about $\mathfrak{Fin}^j_{\mathcal{J}} S_{-\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$ boils down to a similar lemma which we omit, and the spectral sequence degenerates because Theorem 12 provides a splitting of $S_{-\log,b}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$ into submodules for which $\mathfrak{Fin}^j_{\mathcal{J}}$ vanishes for all but one value of j. The proof of the theorem is complete. □

Proof of Lemma 1. – The Koszul complex inside the inductive limit is the nuclear tensor product of m copies of the comple x

$$D'_c(\boldsymbol{R}) \xrightarrow{x^k} D'_c(\boldsymbol{R})$$

and $n-m$ copies of the comple x

$$\boldsymbol{C} \xrightarrow{0} \boldsymbol{C}.$$

It is well known that the cohomology of the first complex is zero in degree one and spanned by the first k derivatives in degree zero, and that it continuously splits away from the complex. Consequently, the cohomology of the Koszul complex is

$$\boldsymbol{C}\left[\left\{\frac{\partial^{i_1}}{\partial x_1^{i_1}}\cdots\frac{\partial^{i_m}}{\partial x_m^{i_m}}\delta_0 \middle| 0 \le i_j < k\right\}\right] \otimes \Lambda^*\left(\boldsymbol{C}^{n-m}\right),$$

where the first factor is concentrated in degree zero, δ_0 is the Dirac distribution, and the second factor is the exterior algebra. The transition homomorphism is the natural inclusion on the first factor, the identity on the degree zero component of the second factor, and zero on $\Lambda^{>0}\left(\boldsymbol{C}^{n-m}\right)$.

The proof of Lemma 1 is complete. Q.E.D.

6. A filtration on spaces of automorphic forms

The aim of this section is to express the space of all automorphic forms on $\mathcal{G}$ in terms of the space of square integrable automorphic forms on Levi components of $\mathcal{G}$. This cannot be done completely. However, it is possible to define a filtration on the space of all automorphic forms on $\mathcal{G}$ whose successive quotients can be expressed as induced representations from the space of all square integrable automorphic forms on Levi components of $\mathcal{G}$.

Let w_n be the same as in 5. For any weight function ρ, we put

(1)
$$S_{\rho+\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G}) = \bigcup_{n=0}^{\infty} S_{w_{-n}\rho}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$$
$$S_{\rho-\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G}) = \bigcap_{n=0}^{\infty} S_{w_n\rho}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G}).$$

As usual, $\mathcal{J}$ is an ideal of finite codimension in $\mathfrak{Z}(\mathfrak{m}_{\mathcal{G}})$. For every standard parabolic subgroup $\mathcal{P}$ and every $f \in \mathfrak{Fin}_{\mathcal{J}} S_{\rho_{-\tau}+\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$, the constant term of f has the following form:

(2)
$$f_{\mathcal{N}_{\mathcal{P}}}(g) = \sum_{\lambda\in(\check{\mathfrak{a}}_{\mathcal{P}})_{\boldsymbol{C}}} \mathrm{e}^{\langle H_{\mathcal{P}}(g),\rho_{\mathcal{P}}+\lambda\rangle} f_{\mathcal{P},\lambda}(H_{\mathcal{P}}(g))(g)$$

for $g \in G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G}$. Here

(3)
$$f_{\mathcal{P},\lambda} \in \boldsymbol{S}\Big(\big(\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{G}}\big)_{\boldsymbol{C}}\Big) \otimes S_{\infty}(\mathbb{N}_{\mathcal{P}}\mathcal{A}_{\mathcal{P}}(\boldsymbol{R})^o P\backslash\mathbb{G}).$$

To explain what the second factor in the tensor product is, we note that

(4)
$$\mathbb{N}_{\mathcal{P}}\mathcal{A}_{\mathcal{P}}(\boldsymbol{R})^o P\backslash\mathbb{G} = L_{\mathcal{P}}\mathcal{A}_{\mathcal{P}}(\boldsymbol{R})^o\backslash\mathbb{L}_{\mathcal{P}}\mathbb{K}$$

and define $S_{\infty}(\mathbb{N}_{\mathcal{P}}\mathcal{A}_{\mathcal{P}}(\boldsymbol{R})^o P\backslash\mathbb{G})$ as the space of those $\mathbb{K}$-finite functions g on (4) such that

$$g(lk) \in S_{\infty}(L_{\mathcal{P}}\mathcal{A}_{\mathcal{P}}(\boldsymbol{R})^o\backslash\mathbb{L}_{\mathcal{P}})$$

for all $k \in \mathbb{K}$. Elements of $\boldsymbol{S}\Big(\big(\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{G}}\big)_{\boldsymbol{C}}\Big)$ can be viewed as polynomials on $\mathfrak{a}_{\mathcal{P}}$. Thus, for every $x \in \mathfrak{a}_{\mathcal{P}}$, $f_{\mathcal{P},\lambda}(x)$ is a function in $S_{\infty}(\mathbb{N}_{\mathcal{P}}\mathcal{A}_{\mathcal{P}}(\boldsymbol{R})^o P\backslash\mathbb{G})$, and $f_{\mathcal{P},\lambda}(x)(g)$ is the value of that function at g. This completes our explanation of the right hand side of (2). It is easy to see that $f_{\mathcal{P},\lambda}$ can be different from zero only for finitely many values of λ. If $\mathcal{P} \in \{P\}$ and $f \in S_{\rho_{-\tau}+\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})_{\{}P\}$, then $f_{\mathcal{P},\lambda}$ has values in a space of cusp forms.

We will need the following geometrical lemma of Langlands

LEMMA 1. – *For every standard parabolic subgroups $\mathcal{P} \subset \mathcal{R}$ and every $\lambda \in \check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{R}}$ there is a unique subset $P(\lambda) \subseteq \Delta_{\mathcal{P}}^{\mathcal{R}}$ such that*

$$\lambda = \sum_{\alpha\in\Delta_{\mathcal{P}}^{\mathcal{R}}-P(\lambda)} a_{\alpha}\omega_{\alpha}^{\mathcal{R}} - \sum_{\beta\in P(\lambda)} b_{\beta}\beta$$

with $a_\alpha > 0$ *and* $b_\beta \geq 0$. *Let*

$$(5) \qquad \begin{aligned}(\lambda)_+ &= \sum_{\alpha\in\Delta_{\mathcal{P}}^{\mathcal{R}}-P(\lambda)} a_\alpha \omega_\alpha^{\mathcal{R}}\\ (\lambda)_- &= -\sum_{\beta\in P(\lambda)} b_\beta \beta\end{aligned}$$

This decomposition has the property

$$(6) \qquad \mathrm{i}_{\check{\mathfrak{a}}_{\mathcal{Q}}^{\mathcal{R}}\hookrightarrow\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{R}}}(\lambda_+) = \left(\mathrm{i}_{\check{\mathfrak{a}}_{\mathcal{Q}}^{\mathcal{R}}\hookrightarrow\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{R}}}\lambda\right)_+$$

for standard parabolic subgroups $\mathcal{P} \subset \mathcal{Q} \subset \mathcal{R}$ *and* $\lambda \in \check{\mathfrak{a}}_{\mathcal{Q}}^{\mathcal{R}}$. *Furthermore,*

$$(7) \qquad \vartheta_+ \in \lambda_+ - \overline{{}^+\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{R}}} \text{ if } \vartheta \in \lambda - \overline{{}^+\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{R}}}$$

Proof. – For a proof of the existence and uniqueness of the decomposition, we refer to [BW80], Lemma IV.6.11., p. 143 or [Wal88], 5.A.1 or to [Art], Lemma 6.3, which for Λ in the positive Weyl chamber boils down to a decomposition (5) for elements of $\mathfrak{a}_{\mathcal{P}}^{\mathcal{R}}$. The equality (6) is a consequence of the well-known inclusion ${}^+\check{\mathfrak{a}}_{\mathcal{Q}}^{\mathcal{R}} \subset {}^+\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{R}}$. Finally, (7) is [BW80], Lemma IV.6.13 or [Wal88], 5.A.1.2.(4). □

The filtration can be defined on $\mathfrak{Fin}_{\mathcal{J}} S_{\rho_{-\tau}+\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$ with $\tau \in \overline{\check{\mathfrak{a}}_o^{\mathcal{G}+}}$, the closure of the positive Weyl chamber. It depends on the choice of a certain function T. To explain what T is, consider the expressions (2). There are only finitely many values of λ for which $f_{\mathcal{P},\lambda}$ may possibly be different from zero. Let $F_{\mathcal{J}}$ be the finite set of possible values for $\Re(\lambda)_+$. We assume that T is a function from $F_{\mathcal{J}}$ to the set of integers such that

$$(8) \qquad T(\lambda) < T(\vartheta) \text{ if } \lambda \neq \vartheta \text{ and } \vartheta \in \lambda - \overline{{}^+\check{\mathfrak{a}}_o}.$$

Let

$$(9) \quad \begin{aligned}&\mathfrak{Fin}_{\mathcal{J}} S_{\rho_{-\tau}+\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})_{\{}P\}^{T,i}\\ &\quad = \left\{f \in \mathfrak{Fin}_{\mathcal{J}} S_{\rho_{-\tau}+\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})_{\{}P\} \;\middle|\; \text{such that } T\big(\Re(\lambda)_+\big) \geq i \text{ if } f_{\mathcal{P},\lambda} \neq 0\right\}.\end{aligned}$$

This is the filtration we announced at the beginning of this chapter. It is clear that this filtration has finite length.

Let $M^k_{\mathcal{J},\{P\}}$ be the set of triples $t = (\mathcal{R}, \Lambda, \chi)$ with the following properties:

- $\mathcal{R} = \mathcal{M}_{\mathcal{R}}\mathcal{A}_{\mathcal{R}}\mathcal{N}_{\mathcal{R}}$ is a standard parabolic subgroup containing an element of $\{P\}$, and rank$\mathcal{R} = k$.
- $\Lambda\colon \mathbb{A}_{\mathcal{R}}/\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o A_{\mathcal{R}} \to \boldsymbol{C}^*$ is a continuous character. Let $\lambda_t \in \left(\check{\mathfrak{a}}_{\mathcal{R}}^{\mathcal{G}}\right)_{\boldsymbol{C}}$ be the differential of the archimedean component of Λ. We assume $\Re(\lambda_t) \in \overline{\check{\mathfrak{a}}_{\mathcal{R}}^+}$, the closure of the positive Weyl chamber in $\check{\mathfrak{a}}_{\mathcal{R}}$. By $M^k_{\mathcal{J},\{P\},\tau}$ we will denote the subset of those elements of $M^k_{\mathcal{J},\{P\}}$ for which

$$\Re(x) \in \tau - \overline{{}^+\check{\mathfrak{a}}_o^{\mathcal{G}}}.$$

Also, $M^{k,T,i}_{\mathcal{J},\{P\}}$ (resp. $M^{k,T,i}_{\mathcal{J},\{P\},\tau}$) is the set of those elements of $M^{k}_{\mathcal{J},\{P\}}$ (resp. $M^{k}_{\mathcal{J},\{P\},\tau}$) for which $T(\Re\lambda_t) = i$.

- The last entry is a character $\chi\colon \mathfrak{Z}(\mathfrak{m}) \to \boldsymbol{C}^*$ of the center of the universal enveloping algebra of $\mathfrak{m}$. If

$$\tilde{\Lambda}(a) = \Lambda(a)\, \mathrm{e}^{-\langle \lambda_t, H(a)\rangle},$$

then

$$u_t = (\mathcal{R}, \tilde{\Lambda}, \chi) \tag{10}$$

is an element of $U^k_{\{}P\}$. We assume that $\lambda_t \in \operatorname{supp}_{u_t}\mathcal{J}$.

For $t \in M^{k}_{\mathcal{J},\{P\}}$ let D_t be the symmetric algebra $S\Big(\big(\mathfrak{a}^{\mathcal{G}}_{\mathcal{R}}\big)_{\boldsymbol{C}}\Big)$ on which $\xi \in \mathfrak{a}_{\mathcal{R}}$ acts by multiplication (in the symmetric algebra) by

$$\mathrm{pr}_{\mathfrak{a}_{\mathcal{R}}\to\mathfrak{a}^{\mathcal{G}}_{\mathcal{R}}}(\xi) + \langle \xi, \rho_{\mathcal{R}_t} + \lambda_t\rangle.$$

D_t can be identified with the space of holomorphic functionals with support in λ_t, *i.e.*, with the set of linear applications

$$\{\text{holomorphic functions in a neighbourhood of } \lambda_t\} \to \boldsymbol{C}$$

which can be expressed as a finite sum of iterated derivatives at λ_t. Thus, for $\delta \in D_t$ and a holomorphic function f, $\delta(f)$ is a complex number. We put

$$\begin{aligned} M(t) &= W(u_t) \otimes D_t \\ &= \mathrm{Ind}^{\mathcal{G}}_{\mathcal{R}} V(u_t) \otimes D_t \end{aligned} \tag{11}$$

These are isomorphic $(\mathfrak{g}, K, \mathbb{G}_f)$-modules, where the $(\mathfrak{g}, K, \mathbb{G}_f)$-structure on $W(u_t) \otimes D_t$ is defined analogously to (5.2.2).

Now we want to turn $M^{k}_{\mathcal{J},\{P\}}$ into a groupoid and $M(t)$ into a functor from that groupoid into the category of $(\mathfrak{g}, K, \mathbb{G}_f)$-modules. Of course, objects of $\mathcal{M}^{k}_{\mathcal{J},\{P\}}$ are elements t of $M^{k}_{\mathcal{J},\{P\}}$. A morphism from $t = (\mathcal{R}, \Lambda, \chi)$ to $\tilde{t} = (\tilde{\mathcal{R}}, \tilde{\Lambda}, \tilde{\chi})$ is an element of the Weyl set $\Omega(\mathfrak{a}_{\mathcal{R}}, \mathfrak{a}_{\tilde{\mathcal{R}}})$ which maps χ to $\tilde{\chi}$ and Λ to $\tilde{\Lambda}$. Let $\mathcal{M}^{k}_{\mathcal{J},\{P\},\tau}$, $\mathcal{M}^{k,T,i}_{\mathcal{J},\{P\}}$, and $\mathcal{M}^{k,T,i}_{\mathcal{J},\{P\},\tau}$ be the full subcategories of $\mathcal{M}^{k}_{\mathcal{J},\{P\}}$ defined by the corresponding subsets of $M^{k}_{\mathcal{J},\{P\}}$. Then $M(.)$ becomes a functor in the same way as in (5.2.6). Thus, if $\frac{\partial^\alpha}{\partial\lambda^\alpha} \in D_t$ is given by a multi-index α, then

$$w_*\left(\phi \otimes \frac{\partial^\alpha}{\partial\lambda^\alpha}\right) = \sum_{\beta+\gamma=\alpha} \binom{\alpha}{\beta} \left(\frac{\partial^\gamma}{\partial\lambda^\gamma} M(w,\lambda)\phi\right)_{\lambda=\lambda_t} \otimes \frac{\partial^\beta}{\partial(w\lambda)^\beta}. \tag{12}$$

This is justified by the following lemma.

LEMMA 2. – *If $w \in Hom_{\mathcal{M}^{k}_{\mathcal{J},\{P\}}}(t, \tilde{t})$, then $M(w, \lambda)$ is holomorphic at $\lambda = \lambda_t$.*

Proof. – If both λ_t and $\lambda_{\tilde{t}}$ are viewed as elements of $(\check{\mathfrak{a}}_o)_{\boldsymbol{C}}$, then $\Re\lambda_t$ and $\Re\lambda_{\tilde{t}}$ both belong to $\overline{\check{\mathfrak{a}}_o^+}$, and w is the restriction to $\check{\mathfrak{a}}_{\mathcal{R}}$ of an element $\overline{w}$ of $\Omega(\check{\mathfrak{a}}_o, \check{\mathfrak{a}}_o)$ with $\overline{w}\lambda_t = \lambda_{\tilde{t}}$. By the theory of root systems, this implies $\Re\lambda_t = \Re\lambda_{\tilde{t}}$ in $\check{\mathfrak{a}}_o$. More precisely, there exists a parabolic subgroup $\mathcal{P} \supseteq \mathcal{R} \bigcup \tilde{\mathcal{R}}$ such that $\Re\lambda_t = \Re\lambda_{\tilde{t}} \in \check{\mathfrak{a}}_{\mathcal{P}}$ and $\overline{w}$ identically acts on $\check{\mathfrak{a}}_{\mathcal{P}}$. Then $M(w, x + \lambda_t) = M(x)$ by **Fun 3**. Since it is known that $M(w, x)$ is holomorphic at $\Re(x) = 0$, this completes the proof. □

If the Eisenstein series $E(\phi, \lambda)$ belonging to $\phi \in W(u_t)$ is regular at $\lambda = \lambda_t$, we can define an homomorphism of $(\mathfrak{g}, K, \mathbb{G}_f)$-modules

$$M(t) = W(u_t) \otimes D_t \to S_\infty(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o \backslash \mathbb{G})_{\{P\}}$$
$$\phi \otimes \delta \to \delta E(\phi, \lambda).$$

However, in general the Eisenstein series $E(\phi, \lambda)$ will have singularities at λ_t. In this case, we will have to evaluate a mean value. Let

$$\sigma_i : \langle \check{\alpha}_i, \quad \lambda - \lambda_t \rangle = 0$$

$0 < i \leq i_t$, be the singular hyperplanes of $E(\phi, \lambda)$ passing through λ_t. The space $\boldsymbol{G}$ of germs at λ_t of meromorphic functions which have singularities only among these hyperplanes can be given a nuclear locally convex topology such that it contains the space $\boldsymbol{G}_o$ of germs of holomorphic functions as a closed subspace. Let **MW** be the following linear functional on $\boldsymbol{G}$ extending the functional $f \to f(\lambda_t)$ on $\boldsymbol{G}_o$. Let $\xi \in \check{\mathfrak{a}}_{\mathcal{R}}^{\mathcal{G}}$ be a vector with $\langle \check{\alpha}_i, \xi \rangle \neq 0$. If $f \in \boldsymbol{G}$, we have a Laurent series

$$f(\lambda_t + T\xi) = \sum_{k \gg -\infty} f_k T^k$$

and put $\mathbf{MW}f = f_0$. For any complete locally convex vector space $\boldsymbol{B}$, **MW** defines a continuous operator $\boldsymbol{B}\widehat{\otimes}\boldsymbol{G} \to \boldsymbol{B}$ on the space of germs of meromorphic functions with values in $\boldsymbol{B}$ and singularities only along the σ_i. This operator will also be denoted by **MW**. It commutes with continuous homomorphisms of complete locally convex spaces, in particular with evaluating the constant term of functions in $S_{\rho_{-\tau}+\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o \backslash \mathbb{G})$. Of course there is an ambiguity in the definition of **MW**, but we will see in the next theorem that this does not matter for the goals of this section.

Now we consider the map

$$M(t) = W(u_t) \otimes D_t \to S_\infty(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o \backslash \mathbb{G})_{\{P\}} \tag{13}$$
$$\phi \otimes \delta \to \mathbf{MW}\delta E(\phi, \lambda).$$

Unless the Eisenstein series is regular at λ_t, this is not a homomorphism of $(\mathfrak{g}, K, \mathbb{G}_f)$-modules. However, we will see that it becomes a homomorphism of $(\mathfrak{g}, K, \mathbb{G}_f)$-modules if a certain quotient of the filtration (9) is taken as its target. This allows us to express all the quotients of (9) as induced representations from unitary representations of the Levi components of parabolic subgroups of $\mathcal{G}$.

THEOREM 14. – *If $\tau \in \overline{\check{\mathfrak{a}}_o^{\mathcal{G}+}}$, then for every $i \geq 0$,* (13) *induces an isomorphism*

$$(14)\qquad \bigoplus_{k=0}^{\operatorname{rank}\{P\}} \operatorname*{colim}_{\mathcal{M}^{k,T,i}_{\mathcal{J},\{P\},\tau}} M(t) \cong$$

$$\cong \mathfrak{Fin}_{\mathcal{J}} S_{\rho_{-\tau}+\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})^{T,i}_{\{P\}} \Big/ \mathfrak{Fin}_{\mathcal{J}} S_{\rho_{-\tau}+\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})^{T,i+1}_{\{P\}}$$

of $(\mathfrak{g}, K, \mathbb{G}_f)$-modules. This isomorphism is independent of the choice of **MW**.

The following corollary is an immediate consequence of the theorem. It is the number field version of a result of J.-L. Waldspurger [MW95], Appendice II in the function field case.

COROLLARY 1. – *Any automorphic form, i.e., any element of $\mathfrak{Fin}_{\mathcal{J}} S_\infty(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$, can be written as a sum of principal values of derivatives of cuspidal and residual Eisenstein series. Since the residual Eisenstein series are residues of cuspidal Eisenstein series, this also means that any automorphic form can be written as a sum of Laurent coefficients of cuspidal Eisenstein series.*

Proof. – To prove the theorem, we will proceed in several steps.

STEP 1. – We want to verify that the image (13) is contained in $S_{\rho_{-\tau}+\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$ if $t \in M^k_{\mathcal{J},\{P\},\tau}$. To do this, we first mention that [Lan76], Lemma 5.1 can be reformulated as one half of the following theorem.

THEOREM 15. – *Let $f \in \mathfrak{Fin}_{\mathcal{J}} S_\infty(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})_{\{P\}}$ and $\tau \in \overline{\check{\mathfrak{a}}_o^{\mathcal{G}+}}$, then the following assertions are equivalent:*

(1) *Whenever $f_{\mathcal{P},\lambda}$ does not identically vanish in* (2), *for $\mathcal{P} \in \{P\}$, then we have $\Re(\lambda) \in \tau - \overline{{}^+\check{\mathfrak{a}}_o^{\mathcal{G}}}$, where we use the embedding $\check{\mathfrak{a}}_{\mathcal{P}} \subset \check{\mathfrak{a}}_o$.*
(2) *The same assertion, but for all standard parabolic subgroups $\mathcal{P}$.*
(3) *$f \in S_{\rho_{-\tau}+\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$.*

We will defer the (less difficult) proof of the other half until we have finished the proof of Theorem 14. It should be mentioned that this half of the proof of Theorem 15 is the only place in this chapter where the condition $\tau \in \overline{\check{\mathfrak{a}}_o^{\mathcal{G}+}}$ comes in.

By Theorem 15 and **Fun 4**, the image of (13) is contained in $S_{\rho_{-\tau}+\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$ if we can show that $\omega\Re(\lambda_t) \in \tau - \overline{{}^+\check{\mathfrak{a}}_o^{\mathcal{G}}}$ for all standard parabolic subgroups $\mathcal{P} \in \{P\}$ and all $\omega \in \Omega(u_t, \mathcal{P})$. To do this, note that

$$(15)\qquad y = \Re\lambda_t - \omega\Re\lambda_t = \Re\lambda_t - \hat{\omega}\Re\lambda_t - \omega 0 \in \overline{{}^+\check{\mathfrak{a}}_o^{\mathcal{G}}}$$

because $\Re\lambda_t - \hat{\omega}\Re\lambda_t \in \overline{{}^+\check{\mathfrak{a}}_o^{\mathcal{G}}}$ by the theory of root systems since $\Re\lambda_t \in \overline{\check{\mathfrak{a}}_o^+}$ and $\omega 0 \in -\overline{{}^+\check{\mathfrak{a}}_o^{\mathcal{G}}}$ by **Fun 6**. We have $\Re\lambda_t \in \tau - \overline{{}^+\check{\mathfrak{a}}_o^{\mathcal{G}}}$ by the definition of $M^k_{\mathcal{J},\{P\},\tau}$, hence $\omega\Re\lambda_t = \Re\lambda_t - y \in \tau - \overline{{}^+\check{\mathfrak{a}}_o^{\mathcal{G}}}$.

STEP 2. – Our next aim is to show that the image of (13) is even contained in the filtration step $S_{\rho_{-\tau}+\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})^{T,i}$. By **Fun 4**, this is equivalent to $T((\omega\Re\lambda_t)_+) \geq i$ for all standard parabolic subgroups $\mathcal{P} \in \{P\}$ and all $\omega \in \Omega(u_t, \mathcal{P})$. Because of $\Re\lambda_t - \omega\Re\lambda_t = y \in \overline{{}^+\check{\mathfrak{a}}_o^{\mathcal{G}}}$ and (7), we have $(\Re\lambda_t)_+ - (\omega\Re\lambda_t)_+ \in \overline{{}^+\check{\mathfrak{a}}_o^{\mathcal{G}}}$ and hence by (8)

$$(16)\qquad T((\omega\Re\lambda_t)_+) \geq T((\Re\lambda_t)_+) = T(\Re\lambda_t) = i$$

as required. The last equality $T(\Re\lambda_t) = i$ holds by the definition of $M^{k,T,i}_{\mathcal{J},\{P\},\tau}$.

STEP 3. – Our next aim is to show that the image of (13) in

$$\mathfrak{Fin}_{\mathcal{J}} S_{\rho_{-\tau}+\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})^{T,i}_{\{P\}} \;/\; \mathfrak{Fin}_{\mathcal{J}} S_{\rho_{-\tau}+\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})^{T,i+1}_{\{P\}}$$

is independent of the choice of **MW**. To do this, let $\mathcal{Q}_t \supseteq \mathcal{R}$ be the largest parabolic subgroup such that $\Re\lambda_t \in \check{\mathfrak{a}}_{\mathcal{Q}_t}$. We claim that for every standard parabolic subgroup $\mathcal{P} \in \{P\}$, for every $\phi \in W(u)$, λ near λ_t, and every differential operator D_λ with constant coefficients in λ we have

$$(17)\qquad \begin{cases} D_\lambda\big(E(\phi,\lambda) - E^{\mathcal{Q}_t}_{\mathcal{R}}(\phi,\lambda)\big)_{\mathcal{N}_\mathcal{P}}(g) & \text{if } \mathcal{Q}_t \supseteq \mathcal{P} \\ D_\lambda(E(\phi,\lambda))_{\mathcal{N}_\mathcal{P}}(g) & \text{otherwise} \end{cases}$$
$$= D_\lambda \sum_{T(\Re\theta(\lambda)_+)>i} \mathrm{e}^{\langle H_\mathcal{P}(g),\rho_\mathcal{P}+\theta(\lambda)\rangle} h_{\mathcal{P},\theta(\lambda),\phi,\lambda}(H_\mathcal{P}(g))(g)$$

with

$$h_{\mathcal{P},\theta(\lambda),\phi,\lambda} \in \boldsymbol{S}\Big(\big(\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{P}}\big)_{\boldsymbol{C}}\Big) \otimes S_\infty(\mathbb{N}_\mathcal{P}\mathcal{A}_\mathcal{P}(\boldsymbol{R})^o P\backslash\mathbb{G}).$$

Of course $h_{\mathcal{P},g,\theta(\lambda),\phi,\lambda}$ meromorphically depends on λ and $\theta(\lambda)$ is an affine function of λ. To verify (17), let us consider the contribution of $\omega \in \Omega(u_t,\mathcal{P})$ to the constant terms of $E(\phi,\lambda)$ and $E^{\mathcal{Q}_t}_{\mathcal{R}}(\phi,\lambda)$. If $\hat{\omega}$ identically acts on $\check{\mathfrak{a}}_{\mathcal{Q}_t}$, then $N(\omega,\lambda)$ gives the same contribution to the constant terms of $E(\phi,\lambda)$ and $E^{\mathcal{Q}_t}_{\mathcal{R}}(\phi,\lambda)$. Otherwise, let v be any vector in the interior of $\mathfrak{a}^{\mathcal{G}+}_{\mathcal{Q}_t}$. Then $\langle\Re(\lambda_t) - \hat{\omega}\Re(\lambda_t), v\rangle > 0$ because $\Re(\lambda_t) \in \check{\mathfrak{a}}^{\mathcal{G}+}_{\mathcal{Q}_t}$. Since $-\omega 0 \in \overline{{}^+\check{\mathfrak{a}}^{\mathcal{G}}_o}$, this implies $\langle y, v\rangle > 0$, where y is the same as in (15). This implies $(\omega\Re(\lambda_t))_+ \neq \Re(\lambda_t)$. Indeed, if we assume $(\omega\Re(\lambda_t))_+ = \Re(\lambda_t)$ we get the contradiction

$$\begin{aligned} \langle\Re(\lambda_t), v\rangle &= \langle\omega\Re(\lambda_t), v\rangle \\ &= \langle\Re(\lambda_t) - y, v\rangle \\ &< \langle\Re(\lambda_t), v\rangle. \end{aligned}$$

In the first line, we have used the fact that $\Re(\lambda_t) \in \check{\mathfrak{a}}^{\mathcal{G}+}_{\mathcal{Q}_t}$, which together with our assumption $\big(\omega\Re(\lambda_t)\big)_+ = \Re(\lambda_t)$ implies $P(\omega\lambda_t) = \Delta_o - \Delta^{\mathcal{Q}_t}_o$ and $v \perp \big(\Re(\omega\lambda_t)\big)_-$. The contradiction we have obtained proves $(\omega\Re(\lambda_t))_+ \neq \Re(\lambda_t)$.

Therefore and by (8), the inequality in (16) becomes strict, hence $T(\omega\Re(\lambda_t)_+) > i$. This remains true if λ_t is replaced by some λ which is near λ_t. Consequently, the contribution of $N(\omega,\lambda)$ to the constant term of $E(\phi,\lambda)$ fits into the right hand side of (17).

Remark 1. – Let us point our for future reference in another paper that we have shown the following fact: Let $u = (\mathcal{R},\Lambda,\chi) \in U^k_{\{P\}}$, $\lambda \in \overline{\check{\mathfrak{a}}^+_{\mathcal{R}}}$, and $\omega \in \Omega(u,\check{\mathfrak{a}}_\mathcal{P})$ such that $\hat{\omega}(\lambda) \neq \lambda$, then $(\omega\lambda)_+ \neq \lambda$ and $(\omega\lambda)_+ \in \lambda - \overline{{}^+\check{\mathfrak{a}}_o}$.

Let $\widetilde{\mathbf{MW}}$ be another choice for **MW**. Then since

$$E^{\mathcal{Q}_t}_{\mathcal{R}}(\phi,\lambda_t) = \mathrm{e}^{\langle\lambda_t, H_{\mathcal{Q}_t}(g)\rangle} E^{\mathcal{Q}_t}_{\mathcal{R}}(\phi,0)$$

and since $E_{\mathcal{R}}^{\mathcal{Q}_t}$ is holomorphic near 0, applying $\mathbf{MW}-\widetilde{\mathbf{MW}}$ to (17) gives

$$\begin{aligned}&\left(\left(\mathbf{MW}-\widetilde{\mathbf{MW}}\right)_{\lambda=\lambda_t}\left(D_\lambda E(\phi,\lambda)\right)\right)_{\mathcal{N}_\mathcal{P}}(g)\\&\quad=\left(\mathbf{MW}-\widetilde{\mathbf{MW}}\right)_{\lambda=\lambda_t}D_\lambda\sum_{T(\theta(\lambda)_+)>i}\mathrm{e}^{\langle H_\mathcal{P}(g),\rho_\mathcal{P}+\theta(\lambda)\rangle}h_{\mathcal{P},\theta(\lambda),\phi,\lambda}(H_\mathcal{P}(g))(g).\end{aligned}$$

It is clear that the left hand side is of the form (2) such that the sum is only over $T(\Re(\lambda)_+)>i$. This concludes the third step.

STEP 4. – Since $E_{\mathcal{R}}^{\mathcal{Q}_t}(\phi,\lambda)$ is holomorphic near λ_t and since

$$E_{\mathcal{R}}^{\mathcal{Q}_{\tilde{t}}}(M(w,\lambda)\phi,w\lambda)=E_{\mathcal{R}}^{\mathcal{Q}_t}(\phi,\lambda),$$

it is clear from (17) that (13) defines a homomorphism of vector spaces (14). By a similar argument, this homomorphism is also a morphism of $(\mathfrak{g},K,\mathbb{G}_f)$-modules.

STEP 5. – It remains to verify that the homomorphism (14) is an isomorphism. We first express the left hand side of (14) in terms of $(\mathfrak{g},K,\mathbb{G}_f)$-modules which are induced from the modules which occur on the right hand side of (5.4.1).

Recall that for every $t\in M^k_{\mathcal{J},\{P\},\tau}$, $\mathcal{Q}_t$ is the largest standard parabolic subgroup such that $\Re(\lambda_t)\in\check{\mathfrak{a}}_{\mathcal{Q}_t}$. Since (λ_t), viewed as an element of $\check{\mathfrak{a}}_o$, is the same for any two elements of $M^k_{\mathcal{J},\{P\},\tau}$ which can be connected by a morphism in $\mathcal{M}^k_{\mathcal{J},\{P\},\tau}$, it follows that $\mathcal{Q}_t=\mathcal{Q}_{\tilde{t}}$ if there exists a morphism from t to $\tilde{t}$ in $\mathcal{M}^k_{\mathcal{J},\{P\},\tau}$. Consequently, $\mathcal{M}^{k,T,i}_{\mathcal{J},\{P\},\tau}$ is the disjoint union of its full subcategories $\mathcal{M}^{k,T,i}_{\mathcal{J},\{P\},\tau,\mathcal{Q},\ell}$ consisting of those objects t for which $\mathcal{Q}_t=\mathcal{Q}$ and ℓ is the projection of λ_t to $(\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{Q}_t})_{\boldsymbol{C}}$. Here $\mathcal{Q}$ varies over all standard parabolic subgroups of rank $\leq k$, and $\ell\in\check{\mathfrak{a}}^{\mathcal{G}+}_{\mathcal{Q}}+i\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{Q}}$.

Let $\mathcal{Q}=\mathcal{M}_\mathcal{Q}\mathcal{A}_\mathcal{Q}\mathcal{N}_\mathcal{Q}$ be a standard parabolic subgroup of $\mathcal{G}$. If $\{{}^\psi P\}$ is a class of associate parabolic subgroups of $\mathcal{L}_\mathcal{Q}$ and if $\{P\}$ is a class of associate parabolic subgroups of $\mathcal{G}$, we will say that $\{{}^\psi P\}$ extends to $\{P\}$ and write $\{{}^\psi P\}\mid\{P\}$ if for every parabolic subgroup ${}^\psi\mathcal{P}\in\{{}^\psi P\}$ of $\mathcal{L}_\mathcal{Q}$ the parabolic subgroup ${}^\psi\mathcal{P}\mathcal{N}_\mathcal{Q}$ belongs to $\{P\}$. Let $U^l_{\{{}^\psi P\}}$ and $\mathcal{U}^l_{\{{}^\psi P\}}$ be defined in the same way as $U^l_{\{P\}}$ and $\mathcal{U}^l_{\{P\}}$, but with $\mathcal{G}$ replaced by $\mathcal{L}$. Let $u=(\mathcal{R},\Lambda,\chi)\in U^k_{\{P\}}$ and ${}^\psi u=({}^\psi\mathcal{R},{}^\psi\Lambda,{}^\psi\chi)\in U^{k-\operatorname{rank}\mathcal{Q}}_{\{{}^\psi P\}}$. We say that ${}^\psi u$ extends to u and write ${}^\psi u\mid u$, if and only if $\mathcal{R}=\mathcal{N}_\mathcal{Q}{}^\psi\mathcal{R}$, $\Lambda={}^\psi\Lambda$, and $\chi={}^\psi\chi$. These conditions characterise ${}^\psi u$ uniquely if u and $\{P\}$ are given. We have

$$V(u)=\bigoplus_{\substack{\{{}^\psi P\}\mid\{P\}\\ {}^\psi u\mid u}}V({}^\psi u),\tag{18}$$

where $V({}^\psi u)$ is defined in the same way as $V(u)$, but with $\mathcal{G}$ replaced by $\mathcal{L}$.

Let

$$D_\ell=\boldsymbol{S}\Big((\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{Q}})_{\boldsymbol{C}}\Big)\otimes\boldsymbol{C}_\ell,\tag{19}$$

where $\alpha \in \mathfrak{a}_{\mathcal{Q}}$ acts by contraction on the first factor and by multiplication by $e^{\langle \ell, \alpha \rangle}$ on the second factor. If $t \in M^k_{\mathcal{J},\{P\},\tau,\mathcal{Q},\ell}$ and if $\tilde{\lambda}_t$ is the projection of λ_t to $(\check{\mathfrak{a}}^{\mathcal{Q}_t}_{\mathcal{R}})_{\boldsymbol{C}}$, then

$$D_t \cong D_\ell \otimes D'(i\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{R}})_{\tilde{\lambda}_t}$$

and hence we have for the summand of the source of (14) a decomposition

$$D_t \otimes W(u_t) \cong \mathrm{Ind}^{\mathcal{G}}_{\mathcal{Q}} D_\ell \otimes \sum_{\substack{\{^\psi P\}|\{P\} \\ ^\psi u|u}} \mathrm{Ind}^{\mathcal{Q}}_{\mathcal{R}_t} D'(i\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{R}})_{\tilde{\lambda}_t} \otimes V(^\psi u), \tag{20}$$

where we have used (18).

Let $\Lambda^{T,i}_{\mathcal{Q},\tau}$ be the set of those $\ell \in \check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{Q}}$ for which there exists a k such that $M^{k,T,i}_{\mathcal{J},\{P\},\tau,\mathcal{Q},\ell}$ is not empty. For such ℓ, let $\mathcal{J}_\ell \subseteq \mathfrak{Z}(\mathfrak{m}_{\mathcal{Q}})$ be the ideal

$$\begin{aligned}\mathcal{J}_\ell = \{&A \in \mathfrak{Z}(\mathfrak{m}_{\mathcal{Q}}) \mid \xi_{\mathcal{Q}}(A)(\mu) = 0 \\ &\text{for all } \mu \in \check{\mathfrak{h}} \cap \check{\mathfrak{m}}_{\boldsymbol{C}} \text{ such that } \xi(B)(\mu + \ell) = 0 \text{ for all } B \in \mathcal{J}\},\end{aligned} \tag{21}$$

where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}_{\boldsymbol{C}}$ containing $(\mathfrak{a}_o)_{\boldsymbol{C}}$ and ξ and $\xi_{\mathcal{Q}}$ are the Harish-Chandra isomorphisms for $\mathfrak{m}_{\mathcal{G}}$ and $\mathfrak{m}_{\mathcal{Q}}$ (*cf.* [Wal88], Theorem 3.2.3.). Summing up (20) over all t, we get the following isomorphism for the left hand side of (14):

$$\begin{aligned}&\bigoplus_{k=0}^{\mathrm{rank}\{P\}} \operatorname*{colim}_{\mathcal{M}^{k,T,i}_{\mathcal{J},\{P\},\tau}} D_t \otimes W(u_t) \cong \\ &\cong \bigoplus_{\mathcal{Q}} \bigoplus_{\ell \in \Lambda^{T,i}_{\mathcal{Q},\tau}} \bigoplus_{\{^\psi P\}|\{P\}} \bigoplus_{l=0}^{\mathrm{rank}\{^\psi P\}} \\ &\quad \times \mathrm{Ind}^{\mathcal{G}}_{\mathcal{Q}} D_\ell \otimes \operatorname*{colim}_{^\psi u \in \mathcal{U}^l_{\{^\psi P\}}} \mathrm{Ind}^{\mathcal{Q}}_{\mathcal{R}_{\psi_u}} D'(i\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{R}_{\psi_u}})_{\mathrm{supp}_{\psi_u}\mathcal{J}_\ell} \otimes V(^\psi u).\end{aligned} \tag{22}$$

By (5.4.1), the right hand side of (22) is isomorphic to a sum of representations induced from automorphic forms of logarithmic growth on $\mathcal{L}$:

$$\begin{aligned}&\bigoplus_{\mathcal{Q}} \bigoplus_{\ell \in \Lambda^{T,i}_{\mathcal{Q},\tau}} \bigoplus_{\{^\psi P\}|\{P\}} \bigoplus_{l=0}^{\mathrm{rank}\{^\psi P\}} \mathrm{Ind}^{\mathcal{G}}_{\mathcal{Q}} D_\ell \otimes \operatorname*{colim}_{^\psi u \in \mathcal{U}^l_{\{^\psi P\}}} \mathrm{Ind}^{\mathcal{Q}}_{\mathcal{R}_{\psi_u}} D'(i\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{R}_{\psi_u}})_{\mathrm{supp}_{\psi_u}\mathcal{J}_\ell} \otimes V(^\psi u) \\ &\quad \cong \bigoplus_{\mathcal{Q}} \bigoplus_{\ell \in \Lambda^{T,i}_{\mathcal{Q},\tau}} \mathrm{Ind}^{\mathcal{G}}_{\mathcal{Q}} D_\ell \otimes \bigoplus_{\{^\psi P\}|\{P\}} \mathfrak{Fin}_{\mathcal{J}_\ell} S_{\log}(L_{\mathcal{Q}} \mathcal{A}_{\mathcal{Q}}(\boldsymbol{R})^o \backslash \mathbb{L}_{\mathcal{Q}})_{\{^\psi P\}}.\end{aligned} \tag{23}$$

STEP 6. – Now we want to construct an injective homomorphism Δ from the target of (14) to the target of (23)

$$\begin{aligned}&\mathfrak{Fin}_{\mathcal{J}} S_{\rho_{-\tau}+\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o \backslash \mathbb{G})^{T,i}_{\{P\}} / \mathfrak{Fin}_{\mathcal{J}} S_{\rho_{-\tau}+\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o \backslash \mathbb{G})^{T,i+1}_{\{P\}} \\ &\quad \xrightarrow{\Delta} \bigoplus_{\mathcal{Q}} \bigoplus_{\ell \in \Lambda^{T,i}_{\mathcal{Q},\tau}} \mathrm{Ind}^{\mathcal{G}}_{\mathcal{Q}} D_\ell \otimes \bigoplus_{\{^\psi P\}|\{P\}} \mathfrak{Fin}_{\mathcal{J}_\ell} S_{\log}(L_{\mathcal{Q}} \mathcal{A}_{\mathcal{Q}}(\boldsymbol{R})^o \backslash \mathbb{L})_{\{^\psi P\}}.\end{aligned} \tag{24}$$

To do this, consider (2) with $\mathcal{P} = \mathcal{Q}$. It is clear that the right hand side of (3) can be given a structure of $(\mathfrak{g}, K, \mathbb{G}_f)$-module such that the association

$$\text{(25)} \qquad \mathfrak{Fin}_{\mathcal{J}} S_{\rho_{-\tau}+\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G}) \to \boldsymbol{S}\Big(\big(\check{\mathfrak{a}}_{\mathcal{Q}}^{\mathcal{G}}\big)_{\boldsymbol{C}}\Big) \otimes S_\infty(\mathbb{N}_{\mathcal{Q}}\mathcal{A}_{\mathcal{Q}}(\boldsymbol{R})^o Q\backslash\mathbb{G})$$
$$f \to f_{\mathcal{Q},\ell}$$

defines a morphism of $(\mathfrak{g}, K, \mathbb{G}_f)$-modules. The target of (25) is then isomorphic to

$$\mathrm{Ind}_{\mathcal{Q}}^{\mathcal{G}} D_\ell \otimes S_\infty(L_{\mathcal{Q}}\mathcal{A}_{\mathcal{Q}}(\boldsymbol{R})^o\backslash\mathbb{L}).$$

We claim that for $f \in \mathfrak{Fin}_{\mathcal{J}} S_{\rho_{-\tau}+\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})^{T,i}_{\{P\}}$, $\ell \in \check{\mathfrak{a}}_{\mathcal{Q}}^{\mathcal{G}+}$ and $T(\ell) = i$ we actually have

$$\text{(26)} \qquad f_{\mathcal{Q},\ell} \in \mathrm{Ind}_{\mathcal{Q}}^{\mathcal{G}} D_\ell \otimes \mathfrak{Fin}_{\mathcal{J}_\ell} S_{\log}(L_{\mathcal{Q}}\mathcal{A}_{\mathcal{Q}}(\boldsymbol{R})^o\backslash\mathbb{L}).$$

If $\mathcal{P} \in \{P\}$ is a standard parabolic subgroup contained in $\mathcal{Q}$ and ${}^{\psi}\mathcal{P} = \mathcal{L}_{\mathcal{Q}} \cap \mathcal{P}$, then for $x \in \mathfrak{a}_{\mathcal{Q}}$ the constant term of $f_{\mathcal{Q},\ell}(x)$ along $\mathcal{N}_{\psi\mathcal{P}}$ is given similarly as in (2) by polynomials $(f_{\mathcal{Q},\ell}(x))_{\psi\mathcal{P},\vartheta}$ on $\mathfrak{a}_{\mathcal{P}}$ with values in the space of cusp forms on $P\mathbb{N}_{\mathcal{P}}\mathcal{A}_{\mathcal{P}}(\boldsymbol{P})^+\backslash\mathbb{G}$. It is clear that $f_{\mathcal{Q},\ell}$ belongs to

$$\mathrm{Ind}_{\mathcal{Q}}^{\mathcal{G}} D_\ell \otimes \mathfrak{Fin}_{\mathcal{J}_\ell} S_\infty\big(L_{\mathcal{Q}}\mathcal{A}_{\mathcal{Q}}(\boldsymbol{P})^+\backslash\mathbb{L}\big).$$

By Theorem 15, applied with $\tau = 0$ (or by [Lan76], Lemma 5.1), (26) is equivalent to

$$(f_{\mathcal{Q},\ell}(x))_{\psi\mathcal{P},\vartheta} = 0 \text{ unless } \Re(\vartheta) \in -\overline{{}^+\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{Q}}}.$$

To prove this inequality, note that

$$\text{(27)} \qquad (f_{\mathcal{Q},\ell}(x))_{\psi\mathcal{R},\vartheta}(y)(g) = f_{\mathcal{R},\ell+\vartheta}(x+y)(g),$$

hence

$$\text{(28)} \qquad (f_{\mathcal{Q},\ell}(x))_{\psi\mathcal{R},\vartheta} = 0 \text{ unless } T(\Re(\ell+\vartheta)_+) \geq i.$$

We want to verify that

$$\text{(29)} \qquad (\Re\ell + \theta)_+ - \Re\ell \in \overline{{}^+\check{\mathfrak{a}}_o^{\mathcal{G}}}$$

for $\theta \in \check{\mathfrak{a}}_o^{\mathcal{Q}}$. In fact, if $\alpha \in \Delta_o^{\mathcal{Q}}$, then

$$\langle\check{\alpha}, (\Re\ell+\theta)_+\rangle \geq 0 = \langle\check{\alpha}, \Re\ell\rangle,$$

otherwise

$$\langle\check{\omega}_\alpha, (\Re\ell+\theta)_+\rangle \geq \langle\check{\omega}_\alpha, \Re\ell+\theta\rangle = \langle\check{\omega}_\alpha, \Re\ell\rangle.$$

Since the set of the non-negative linear combinations of the $\check{\alpha}$ and $\check{\omega}_\alpha$ occurring in the last two inequalities contains $\overline{\check{\mathfrak{a}}_o^{\mathcal{G}+}}$, (29) holds. Now if $\Re\vartheta \notin -\overline{{}^+\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{Q}}}$, then we have

$$(\Re(\ell+\vartheta))_+ \neq \Re\ell.$$

Then

$$T((\Re(\ell+\vartheta))_+) < T(\Re\ell) = i$$

by (8) and (29) applied with $\theta = \Re\vartheta$. By (28), this implies that $f_{\mathcal{R},\ell+\vartheta}$ vanishes unless $\vartheta \in -\overline{\check{\mathfrak{a}}_o^{\mathcal{G}}}$. This proves (26). By Theorem 15, $f_{\mathcal{Q},\ell} = 0$ unless $\Re(\ell) \in \tau - \overline{{}^+\check{\mathfrak{a}}_o^{\mathcal{G}}}$. Consequently, the collection $f_{\mathcal{Q},\ell}$ for $\Re(\ell) \in \check{\mathfrak{a}}_{\mathcal{Q}}^{\mathcal{G}+}$ and $T(\Re\ell) = i$ defines a homomorphism of $(\mathfrak{g}, K, \mathbb{G}_f)$-modules

$$(30) \qquad \mathfrak{Fin}_{\mathcal{J}} S_{\rho_{-\tau}+\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})^{T,i}_{\{P\}} \\ \to \bigoplus_{\mathcal{Q}} \bigoplus_{\substack{\Re(\ell)\in\check{\mathfrak{a}}_{\mathcal{Q}}^{\mathcal{G}+} \\ T(\Re\ell)=i \\ \Re(\ell)\in\tau-{}^+\check{\mathfrak{a}}_o^{\mathcal{G}}}} \operatorname{Ind}_{\mathcal{Q}}^{\mathcal{G}} D_\ell \otimes \bigoplus_{\{^\psi P\}|\{P\}} \mathfrak{Fin}_{\mathcal{J}_\ell} S_{\log}(L_{\mathcal{Q}}\mathcal{A}_{\mathcal{Q}}(\boldsymbol{R})^o\backslash\mathbb{L})_{\{^\psi P\}}.$$

It is easy to see that the summand belonging to ℓ and $\mathcal{Q}$ vanishes unless $\ell \in \Lambda^{T,i}_{\mathcal{Q},\tau}$. Consequently, the targets of (30) and (24) agree. Now if $f \in \mathfrak{Fin}_{\mathcal{J}} S_{\rho_{-\tau}+\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})^{T,i+1}_{\{P\}}$, then (27) proves that $(f_{\mathcal{Q},\ell}(x))_{\psi\mathcal{R},\vartheta}$ vanishes unless $T(\Re(\ell+\vartheta))_+ \geq i+1$. For $T(\Re\ell) = i$, this is never the case because of (29) and (8), and the image of f by (30) is zero. It follows that (30) factorises over a unique morphism (24). It is clear from (9) that (24) is injective.

STEP 7. – Let Γ be the homomorphism (14) constructed in the first four steps. We have to show that this is an isomorphism. Let A and B be the isomorphisms (22) and (23), and let Δ be the monomorphism (24). If follows from (17) that $\Delta\Gamma = BA$. Since BA is an isomorphism and Δ is a monomorphism, Γ is an isomorphism. The proof of Theorem 14 is complete. □

Proof of Theorem 15. – As we already mentioned, the implication 1 $\Rightarrow$ 3 is a consequence of [Lan76], Lemma 5.1. The implication 2 $\Rightarrow$ 1 is trivial. It remains to prove 3 $\Rightarrow$ 2.

Let $f \in \mathfrak{Fin}_{\mathcal{J}} S_{\rho_{-\tau}+\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})_{\{P\}}$. Let $\mathcal{R}$ be a standard parabolic subgroup, and let $\tau_{\mathcal{R}}$ be the projection of τ to $\check{\mathfrak{a}}_{\mathcal{R}}^{\mathcal{G}}$. We first want to verify that

$$(31) \qquad f_{\mathcal{R},\lambda} = 0 \text{ unless } \Re(\lambda) \in \tau_{\mathcal{R}} - \overline{{}^+\check{\mathfrak{a}}_{\mathcal{R}}^{\mathcal{G}}}.$$

In fact, it is easy to see that

$$\int_{R\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G}} w_{-n}(g)(\rho_{-\tau}(g) \mid f_{\mathcal{N}_{\mathcal{R}}}(g) \mid)^2\, dg < \infty$$

for some n. It is easy to see from (2) that the integral is infinite if (31) is violated.

Now we prove that because of $\tau \in \check{\mathfrak{a}}_o^{\mathcal{G}+}$, for all $\vartheta \in \check{\mathfrak{a}}_{\mathcal{R}}^{\mathcal{G}}$, the inequality

$$(32) \qquad \vartheta \in \tau_{\mathcal{R}} - \overline{{}^+\check{\mathfrak{a}}_{\mathcal{R}}^{\mathcal{G}}}$$

implies the stronger inequality

$$(33) \qquad \vartheta \in \tau - \overline{{}^+\check{\mathfrak{a}}_o^{\mathcal{G}}}.$$

This means that (31) implies 2 and hence finishes the proof.

If (32) is satisfied, then

$$\langle \omega_\alpha, \tau - \vartheta \rangle \geq 0$$

for all $\alpha \in \Delta_o - \Delta_o^{\mathcal{R}}$. For $\alpha \in \Delta_o^{\mathcal{R}}$, we have

$$\langle \check{\alpha}, \tau - \vartheta \rangle = \langle \check{\alpha}, \tau \rangle \geq 0.$$

Because every element of $\check{\mathfrak{a}}_o^{\mathcal{G}+}$ is a non-negative linear combination of the ω_α for $\alpha \in \Delta_{\mathcal{R}}$ and of the elements of $\Delta_o^{\mathcal{R}}$, the inequality (33) follows.

The proof of Theorem 15 is complete. Q.E.D.

For every class $\{P\}$ of associate parabolic subgroups, there is a filtration

$$\mathfrak{Fin}_{\mathcal{J}} S_{\rho_{-\tau}-\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o \backslash \mathbb{G})^{T,i}_{\{P\}} \tag{34}$$

consisting of all functions f for which $f_{\mathcal{P},\lambda}$ vanishes unless $\mid \lambda_+ \mid$ is less than or equal to its i-th possible value. Let us assume that $\tau \in \overline{\check{\mathfrak{a}}_o^{\mathcal{G}+}} \cap {}^+\check{\mathfrak{a}}_o^{\mathcal{G}}$. If $\mathcal{M}^{k,T,i}_{\mathcal{J},\{P\},\tau-}$ denotes the union of $\mathcal{M}^{k,T,i}_{\mathcal{J},\{P\},\tau-\varepsilon}$ over all ε in the interior of the positive Weyl chamber, then we have an isomorphism

$$\begin{aligned} &\bigoplus_{k=0}^{\operatorname{rank}(\{P\})} \operatorname*{colim}_{\mathcal{M}^{k,T,i}_{\mathcal{J},\{P\},\tau-}} M(t) \\ &\cong \mathfrak{Fin}_{\mathcal{J}} S_{\rho_{-\tau}-\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o \backslash \mathbb{G})^{T,i}_{\{P\}} \Big/ \mathfrak{Fin}_{\mathcal{J}} S_{\rho_{-\tau}-\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o \backslash \mathbb{G})^{T,i+1}_{\{P\}} \end{aligned} \tag{35}$$

which is constructed in a similar way as Theorem 14. The condition $\tau \in {}^+\check{\mathfrak{a}}_o^{\mathcal{G}}$ is needed in this case because otherwise Theorem 15 will fail. Otherwise, the prove is the same as above for $S_{\rho_{-\tau}+\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o \backslash \mathbb{G})$.

Remark 2. – These are some remarks about why the filtration on the space of automorphic forms is defined in precisely the preceding way. The problem is to define a suitable filtration on the space of automorphic forms, such that the quotients are given by Eisenstein series. Although the filtration really has to be defined in terms of conditions on the constant term of the automorphic forms, since I think it is a non-trivial fact that all automorphic forms are sums of derivatives of Eisenstein series, a convenient way to think about the filtration is to define for each Eisenstein series $E(\phi, \lambda)$ (with λ in the closure of the positive Weyl chamber) the $k(\phi, \lambda)$ such that it contributes to the k-th grading $\mathrm{Gr}^k \mathfrak{Fin}_{\mathcal{J}} S_\infty(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o \backslash \mathbb{G})$ of a descending filtration $F^k \mathfrak{Fin}_{\mathcal{J}} S_\infty(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o \backslash \mathbb{G})$ of the space $\mathfrak{Fin}_{\mathcal{J}} S_\infty(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o \backslash \mathbb{G})$ of automorphic forms annihilated by a power of $\mathcal{J}$. The filtration on the space of automorphic forms orders these Eisenstein series in the same way as the composition factors of an induced representation in the proof of the Langlands classification (*cf.* for instance [Wal88], Corollary 5.5.3 or other expositions of the various cases of the Langlands classification theorem).

The first idea is to put $k(\phi, \lambda)$ equal to the rank of the Levi component on which ϕ lives. Thus Gr^0 is spanned by main values of Eisenstein series starting from $P_o, \ldots, \mathrm{Gr}^{\operatorname{rank}(\mathcal{P}_o)-1}$

is spanned by maximally residual Eisenstein series starting from maximal parabolic subgroups, $F^{\mathrm{rank}(\mathcal{P}_o)}$ is the space of square integrable maximally residual automorphic forms, and $F^{\mathrm{rank}(\mathcal{P}_o)+1} = 0$. Unfortunately, this definition seems to fail in certain cases. It works for rank one groups, and also in some of the rank two cases. It also seems to work for GL_n because of the result of Mœglin and Waldspurger. One gets problems with this approach in the rank two case if there are Eisenstein series from a maximal parabolic subgroup whose residue at a point in the positive Weyl chamber is not square integrable. This never happens for cuspidal Eisenstein series, and for residual Eisenstein series the only example of this kind which I know, and which I will explain in more detail below, is the example of G_2 described in the appendix in Langlands' book.

Example 1. – A useful special case of the class of filtrations considered in this section is obtained as follows: Put $R_1 > R_2 \ldots > R_N \geq 0$ equal to the possible absolute values of $(\Re\lambda)_+$ such that $f_{\mathcal{P},\lambda}$ may be non-zero for a function f which is annihilated by a power of $\mathcal{J}$. Let $E(\phi, \lambda)$ contribute to Gr^i, where i is such that the absolute value of $\Re\lambda$ is equal to R_i. This is achieved by the filtration in the special case of (9) where one puts $T(\lambda)$ to be equal to the i with $\mid \lambda \mid = R_i$.

One problem is that the filtrations on automorphic forms on Levi components of $\mathcal{G}$ defined in this way do not patch together nicely. Thus, for the application in 7 one needs the general class of filtrations considered in (9).

Example 2. – Here we describe the example of a one-parameter residual Eisenstein series whose residue at a point of the positive half-axis is not square integrable. As was explained above, the existence of such residues was one motivation for our choice of the definition of the filtration on the space of automorphic forms in this section. The example is based on Langlands' calculation for the Chevalley group G_2 in [Lan76], Appendix III. We will adopt the notations of this appendix. For instance $\beta_1, \ldots, \beta_6$ are the positive roots of G_2 numbered in the same way as in [Lan76], Appendix III. The roots β_1 and β_6 are simple. Let $\mathcal{P}$ be the parabolic subgroup such that $\Delta_o^{\mathcal{P}} = \{\beta_6\}$, and let $\mathbf{1}$ be the constant function on $\mathbb{P}\backslash\mathbb{G}$. We will consider the residual Eisenstein series $E_{\mathcal{P}}^{\mathcal{G}}(\mathbf{1}, \lambda)$, where the parameter λ lives in the one-dimensional vector space $\check{\mathfrak{a}}_{\mathcal{P}}$. Obviously,$\beta_4 - \frac{1}{2}\beta_6 \in \check{\mathfrak{a}}_{\mathcal{P}}^+$. Our claim is that

$$\operatorname*{Res}_{\lambda = \beta_4 - \frac{1}{2}\beta_6} E_{\mathcal{P}}^{\mathcal{G}}(\mathbf{1}, \lambda) \tag{36}$$

is not square integrable.

Proof. – Let $E_{\mathcal{P}_o}^{\mathcal{G}}(\mathbf{1}, \lambda)$ be the cuspidal Eisenstein series. It has a singular hyperplane σ_6, and we have

$$E_{\mathcal{P}}^{\mathcal{G}}(\mathbf{1}, \lambda) = c\mathrm{Res}_{\sigma_6} E_{\mathcal{P}_o}^{\mathcal{G}}\left(\mathbf{1}, \lambda + \frac{1}{2}\beta_6\right). \tag{37}$$

We will use this formula and the formula for the constant term of the Eisenstein system on the right hand side of (37) to examine the summand belonging to the exponential $e^{\langle \beta_6 + \rho_o, H(g)\rangle}$ in the expansion of the constant term of the residue (36). The relevant residues occurring in the expansion of the constant term of (37) are parametrised by elements of the sets $\Omega(\sigma_6, \sigma_j)$ which are listed in the lowest row of table (5) in [Lan76], Appendix III. The only element of $\Omega(\sigma_6, \sigma_j)$ which maps β_4 to β_6 is the element $\sigma(\frac{5\pi}{3})$ in the lower left corner of table (5) in [Lan76], Appendix III. By the corresponding entry in table (7) in [Lan76], Appendix III, we have

$$\mathrm{Res}_{\sigma_6} M(\frac{5\pi}{3}, \lambda) = \tilde{c}\frac{\xi(z+\frac{1}{2})}{\xi(z+\frac{3}{2})}, \tag{38}$$

with $\tilde{c} \neq 0$, where the parameter z is given by

$$z = \frac{1}{2} + 2\frac{\langle \lambda, \beta_1 \rangle}{\langle \beta_1, \beta_1 \rangle}$$

and ξ has the same meaning as in [Lan76], Appendix III. Since β_1 and β_4 are orthogonal, the value $\lambda = \beta_4$ on the left hand side of (38) corresponds to the value $z = \frac{1}{2}$ on the right hand side. Obviously, the right hand side of (38) has a residue at this value of z. It follows that there is a summand $\mathrm{e}^{\langle \beta_6 + \rho_o, H(g) \rangle}$ in the expansion of the constant term of (38), proving our claim. □

Note that this example is no contradiction to the square integrability result in [Lan76], Lemma 7.5, because the final rank zero Eisenstein system supported at β_4 will contain one more summand besides (36), and this summand will cancel the non-square integrable part of the constant term of (36). In the calculations in [Lan76], Appendix III, this cancellation reveals itself as the cancellation of the term

$$\frac{-1}{\xi^2(2)}\Phi(\beta_4)\overline{\Psi}(-\beta_6),$$

on line 4 of page 301 of [Lan76], Appendix III by the similar summand on line -4 of page 303.

It should also be mentioned that in this example the ideal $\mathcal{J} \subset \mathfrak{Z}(\mathfrak{g})$ is defined by β_4, which is in the boundary of the positive Weyl chamber and cannot be the infinitesimal character of a finite-dimensional representation. There is the possibility that the definition of the filtration can be simplified in all examples which are relevant to the computation of the cohomology of automorphic local systems.

It is also useful to note that we have proved more than we claimed. By an associate set of characters χ for $\{P\}$, we understand a collection of finite sets of characters $\chi_{\mathcal{P}}$ of $\mathcal{Z}(\mathfrak{m}_{\mathcal{P}})$ for each $\mathcal{P} \in \{P\}$, such that

- If $g \in G$, $\mathcal{P}, \mathcal{Q} \in \{P\}$ and $\mathrm{Int}(g)\mathcal{L}_{\mathcal{P}} = \mathcal{L}_{\mathcal{Q}}$, then $\mathrm{Ad}(g)^*\chi_{\mathcal{Q}} = \chi_{\mathcal{P}}$.
- If $\mathcal{P}, \mathcal{Q} \in \{P\}$, $\chi \in \chi_{\mathcal{P}}$ and $\tilde{\chi} \in \chi_{\mathcal{Q}}$, then there exists $g \in G$ with $\mathrm{Int}(g)\mathcal{L}_{\mathcal{Q}} = \mathcal{L}_{\mathcal{P}}$ and $\mathrm{Ad}(g)^*\chi = \tilde{\chi}$.

Let $S_{\rho_{-\tau}\pm\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})_{\{P\},\chi}$ be the set of all $f \in S_{\rho_{-\tau}\pm\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})_{\{P\}}$ which satisfy the additional assumption that for every $\mathcal{P} \in \{P\}$ and every $g \in \mathbb{G}$, the function on $\mathbb{M}_{\mathcal{P}}$ $m \to f_{\mathcal{N}_{\mathcal{P}}}(mg)$ is annihilated by some power of the ideal

$$\mathcal{I}_{\mathcal{P},\chi} = \{D \in \mathcal{Z}(\mathfrak{m}_{\mathcal{P}}) \mid \chi(D) = 0 \text{ for } \chi \in \chi_{\mathcal{P}}\}.$$

Similarly, if $t \in M^{k,T,i}_{\mathcal{J},\{P\},\tau}$, let $W(u_t)_\chi$ be the set of all $f \in W(u_t)$ such that for $g \in \mathbb{G}$ and $\mathcal{P} \in \{P\}$, the function on $\mathbb{M}_\mathcal{P}$ $f_{\mathcal{N}_\mathcal{P}}(mg)$ is killed by some power of $\mathcal{I}_{\mathcal{P},\chi}$, and let $M(t)_\chi = W(u_t)_\chi \otimes D_t$ (*cf.* (11)). Let $\mathcal{J} \subset \mathcal{Z}(\mathfrak{g})$ be an ideal of finite codimension. If $\mathcal{P} \in \{P\}$ and if $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{l}_\mathcal{P}$, then there is a finite union Z of Weyl group orbits on $\check{\mathfrak{h}}$ such that the radical of $\mathcal{J}$ consists of all elements of $\mathcal{Z}(\mathfrak{g})$ whose image under the Harish-Chandra isomorphism vanishes on Z. Let $C_{\mathcal{P},\mathcal{J}}$ be the set of all characters of $\mathcal{Z}(\mathfrak{m}_\mathcal{P})$ which can be defined by evaluation of the Harish-Chandra symbol at an element of Z. It is easy to see that this set of infinitesimal characters is independent of the choice of a Cartan subalgebra and that there exists a finite set $\boldsymbol{X}_{\mathcal{J},\{P\}}$ of associate sets of characters on $\{P\}$ such that $C_{\mathcal{P},\mathcal{J}} = \bigcup_{\chi \in \boldsymbol{X}_{\mathcal{J},\{P\}}} \chi_\mathcal{P}$.

PROPOSITION 1.
- *We have* $M(t) = \oplus_{\chi \in \boldsymbol{X}_{\mathcal{J},\{P\}}} M(t)_\chi$ *for* $t \in M^{k,T,i}_{\mathcal{J},\{P\},\tau}$.
- *We have*

$$\mathfrak{Fin}_\mathcal{J} S_{\rho_{-\tau}+\log}(G\mathcal{A}_G(\boldsymbol{R})^o\backslash\mathbb{G})_{\{P\}} = \oplus_{\chi \in \boldsymbol{X}_{\mathcal{J},\{P\}}} \mathfrak{Fin}_\mathcal{J} S_{\rho_{-\tau}+\log}(G\mathcal{A}_G(\boldsymbol{R})^o\backslash\mathbb{G})_{\{P\},\chi}.$$

- *If in addition the assumptions of Theorem* 14, *then*

$$\mathfrak{Fin}_\mathcal{J} S_{\rho_{-\tau}+\log}(G\mathcal{A}_G(\boldsymbol{R})^o\backslash\mathbb{G})^{T,i}_{\{P\}} = \oplus_{\chi \in \boldsymbol{X}_{\mathcal{J},\{P\}}} \mathfrak{Fin}_\mathcal{J} S_{\rho_{-\tau}+\log}(G\mathcal{A}_G(\boldsymbol{R})^o\backslash\mathbb{G})^{T,i}_{\{P\},\chi}.$$

Moreover, the assertion of Theorem 14, *remains true with* $S_{\rho_{-\tau}+\log}(G\mathcal{A}_G(\boldsymbol{R})^o\backslash\mathbb{G})_{\{P\}}$ *replaced by* $S_{\rho_{-\tau}+\log}(G\mathcal{A}_G(\boldsymbol{R})^o\backslash\mathbb{G})_{\{P\},\chi}$ *and* $M(t)$ *and* $W(u_t)$ *replaced by* $M(t)_\chi$ *and* $W(u_t)_\chi$.

Proof. – Both the first and the second point follow from [MW95], Proposition III.2.1 and Theorem 15. The first point also follows from Langlands' result about Eisenstein systems, which implies that every square integrable automorphic form is a sum of iterated residues of cuspidal Eisenstein series, which we may assume to start from cusp forms which have an infinitesimal character.

The third point follows from the first point, the proof of Theorem 14 and from the fact that if $t \in M^{k,T,i}_{\mathcal{J},\{P\},\tau}$ and $\mathcal{P} \in \{P\}$, then the term of an Eisenstein series starting from an element of $W(u_t)_\chi$ is a meromorphic function with values in the space of function which are annihilated by some power of $\mathcal{I}_{\mathcal{P},\chi}$. In the case where τ satisfies the assumption of Theorem 14, this argument can also be used to derive the second point from the first. □

It is possible to refine the above decomposition even further, *cf.* [MW], Proposition III.2.1 or [FS91].

7. $\mathfrak{Fin}$-acyclicity of $S_{\rho_{-\tau}+\log}(G\mathcal{A}_G(\boldsymbol{R})^o\backslash\mathbb{G})$

As usual, we assume that $\mathcal{J}$ is an ideal of finite codimension in $\mathfrak{Z}(\mathfrak{m}_G)$. Our aim is to prove that $S_{\rho_{-\tau}+\log}(G\mathcal{A}_G(\boldsymbol{R})^o\backslash\mathbb{G})$ is $\mathfrak{Fin}$-acyclic if τ is in the closure of the positive Weyl chamber. In the inductive limit $\tau \to \infty$ in the positive Weyl chamber, this implies a conjecture of Borel about the cohomology of arithmetic subgroups (*cf.* Theorem 18). We will also prove the $\mathfrak{Fin}$-acyclicity of $S_{\rho_{-\tau}-\log}(G\mathcal{A}_G(\boldsymbol{R})^o\backslash\mathbb{G})$ when τ is in the intersection of closure of the positive Weyl chamber and the interior of the positive cone.

The main theorem of this section is:

THEOREM 16. – *If* $\tau \in \overline{\check{\mathfrak{a}}_o^{\mathcal{G}+}}$, *then* $\mathfrak{Fin}^i_{\mathcal{J}} S_{\rho_{-\tau}+\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$ *vanishes for* $i > 0$. *If in addition* $\tau \in {}^+\check{\mathfrak{a}}_o^{\mathcal{G}}$, *then the same is true for* $\mathfrak{Fin}^i_{\mathcal{J}} S_{\rho_{-\tau}-\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$.

The proof will proceed by induction on the rank of $\mathcal{G}$. In the cocompact case we have $S_\rho(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G}) = S_{\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$ for all weight functions and the result follows from Theorem 13. Let us assume that the theorem is true for the Levi components of proper parabolic subgroups of $\mathcal{G}$. We will first construct a resolution of

$$S_{\rho_{-\tau}+\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})/S_{\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$$

by $(\mathfrak{g}, K, \mathbb{G}_f)$-modules which are induced from weighted L_2-spaces on Levi components of $\mathcal{G}$. The induction assumption can then be used to show that all members of the resolution are $\mathfrak{Fin}$-acyclic. Consequently, applying $\mathfrak{Fin}_{\mathcal{J}}$ to the resolution we get a complex of automorphic forms whose cohomology is isomorphic to

$$\mathfrak{Fin}^i_{\mathcal{J}}\big(S_{\rho_{-\tau}+\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})/S_{\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})\big),$$

which for $i > 0$ is isomorphic to

$$\mathfrak{Fin}^i_{\mathcal{J}} S_{\rho_{-\tau}+\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$$

by the Theorem 13. Then we use the result of section 6 to show that the complex of automorphic forms is acyclic in positive dimensions.

We will give the details of the proof only for $S_{\rho_{-\tau}+\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$ since the other case is similar.

7.1. A resolution for $S_\rho(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$

Here we consider arbitrary weight functions ρ satisfying (2.1.3). Recall that for given real constants $T > D$ the set $\mathfrak{S}(\mathcal{P}, D, T)$ consists of all elements of the Siegel-like domain $\mathfrak{S}(D)$ such that $\langle\check{\alpha}, H_o(g)\rangle > T$ for all roots $\alpha \in \Delta_o - \Delta_o^{\mathcal{P}}$, *cf.* (2.1.3). Let us denote by $S_c(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$ the space of all $f \in S_\rho(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$ with the following property: For every standard parabolic subgroup $\mathcal{P}$ and every D there exists a constant T such that $f_{\mathcal{N}_{\mathcal{P}}}$ vanishes on $\mathfrak{S}(\mathcal{P}, D, T)$. By Theorem 5, $S_c(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$ is independent of the choice of ρ. Our aim is to find a resolution of

$$S_\rho(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})/S_c(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$$

by $(\mathfrak{g}, K, \mathbb{G}_f)$-modules which are induced from parabolic subgroups of $\mathcal{G}$.

Let $\mathfrak{P}$ be the partially ordered (by the opposite of inclusion) set of standard parabolic subgroups of $\mathcal{G}$, excluding $\mathcal{G}$ itself. For every $\mathcal{P} \in \mathfrak{P}$, let

$$S_\rho[\mathcal{P}] = \operatorname*{colim}_{T\to\infty} S_\rho(P\mathbb{N}_{\mathcal{P}}\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash P\mathfrak{S}(\mathcal{P}, D, T)), \tag{1}$$

where $S_\rho(\boldsymbol{X})$ for a $\mathbb{K}$-invariant subset $\boldsymbol{X}$ of $P\mathbb{N}_{\mathcal{P}}\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G}$ denotes the space of all $\mathbb{K}$-finite C^∞-functions on $\boldsymbol{X}$ with

$$\int_X (\rho(g)|(Af)(g)|)^2\, dg < \infty$$

for all $A \in \mathfrak{U}(\mathfrak{g})$. In (1), D is assumed to be so small that $G\mathfrak{S}(D) = \mathbb{G}$. If this assumption is satisfied, the inductive limit is independent of D.

It is clear that $S_\rho[\cdot]$ is a functor from $\mathfrak{P}$ to $(\mathfrak{g}, K, \mathbb{G}_f)$-modules if the transition homomorphism from $\mathcal{P}$ to $\mathcal{Q}$ is given by the constant term along $\mathcal{N}_\mathcal{Q}$. For every $f \in S_\rho(G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G})$ and every $\mathcal{P} \in \mathfrak{P}$, we have $f_{\mathcal{N}\mathcal{P}} \in S_\rho[\mathcal{P}]$, defining a homomorphism of $(\mathfrak{g}, K, \mathbb{G}_f)$-modules

$$S_\rho(G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G}) \;/\; S_c(G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G}) \to \lim_{\mathfrak{P}} S_\rho[\cdot]. \tag{2}$$

Our aim is to prove

THEOREM 17. – *The map* (2) *is an isomorphism, and we have*

$$\lim\nolimits^i_{\mathfrak{P}} S_\rho[\cdot] = 0 \tag{3}$$

for $i > 0$.

We first define a sufficiently large class of acyclic functors from $\mathfrak{P}$ to the category of vector spaces. Then we show that the functor $S_\rho[\cdot]$ has a filtration whose successive quotients belong to that class.

If $\mathcal{R} \subseteq \mathcal{Q}$ are standard parabolic subgroups, let $\mathcal{F}^\mathcal{Q}_\mathcal{R}[\cdot]$ be the functor from $\mathfrak{P}$ to complex vector spaces defined by

$$\mathcal{F}^\mathcal{Q}_\mathcal{R}[\mathcal{P}] = \begin{cases} \boldsymbol{C} & \text{if } \mathcal{R} \subseteq \mathcal{P} \subseteq \mathcal{Q} \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

where the transition homomorphism is the identity if both its source and its target are different from zero. We have

PROPOSITION 1. – *If* $\mathcal{R} \subset \mathcal{Q}$, *then*

$$\lim\nolimits^i_{\mathfrak{P}} \mathcal{F}^\mathcal{Q}_\mathcal{R}[\cdot] = \begin{cases} \boldsymbol{C} & \textit{if } i = 0 \textit{ and } \mathcal{Q} = \mathcal{G} \\ 0 & \textit{if } i > 0 \textit{ or } \mathcal{Q} \subset \mathcal{G}. \end{cases}$$

If $\mathcal{R} = \mathcal{Q}$, *then*

$$\lim\nolimits^i_{\mathfrak{P}} \mathcal{F}^\mathcal{Q}_\mathcal{Q}[\cdot] = \begin{cases} \boldsymbol{C} & \textit{if } i = \operatorname{rank}(\mathcal{Q}) - 1 \\ 0 & \textit{if } i \neq \operatorname{rank}(\mathcal{Q}) - 1. \end{cases}$$

Proof. – Let us chose a total ordering of Δ_o, and let us order predecessors $\mathcal{Q}$ of $\mathcal{P}$ in $\mathfrak{P}$ according to the order of the unique element of $\Delta^\mathcal{Q}_o - \Delta^\mathcal{P}_o$. We denote the i-th predecessor of $\mathcal{P}$ by $\mathcal{P}_i$. For a functor $\mathcal{F}[\cdot]$ from $\mathfrak{P}$ to complex vector spaces, let

$$C^i(\mathfrak{P}, \mathcal{F}) = \bigoplus_{\operatorname{rank}(\mathcal{P})=i+1} \mathcal{F}[\mathcal{P}]$$

be the complex with differential

$$d\Big(f_\mathcal{Q}\Big)_{\{\operatorname{rank}(\mathcal{Q})=i\}} = \bigg(\sum_{j=0}^{i} (-1)^j \mathcal{F}[\mathcal{P}_j \supset \mathcal{P}]\big(f_{\mathcal{P}_j}\big)\bigg)_{\{\operatorname{rank}(\mathcal{P})=i+1\}}$$

Then it is easy to see that the assertions of the proposition are true if $\lim^i_{\mathfrak{P}} \mathcal{F}[\cdot]$ can be computed as the i-th cohomology of $C^*(\mathfrak{P}, \mathcal{F})$. In particular, if $\boldsymbol{T}$ denotes the class of all functors $\mathcal{F}$ from $\mathfrak{P}$ to vector spaces for which $C^*(\mathfrak{P}, \mathcal{F})$ is acyclic in positive dimension, then $\boldsymbol{T}$ contains all functors $\mathcal{F}^{\mathcal{G}}_{\mathcal{R}}$. It is easy to see that every functor from $\mathfrak{P}$ to vector spaces can be embedded into a sum of copies of $\mathcal{F}^{\mathcal{G}}_{\mathcal{R}}$, consequently $\boldsymbol{T}$ satisfies the assumptions of a well known lemma from Grothendieck's Tôhoku paper [Gro57], Lemme 3.3.1 and hence $\lim^i_{\mathfrak{P}} \mathcal{F}[\cdot]$ can be computed using $C^*(\mathfrak{P}, \mathcal{F})$. $\square$

Let $S^i_\rho[\mathcal{P}]$ (resp. $S^i_\rho(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$) be the set of all $f \in S_\rho[\mathcal{P}]$ (resp. $f \in S_\rho(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$) such that the image of f in $S_\rho[\mathcal{Q}]$ vanishes for all standard parabolic subgroups $\mathcal{Q}$ with $\operatorname{rank}(\mathcal{Q}) > \operatorname{rank}(\mathcal{P}_o) - i$.

PROPOSITION 2. – *The natural mappings*

$$S^i_\rho[\mathcal{Q}] \;/\; S^{i+1}_\rho[\mathcal{Q}] \to \bigoplus_{\substack{\mathcal{P}\subseteq\mathcal{Q}\\ \operatorname{rank}(\mathcal{P})=\operatorname{rank}(\mathcal{P}_o)-i}} S^i_\rho[\mathcal{P}] \tag{5}$$

$$S^i_\rho(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G}) \;/\; S^{i+1}_\rho(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G}) \to \bigoplus_{\operatorname{rank}(\mathcal{P})=\operatorname{rank}(\mathcal{P}_o)-i} S^i_\rho[\mathcal{P}]. \tag{6}$$

are isomorphisms.

Proof. – The injectivity of (5) and (6) is obvious, and the surjectivity of (6) implies the surjectivity of (5). To prove the surjectivity of (6), let $\overline{f} \in S_\infty(P\mathbb{N}_{\mathcal{P}}\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash P\mathfrak{S}(\mathcal{P}, D, T))$ be a representative of $f \in S_\rho[P]$, and let $\varphi_{\mathcal{P}}$ be the same as in (3.5), where v is sufficiently small such that $\operatorname{supp}\varphi \subset P\mathfrak{S}(\mathcal{P}, D, T+1)$. Let

$$h(g) = \sum_{\gamma\in P\backslash G} (\varphi_{\mathcal{P}}\overline{f})(\gamma g),$$

where $\varphi_{\mathcal{P}}\overline{f}$ has been extended by zero from

$$P\mathbb{N}_{\mathcal{P}}\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash P\mathfrak{S}(\mathcal{P}, D, T)$$

to

$$P\mathbb{N}_{\mathcal{P}}\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G}.$$

Then step 3 of the proof of Theorem 6 proves that for sufficiently small v, the image of h in $S_\rho[\mathcal{R}]$ is f if $\mathcal{R} = \mathcal{P}$ and vanishes if $\mathcal{R}$ is of rank $\operatorname{rank}(\mathcal{P}_o) - i$ and different from $\mathcal{P}$. This proves the surjectivity of (6) and completes the proof. $\square$

Proof of Theorem 17. – By Proposition 2, we have

$$\left(S^i_\rho/S^{i+1}_\rho\right)[\cdot] \cong \bigoplus_{\operatorname{rank}(\mathcal{Q})=\operatorname{rank}(\mathcal{P}_o)-i} \mathcal{F}^{\mathcal{G}}_{\mathcal{Q}}[\cdot] \otimes S^i_\rho[\mathcal{Q}].$$

By Proposition 1, this implies

$$\lim\nolimits^j_{\mathfrak{P}} \left(S^i_\rho/S^{i+1}_\rho\right)[\cdot] = 0$$

for $j > 0$ and

$$\begin{aligned}\lim\nolimits_{\mathfrak{P}} \left(S^i_\rho/S^{i+1}_\rho\right)[\cdot] &\cong \bigoplus_{\operatorname{rank}(\mathcal{Q})=\operatorname{rank}(\mathcal{P}_o)-i} S^i_\rho[\mathcal{Q}] \\ &\cong \left(S^i_\rho \;/\; S^{i+1}_\rho\right)(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})\end{aligned}$$

in view of (6).

The proof of Theorem 17 is complete. Q.E.D.

7.2. $\mathfrak{Fin}$-acyclicity of $S_{\rho_{-\tau}+\log}[\mathcal{P}]$

For a standard parabolic subgroup $\mathcal{P}$ and $\ell \in \left(\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{P}}\right)_{\boldsymbol{C}}$, let $\mathcal{J}_\ell$ and D_ℓ be defined by (6.21) and (6.19). Let

$$(1) \qquad \tau_\ell = \mathrm{pr}_{\check{\mathfrak{a}}^{\mathcal{G}}_o \to \check{\mathfrak{a}}^{\mathcal{P}}_o}\Big(\tau - \sum_{\alpha\in\Delta_o-\Delta^{\mathcal{P}}_o} \langle \check{\omega}_\alpha, \tau_{\mathcal{P}} - \Re(\ell)\rangle \alpha\Big).$$

If $\tau \in \overline{\check{\mathfrak{a}}^{\mathcal{G}+}_o}$ and $\Re\ell \in \tau - \overline{{}^+\check{\mathfrak{a}}^{\mathcal{G}}_o}$, then $\tau_\ell \in \overline{\check{\mathfrak{a}}^{\mathcal{P}+}_o}$. There is an obvious homomorphism

$$(2) \qquad \bigoplus_{\substack{\Re(\ell)\in\tau-\overline{{}^+\check{\mathfrak{a}}^{\mathcal{G}}_o} \\ \mathcal{J}_\ell\subset\mathfrak{Z}(\mathfrak{m}_{\mathcal{P}})}} \mathrm{Ind}^{\mathcal{G}}_{\mathcal{P}} D_\ell \otimes \mathfrak{Fin}_{\mathcal{J}_\ell} S_{\rho_{-\tau_\ell}+\log}(L_{\mathcal{P}}\mathcal{A}_{\mathcal{P}}(\boldsymbol{R})^o\backslash\mathbb{L}_{\mathcal{P}}) \to \mathfrak{Fin}_{\mathcal{J}} S_{\rho_{-\tau}+\log}[\mathcal{P}]$$

which sends (f_ℓ) to

$$(3) \qquad h(g) = \sum_{\substack{\Re(\ell)\in\tau-\overline{{}^+\check{\mathfrak{a}}^{\mathcal{G}}_o} \\ \mathcal{J}_\ell\subset\mathfrak{Z}(\mathfrak{m}_{\mathcal{P}})}} \mathrm{e}^{\langle H_{\mathcal{P}}(g),\ell+\rho_{\mathcal{P}}\rangle} f_\ell(H_{\mathcal{P}}(g))(g),$$

where f_ℓ is viewed as a polynomial on $\check{\mathfrak{a}}_{\mathcal{P}}$ with values in functions on

$$P\mathcal{A}_{\mathcal{P}}(\boldsymbol{R})^o\mathbb{N}_{\mathcal{P}}\backslash\mathbb{G}.$$

In order to make (3) an homomorphism of $(\mathfrak{g}, K, \mathbb{G}_f)$-modules, the first tensor factor in (2) has to be twisted by $\rho_{\mathcal{P}}$ similar to the twist used for $D'(i\mathfrak{a})_c$ before we formulated (5.2.3). There is a similar map

$$(4) \qquad \bigoplus_{\substack{\Re(\ell)\in\tau-{}^+\check{\mathfrak{a}}^{\mathcal{G}}_o \\ \mathcal{J}_\ell\subset\mathfrak{Z}(\mathfrak{m}_{\mathcal{P}})}} \mathrm{Ind}^{\mathcal{G}}_{\mathcal{P}} D_\ell \otimes \mathfrak{Fin}_{\mathcal{J}_\ell} S_{\rho_{-\tau_\ell}-\log}(L_{\mathcal{P}}\mathcal{A}_{\mathcal{P}}(\boldsymbol{R})^o\backslash\mathbb{L}_{\mathcal{P}}) \to \mathfrak{Fin}_{\mathcal{J}} S_{\rho_{-\tau}-\log}[\mathcal{P}]$$

Now we use the induction assumption to prove the following fact.

PROPOSITION 1.

- *If* $\tau \in \overline{\check{\mathfrak{a}}^{\mathcal{G}+}_o}$, *then the homomorphisms* (2) *and* (4) *are isomorphisms.*
- *Under the same assumption to* τ, $\mathfrak{Fin}^i_{\mathcal{J}} S_{\rho_{-\tau}+\log}[\mathcal{P}]$ *vanishes for* $i > 0$.
- *The same is true for* $\mathfrak{Fin}^i_{\mathcal{J}} S_{\rho_{-\tau}-\log}[\mathcal{P}]$ *provided that* τ *also belongs to* ${}^+\check{\mathfrak{a}}^{\mathcal{G}}_o$.

Proof. STEP 1. – It is clear that every function f in $\mathfrak{Fin}_{\mathcal{J}} S_\rho[\mathcal{P}]$ has an expansion of type (3) with

$$(f_\ell) \in \bigoplus_{\mathcal{J}_\ell\subset\mathfrak{Z}(\mathfrak{m}_{\mathcal{P}})} \mathrm{Ind}^{\mathcal{G}}_{\mathcal{P}} D_\ell \otimes \mathfrak{Fin}_{\mathcal{J}_\ell} S_\infty(L_{\mathcal{P}}\mathcal{A}_{\mathcal{P}}(\boldsymbol{R})^o\backslash\mathbb{L}_{\mathcal{P}}).$$

Using the same arguments as in the proof of Theorem 15, we see that the sum is actually over the same range as in (2) and that f_ℓ is square integrable with weight $\rho_{-\tau_\ell} + \log$. This proves the first assertion.

STEP 2. – To prove the acyclicity, we first note that

$$S_{\rho_{-\tau}+\log}(P\mathbb{N}_{\mathcal{P}}\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash P\mathfrak{S}(\mathcal{P},D,T)) \cong \mathrm{Ind}_{\mathcal{P}}^{\mathcal{G}} S_{\rho_{-\tau}+\log}(L_{\mathcal{P}}\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash(\mathbb{P}\cap P\mathfrak{S}(\mathcal{P},D,T))).$$

By Theorem 7, it suffices to prove the vanishing of

$$\mathfrak{Fin}^i_{\mathcal{J}'} \operatorname*{colim}_{T\to\infty} S_{\rho_{-\tau}+\log}(L_{\mathcal{P}}\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash(\mathbb{L}\cap P\mathfrak{S}(\mathcal{P},D,T))),$$

for $i > 0$, where $\mathcal{J}'$ is related to $\mathcal{J}$ by (4.5). The radical of $\mathcal{J}'$ is equal to the radical of the product

$$\prod_{\substack{\ell\in(\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{P}})_{\boldsymbol{C}} \\ \mathcal{J}_\ell\subset\mathfrak{Z}(\mathfrak{l}_{\mathcal{P}})}} \mathcal{J}'_\ell,$$

where $\mathcal{J}'_\ell \subset \mathfrak{Z}(\mathfrak{l}_{\mathcal{P}})$ is the ideal generated by $\mathcal{J}'$ and

$$\mathcal{I}_\ell = \{A \in \mathfrak{Z}(\mathfrak{a}^{\mathcal{G}}_{\mathcal{P}}) = \boldsymbol{S}(\mathfrak{a}^{\mathcal{G}}_{\mathcal{P}}) | A(\ell+\rho_{\mathcal{P}}) = 0\}.$$

Consequently, by part 6 of Theorem 7, it is sufficient to verify the vanishing of

$$\mathfrak{Fin}^i_{\mathcal{J}'_\ell} \operatorname*{colim}_{T\to\infty} S_{\rho_{-\tau}+\log}(L_{\mathcal{P}}\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash(\mathbb{L}\cap P\mathfrak{S}(\mathcal{P},D,T))) \tag{5}$$

for $i > 0$. The family of domains

$$X_T = \left\{g\in L\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{L} \;\middle|\; \langle H_{\mathcal{P}}(g),\alpha\rangle \geq \log\big(\rho_{-\mathrm{pr}_{\check{\mathfrak{a}}^{\mathcal{G}}_o\to\check{\mathfrak{a}}^{\mathcal{P}}_o}\alpha}(\hat{g})\big)+T \text{ for all } \alpha\in\Delta_o-\Delta^{\mathcal{P}}_o\right\},$$

where $\hat{g}$ is the image of g in $L\mathcal{A}_{\mathcal{P}}(\boldsymbol{R})^o\backslash\mathbb{L}$, is cofinal with the family

$$L\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o \backslash (\mathbb{L}\cap P\mathfrak{S}(\mathcal{P},D,T)).$$

Consequently, it is sufficient to prove the vanishing of

$$\operatorname*{colim}_{T\to\infty} \mathfrak{Fin}^i_{\mathcal{J}'_\ell} S_{\rho_{-\tau}+\log}(X_T) \tag{6}$$

for positive i. We first verify that

$$\mathfrak{Fin}^i_{\mathcal{I}_\ell} S_{\rho_{-\tau}+\log}(X_T) = \begin{cases} 0 & \text{if } i>0 \text{ or } \Re\ell \notin \tau - \overline{{}^+\check{\mathfrak{a}}^{\mathcal{G}}_o} \\ D_\ell\otimes S_{\rho_{-\tau_\ell}}(L\mathcal{A}_{\mathcal{P}}(\boldsymbol{R})^o\backslash\mathbb{L}) & \text{if } i=0 \text{ and } \Re\ell\in\tau-\overline{{}^+\check{\mathfrak{a}}^{\mathcal{G}}_o}. \end{cases} \tag{7}$$

STEP 3. – To prove (7), we note that the substitution

$$f(g) \to f\Big(g\cdot\exp\Big(\sum_{\alpha\in(\Delta_o-\Delta^{\mathcal{P}}_o)} (T+\log(\rho_{\mathrm{pr}_{\check{\mathfrak{a}}^{\mathcal{G}}_o\to\check{\mathfrak{a}}^{\mathcal{P}}_o}\alpha}(\hat{g})))\check{\omega}_\alpha\Big)\Big)$$

identifies $C^\infty(X_T)$ with

$$C^\infty\Big(\{g \in L\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{L} \mid H_\mathcal{P}(g) \in \overline{\mathfrak{a}_\mathcal{P}^{\mathcal{G}+}}\}\Big) = C^\infty\Big(L\mathcal{A}_\mathcal{P}(\boldsymbol{R})^o\backslash\mathbb{L}\Big)\widehat{\otimes}C^\infty\Big(\overline{\mathfrak{a}_\mathcal{P}^{\mathcal{G}+}}\Big)$$

as an $\mathfrak{a}_\mathcal{P}$-module and defines an isomorphism

$$S_{\rho_{-\tau}+\log}(X_T) \cong S_{-\tau-\rho_\mathcal{P}+\log}\Big(\overline{\mathfrak{a}_\mathcal{P}^{\mathcal{G}+}}\Big)\widehat{\otimes}S_{\rho_{-\tau}+\log}(L\mathcal{A}_\mathcal{P}(\boldsymbol{R})^o\backslash\mathbb{L}),$$

where $S_{-\tau-\rho_\mathcal{P}+\log}\Big(\overline{\mathfrak{a}_\mathcal{P}^{\mathcal{G}+}}\Big)$ is the set of those $f \in C^\infty\Big(\overline{\mathfrak{a}_\mathcal{P}^{\mathcal{G}+}}\Big)$ for which there exists an m with

$$\int_{\overline{\mathfrak{a}_\mathcal{P}^{\mathcal{G}+}}} \Big(\mathrm{e}^{-\langle x,\tau+\rho_\mathcal{P}\rangle}(1+\mid x\mid)^{-m} \mid (Af)(x)\mid\Big)^2 dx < \infty$$

for all differential operators $A \in \mathfrak{Z}(\mathfrak{a}_\mathcal{P}^\mathcal{G})$. As in the proof of Theorem 7.2, $\mathfrak{Fin}^i_{\mathcal{I}_\ell}$ of the last $\mathfrak{a}$-module can be computed using the limit of the i-th cohomology of the Koszul complex

$$C^*\left(\left\{\left(\frac{\partial}{\partial\hat{\omega}_\alpha} - \langle\hat{\omega}_\alpha, \ell+\rho_\mathcal{P}\rangle\right)^n\right\}_{\alpha\in\Delta_\mathcal{P}}, S_{-\tau-\rho_\mathcal{P}+\log}\Big(\overline{\mathfrak{a}_\mathcal{P}^{\mathcal{G}+}}\Big)\widehat{\otimes}S_{\rho_{-\tau_\ell}+\log}(L\mathcal{A}_\mathcal{P}(\boldsymbol{R})^o\backslash\mathbb{L})\right)$$
$$= C^*\left(\left\{\left(\frac{\partial}{\partial\hat{\omega}_\alpha} - \langle\hat{\omega}_\alpha, \ell+\rho_\mathcal{P}\rangle\right)^n\right\}_{\alpha\in\Delta_\mathcal{P}}, S_{-\tau-\rho_\mathcal{P}+\log}(\overline{\mathfrak{a}_\mathcal{P}^{\mathcal{G}+}})\right)\widehat{\otimes}S_{\rho_{-\tau_\ell}+\log}(L\mathcal{A}_\mathcal{P}(\boldsymbol{R})^o\backslash\mathbb{L})$$

for $n \to \infty$. By standard properties of the nuclear tensor product, it is sufficient to investigate the cohomology of the first factor provided that it is finite-dimensional. The first factor is a tensor product

$$\widehat{\bigotimes_{\alpha\in\Delta_\mathcal{P}}}\left(S_{\langle\hat{\omega}_\alpha,\ell+\rho_\mathcal{P}\rangle+\log}([0,\infty)) \xrightarrow{\left(\frac{d}{dt} - \langle\hat{\omega}_\alpha, \ell+\rho_\mathcal{P}\rangle\right)^n} S_{\langle\hat{\omega}_\alpha,\ell+\rho_\mathcal{P}\rangle+\log}([0,\infty))\right).$$

Here $S_{-\lambda+\log}([0,\infty))$ is the set of those $f \in C^\infty([0,\infty))$ for which there exists an m with

$$\int_0^\infty \mathrm{e}^{-2\lambda t}(1+t)^{-m}\left|\frac{d^k f}{dt^k}\right|^2 dt < \infty$$

for all $k \geq 0$. It is easy to see that the cokernel of the map

$$S_{-\lambda+\log}([0,\infty)) \xrightarrow{\left(\frac{d}{dt} - \vartheta\right)^n} S_{-\lambda+\log}([0,\infty))$$

is always zero while its kernel vanishes if $\Re\vartheta > \lambda$ but is equal to

$$\bigoplus_{j=0}^n \mathrm{e}^{t\vartheta}t^j\boldsymbol{C}$$

if $\Re\vartheta \leq \lambda$. This finishes the proof of (7).

STEP 4. – By (7) and (4.6), we have

$$\mathfrak{Fin}^i_{\mathcal{J}'_\ell} S_{\rho_{-\tau}+\log}(X_T) \cong 0$$

if $\Re\ell \notin \tau - \overline{{}^+\check{\mathfrak{a}}^{\mathcal{G}}_o}$ and

$$\mathfrak{Fin}^i_{\mathcal{J}'_\ell} S_{\rho_{-\tau}+\log}(X_T) \cong D_\ell \otimes \mathfrak{Fin}^i_{\mathcal{J}_\ell} S_{\rho_{-\tau_\ell}+\log}(L\mathcal{A}_{\mathcal{P}}(\boldsymbol{R})^o\backslash\mathbb{L})$$

if $\Re\ell \in \tau - \overline{{}^+\check{\mathfrak{a}}^{\mathcal{G}}_o}$. In the first case, (6) is clear. In the second case, we have $\tau_\ell \in \overline{\check{\mathfrak{a}}^{\mathcal{P}}_o}$ and (6) follows because the induction assumption implies the vanishing of

$$\mathfrak{Fin}^i_{\mathcal{J}_\ell} S_{\rho_{-\tau_\ell}+\log}(L\mathcal{A}_{\mathcal{P}}(\boldsymbol{R})^o\backslash\mathbb{L})$$

for positive i. This completes the proof of the second assertion. The proof of the third assertion is similar. □

7.3. End of the induction argument

By the results of the previous two subsections, and by Theorem 13, we have an isomorphism

$$\mathfrak{Fin}^i_{\mathcal{J}} S_{\rho_{-\tau}+\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G}) \cong \lim\nolimits^i_{\mathfrak{P}} \mathfrak{Fin}_{\mathcal{J}}\big(S_{\rho_{-\tau}+\log}/S_{\log}\big)[\cdot].$$

In order to understand the cohomology groups on the right hand side, we introduce a filtration of the functor $\mathfrak{Fin}_{\mathcal{J}} S_{\rho_{-\tau}+\log}[\cdot]$. For every $f \in \mathfrak{Fin}_{\mathcal{J}} S_{\rho_{-\tau}+\log}[\mathcal{Q}]$ and every standard parabolic subgroup $\mathcal{P}$ of $\mathcal{Q}$, there is an expansion (6.2) for the constant term of f along $\mathcal{N}_{\mathcal{P}}$. For $\lambda \in \check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{P}}$, let $\lambda_{\mathcal{Q}}$ and $\lambda^{\mathcal{Q}}$ be the projections of λ to $\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{Q}}$ and $\check{\mathfrak{a}}^{\mathcal{Q}}_{\mathcal{P}}$, and let $\lambda^{\mathcal{Q}} = \lambda^{\mathcal{Q}}_+ - \lambda^{\mathcal{Q}}_-$ be the decomposition of $\lambda^{\mathcal{Q}}$ by (6.5), using the roots of $\mathcal{M}_{\mathcal{P}}$. Let X be the set of those $\lambda \in \check{\mathfrak{a}}^{\mathcal{G}}_o$ such that there exist parabolic subgroups $\mathcal{P} \supseteq \mathcal{R}$, $f \in \mathfrak{Fin}_{\mathcal{J}} S_{\rho_{-\tau}+\log}[\mathcal{P}]$ and $x \in \big(\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{R}}\big)_{\boldsymbol{C}}$ with $f_{\mathcal{R},x} \neq 0$ and $\Re(x)_{\mathcal{P}} + \Re(x)^{\mathcal{P}}_+ = \lambda$. Let $T(\lambda)$ be an integer-valued function on X such that $T(\lambda) = T(\vartheta)$ implies $\lambda = \vartheta$ and such that $T(\lambda) \leq T(\vartheta)$ if $\vartheta \in \lambda - \overline{{}^+\check{\mathfrak{a}}^{\mathcal{G}}_o}$.

We define a filtration on $\mathfrak{Fin}_{\mathcal{J}} S_{\rho_{-\tau}+\log}[\mathcal{P}]$ by

$$(1)\quad \mathfrak{Fin}_{\mathcal{J}} S^i_{\rho_{-\tau}+\log}[\mathcal{P}] = \Big\{ f \in \mathfrak{Fin}_{\mathcal{J}} S_{\rho_{-\tau}+\log}[\mathcal{P}] \;\Big|\; \text{such that } f_{\mathcal{R},\lambda} \text{ is orthogonal to the space of cusp forms if } T\big(\Re(\lambda)_{\mathcal{P}} + \Re(\lambda)^{\mathcal{P}}_+\big) < i \Big\}.$$

To verify that $\mathfrak{Fin}_{\mathcal{J}} S^i_{\rho_{-\tau}+\log}[\cdot]$ is a subfunctor of the functor $\mathfrak{Fin}_{\mathcal{J}} S_{\rho_{-\tau}+\log}[\cdot]$ from $\mathfrak{P}$ to $(\mathfrak{g}, K, \mathbb{G}_f)$-modules, it suffices to prove the following lemma:

LEMMA 1. – *If* $\lambda \in \check{\mathfrak{a}}^{\mathcal{G}}_o$ *and* $\mathcal{P} \supset \mathcal{Q}$, *then*

$$\lambda_{\mathcal{P}} + \lambda^{\mathcal{P}}_+ - \lambda_{\mathcal{Q}} - \lambda^{\mathcal{Q}}_+ \in \overline{{}^+\check{\mathfrak{a}}^{\mathcal{P}}_o}.$$

Proof. – This is similar to the proof of (6.7). If α belongs to $\Delta^{\mathcal{P}}_o - P(\lambda^{\mathcal{Q}})$, then

$$\langle \check{\omega}^{\mathcal{P}}_\alpha, \lambda_{\mathcal{P}} + \lambda^{\mathcal{P}}_+ \rangle \geq \langle \check{\omega}^{\mathcal{P}}_\alpha, \lambda \rangle = \langle \check{\omega}^{\mathcal{P}}_\alpha, \lambda_{\mathcal{Q}} + \lambda^{\mathcal{Q}}_+ \rangle,$$

otherwise (if $\alpha \in P(\lambda^{\mathcal{Q}}) \subseteq \Delta_o^{\mathcal{Q}}$)

$$\langle \check{\alpha}, \lambda_{\mathcal{P}} + \lambda_+^{\mathcal{P}} \rangle \geq 0 = \langle \check{\alpha}, \lambda_{\mathcal{Q}} + \lambda_+^{\mathcal{Q}} \rangle.$$

Since $\mathfrak{a}_o^{\mathcal{P}}$ is contained in the cone spanned by $\{\alpha | \alpha \in (\Delta_o^{\mathcal{P}} - \Delta_o^{\mathcal{Q}}) \bigcup P(\lambda^{\mathcal{Q}})\}$ and $\{\check{\omega}_\alpha^{\mathcal{P}} | \alpha \in \Delta_o^{\mathcal{Q}} - P(\lambda^{\mathcal{Q}})\}$, this proves the assertion. □

We put

$$\mathfrak{Fin}_{\mathcal{J}} S_{\log}^i[\cdot] = \mathfrak{Fin}_{\mathcal{J}} S_{\rho_- \tau + \log}^i[\cdot] \cap S_{\log}[\cdot].$$

The induction step will be completed if we show that

$$\lim\nolimits_{\mathfrak{P}}^k \left(\mathfrak{Fin}_{\mathcal{J}} S_{\rho_- \tau + \log}^i[\cdot] \Big/ \left(\mathfrak{Fin}_{\mathcal{J}} S_{\rho_- \tau + \log}^{i+1}[\cdot] + \mathfrak{Fin}_{\mathcal{J}} S_{\log}^i[\cdot] \right) \right) \tag{2}$$

vanishes for positive k. The quotient can be determined using Theorem 14. Let $\lambda \in \check{\mathfrak{a}}_o^{\mathcal{G}}$ be the unique λ with $T(\lambda) = i$ (if it does not exist, then the quotient vanishes for obvious reasons). Let $\mathcal{P}_1 \supseteq \mathcal{P}_2$ be the standard parabolic subgroups defined by

$$\Delta_o^{\mathcal{P}_1} = \{\alpha \in \Delta_o \mid \langle \check{\alpha}, \lambda \rangle \geq 0\}$$
$$\Delta_o^{\mathcal{P}_2} = \{\alpha \in \Delta_o \mid \langle \check{\alpha}, \lambda \rangle = 0\}$$

Let $\mathcal{M}_\lambda$ be the following groupoid: Objects are pairs $(\mathcal{R}, \tilde{\lambda})$ with $\mathcal{R} \subseteq \mathcal{P}_2$ and $\tilde{\lambda} \in (\check{\mathfrak{a}}_{\mathcal{R}}^{\mathcal{G}})_{\boldsymbol{C}}$ such that $\Re(\tilde{\lambda}) = \lambda$. Morphisms are Weyl set elements $w \in \Omega(\mathcal{R}_1, \mathcal{R}_2)$ with $w\tilde{\lambda}_1 = \tilde{\lambda}_2$ which identically act on $\check{\mathfrak{a}}_{\mathcal{P}_1}$ (and hence on $\check{\mathfrak{a}}_{\mathcal{P}_2}$ too, because $\lambda \in \check{\mathfrak{a}}_{\mathcal{P}_2}^{\mathcal{P}_1 +}$). Let $\mathcal{M}_\lambda^{\mathcal{P}}$ be the subcategory consisting of those objects $(\mathcal{R}, \tilde{\lambda})$ for which $\mathcal{R} \subseteq \mathcal{P}$ and those morphisms w which identically act on $\check{\mathfrak{a}}_{\mathcal{P}}$.

Let E be a functor from $\mathcal{M}_\lambda$ to $(\mathfrak{g}, K, \mathbb{G}_f)$-modules. We define a class of functors $E^\clubsuit[\cdot]$ from $\mathfrak{P}$ to $(\mathfrak{g}, K, \mathbb{G}_f)$-modules as follows: For a standard parabolic subgroup $\mathcal{P}$ which is not contained in $\mathcal{P}_1$, we put $E^\clubsuit[\mathcal{P}] = 0$. Otherwise, we put

$$E^\clubsuit[\mathcal{P}] = \operatorname*{colim}_{\mathcal{M}_\lambda^{\mathcal{P}}} E.$$

If $\mathcal{P} \cap \mathcal{P}_2 = \mathcal{Q} \cap \mathcal{P}_2$, then $\mathcal{M}_\lambda^{\mathcal{P}} = \mathcal{M}_\lambda^{\mathcal{Q}}$. Let us assume that the structure of a functor from $\mathfrak{P}$ to $(\mathfrak{g}, K, \mathbb{G}_f)$-modules on $E^\clubsuit[\cdot]$ has the property that for such an inclusion the transition homomorphism

$$E^\clubsuit[\mathcal{P}] \to E^\clubsuit[\mathcal{Q}] \tag{3}$$

is the identity.

LEMMA 2. – *If $\mathcal{P}_1 \supset \mathcal{P}_2$, then $\lim_{\mathfrak{P}}{}^k E^\clubsuit[\cdot]$ vanishes for positive k.*

Proof. – We define a filtration on $E^\clubsuit$ such that Proposition 7.1.1 can be applied to the quotients. Let

$$E^{\clubsuit i}[\mathcal{P}] = \begin{cases} 0 & \text{if } \operatorname{rank}(\mathcal{P} \cap \mathcal{P}_2) < i \\ E^\clubsuit[\mathcal{P}] & \text{otherwise} \end{cases}$$

Let $\mathcal{R}_1^i, \ldots, \mathcal{R}_{M_i}^i$ be the standard parabolic subgroups of rank i contained in $\mathcal{P}_2$, and let $\mathcal{Q}_j^i$ be the largest parabolic subgroup of $\mathcal{P}_1$ with $\mathcal{Q}_j^i \cap \mathcal{P}_2 = \mathcal{R}_j^i$. Then $\mathcal{Q}_j^i \supset \mathcal{R}_j^i$ because $\mathcal{P}_1 \supset \mathcal{P}_2$. There is an obvious isomorphism

$$\left(E^{\clubsuit i}/E^{\clubsuit i+1}\right)[\cdot] \cong \bigoplus_{j=1}^{M_i} E^{\clubsuit}[\mathcal{R}_j^i] \otimes F_{\mathcal{R}_j^i}^{\mathcal{Q}_j^i}[\cdot]$$

and our assertion follows from Proposition 7.1.1. □

Let $V_{(\mathcal{R},\tilde{\lambda})}$ be the space of all square integrable $\mathbb{K}$-finite C^∞-functions on $R\mathbb{N}_\mathcal{R}\mathcal{A}_\mathcal{R}(\boldsymbol{R})^o\backslash\mathbb{G}$ such that

$$\mathrm{e}^{\langle\tilde{\lambda}+\rho_\mathcal{R}, H(g)\rangle} f(g)$$

is annihilated by $\mathcal{J}$. Then as in Lemma 6.2, the intertwining operator

$$M(w,\vartheta)\colon V_{\mathcal{R}_1,\tilde{\lambda}_1} \to V_{\mathcal{R}_2,\tilde{\lambda}_2}$$

is holomorphic near $\tilde{\lambda}$. If $D_{\mathcal{R},\tilde{\lambda}}$ denotes holomorphic differential operators on $(\check{\mathfrak{a}}_\mathcal{R}^\mathcal{G})_{\boldsymbol{C}}$ with support at $\tilde{\lambda}$, then (6.12) defines a structure of a functor from $\mathcal{M}_\lambda$ to $(\mathfrak{g}, K, \mathbb{G}_f)$-modules on

$$E_\lambda(\mathcal{R},\tilde{\lambda}) = D_{\mathcal{R},\tilde{\lambda}} \otimes V_{\mathcal{R},\tilde{\lambda}}.$$

Now we are ready to compute the quotient

$$\mathfrak{Fin}_\mathcal{J} S^i_{\rho_{-\tau}+\log}[\mathcal{P}] \Big/ \mathfrak{Fin}_\mathcal{J} S^{i+1}_{\rho_{-\tau}+\log}[\mathcal{P}]$$

The quotient vanishes if $\mathcal{P} \not\subseteq \mathcal{P}_1$ because then λ cannot be of the form $x_\mathcal{P} + x_+^\mathcal{P}$ with $x_\mathcal{P} \in \check{\mathfrak{a}}_\mathcal{P}$ and $x_+^\mathcal{Q} \in \overline{\check{\mathfrak{a}}_o^{\mathcal{P}+}}$. Otherwise, Theorem 14 and Proposition 7.2.1 provide an isomorphism

$$\operatorname*{colim}_{\mathcal{M}_\lambda^\mathcal{P}} E_\lambda \cong \mathfrak{Fin}_\mathcal{J} S^i_{\rho_{-\tau}+\log}[\mathcal{P}] \Big/ \mathfrak{Fin}_\mathcal{J} S^{i+1}_{\rho_{-\tau}+\log}[\mathcal{P}] \tag{4}$$

which sends $D \otimes \phi$ to

$$\mathbf{MW}_{\vartheta=\tilde{\lambda}} D E_\mathcal{R}^\mathcal{P}(\phi,\vartheta). \tag{5}$$

Let us assume that ϕ belongs to a fixed class of associate parabolic subgroups of $\mathcal{P}$. Then for $\hat{\mathcal{Q}}$ in that class, the constant term of (5) along $\mathcal{N}_{\hat{\mathcal{Q}}}$ is given by

$$\mathbf{MW}_{\vartheta=\tilde{\lambda}} \left(\sum_{\hat{\omega}=\text{identity on } \check{\mathfrak{a}}_{\mathcal{P}\cap\mathcal{P}_2}} N(\omega,\vartheta)\phi \mathrm{e}^{\langle\vartheta+\rho_{\hat{\mathcal{Q}}}, H(g)\rangle} \right) \tag{6}$$

modulo terms which are only essential for the next step of the filtration of $\mathfrak{Fin}_\mathcal{J} S_{\rho_{-\tau}+\log}[\mathcal{P}]$. Let $E_\lambda^\clubsuit$ be given the structure of a functor from $\mathfrak{P}$ to the category of $(\mathfrak{g}, K, \mathbb{G}_f)$-modules such that (4) is an isomorphism of functors from $\mathfrak{P}$ to $(\mathfrak{g}, K, \mathbb{G}_f)$-modules. If

$\mathcal{P} \cap \mathcal{P}_2 = \mathcal{Q} \cap \mathcal{P}_2$, then the sum in (6) does not change, consequently (3) is the identity for such an inclusion.

If $\lambda \notin -\overline{{}^+\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{P}_2}}$, then Theorem 14 implies

$$\mathfrak{Fin}_{\mathcal{J}} S^i_{\log}[\cdot] = \mathfrak{Fin}_{\mathcal{J}} S^{i+1}_{\log}[\cdot],$$

hence

$$\mathfrak{Fin}_{\mathcal{J}} S^i_{\rho_{-\tau}+\log}[\mathcal{P}] \Big/ \Big(\mathfrak{Fin}_{\mathcal{J}} S^{i+1}_{\rho_{-\tau}+\log}[\mathcal{P}] + \mathfrak{Fin}_{\mathcal{J}} S^i_{\log}[\mathcal{P}]\Big) \tag{7}$$

is isomorphic to

$$\mathfrak{Fin}_{\mathcal{J}} S^i_{\rho_{-\tau}+\log}[\mathcal{P}] \Big/ \mathfrak{Fin}_{\mathcal{J}} S^{i+1}_{\rho_{-\tau}+\log}[\mathcal{P}] \cong E^{\clubsuit}_{\lambda}[\mathcal{P}].$$

Since $\mathcal{P}_1 = \mathcal{P}_2$ would imply $\lambda \in -\overline{\check{\mathfrak{a}}^{\mathcal{G}+}_{\mathcal{P}_2}}$, Lemma 2 can be applied. Consequently, (2) vanishes for positive k.

If $\lambda \in -\overline{{}^+\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{P}_2}}$, then Theorem 16 and Proposition 7.2.1 imply

$$\mathfrak{Fin}_{\mathcal{J}} S^i_{\log}[\mathcal{P}]/\mathfrak{Fin}_{\mathcal{J}} S^{i+1}_{\log}[\mathcal{P}] \cong E^{\clubsuit}_{\lambda}[\mathcal{P}] \cong \mathfrak{Fin}_{\mathcal{J}} S^i_{\rho_{-\tau}+\log}[\mathcal{P}] \Big/ \mathfrak{Fin}_{\mathcal{J}} S^{i+1}_{\rho_{-\tau}+\log}[\mathcal{P}],$$

hence (7) vanishes and (2) vanishes for all k.

The proof of Theorem 16 is complete.

7.4. Borel's conjecture

Let $C^\infty(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$ be the $(\mathfrak{g}, K, \mathbb{G}_f)$-module of $\mathbb{K}$-finite C^∞-functions on $G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G}$. By (2.3.16), the $(\mathfrak{m}_{\mathcal{G}}, K)$-cohomology of $C^\infty(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G}) \otimes \boldsymbol{E}$ is isomorphic to the cohomology of the de Rham complex of $G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G}/K$ with coefficients in the local system $\mathbb{E}$ defined by the finite dimensional representation $\boldsymbol{E}$ of $\mathcal{G}(\boldsymbol{R})$, which computes the inductive limit of the cohomology of the arithmetic subgroups of $\mathcal{G}$ with coefficients in $\boldsymbol{E}$. From Theorem 2, we can derive Borel's result that the inclusion

$$S_\infty(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G}) \to C^\infty(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$$

induces in isomorphism on the $(\mathfrak{m}_{\mathcal{G}}, K)$-cohomology with coefficients in any finite-dimensional representation. Since $S_\infty(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$ is the union of the $S_{\rho_{-\tau}+\log}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$ over all possible τ, Theorem 16 implies the vanishing of $\mathfrak{Fin}^i_{\mathcal{J}} S_\infty(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$ for positive i. Taking $\mathcal{J}$ to be the annihilator of $\check{\boldsymbol{E}}$ and using (4.4), we get the following theorem, which establishes a conjecture of Borel.

THEOREM 18. – *The inclusions*

$$\mathfrak{Fin}_{\mathcal{J}} S_\infty(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G}) \to S_\infty(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G}) \to C^\infty(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})$$

define an isomorphism on the $(\mathfrak{m}_{\mathcal{G}}, K)$-cohomology with coefficients in $\boldsymbol{E}$. *In particular,*

$$H^*\Big(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G}/K, \mathbb{E}\Big) \cong H^*_{(\mathfrak{g},K)}\Big(\mathfrak{Fin}_{\mathcal{J}}\big(S_\infty(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})\big) \otimes \boldsymbol{E}\Big)(\zeta_{\boldsymbol{E}}),$$

where the twist $(\zeta_{\boldsymbol{E}})$ *has the same meaning and occurs for the same reason as in* (2.3.17).

Using Theorem 14, we get the following spectral sequence of $(\mathfrak{g}, K, \mathbb{G}_f)$-modules which, at least theoretically, computes the cohomology of $C^\infty(G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G})$ in terms of the cohomology of automorphic representations of $\mathcal{G}$ and its Levi components. The term $E_1^{p,q}$ is given by

$$\bigoplus_{\{P\}} \bigoplus_{k=0}^{\operatorname{rank}\{P\}} \operatorname*{colim}_{\mathcal{M}^{k,T,p}_{\mathcal{J},\{P\},\infty}} H^{p+q}_{(\mathfrak{m}_\mathcal{G},K)}(W(u_t)\otimes D_t\otimes \boldsymbol{E}), \tag{1}$$

and the limit is

$$H^{p+q}_{(\mathfrak{m}_\mathcal{G},K)}(C^\infty(G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G})\otimes \boldsymbol{E}).$$

Here $\mathcal{M}^{k,T,p}_{\mathcal{J},\{P\},\infty}$ is the union of $\mathcal{M}^{k,T,p}_{\mathcal{J},\{P\},\tau}$ over all τ, and T is a function satisfying (6.8). The E_1-term of this spectral sequence is easily computed. We have

(2)
$$\begin{aligned} H^r_{(\mathfrak{m}_\mathcal{G},K)}\Big(W(u_t)\otimes D_t\otimes \boldsymbol{E}\Big) &= H^r_{(\mathfrak{m}_\mathcal{G},K)}\Big(\operatorname{Ind}^{\mathcal{G}}_{\mathcal{R}_t}(\boldsymbol{E}\otimes V(u_t)\otimes D_t)\Big)\\ &= \operatorname{Ind}^{\mathbb{G}_f}_{\mathbf{R}_{t,f}} H^r_{(\mathfrak{r}\cap\mathfrak{m}_\mathcal{G},K\cap\mathcal{R}(\mathbf{R}))}\big(V(u_t)\otimes \boldsymbol{E}\otimes D_t\big)\\ &= \bigoplus_{r=i+j} \operatorname{Ind}^{\mathbb{G}_f}_{\mathbf{R}_{t,f}} H^i_{(\mathfrak{l}_{\mathcal{R}_t},K\cap\mathcal{R}_t(\boldsymbol{R}))}\big(V(u_t)\otimes H^j(\mathfrak{n}_{\mathcal{R}_t},\boldsymbol{E})\otimes D_t\big), \end{aligned}$$

where the first line uses the interchangeability of Ind and the tensor product by a finite-dimensional representation, the second line uses Frobenius reciprocity, and the third line will be explained below.

Let $\mathfrak{h}\supset\mathfrak{a}_o$ be a Cartan subalgebra of $\mathfrak{g}$, and let us assume that $\boldsymbol{E}=\boldsymbol{E}_{\mathcal{G},\Lambda}$ is irreducible with highest weight $\Lambda\in\overline{\mathfrak{h}^+}$. By Kostant's theorem on $\mathfrak{n}$-cohomology ([Wal88], Theorem 9.6.2 or [Vog81], Theorem 3.2.3), the $\mathfrak{n}_{\mathcal{R}_t}$-homology groups in this formula can be computed as a sum

$$H^j(\mathfrak{n}_{\mathcal{R}_t},\boldsymbol{E}_{\mathcal{G},\Lambda}) = \bigoplus_{\operatorname{length}(w)=j} \boldsymbol{E}_{\mathcal{L}_{\mathcal{R}_t},w(\Lambda+\rho_\mathfrak{h})-\rho_\mathfrak{h}} \tag{3}$$

of duals of $\mathcal{L}_{\mathcal{R}_t}$-representations with highest weights $w\Lambda$ over those elements w of the Weyl group of $\mathfrak{h}$ which satisfy

$$w^{-1}\alpha>0 \tag{4}$$

for all positive roots α of $\mathfrak{h}$ in $\mathfrak{l}_{\mathcal{R}_t}$. Here $\rho_\mathfrak{h}$ is one half the sum of the positive roots of $\mathfrak{h}$ in $\mathfrak{g}$. Actually, the proof of Kostant's theorem as given for instance in Vogan's book also proves that (3) is a splitting in the derived category, whence the third line in (2).

Let us also note that for $\Theta\in\check{\mathfrak{a}}_{\mathcal{R}_t}$, we have

$$H^k_{\mathfrak{a}_{\mathcal{R}_t}}(\boldsymbol{C}_\Theta\otimes D_t)=\begin{cases}0 & \text{if } k>0 \text{ or if } \Theta\neq-\rho_{\mathcal{R}_t}-\lambda_t\\ \boldsymbol{C} & \text{if } k=0 \text{ and } \Theta=-\rho_{\mathcal{R}_t}-\lambda_t,\end{cases}$$

where $\boldsymbol{C}_\Theta$ is the one-dimensional vector space $\boldsymbol{C}$ on which $x\in\mathfrak{a}_{\mathcal{R}_t}$ acts by multiplication by $\langle x,\Theta\rangle$. That there is no higher cohomology is due to the fact that the Koszul complex for the coordinate functions and the space of distributions supported at 0 is acyclic in higher dimensions. Together with (2) and (3) this gives our final formula

$$(5)\quad H^r_{(\mathfrak{m}_\mathcal{G},K)}(W(u_t)\otimes D_t\otimes \boldsymbol{E})$$
$$= \bigoplus_{\substack{w \text{ satisfies } (4)\\ \mathrm{pr}_{\mathfrak{h}\to\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{R}_t}} w(\Lambda+\rho_\mathfrak{h})=-\lambda_t}} \mathrm{Ind}^{\mathbb{G}_f}_{(\mathbf{R}_t)_f}\left(H^{r-\mathrm{length}(w)}_{(\mathfrak{m}_{\mathcal{R}_t},K\cap\mathcal{R}_t(\boldsymbol{R}))}\big(\boldsymbol{E}_{\mathcal{L}_{\mathcal{R}_t},w(\Lambda+\rho_\mathfrak{h})-\rho_h}\otimes V(u_t)\big)\otimes \boldsymbol{C}_{\lambda_t+\rho_{\mathcal{R}_t}}\right).$$

To compute the E_1-term of the spectral sequence (1), it remains to compute the colimit. This turns out to be very easy, since no object of the groupoid over which it is taken has non-trivial automorphisms. The remaining problem is to compute the differentials of the spectral sequence, and this will be quite difficult in the general (= non-generic) since it requires a complete knowledge of the residues of Eisenstein series. In the special (= generic) case where Λ is regular, the spectral sequence degenerates.

THEOREM 19. **I.** – *If the ideal $\mathcal{J}\subset \mathfrak{Z}(\mathfrak{m}_\mathcal{G})$ corresponds to a set of non-singular elements of $\check{\mathfrak{h}}$, then no object of the groupoid $\mathcal{M}^{k,T,p}_{\mathcal{J},\{P\},\infty}$ has non-trivial automorphisms. Since this is the case if $\mathcal{J}$ is the annihilator of $\check{\boldsymbol{E}}_{\mathcal{G},\Lambda}$, the initial term of the spectral sequence* (1) *can be computed as follows. For each k, each $\{P\}$ and each p, choose a set $\mathcal{C}^T_{\{P\},k,p}$ of representatives for the isomorphism classes of objects of the groupoid $\mathcal{M}^{k,T,p}_{\mathcal{J},\{P\},\infty}$. Then the E_1-term is given by*

$$(6)\qquad E_1^{p,q} = \bigoplus_{\{P\},k}\ \bigoplus_{t\in\mathcal{C}^T_{\{P\},k,p}} H^r_{(\mathfrak{m}_\mathcal{G},K)}(W(u_t)\otimes D_t\otimes \boldsymbol{E}),$$

where the summands in 6 *are given by* (5).

II. *If the highest weight Λ is regular with respect to the root system of $\mathfrak{h}$ in $\mathfrak{g}$, then the summands in* (6) *vanish unless $k=\mathrm{rank}\{P\}$. In other words, residual Eisenstein series do not contribute to the Eisenstein cohomology of* **E**. *The spectral sequence* (1) *degenerates, and the cohomology of* **E** *can be computed as follows. Let*

$$\mathcal{C} = \bigcup_{\{P\},p} \mathcal{C}^T_{\{P\},\mathrm{rank}\{P\},p}$$

be a set of representatives (modulo the functional equations of the Eisenstein series) for the cuspidal Eisenstein series which contribute to the cohomology of **E**. *Then*

$$(7)\quad H^r_{(\mathfrak{m}_\mathcal{G},K)}(C^\infty(G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G})\otimes\boldsymbol{E}) = \bigoplus_{t\in\mathcal{C}}$$
$$\bigoplus_{\substack{\mathrm{pr}_{\mathfrak{h}\to\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{R}_t}} w(\Lambda+\rho_\mathfrak{h})=-\lambda_t\\ w \text{ satisfies } (4)}} \mathrm{Ind}^{\mathbb{G}_f}_{(\mathbf{R}_t)_f}\left(H^{r\text{-length}(w)}_{(\mathfrak{m}_{\mathcal{R}_t},K\cap\mathcal{R}_t(\boldsymbol{R}))}\big(\boldsymbol{E}_{\mathcal{L}_{\mathcal{R}_t},w(\Lambda+\rho_\mathfrak{h})-\rho_h}\otimes V(u_t)\big)\otimes \boldsymbol{C}_{\lambda_t+\rho_{\mathcal{R}_t}}\right).$$

Proof. – To prove the first part of the theorem, let us assume that $t=(R,\Xi,\chi)$ be an object of $\mathcal{M}^{k,T,p}_{\mathcal{J},\{P\},\infty}$ which has a non-trivial automorphism $w\in\Omega(\mathfrak{a}_\mathcal{R},\mathfrak{a}_\mathcal{R})$. Let the character χ of $\mathcal{Z}(\mathfrak{m}_\mathcal{R})$ be represented by a vector $x\in\check{\mathfrak{h}}^+_{\mathfrak{m}_\mathcal{R}}$, where $\mathfrak{h}_{\mathfrak{m}_\mathcal{R}}=\mathfrak{h}\cap\mathfrak{m}_\mathcal{R}$. Then by our assumptions on $\mathcal{J}$ the vector $\lambda_t + x \in \check{\mathfrak{h}}$ is regular. The Weyl group element w has a representative $\tilde{w} \in \Omega(\mathfrak{h}, \mathfrak{h})$ with $\tilde{w}\check{\mathfrak{h}}^+_{\mathfrak{m}_{\mathcal{R}}} = \check{\mathfrak{h}}^+_{\mathfrak{m}_{\mathcal{R}}}$. Then $\tilde{w}(\lambda_t + x) = \lambda_t + x$ since w is an automorphism of t, and this is a contradiction to the regularity of $\lambda_t + x$.

The first ingredient in our proof of the second remark is the following fact: If $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra and if F is a finite-dimensional irreducible $(\mathfrak{g}, K)$-module whose extremal $\mathfrak{h}$-weights are regular (for the root system of $\mathfrak{h}$ in $\mathfrak{g}$), then any unitary $(\mathfrak{m}_{\mathcal{G}}, K)$-module which has non-trivial cohomology with coefficients in F must be tempered. This is based on the Vogan-Zuckerman classification of unitary representations with non-zero cohomology (*see* the textbook [Wal88], §9 or the original publication [VZ83]) ([3]) with coefficients in F. The representatives of isomorphism classes of such unitary representations are parametrised $A_{\mathfrak{q}}(\lambda)$. Here $\mathfrak{q}$ is a θ-stable parabolic subalgebra of $\mathfrak{g}_{\boldsymbol{C}}$, which must have the property that $F/\mathfrak{n}_{\mathfrak{q}}F$ is one dimensional. In particular since the extremal weights of F are regular $\mathfrak{q}$ must be a Borel subalgebra. It is known that in this case the representation $A_{\mathfrak{q}}(\lambda)$ is tempered (*cf.* [Wal88], Theorem 6.8.1), which proves our claim.

Now let us assume $t \in C^T_{\{P\},k,p}$ contributes to the sum in (6). We claim that the $(\mathfrak{m}_{\mathcal{R}_t}, K \cap \mathcal{R}_t(\boldsymbol{R}))$-module $V(u_t)$ must be tempered. By our previous remark, it suffices to show that

$$\langle \check{\alpha}, w(\Lambda + \rho_{\mathfrak{h}}) - \rho_h \rangle > 0$$

for every simple positive root of $\mathfrak{h}$ in $\mathfrak{m}_{\mathcal{R}_t}$. Since Λ was supposed to be regular and since $w^{-1}\alpha > 0$, we have

$$\langle \check{\alpha}, w(\Lambda + \rho_{\mathfrak{h}}) \rangle > 1,$$

hence

$$\langle \check{\alpha}, w(\Lambda + \rho_{\mathfrak{h}}) - \rho_h \rangle = \langle \check{\alpha}, w(\Lambda + \rho_{\mathfrak{h}}) \rangle - 1 > 0,$$

which proves our claim.

Finally, we use an observation of Wallach [Wal88], Theorem 4.3 according to which a discrete non-cuspidal automorphic representation cannot be tempered. Applying this to $V(u_t)$, this forces $\mathcal{R}_t \in \{P\}$ and hence $k = \operatorname{rank}\mathcal{R}_t = \operatorname{rank}\{P\}$ in the above situation. The essential point in Wallach's argument is that temperedness of the representation to which f belongs forces the coefficients $f_{\mathcal{P},\lambda}$ of the constant term of f to be zero unless $\lambda \in \overline{{}^+\check{\mathfrak{a}}}$, while square integrability of f forces them to vanish unless $\lambda \in -{}^+\check{\mathfrak{a}}$.

These arguments prove our claim that the summands in (6) vanish unless $k = \operatorname{rank}\{P\}$. This makes the degeneracy of the spectral sequence highly plausible. To give an exact proof of this fact, we note that the spectral sequence we are investigating is the sum of similar spectral sequences

$$E_1^{p,q} = \bigoplus_{\{P\},k} \bigoplus_{t \in C^T_{\{P\},k,p}} H^{p+q}_{(\mathfrak{m}_{\mathcal{G}},K)}(W(u_t)_\chi \otimes D_t \otimes \boldsymbol{E}) \Rightarrow H^{p+q}_{(\mathfrak{m}_{\mathcal{G}},K)}\left(\mathfrak{Fin}_{\mathcal{J}} S_\infty(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o \backslash \mathbb{G})_{\{P\},\chi} \otimes \boldsymbol{E}\right) \tag{8}$$

([3]) In these references, the result is formulated for connected real reductive groups. However, in general the temperedness assertion we need reduces to the connected case since for an open subgroup $\tilde{K} \subseteq K$ a $(\mathfrak{g}, K)$-module X with cohomology is a direct summand of $\operatorname{Ind}^{(\mathfrak{g},K)}_{(\mathfrak{g},\tilde{K})} X\big|_{(\mathfrak{g},\tilde{K})}$, and the restriction still has cohomology. *cf.* also [Sch94].

where $W(u_t)_\chi$ has the same meaning as in Proposition 6.1. It follows from Proposition 6.1 that (1) is the sum of the spectral sequences (8) over all $\{P\}$ and all $\chi \in X_{\mathcal{J},\{P\}}$. Therefore, it suffices to prove the degeneracy of these spectral sequences. Moreover, it is easy to see that for an open subgroup $\mathbb{K}^f \subset \mathbb{G}_f$, the sum over all p and q of the dimensions of $E_1^{p,q}$ in (8) is independent of the choice of T. For this reason, it is sufficient to prove the degeneracy of this spectral sequence for one particular T. We may also assume without loosing generality that E is irreducible.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}_{\boldsymbol{C}}$. Let us chose the real structure on $\mathfrak{h}$ such that all roots and all differentials of algebraic characters of $\mathcal{G}$ defined over $\boldsymbol{C}$ are real and a Weyl-group invariant Euclidean metric on the real subspace of $\mathfrak{h}$ such that for the dual of this Euclidean metric the differentials of characters of $\mathcal{G}$ are orthogonal to the roots and such that the restriction of this metric to $\mathfrak{h} \cap \mathfrak{g}^{\mathrm{der}}$ is proportional to the Killing form. Since different Cartan subalgebras are conjugate over $\mathcal{G}(\boldsymbol{C})$, these data determine a real subspace and a metric on every Cartan subalgebra of $\mathfrak{g}_{\boldsymbol{C}}$. By restriction and duality, we also get metrics on $\mathfrak{a}_\mathcal{P}$ and $\check{\mathfrak{a}}_\mathcal{P}$ for every parabolic subgroup $\mathcal{P}$. Let $R_0 > R_1 > \cdots > R_N \geq 0$ be the absolute values of the elements of $F_\mathcal{J}$, where $F_\mathcal{J}$ was defined before formulating (6.8). We will prove the degeneracy of (8) for the function T determined by $|\lambda| = R_{T(\lambda)}$. It is easy to see that with respect to the metric we have chosen, ${}^+\check{\mathfrak{a}}_\mathcal{P}^\mathcal{G}$ is the dual cone to $\check{\mathfrak{a}}_\mathcal{P}^{\mathcal{G}+}$, such that T satisfies (6.8).

The ideal $\mathcal{J}$ consists of all elements of $\mathcal{Z}(\mathfrak{g})$ whose Harish-Chandra symbol vanishes at the elements of a Weyl group orbit in $\check{\mathfrak{h}}$. By our conventions, this Weyl group orbit is real. Let $R_\mathcal{J}$ be the absolute value of its elements. Similarly, for every $\mathcal{P} \in \{P\}$ the elements of $\chi_\mathcal{P}$ are given by a Weyl group orbit in the dual of a Cartan subalgebra of $\mathfrak{m}_\mathcal{P}$. The absolute value R_χ of the elements of these orbits depends only on χ but not on the choice of $\mathcal{P}$ or an element of $\chi_\mathcal{P}$. From the part of the second assertion of the theorem which was already proved, we know that

$$H^{p+q}_{(\mathfrak{m}_\mathcal{G},K)}(W(u_t)_\chi \otimes D_t \otimes \boldsymbol{E}) \tag{9}$$

vanishes unless $\mathcal{R}_t \in \{P\}$. Without loosing generality we may assume $\mathfrak{h} \subseteq \mathfrak{l}_{\mathcal{R}_t}$. Let $\Omega_\mathcal{J}$ be the Weyl group Orbit in $\check{\mathfrak{h}}$ belonging to $\mathcal{J}$ and let $\Omega_\chi \subset \check{\mathfrak{h}}_{\mathfrak{m}_{\mathcal{R}_t}}$ be the union of the Weyl group orbits in $\check{\mathfrak{h}}$ belonging to the elements of $\chi_{\mathcal{R}_t}$. It follows from our choice of the metric that $\mathfrak{h}_{\mathfrak{m}_{\mathcal{R}_t}}$ and $\mathfrak{a}_{\mathcal{R}_t}$ are orthogonal. This allows us to consider $\check{\mathfrak{h}}_{\mathfrak{m}_{\mathcal{R}_t}}$ as a subspace of $\check{\mathfrak{h}}$. By (5) and Wigners lemma about the vanishing of the cohomology of representations with the wrong infinitesimal character, we see that (9) vanishes unless there exist $\xi_\mathcal{J} \in \Omega_\mathcal{J}$ and $\xi_\chi \in \Omega_\chi$ such that $\lambda_t + \xi_\chi = \xi_\mathcal{J}$. But then $|\lambda_t|^2 = R_\mathcal{J}^2 - R_\chi^2$. Since the initial term of (8) is a sum of terms of the form (9), this proves that $E_1^{p,q}$ in (8) vanishes unless $R_p^2 = R_\mathcal{J}^2 - R_\chi^2$. Since there is at most one p with this property, this proves the degeneracy of (8). As was mentioned before, this proves the degeneracy of (1) in full generality. The remaining assertion of the theorem is an easy consequence of this fact. □

In the case of degenerate highest weight, the computation of the differentials of the Eisenstein spectral sequence (1) seems to be a difficult problem. For rank one groups, the spectral sequence gives us a long exact sequence which can usually be understood. This case is also considered in [Har86]. Cases of higher rank are considered in [Sch83] and [Har91]. It seems that the first systematic treatment of an example with an arbitrary number of singular hyperplanes is the author's forthcoming paper [Fra91], where the space of all cohomology classes which have the same Hecke eigenvalue as the constant function is studied.

It seems that the first result in the literature which is related to Borel's conjecture is a result of Harder for the rank 1 case, which states that every de Rham cohomology class has a harmonic representative. Of course, in the real rank one case this implies that every cohomology class has an automorphic representative. I do not know whether in general every cohomology class has a harmonic representative. The full proof of Borel's conjecture in the rank one case is a joint work of Casselman and Speh. A description in the case of $SL_2(\mathbb{Z})$ appeared in [Cas84]. There is also the unpublished paper [Har86] which gives a relatively detailed description of the Eisenstein contribution of systems $\{P\}$ of associate parabolic subgroups of rank one.

7.5. Stable real cohomology

Since only Eisenstein series at parameters $\lambda \in \overline{\check{\mathfrak{a}}_{\mathcal{R}}^{+}}$ in the closure of the positive Weyl chamber contribute to the space of automorphic forms, the summands of the cohomology groups $H^*(\mathfrak{n}_{\mathcal{R}}, E)$ which occur in the computation of these Eisenstein contributions to the cohomology of discrete subgroups must have $\mathfrak{a}_{\mathcal{R}}$-action $-\rho_{\mathcal{R}} - \lambda \in -\overline{\check{\mathfrak{a}}_{\mathcal{R}}^{+}}$. By Kostant's formula, this implies that the $\mathfrak{n}_{\mathcal{R}}$-cohomology is in high dimension. This gives us the result of Borel [Bor74] on stable real cohomology.

For a parabolic subgroup $\mathcal{R}$, let $c(\mathcal{G}, \mathcal{R})$ and $\tilde{c}(\mathcal{G}, \mathcal{R})$ be the largest integers such that for any $i \leq c(\mathcal{G}, \mathcal{R})$ and any $j \leq \tilde{c}(\mathcal{G}, \mathcal{R})$ and any weights μ of $\mathfrak{a}_{\mathcal{R}}$ in $\Lambda^i(\mathfrak{n}_{\mathcal{R}})$ and ν in $\Lambda^j(\mathfrak{n}_{\mathcal{R}})$, we have $\rho_{\mathcal{R}} - \mu \in {}^{+}\check{\mathfrak{a}}_{\mathcal{R}}^{\mathcal{G}}$ and $\rho_{\mathcal{R}} - \nu \notin -\overline{\check{\mathfrak{a}}_{\mathcal{R}}^{\mathcal{G}+}}$. Then $c(\mathcal{G}, \mathcal{R}) \leq \tilde{c}(\mathcal{G}, \mathcal{R})$, and equality occurs for rank one parabolic subgroups. Let $c(\mathcal{G})$ and $\tilde{c}(\mathcal{G})$ be the minima of $c(\mathcal{G}, \mathcal{R})$ and $\tilde{c}(\mathcal{G}, \mathcal{R})$ over all parabolic subgroups $\mathcal{R}$. It is easily seen that the minimum $c(\mathcal{G})$ is attained for rank one parabolic subgroups (*cf.* [Bor74], 7.(4)), therefore $c(\mathcal{G}) = \tilde{c}(\mathcal{G})$.

Let $E_{\mathcal{R}}^{\mathcal{G}}(\lambda, .)$, $\lambda \in \overline{\check{\mathfrak{a}}_{\mathcal{R}}^{\mathcal{G}+}}$, be an Eisenstein series which contributes to the cohomology of the constant local system. In other words, $\lambda = \lambda_t$ and $\mathcal{R} = \mathcal{R}_t$ for some $t \in M_{\mathcal{J},\{P\},\infty}^{k,T,p}$. Let j be the degree of the $\mathfrak{n}_{\mathcal{R}}$-homology in (7.4.2) for a non-trivial summand. Then $-\lambda_t = \rho_{\mathcal{R}} - \nu$ for a weight ν in $\Lambda^j \mathfrak{n}_{\mathcal{R}}$. This implies $j > \tilde{c}(\mathcal{G}, \mathcal{R})$. It follows that the contribution of cuspidal or residual Eisenstein series from proper parabolic subgroups is in dimension greater than $\tilde{c}(\mathcal{G})$, and that the map from discrete L_2-cohomology to de Rham cohomology for the constant local system is an isomorphism in dimension $\leq \tilde{c}(\mathcal{G}) = c(\mathcal{G})$. If $m(\mathcal{G}) + 1$ is the smallest dimension in which a non-constant unitary representation of $\mathcal{G}(\boldsymbol{R})$ may have cohomology, then the map from the cohomology of the constant representation into de Rham cohomology is an isomorphism in dimension $\leq \min(c(\mathcal{G}), m(\mathcal{G}))$. This is Borel's result.

The correct values of $m(\mathcal{G})$ are known by the Vogan-Zuckerman classification ([VZ84] or [Wal88], 9.8.). It would be interesting to see whether a combination of vanishing results for unitary representations of the Levi component can be combined with the fact that the $\mathfrak{n}$-cohomology is in big dimension to get an improvement of Borel's bound $c(\mathcal{G})$ for certain groups.

7.6. Extension of the rationality results of Manin-Drinfel'd and Clozel

Let $\mathbb{K}^f \subset \mathbb{K}_f$ be a sufficiently small open subgroup. IfE is an algebraic representation of $\mathcal{G}$, defined over thecomplex numbers, then we have a local system $\mathbb{E}$ on $G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G}/\mathbb{K}^f K$. By Theorem 6, its cohomology has a decomposition according to classes of associate parabolic subgroups:

$$H^*(G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G}/\mathbb{K}^f K,\mathbb{E}) \cong \mathrm{Ext}^*_{(\mathfrak{m}_\mathcal{G},K)}\left(\check{E}, S_\infty(G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G})^{\mathbb{K}^f}\right)(\zeta_E)$$
$$\cong \bigoplus_{\{P\}} \mathrm{Ext}^*_{(\mathfrak{m}_\mathcal{G},K)}\left(\check{E}, S_\infty(G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G})^{\mathbb{K}^f}_{\{P\}}\right)(\zeta_E).$$

Here (ζ_E) refers to the twist of the $\mathbb{G}_f$-representation by$e^{\langle\zeta_E,H_\mathcal{G}(g)\rangle}$, where $\zeta_E \in \check{\mathfrak{a}}_\mathcal{G}$ corresponds to the action of the centre of $\mathcal{G}$ on E. Since the cohomology on the left hand side can also be computed as Betti cohomology, there is a σ-linear isomorphism

$$\sigma_*: H^*(G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G}/\mathbb{K}^f K,\mathbb{E}) \to H^*(G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G}/\mathbb{K}^f K,\mathbb{E}^\sigma)$$

into the cohomology of the σ-conjugate representation, where σ is any automorphism of $\boldsymbol{C}$ over $\boldsymbol{Q}$. One may ask whether this isomorphism respects the $\{P\}$-decompositions of both cohomology groups. In the case $\mathcal{G} = \mathrm{Res}^{\boldsymbol{K}}_{\boldsymbol{Q}}\mathcal{GL}_n$ for a number field $\boldsymbol{K}$, this has been answered affirmatively by Clozel [Clo90], Théorème 3.19 for the summand given by the space of cusp forms. Our result in this subsection is that Clozel's theorem holds for all summands in the $\{P\}$-decomposition of the cohomology of GL_n. For $n = 2$, this rationality result is closely related to the Drinfel'd-Manin theorem.

THEOREM 20. – *Let $\mathcal{G} = \mathrm{Res}^{\boldsymbol{K}}_{\boldsymbol{Q}}\mathcal{GL}_n$ for a number field $\boldsymbol{K}$, then for every class $\{P\}$ of associate parabolic subgroups and any automorphism σ of $\boldsymbol{C}$, the isomorphism σ_* maps the subspace*

$$\mathrm{Ext}^*_{(\mathfrak{m}_\mathcal{G},K)}\left(\check{E}, S_\infty(G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G})^{\mathbb{K}^f}_{\{P\}}\right) \subset H^*(G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G}/\mathbb{K}^f K,\mathbb{E})$$

to the subspace

$$\mathrm{Ext}^*_{(\mathfrak{m}_\mathcal{G},K)}\left(\check{E}^\sigma, S_\infty(G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G})^{\mathbb{K}^f}_{\{P\}}\right) \subset H^*(G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G}/\mathbb{K}^f K,\mathbb{E}^\sigma).$$

Proof. – Let S be a finite set of places of $\boldsymbol{K}$ containing all archimedean places and all finite places v where $\mathbb{K}_{f,p(v)} \neq \mathbb{K}^f_{p(v)}$, where $p(v)$ is the characteristic of the residue field of v. For $v \notin S$, let $\mathcal{H}_v$ be the space of bi-$\mathbb{K}_v$-invariant $\boldsymbol{Q}$-valued functions with compact support on $GL_n(\boldsymbol{K}_v)$. This is a commutative convolution algebra with unit. One can use these units to form the restricted tensor product

$$\mathcal{H}_S = \bigotimes_{v\notin S} \mathcal{H}_v.$$

This algebra acts on $H^*(G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G}/\mathbb{K}^f K,\mathbb{E})$ by convolution. It suffices to verify that for $\{P\} \neq \{Q\}$, the finite length $\mathcal{H}_S \otimes \boldsymbol{C}$-modules

$$\mathrm{Ext}^*_{(\mathfrak{m}_\mathcal{G},K)}\left(\check{E}, S_\infty(G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G})^{\mathbb{K}^f}_{\{P\}}\right)^\sigma$$

and

$$\mathrm{Ext}^*_{(\mathfrak{m}_{\mathcal{G}},K)}\left(\check{E}^\sigma, S_\infty(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})^{\mathbb{K}^f}_{\{Q\}}\right)$$

have no isomorphic subquotients. Let us call this assertion $A_{E,\sigma,\{P\},\{Q\}}$. Then it is clear that $A_{E,\sigma,\{P\},\{Q\}}$ and $A_{E^\sigma,\sigma^{-1},\{Q\},\{P\}}$ are equivalent.

Let us assume that $\mathcal{P}_o$ is the Weil restriction to $\boldsymbol{Q}$ of the group of upper triangular matrices. Recall that $\mathcal{L}_o$ is the Levi component of $\mathcal{P}_o$, we may assume that this is the group of diagonal matrices. Let the Satake transform be defined by

$$\boldsymbol{S}(\phi)(z_1,\ldots,z_n) = \sum_{k_1=-\infty}^{\infty}\cdots\sum_{k_n=-\infty}^{\infty}\left(\prod_{i=1}^{n} |\,\pi\,|_v^{k_i(n-i)}\, z_i^{k_i}\right)\cdot$$

$$\cdot\int_{K_v^{\frac{n(n-1)}{2}}}\phi\left(\begin{pmatrix}\pi^{k_1} & 0 & \ldots & 0\\ 0 & \pi^{k_2} & \ldots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \ldots & \pi^{k_n}\end{pmatrix}\begin{pmatrix}1 & n_{12} & \ldots & n_{1n}\\ 0 & 1 & \ldots & n_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \ldots & 1\end{pmatrix}\right)dn$$

for $\phi \in \mathcal{H}_v$, where π is an uniformising element of $\mathbb{K}_v$ and $p(v)$ is the characteristic of the residue field of v. The Haar measure is normalised by

$$\int_{\mathcal{O}_v^{\frac{n(n-1)}{2}}} dn = 1,$$

where $\mathcal{O}_v \subset K_v$ is the ring of v-adic integers. As in [Clo90], Definition 3.3, we use an additional factor instead of the usual convention to make sure that we really get a function with rational coefficients. It is known that the Satake transform is an isomorphism between $\mathcal{H}_v$ and the elements of $\boldsymbol{Q}[z_i, z_i^{-1}]$ which are invariant under the symmetric group. Consequently, characters of $\mathcal{H}_v \otimes \boldsymbol{C}$ correspond to n-tuples $z = (z_1,\ldots,z_n)$ of non-vanishing numbers, which are unique up to permutation.

We are no ready to give the proof of the theorem. We will proceed by induction on n, the case $n = 1$ being clear. Let us assume that the assertion is true for $GL_m(\boldsymbol{K})$ with $m < n$.

1 As in Clozel's paper, let the class regular algebraic representations of $\mathbb{G}$ be defined as the class of automorphic representations of $\mathcal{G}$ such that the factor at infinity has the same infinitesimal character has a finite dimensional representation. Obviously, automorphic representations which contribute to the cohomology of an automorphic local system must be regular algebraic. Conversely, if π is a cuspidal regular algebraic representation of $\mathbb{G}$, then Clozel [Clo90], Lemme 3.14 has shown that there exists a character ε of $\mathbb{G}$ with values in $\{\pm 1\}$ such that $\pi\varepsilon$ is a cuspidal automorphic representation which has cohomology with coefficients in some finite-dimensional algebraic representation of $\mathcal{G}(K \bigotimes_{\boldsymbol{Q}} \boldsymbol{C})$.

2 Let $\{P\}$ be a class of associate parabolic subgroups, corresponding to the partition $n = \sum_{i=1}^{k} n_i$. Since

$$\mathrm{Ext}^*_{(\mathfrak{m}_{\mathcal{G}},K)}\left(\check{E}, S_\infty(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G})^{\mathbb{K}^f}_{\{P\}}\right)$$

is a finite-dimensional vector space, it is a sum of associate eigenspaces for finitely many characters of $\mathcal{H}_S \otimes \boldsymbol{C}$. We want to show that the characters which occur are of the form

$z^{(v)} = (z_1^{(v)}, \ldots, z_k^{(v)})$ (*i.e.*, $\mathcal{H}_v$ acts by $z^{(v)}$) such that $z_i^{(v)}$ (itself a tuple) is the Hecke character of a cuspidal regular algebraic representation of $\mathcal{GL}_{n_i}$. It is clear from the spectral sequence (7.4.1) that there is a coarser partition $n = \sum m_i$, $m_i = \sum n_{i,j}$ (the $n_{i,j}$ form a rearrangement of the n_i) such that $z^{(v)}$ has the form $z^{(v)} = (\tilde{z}_1^{(v)}, \ldots, \tilde{z}_l^{(v)})$, where $\tilde{z}_i^{(v)}$ is the Hecke eigenvalue of a representation π_i of $\mathcal{GL}_{m_i}$ which occurs in the discrete spectrum of the summand of the L_2-space on $\mathcal{GL}_{m_i}$ of type $n_{i,1}, \ldots, n_{i,k_i}$ and is regular algebraic. The elements of π_i are sums of residues of cuspidal Eisenstein series starting from automorphic representations $\pi_{i,l}$ of $\mathcal{GL}_{n_{i,l}}$. By a comparison of infinitesimal characters, the $\pi_{i,l}$ must be regular algebraic. It follows that $\tilde{z}_i^{(v)} = (z_{i,1}^{(v)}, \ldots, z_{i,k_i}^{(v)})$, where $z_{i,j}^{(v)}$ is the Hecke character of a cuspidal regular algebraic representation of $\mathcal{GL}_{n_{i,j}}$. Since the $z_i^{(v)}$ can be taken to be a rearrangement of the $z_{i,j}^{(v)}$, the assertion follows.

3 Let $z^{(v)} = (z_1^{(v)}, \ldots, z_m^{(v)})$ be the Hecke character of a cuspidal representation π of GL_m with $m < n$. Let us assume that π has cohomology with coefficients in E. We claim that $z^{(v)^\sigma} = (z^{(v)}{}_1^\sigma, \ldots, z^{(v)}{}_m^\sigma)$ is again the Hecke character of a cuspidal representation with cohomology. Since $z^{(v)}$ is the Hecke eigenvalue of a subspace of

$$\mathrm{Ext}^*_{(\mathfrak{m}_\mathcal{G},K)}\left(\check{E}, S_\infty(G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G})^{\mathbb{K}^f}_{\{\mathcal{G}\}}\right)$$

for $\mathcal{G} = \mathrm{Res}_{\boldsymbol{Q}}^{\boldsymbol{K}}\mathcal{GL}_m$, $z^{(v)^\sigma}$ is the Hecke character for the image by σ_* of that subspace, which by the induction assumption is contained in

$$\mathrm{Ext}^*_{(\mathfrak{m}_\mathcal{G},K)}\left(\check{E}^\sigma, S_\infty(G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G})^{\mathbb{K}^f}_{\{\mathcal{G}\}}\right),$$

and the assertion follows.

More generally, if $z^{(v)}$ is the Hecke character of a regular algebraic representation, then so is the collection $(z^{(v)})^\sigma$. This can be reduced to the special case of cohomological representations, by the result of Clozel quoted in step 1.

This is the only step where the induction assumption is used. We could also have used [Clo90], Théorème 3.19 instead of the induction assumption.

4 Now let $\mathcal{G} = \mathrm{Res}_{\boldsymbol{Q}}^{\boldsymbol{K}}\mathcal{GL}_n$, and let V be a one-dimensional subquotient of the $\mathcal{H}_S$-module

$$\mathrm{Ext}^*_{(\mathfrak{m}_\mathcal{G},K)}\left(\check{E}, S_\infty(G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G})^{\mathbb{K}^f}_{\{P\}}\right),$$

with Hecke character $z^{(v)}$ for $\mathcal{H}_v$. Let us assume that $\{P\} \neq \{\mathcal{G}\}$ is given by the partition $n = \sum_{i=1}^k n_i$. By the result of step 2, $z^v = (z_1^{(v)}, \ldots, z_k^{(v)})$, where $z_i^{(v)}$ is the Hecke character of a cuspidal regular algebraic representation of $\mathcal{GL}_{n_i}$. By Step 3, $z^{(v)^\sigma} = (z_1^{(v)^\sigma}, \ldots, z_k^{(v)^\sigma})$ where $z_1^{(v)^\sigma}$ is the Hecke character of a cuspidal regular algebraic representation.

Now let $\{Q\} \neq \{P\}$, and let $y^{(v)}$ be an associate Hecke eigenvalue occurring in

$$\mathrm{Ext}^*_{(\mathfrak{m}_\mathcal{G},K)}\left(\check{E}^\sigma, S_\infty(G\mathcal{A}_\mathcal{G}(\boldsymbol{R})^o\backslash\mathbb{G})^{\mathbb{K}^f}_{\{Q\}}\right).$$

By step 2, we know that $y^v = (y_1^{(v)}, \ldots, y_l^{(v)})$, where $y_i^{(v)}$ is the Hecke character of a regular algebraic cuspidal representation of $\mathcal{GL}_{m_i}$, and $\{Q\}$ belongs to the partition $n = \sum_{i=1}^l m_i$.

Since $\{Q\} \neq \{P\}$, the partitions $n = \sum_{i=1}^{l} m_i$ and $n = \sum_{i=1}^{k} n_i$ are different even if their order is not considered, hence [JS81], Theorem (4.2) implies that there is no permutation of the n entries which identifies the tuples $z^{(v)}$ and $y^{(v)}$. It follows that the Hecke characters defined by $z^{(v)}$ and $y^{(v)}$ are different.

Recall that the theorem follows if the assertions $A_{E,\sigma,\{P\},\{Q\}}$, which have been formulated in the introduction to the proof, are true for all classes $\{Q\} \neq \{P\}$ of associate parabolic subgroups. Our previous remarks imply that the assertion $A_{E,\sigma,\{P\},\{Q\}}$ is true for $\{P\} \neq \{\mathcal{G}\}$and $\{Q\} \neq \{P\}$. Since $A_{E,\sigma,\{P\},\{Q\}}$ and $A_{E^{\sigma},\sigma^{-1},\{Q\},\{P\}}$ are equivalent, they also prove $A_{E,\sigma,\{P\},\{Q\}}$ in the case $\{Q\} \neq \{\mathcal{G}\}$, hence in the general case for $\{Q\} \neq \{P\}$. The induction argument is complete. □

The proof of the theorem also gives the following result:

COROLLARY 1. – *If $\{P\} \neq \{Q\}$ are different classes of associate parabolic subgroups of $\mathcal{GL}_n$, then for any finite set S of primes of the number field K the supports of the $\mathcal{H}_S$-modules*

$$\mathrm{Ext}^*_{(\mathfrak{m}_{\mathcal{G}},K)}\left(\check{E}, S_{\infty}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^{o}\backslash\mathbb{G})^{\mathbb{K}^f}_{\{P\}}\right) \subset H^*(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^{o}\backslash\mathbb{G}/\mathbb{K}^f K, \mathbb{E})$$

and

$$\mathrm{Ext}^*_{(\mathfrak{m}_{\mathcal{G}},K)}\left(\check{E}, S_{\infty}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^{o}\backslash\mathbb{G})^{\mathbb{K}^f}_{\{Q\}}\right) \subset H^*(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^{o}\backslash\mathbb{G}/\mathbb{K}^f K, \mathbb{E})$$

are disjoint sets of maximal ideals.

It seems that Harder (*cf.* for instance [Har83]) and Clozel first mentioned that the above rationality result can be viewed as the generalisation of the classical Manin-Drinfel'd theorem to $\mathcal{GL}_n$ over a number field. One reason for this is that the argument about separation of Hecke eigenvalues which is used in the proof of rationality of the $\{P\}$-decomposition is the generalisation of the main argument in the proof of the classical Drinfel'd-Manin theorem. Another reason is the following: It is known that for a curve C over the complex numbers and a finite set $Z \subset C$ the divisors of degree zero concentrated in Z are all torsion in the Jacobian if and only if the rational mixed Hodge structure on $H^1(C - Z)$ splits (*cf.* [Elkik:DrinMan]). Moreover, if a rational splitting exists then it coincides with the unique splitting of the real mixed Hodge structure. For quotients of the upper half plane by a Fuchsian subgroup of the first kind, such a splitting is provided by the decomposition into the cuspidal and the Eisenstein summand. Therefore, the difference of any two cusps of a Fuchsian subgroup is torsion in the Jacobian if and only if for this Fuchsian subgroup the decomposition of the cohomology into the cuspidal and Eisenstein summands is rational. Therefore, Theorem 20 for $GL_2(\boldsymbol{Q})$ is equivalent to the usual Manin-Drinfel'd theorem.

Note that even for open Hilbert modular surfaces, the rationality assertion of Theorem 20 does no longer imply a splitting of the mixed Hodge structure. This was first observed in unpublished work of Harder. Therefore, the rationality of the $\{P\}$-decomposition is the only reasonable generalisation of the Manin-Drinfel'd theorem to the higher rank case, even when a holomorphic structure exists. Using the lift between unitary representations of $\mathcal{GL}_n$ and of a division algebra, it is not hard to see that our proof of Theorem 20 generalises to $\mathcal{GL}_n$ over a division algebra. However, Harder expects it to fail for Sp_2 because of the existence of non-tempered cuspidal representation which have the same Hecke character as an Eisenstein class.

7.7. Application to the trace of Hecke operators

Let us apply the Eisenstein spectral sequence (7.4.1) to the trace of a Hecke correspondence $h \in C_c^\infty(\mathbb{G}_f)$ on the cohomology of a local system defined by the finite dimensional representation E. We will derive our formula from Arthur's trace formula [Art89] for L_2-cohomology. A trace formula for the full cohomology of local systems has previously been obtained by Goresky and MacPherson. Their formula involves certain truncated Hecke correspondences. Our formula will be similar to Arthur's formula, but the character of discrete series representations in Arthur's formula is replaced by the character of E, *cf.* (24) below. Since this character is a continuous function, there is a further simplification (26) of the formula as a sum over G-conjugacy classes,

$$(-1)^{2q(\mathcal{G})} \sum_{\gamma \in (\mathcal{G})_{\text{semisimple}}} \varepsilon(\gamma) \frac{\chi(\mathcal{G}_\gamma)}{\iota^{\mathcal{G}}(\gamma)} \mathrm{char}_E(\gamma) O_\gamma(h),$$

where $\varepsilon(\gamma) \in \{0; \pm 1\}$ is a sign factor, O_γ is an orbital integral, and $\chi(\mathcal{G}_\gamma)$ is a Euler-Poincare characteristic.

Throughout this subsection, the trace of an endomorphism of a graded vector space V^* will be the alternating sum over i of its traces on V^i. Let us assume for a moment that $K = K_\infty$.

The trace of h on the limit of (7.4.1) is equal to the trace of h on the E_1-term. By (7.4.6), the colimit in (7.4.1) can be replaced by a sum over representatives of isomorphism classes in the groupoid $\mathcal{M}^{k,T,p}_{\mathcal{J},\{P\},\infty}$. For computing the trace, we sum over all objects of $\mathcal{M}^{k,T,p}_{\mathcal{J},\{P\},\infty}$ but divide each contribution by the cardinality of the corresponding isomorphism class in $\underline{\mathcal{M}}^{k,T,p}_{\mathcal{J},\{P\},\infty}$. The result can be described as follows. For a parabolic subgroup $\mathcal{R}$ and $\lambda \in \check{\mathfrak{a}}^+_{\mathcal{R}}$, let $E^*_{\mathcal{R},\lambda}$ be twist by $\rho_{\mathcal{R}}$ of the piece of $H^*(\mathfrak{n}_{\mathcal{R}}, E)$ on which $\mathfrak{a}_{\mathcal{R}}$ acts by $-\lambda - \rho_{\mathcal{R}}$, and let $n_{\mathcal{R}}(\lambda)$ be the number of Weyl chambers to which λ belongs. Let $\mathcal{L}_{\mathcal{R}}$ be the Levi component of $\mathcal{R}$ and let $h_{\mathcal{L}_{\mathcal{R}}}$ be the following function on $\mathbb{L}_{\mathcal{R}f}$ ([4]):

$$h_{\mathcal{L}_{\mathcal{R}}}(l) = \mathrm{e}^{\langle \rho_{\mathcal{R}}, H_{\mathcal{R}}(l)\rangle} \int_{\mathbb{K}_f} \int_{\mathbb{N}_{\mathcal{R}f}} h(klnk^{-1})\, dn\, dk,$$

where the Haar measure on $\mathbb{K}_f$ is normalised by $\int_{\mathbb{K}_f} dk = 1$ and the Haar measure on $\mathbb{N}_{\mathcal{R}f}$ is normalised by the Haar measures on $\mathbb{G}_f$, $\mathbb{L}_{\mathcal{R}f}$, and the Iwasawa decomposition in such a way that

$$\int_{\mathbb{G}_f} \varphi(g)\, dg = \int_{\mathbb{N}_{\mathcal{R}f}} \int_{\mathbb{L}_{\mathcal{R}f}} \int_{\mathbb{K}_f} \mathrm{e}^{-\langle 2\rho_{\mathcal{R}}, H_{\mathcal{R}}(l)\rangle} \varphi(nlk)\, dn\, dl\, dk.$$

Then the trace of h on the cohomology of E

(1) $$\mathrm{Tr}\Big(h \,\Big|\, H^*_{(\mathfrak{m}_{\mathcal{G}},K)}\big(S_\infty(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G}) \otimes E\big)(\zeta_E)\Big)$$

([4]) Strictly speaking, $h_{\mathcal{L}_{\mathcal{R}}}$ as a function does depend on the choice of a parabolic subgroup $\mathcal{R}$ with Levi component $\mathcal{L}_{\mathcal{R}}$. However, the value of an invariant distribution at $h_{\mathcal{L}_{\mathcal{R}}}$ is independent of $\mathcal{R}$. This follows, for instance, from the formula (25) for regular orbital integrals below.

is equal to

(2)

$$\sum_{\mathcal{R}} \sum_{\lambda\in\overline{\check{\mathfrak{a}}_{\mathcal{R}}^{+}}} n_{\mathcal{R}}(\lambda)^{-1}\mathrm{Tr}\Big(h_{\mathcal{L}_{\mathcal{R}}}\Big| H^{*}_{(\mathfrak{m}_{\mathcal{R}},K\cap\mathcal{L}_{\mathcal{R}}(\boldsymbol{R}))}\big(L_{2,\mathrm{discrete}}(\mathcal{A}_{\mathcal{R}}(\boldsymbol{R})^{o}L_{\mathcal{R}}\backslash\mathbb{L}_{\mathcal{R}})\otimes E^{*}_{\mathcal{R},\lambda}\big)(\lambda)\Big)$$

where the first sum is over all standard parabolic subgroups $\mathcal{R}$ and (λ) refers to the twist of a $\mathbb{L}_{\mathcal{R}f}$-representation by $e^{\langle\lambda,H_{\mathcal{R}}(l)\rangle}$ (*cf.* (2.3.17) for the significance of the twist by ζ_E). The factor $n_{\mathcal{R}}(\lambda)^{-1}$ in this sum stands for the cardinality of an isomorphism class in $\mathcal{M}^{k,T,T(\lambda)}_{\mathcal{J},\{P\},\infty}$.

Let $\mathfrak{L}$ be the set of Levi components of parabolic subgroups of $\mathcal{G}$ which contain $\mathcal{A}_o$. We want to get a sum over all Levi components $\mathcal{L}\in\mathfrak{L}$. It is easy to see that for a standard parabolic subgroup $\mathcal{R}$, the number of parabolic subgroups $\tilde{\mathcal{R}}$ which are conjugate to $\mathcal{R}$ and have a Levi component in $\mathfrak{L}$ is equal to $\frac{w^{\mathcal{L}}}{w^{\mathcal{G}}}$, where $w^{\mathcal{G}}$ is the cardinality of the Weyl group of $\mathcal{A}_o$ in $\mathcal{G}(\boldsymbol{Q})$. For a reductive subgroup $\mathcal{L}\subset\mathcal{G}$, let $\mathfrak{P}(\mathcal{L})$ be the set of rational parabolic subgroups which have $\mathcal{L}$ as a Levi component. Then the sum over standard parabolic subgroups in (2) can be replaced by a sum

(3)

$$\sum_{\mathcal{L}\in\mathfrak{L}} \sum_{\lambda\in\check{\mathfrak{a}}_{\mathcal{L}}} \frac{w^{\mathcal{L}}}{w^{\mathcal{G}}} n_{\mathcal{R}}(\lambda)^{-1} \sum_{\substack{\mathcal{R}\in\mathfrak{P}(\mathcal{L})\\ \lambda\in\overline{\check{\mathfrak{a}}_{\mathcal{R}}^{+}}}} \mathrm{Tr}\Big(h_{\mathcal{L}}\Big| H^{*}_{(\mathfrak{m}_{\mathcal{L}},K\cap\mathcal{L}(\boldsymbol{R}))}\big(L_{2,\mathrm{discrete}}(\mathcal{A}_{\mathcal{R}}(\boldsymbol{R})^{o}L\backslash\mathbb{L})\otimes E^{*}_{\mathcal{R},\lambda}\big)(\lambda)\Big),$$

where $\mathfrak{m}_{\mathcal{L}}$ is the Lie algebra of the anisotropic part of $\mathcal{L}$.

If $\mathcal{L}(\boldsymbol{R})/\mathcal{A}_{\mathcal{L}}(\boldsymbol{R})$ has no compact Cartan subgroup (*i.e.*, no discrete series), then the contribution of $\mathcal{L}$ to (3) is zero. This is due to the following well-known fact: If π is any unitary admissible representation of $\mathcal{L}(\boldsymbol{R})$, then the Euler-Poincare characteristic $\chi_{(\mathfrak{m}_{\mathcal{L}},K\cap\mathcal{L}(\boldsymbol{R}))}(\pi\otimes F)$ for any finite-dimensional representation F is zero if the infinitesimal characters of π and $\check{F}$ disagree and is equal to the coefficient of the constant representation in the formal sum

(4)

$$\pi\cdot F\cdot\Lambda^{*}(\mathcal{M}_{\mathcal{L}}/\mathfrak{k}\cap\mathfrak{m}_{\mathcal{L}})$$

of $K\cap\mathcal{L}(\boldsymbol{R})$-representations if π and F have the same infinitesimal character. Now if $\mathfrak{m}_{\mathcal{L}}$ has no compact Cartan subalgebra, then any element of $K\cap\mathcal{L}(\boldsymbol{R})$ has a non-trivial centraliser in $\mathfrak{m}_{\mathcal{L}}/\mathfrak{k}\cap\mathfrak{m}_{\mathcal{L}}$. Consequently, the virtual $K\cap\mathcal{L}(\boldsymbol{R})$-representation $\Lambda^{*}(\mathfrak{m}_{\mathcal{L}}/\mathfrak{k}\cap\mathfrak{m}_{\mathcal{L}})$ vanishes, and (4) is zero. We will say that a reductive group $\mathcal{L}$ which is defined over $\boldsymbol{Q}$ is $\boldsymbol{R}$-cuspidal if $\mathcal{L}(\boldsymbol{R})/\mathcal{A}_{\mathcal{L}}(\boldsymbol{R})$ has a compact Cartan subalgebra. Let $\mathfrak{L}_c\subset\mathfrak{L}$ be the set of $\boldsymbol{R}$-cuspidal Levi components.

For $\mathcal{L}\in\mathfrak{L}_c$, we invoke Arthur's formula for the trace of Hecke operators in the L_2-cohomology [Art89] to compute the summand in (3). The result is that

$$\mathrm{Tr}\Big(h_{\mathcal{L}}\Big| H^{*}_{(\mathfrak{m}_{\mathcal{L}},K\cap\mathcal{L}(\boldsymbol{R}))}\big(L_{2,\mathrm{discrete}}(\mathcal{A}_{\mathcal{R}}(\boldsymbol{R})^{o}L\backslash\mathbb{L})\otimes E^{*}_{\mathcal{R},\lambda}\big)(\lambda)\Big)$$

is equal to

$$(5)\qquad \sum_{\substack{\mathcal{M}\in\mathfrak{L}\\ \mathcal{M}\subseteq\mathcal{L}}} (-1)^{a^{\mathcal{L}}_{\mathcal{M}}} \frac{w^{\mathcal{M}}}{w^{\mathcal{L}}} \sum_{\gamma\in(\mathcal{M})_{\mathbb{R}\text{-elliptic}}} \frac{\chi(\mathcal{M}_\gamma)}{\iota^{\mathcal{M}}(\gamma)} \Phi_{\mathcal{L},E^*_{\mathcal{R},\lambda},\mathcal{M}}(\gamma) O_\gamma(h_{\mathcal{M}}).$$

The terms in this formula have the following meaning: $a^{\mathcal{L}}_{\mathcal{M}}$ is the dimension of $\mathcal{A}_{\mathcal{M}}/\mathcal{A}_{\mathcal{L}}$. $(\mathcal{M})_{\mathbb{R}\text{-elliptic}}$ is the set of $\mathbb{R}$-elliptic conjugacy classes in M. A semisimple element of $\mathcal{M}(\mathcal{R})$ is called $\mathbb{R}$-elliptic if it is an element of a Cartan subgroup which is compact modulo the center. $\mathcal{M}_\gamma$ is the connected component of the centraliser of γ in $\mathcal{M}$. $\iota^{\mathcal{M}}(\gamma)$ is the number of connected components of the centraliser of γ in $\mathcal{M}$ which have rational points. $\Phi_{\mathcal{L},\mathcal{M},E^*_{\mathcal{R},\lambda}}$ is zero if $\mathcal{M}\notin\mathfrak{L}_c$, otherwise it is a sum of characters of discrete series representations of $\mathcal{L}(\boldsymbol{R})$ and will be described in detail later. $O_\gamma(h_{\mathcal{M}})$ is the usual orbital integral

$$O_\gamma(h_{\mathcal{M}}) = \int_{\mathsf{M}/\mathsf{M}_{\gamma f}} h_{\mathcal{M}}(m\gamma m^{-1})\,dm.$$

For any reductive group $\mathcal{H}$, let $q(\mathcal{H})$ be one half the dimension of the homogeneous space $\mathcal{A}_{\mathcal{H}}(\boldsymbol{R})\backslash\mathcal{H}(\boldsymbol{R})/K_{\mathcal{H}}$, where $K_{\mathcal{H}}$ is a maximal compact subgroup in $\mathcal{H}(\boldsymbol{R})$. This is an integer if $\mathcal{H}(\boldsymbol{R})/\mathcal{A}_{\mathcal{H}}(\boldsymbol{R})$ has a discrete series. Let us assume that this is the case. Let $\mathcal{D}(\mathcal{H})$ be the cardinality of L-packets of discrete series representations of $\mathcal{H}(\mathcal{R})/\mathcal{A}_{\mathcal{H}}(\boldsymbol{R})^o$. The compact modulo $\mathcal{A}_{\mathcal{H}}$ form $\overline{\mathcal{H}}$ is an inner twist of $\mathcal{H}$. We can use this to transfer any Haar measure on $\overline{\mathcal{H}}(\boldsymbol{R})/\mathcal{A}_{\mathcal{H}}(\boldsymbol{R})^o$ to $\mathcal{H}(\boldsymbol{R})/\mathcal{A}_{\mathcal{H}}(\boldsymbol{R})^o$. Then the quotient

$$\chi(\mathcal{H}) = (-1)^{q(\mathcal{H})}\mathcal{D}(\mathcal{H})\frac{|\mathcal{A}_{\mathcal{H}}(\boldsymbol{R})^o H\backslash\mathbb{H}|}{|\mathcal{A}_{\mathcal{H}}(\boldsymbol{R})^o\backslash\overline{H}(\boldsymbol{R})|}$$

depends only on the choice of a Haar measure on $\mathbb{H}_f$. These are the numbers $\chi(\mathcal{M}_\gamma)$, the last remaining term in (5) which we had to explain. It is easy to see that each summand in (5) is independent of the choice that was made for Haar measures on $\mathbb{M}_f$ and $\mathbb{M}_{\gamma f}$. In (5), we have used the fact that $(h_{\mathcal{L}})_{\mathcal{M}} = h_{\mathcal{M}}$.

It follows that (3) is equal to

$$(6)\qquad \sum_{\mathcal{M}\in\mathfrak{L}} (-1)^{a^{\mathcal{G}}_{\mathcal{M}}} \frac{w^{\mathcal{M}}}{w^{\mathcal{G}}} \sum_{\gamma\in(\mathcal{M})_{\mathbb{R}\text{-elliptic}}} \frac{\chi(\mathcal{M}_\gamma)}{\iota^{\mathcal{M}}(\gamma)} \tilde{\Phi}_{\mathcal{G},E,\mathcal{M}}(\gamma) O_\gamma(h_{\mathcal{M}}),$$

where

$$(7)\qquad \tilde{\Phi}_{\mathcal{G},E,\mathcal{M}}(\gamma) = \sum_{\substack{\mathcal{L}\in\mathfrak{L}_c\\ \mathcal{L}\supseteq\mathcal{M}}} \sum_{\substack{\mathcal{R}\in\mathfrak{P}(\mathcal{L})\\ \lambda\in\check{\mathfrak{a}}^+_{\mathcal{R}}}} (-1)^{a^{\mathcal{G}}_{\mathcal{L}}} n_{\mathcal{R}}(\lambda)^{-1} \Phi_{\mathcal{L},E^*_{\mathcal{R},\lambda},\mathcal{M}}(\gamma)$$

is a sum, over all Levi components $\mathcal{L}$ containing $\mathcal{M}$ such that $\mathcal{L}(\boldsymbol{R})/\mathcal{A}_{\mathcal{L}}(\boldsymbol{R})$ has a discrete series, of the factors occurring in Arthur's trace formula. That looks ugly, since these factors are related to characters of discrete series representations at a Cartan subgroup which may not be fundamental. However, the final result turns out to be very simple.

We claim that if $\mathcal{M}$ is $\boldsymbol{R}$-cuspidal, then any $\mathcal{L} \supseteq \mathcal{M}$ which is the Levi component of a real parabolic subgroup is already defined over $\mathbb{Q}$. Let $\mathcal{A}_{\mathcal{L}}$ be the maximal $\boldsymbol{R}$-split torus in the centre of $\mathcal{L}$, and let H be a fundamental (in the sense of [Vog81], Def. 5.1.1) Cartan subgroup of $\mathcal{M}(\mathbb{R})$. Then $\mathcal{A}_{\mathcal{L}} \subset \mathcal{A}_{\mathcal{M}}$ since $H/\mathcal{A}_{\mathcal{M}}(\boldsymbol{R})$ is compact and $\mathcal{A}_{\mathcal{L}}(\boldsymbol{R}) \subset H$. As $\mathcal{A}_{\mathcal{M}}$ is a split torus over the rationals, this means that $\mathcal{A}_{\mathcal{L}}$ is also defined over the rationals. Consequently, $\mathcal{L}$, being the centraliser of $\mathcal{A}_{\mathcal{L}}$ in $\mathcal{G}$, is also defined over the rationals. It follows that the sum over $\mathcal{L}$ in (7) is in fact a sum over the Levi components of cuspidal real parabolic subgroups containing $\mathcal{M}$. Similarly, the fact that $\mathcal{L}$ is $\boldsymbol{R}$-cuspidal implies that any real parabolic subgroup with Levi component $\mathcal{L}$ is already defined over the rational numbers. For real (resp. rational) parabolic subgroups with Levi component $\mathcal{L}$ correspond to chambers in the maximal $\boldsymbol{R}$-split (resp. $\boldsymbol{Q}$-split) torus in the centre of $\mathcal{L}$, and the fact that $\mathcal{L}$ is $\boldsymbol{R}$-cuspidal implies that the maximal $\mathcal{R}$-split and $\boldsymbol{Q}$-split tori in the centre of $\mathcal{L}$ coincide. Consequently, the sum (7) depends only on the real structure of $\mathcal{G}$.

Let H be a fundamental Cartan subgroup of $\mathcal{M}(\boldsymbol{R})$. For $\mathcal{L} \supseteq \mathcal{M}$ let $V_{\mathcal{L},H}$ be a family of elements $v \in \mathcal{G}(\boldsymbol{R})$ such that vHv^{-1} is a complete set of representatives for the $\mathcal{L}(\boldsymbol{R})$-conjugacy classes of Cartan subgroups of $\mathcal{L}(\boldsymbol{R})$ which are $\mathcal{G}(\boldsymbol{R})$-conjugate to H. For $v \in V_{\mathcal{L},H}$, let $W(vHv^{-1}: \mathcal{L}(\boldsymbol{R}))$ and $W(H: \mathcal{G}(\boldsymbol{R})) \cong W(vHv^{-1}: \mathcal{G}(\boldsymbol{R}))$ be the Weyl groups of vHv^{-1} in $\mathcal{L}(\boldsymbol{R})$ and in $\mathcal{G}(\boldsymbol{R})$. For $\mathcal{M} \in \mathfrak{L}$, let $\mathfrak{R}(\mathcal{M})$ be a set of representatives for the $\mathcal{G}$-conjugacy classes of parabolic subgroups $\mathcal{R}$ which contain $\mathcal{M}$.

Let $\mathcal{R} \in \mathfrak{R}(\mathcal{M})$ and let $\mathcal{L} = \mathcal{L}_{\mathcal{R}}$ be the Levi component containing $\mathcal{M}$. We claim that any parabolic subgroup $\tilde{\mathcal{R}} \supset \mathcal{M}$ which is $\mathcal{G}$-conjugate to $\mathcal{R}$ must be of the the form

$$\tilde{\mathcal{R}} = \mathrm{Int}(wv^{-1})\mathcal{R} \tag{8}$$

for some $v \in V_{\mathcal{L},H}$ and $w \in W(H: \mathcal{G}(\boldsymbol{R}))$. Indeed, let $g \in \mathcal{G}(\boldsymbol{R})$ such that $\mathrm{Int}(g)\mathcal{R} = \tilde{\mathcal{R}}$ and $\mathrm{Int}(g)\mathcal{L} \supseteq \mathcal{M}$. Then $\mathrm{Int}(g^{-1})H \subseteq \mathcal{L}(\boldsymbol{R})$, hence there exist $l \in \mathcal{L}(\boldsymbol{R})$ and $v \in V_{\mathcal{L},H}$ with $\mathrm{Int}(g^{-1})H = \mathrm{Int}(lv)H$. Then g has the form $g = wv^{-1}l^{-1}$ with $w \in N_{\mathcal{G}(\boldsymbol{R})}(H)$, hence $\tilde{\mathcal{R}}$ is of the form (8).

If $\tilde{\mathcal{R}}$ is represented by (8) with $v \in V_{\mathcal{L},H}$ and $w \in W(H: \mathcal{G}(\boldsymbol{R}))$, then the number solutions to (8) is $\#(W(vHv^{-1}: \mathcal{L}(\boldsymbol{R})))$. Indeed, let $\mathrm{Int}(wv^{-1})\mathcal{R} = \mathrm{Int}(\tilde{w}\tilde{v}^{-1})\mathcal{R}$ with $v, \tilde{v} \in V_{\mathcal{L},H}$ and $w, \tilde{w} \in W(H: \mathcal{G}(\boldsymbol{R}))$. Then $\mathrm{Int}(wv^{-1})\mathcal{L} = \mathrm{Int}(\tilde{w}\tilde{v}^{-1})\mathcal{L}$ since these two Levi components of $\tilde{\mathcal{R}}$ both contain H. It follows that $l = vw^{-1}\tilde{w}\tilde{v}^{-1} \in \mathcal{L}(\boldsymbol{R})$ since l normalises both $\mathcal{R}$ and $\mathcal{L}$. Then $\mathrm{Int}(l\tilde{v})H = \mathrm{Int}(vw^{-1}\tilde{w})H = \mathrm{Int}(v)H$ and hence $v = \tilde{v}$ by the definition of $V_{\mathcal{L},H}$. It follows that $vw^{-1}\tilde{w}v^{-1} \in W(vHv^{-1}: \mathcal{L}(\boldsymbol{R}))$, proving our claim about the number of solutions to (8).

Assume that H is a fundamental Cartan subgroup of $\mathcal{M}(\boldsymbol{R})$ containing γ and that $V_{\mathcal{L},H}$ was defined above. Our previous considerations imply that (7) is equal to

$$\sum_{\substack{\mathcal{R} \in \mathfrak{R}(\mathcal{M}) \\ \lambda \in \check{\mathfrak{a}}_{\mathcal{R}}^{+}}} \frac{(-1)^{a_{\mathcal{R}}^{\mathcal{G}}}}{n_{\mathcal{R}}(\lambda)} \sum_{v \in V_{\mathcal{L}_{\mathcal{R}},H}} \sum_{w \in W(H: \mathcal{G}(\boldsymbol{R}))} \frac{1}{\# W(vHv^{-1}: \mathcal{L}_{\mathcal{R}}(\boldsymbol{R}))} \cdot \Phi_{\mathrm{Int}(wv^{-1})\mathcal{L}_{\mathcal{R}}, E^{*}_{\mathrm{Int}(wv^{-1})\mathcal{R}, \mathrm{Ad}(wv^{-1})^{*}\lambda, \mathcal{M}}}(\gamma) \tag{9}$$

$$= \sum_{\substack{\mathcal{R}\in\Re(\mathcal{M}) \\ \lambda\in\overline{\mathfrak{d}_{\mathcal{R}}^{+}}}} \frac{(-1)^{a_{\mathcal{R}}^{\mathcal{G}}}}{n_{\mathcal{R}}(\lambda)} \sum_{v\in V_{\mathcal{L}_{\mathcal{R}},\mathcal{M}}} \sum_{w\in W(H:\mathcal{G}(\boldsymbol{R}))} \times \frac{1}{\#W(vHv^{-1}:\mathcal{L}_{\mathcal{R}}(\boldsymbol{R}))} \Phi_{\mathcal{L}_{\mathcal{R}},E^*_{\mathcal{R},\lambda},v\mathcal{M}v^{-1}}(\mathrm{Int}(vw)\gamma)$$

Here $\mathcal{L}_{\mathcal{R}}$ is the Levi component of $\mathcal{R}$ containing $\mathcal{M}$.

Now we rewrite the summands in (9) as characters of induced representations. For that purpose, we recall Arthur's definition of the factors $\Phi_{\mathcal{L},F^*,\mathcal{M}}(\gamma)$ in [Art89], where F^* is a graded finite-dimensional algebraic representation of $\mathcal{L}(\boldsymbol{C})$. Actually, only the case of an ungraded irreducible representation is considered in [Art89], but the general case easily follows from this special case. Let

$$D^{\mathcal{L}}_{\mathcal{M}}(\gamma) = \det\left(1 - Ad(\gamma)|\mathfrak{l}/\mathfrak{m}\right), \tag{10}$$

and let Π_{F^*} be the set of discrete series representations of $\mathcal{L}$ which have cohomology with coefficients in F^*, with signed multiplicity equal to the multiplicity of the corresponding summand in F^* multiplied by the parity of the dimension in which it occurs. Let n_{F^*} be the multiplicity of π. Then the summands in (9) are defined by

$$\Phi_{\mathcal{L},E^*_{\mathcal{R},\lambda},\mathcal{M}}(\gamma) = (-1)^{q(\mathcal{L})}\left|D^{\mathcal{L}}_{\mathcal{M}}(\gamma)\right|^{1/2} \sum_{\pi\in\Pi_{E^*_{\mathcal{R},\lambda}}} n_{E^*_{\mathcal{R},\lambda}}\pi \mathrm{char}_{\pi}(\gamma). \tag{11}$$

Actually, the last expression is well-defined for regular γ. Arthur shows that it has a continuous expansion to all γ in the fundamental Cartan subgroup of $\mathcal{M}(\boldsymbol{R})$.

Let

$$I^{\mathcal{G}}_{\mathcal{L}}(\pi) = \left[\mathrm{Ind}^{\mathcal{G}(\boldsymbol{R})}_{\mathcal{R}(\boldsymbol{R})}\pi \otimes \boldsymbol{C}_{\rho_{\mathcal{R}}}\right], \tag{12}$$

where $\mathcal{R}$ is a parabolic subgroup with Levi component $\mathcal{L}$, and $[x]$ is the class of x in the Grothendieck group of admissible representations of $\mathcal{G}(\boldsymbol{R})$. It is well-known (for instance, by the character formula for induced representations) that the right hand side of (12) is indeed independent of the choice of a parabolic subgroup $\mathcal{R}$ with Levi component $\mathcal{L}$. Recall the formula for the character of a induced representation

$$\mathrm{char}_{I^{\mathcal{G}}_{\mathcal{L}}(\pi)}(\gamma) = \frac{\displaystyle\sum_{v\in V_{\mathcal{L},H}} \sum_{w\in W(H:\mathcal{G}(\boldsymbol{R}))} \frac{\left|D^{\mathcal{L}}_{v\mathcal{M}v^{-1}}(\mathrm{Int}(vw)\gamma)\right|^{\frac{1}{2}}}{\#W(vHv^{-1}:\mathcal{L}(\boldsymbol{R}))}\mathrm{char}_{\pi}(\mathrm{Int}(vw)\gamma)}{\left|D^{\mathcal{G}}_{\mathcal{M}}(\gamma)\right|^{\frac{1}{2}}},$$

where $\mathcal{M} \subseteq \mathcal{L}$ is $\boldsymbol{R}$-cuspidal with fundamental Cartan subgroup H and $\gamma \in H$. This follows from [Kna86], §X.3., (10.27).

Using (11) and this formula, we conclude that (9) is equal to

$$(13)\qquad \sum_{\substack{\mathcal{R}\in\mathfrak{R}(\mathcal{M})\\ \lambda\in\overline{\check{\mathfrak{a}}_{\mathcal{R}}^{+}}}} \sum_{v\in V_{\mathcal{L}_{\mathcal{R}},H}} \sum_{w\in W(H:\mathcal{G}(\boldsymbol{R}))} \frac{(-1)^{a_{\mathcal{L}}^{\mathcal{G}}+q(\mathcal{L}_{\mathcal{R}})}}{n_{\mathcal{R}}(\lambda)\#W(vHv^{-1}:\mathcal{L}_{\mathcal{R}}(\boldsymbol{R}))} \cdot \left|D_{v\mathcal{M}v^{-1}}^{\mathcal{L}}(\mathrm{Int}(vw)\gamma)\right|^{1/2} \sum_{\pi\in\Pi_{E_{\mathcal{R},\lambda}}} n_{\pi}\mathrm{char}_{\pi}(\mathrm{Int}(vw)\gamma)$$

$$= \sum_{\substack{\mathcal{R}\in\mathfrak{R}(\mathcal{M})\\ \lambda\in\overline{\check{\mathfrak{a}}_{\mathcal{R}}^{+}}}} \frac{(-1)^{a_{\mathcal{L}}^{\mathcal{G}}+q(\mathcal{L})}}{n_{\mathcal{R}}(\lambda)} \left|D_{\mathcal{M}}^{\mathcal{G}}(\gamma)\right|^{1/2} \sum_{\pi\in\Pi_{E_{\mathcal{R},\lambda}}} n_{\pi}\mathrm{char}_{I_{\mathcal{L}_{\mathcal{R}}}^{\mathcal{G}}(\pi)}(\gamma),$$

where H is a fundamental Cartan subgroup of $\mathcal{M}(\boldsymbol{R})$ containing γ and $I_{\mathcal{L}}^{\mathcal{G}}$ has been defined in (12).

The set $\mathfrak{R}(\mathcal{M})$ contains precisely one representative out of each conjugacy class of real parabolic subgroups which has a representative containing $\mathcal{M}$. However, if the conjugacy class of a real Levi component $\mathcal{L}$ has no representative containing $\mathcal{M}$, then for any representation π of $\mathcal{L}$ the character of $I_{\mathcal{L}}^{\mathcal{G}}(\pi)$ vanishes at the fundamental Cartan subalgebra of $\mathcal{M}$. We arrive at

$$(14)\qquad \tilde{\Phi}_{\mathcal{G},E,\mathcal{M}}(\gamma) = \left|D_{\mathcal{M}}^{\mathcal{G}}(\gamma)\right|^{1/2} \sum_{\substack{\mathcal{R}\\ \lambda\in\overline{\check{\mathfrak{a}}_{\mathcal{R}}^{+}}}} \frac{(-1)^{a_{\mathcal{L}}^{\mathcal{G}}+q(\mathcal{L})}}{n_{\mathcal{R}}(\lambda)} \sum_{\pi\in\Pi_{E_{\mathcal{R},\lambda}}} n_{\pi}\mathrm{char}_{I_{\mathcal{L}}^{\mathcal{G}}(\pi)}(\gamma),$$

where the sum over $\mathcal{R}$ is over a set of representatives of the conjugacy classes of cuspidal real parabolic subgroups of $\mathcal{G}$, and $\mathcal{L}$ is a Levi component of $\mathcal{R}$.

It turns out that the sum over characters in (14) is a Kazhdan-Lusztig expansion. Fortunately, only the easiest case of the Kazhdan-Lusztig conjecture is needed, which for real reductive groups is due to Zuckerman. Let us get the necessary fact from Vogan's book.

LEMMA 1. – **i.** *For $\mathcal{L} \in \mathfrak{L}_c$ and $\lambda \in \check{\mathfrak{a}}_{\mathcal{L}}$, the $\mathbb{Z}/2\mathbb{Z}$-graded vector space $E_{\mathcal{R},\lambda}^{*}$ is independent of the choice of $\mathcal{R}$ with $\lambda \in \overline{\check{\mathfrak{a}}_{\mathcal{R}}^{+}}$. Let*

$$(15)\qquad E_{\mathcal{L}}^{*} = \bigoplus_{\lambda\in\check{\mathfrak{a}}_{\mathcal{L}}} E_{\mathcal{R}_{\lambda},\lambda}^{*},$$

where $\mathcal{R}_{\lambda} \in \mathfrak{P}(\mathcal{L})$ has been chosen such that $\lambda \in \overline{\check{\mathfrak{a}}_{\mathcal{R}_{\lambda}}^{+}}$.

ii. *We have an equality in the Grothendieck group of admissible $\mathcal{G}(\boldsymbol{R})$-representations*

$$(16)\qquad E = (-1)^{2q(\mathcal{G})} \sum_{(\mathcal{L})} (-1)^{a_{\mathcal{L}}^{\mathcal{G}}+q(\mathcal{L})} \sum_{\pi\in\Pi_{E_{\mathcal{L}}^{*}}} n_{\pi} I_{\mathcal{L}}^{\mathcal{G}}(\pi),$$

where $E_{\mathcal{L}}^{*}$ was defined in **i.** *and the signed multiplicity n_{π} has the same meaning as in* (11).

iii. *We have a similar formula*

$$E = (-1)^{2q(\mathcal{G})} \sum_{\substack{\mathcal{R} \\ \lambda \in \breve{\mathfrak{a}}^+_{\mathcal{R}}}} \frac{(-1)^{a^{\mathcal{G}}_{\mathcal{L}} + q(\mathcal{L})}}{n_{\mathcal{R}}(\lambda)} \sum_{\pi \in \Pi_{E^*_{\mathcal{R},\lambda}}} n_\pi I^{\mathcal{G}}_{\mathcal{L}}(\pi), \tag{17}$$

where the sum over $\mathcal{R}$ is over a set of representatives of the conjugacy classes of cuspidal real parabolic subgroups of $\mathcal{G}$, and $\mathcal{L}$ is a Levi component of $\mathcal{R}$.

Proof. – Obviously, it suffices to consider the case of an irreducible representation E. Moreover, part **iii** is a consequence of **i** and **ii** For $\mathcal{L} \in \mathfrak{L}_c$ and $\lambda \in \breve{\mathfrak{a}}_{\mathcal{L}}$, choose $\mathcal{R}_\lambda \in \mathfrak{P}(\mathcal{L})$ such that $\lambda \in \breve{\mathfrak{a}}^+_{\mathcal{R}_\lambda}$. We will prove **ii** with $E_{\mathcal{L}}$ defined by (15). This proof will give us a formula for n_π which does not depend on $\mathcal{R}_\lambda$, proving **i**.

Let ϑ be the Cartan involution. The statement in Vogan's book which we have to use is [Vog81], Proposition 9.4.16. It expresses the constant representation of $\mathcal{G}(\boldsymbol{R})$ as a sum over representations $\Theta(\gamma)$ attached to conjugacy classes $\gamma = (H, \Delta^+)$, where $H \subset \mathcal{G}(\boldsymbol{R})$ is a Cartan subalgebra and Δ^+ is a positive root system for $\mathfrak{h}$ (the Lie algebra of H) in $\mathfrak{g}$. Let γ_o be the conjugacy class of the maximally split Cartan subalgebra, with the root system belonging to a minimal real parabolic subgroup. For each conjugacy class γ, Vogan defines an integer $l^I(\gamma)$. Then Zuckerman's theorem ([Vog81], Proposition 9.4.16) says that the coefficient of $\Theta(\gamma)$ in the Kazhdan-Lusztig expression for the constant representation is $(-1)^{l^I(\gamma) - l^I(\gamma_o)}$:

$$\boldsymbol{C} = \sum_{\gamma} (-1)^{l^I(\gamma) - l^I(\gamma_o)} \Theta(\gamma). \tag{18}$$

A more conceptual way to understand (18) uses the Beilinson-Bernstein localisation principle to interpret $(\mathfrak{g}, K)$-modules as $\mathcal{K}(\boldsymbol{C})$-homogeneous $\mathcal{D}$-modules on the flag variety of $\mathcal{G}(\boldsymbol{C})$ and the correspondence between regular holonomic $\mathcal{D}$-modules and perverse sheaves. The standard representations in (18) correspond to the extension by zero of the constant sheaves along the strata (*cf.* [HMSW87]), and the exponent in the sign factor has the same parity as the codimension of the corresponding stratum, which is the shift which is needed to perversify the constant shaef along a stratum.

Since we have to work with an arbitrary irreducible representation E, we need the Jantzen-Zuckerman translation functor T_E. If ν is an extremal weight of E, then T_E is an equivalence from the category of $(\mathfrak{g}, K)$-modules which have a non-singular infinitesimal character given by a vector μ such that ν belongs to the Weyl chamber of μ to the category of $(\mathfrak{g}, K)$-modules whose infinitesimal character is given by $\nu + \mu$ (*cf.* [Vog81], chapter 7.3 or [Wal88], Chapter 6.A.3). Therefore, from (18) we get

$$E = \sum_{\gamma} (-1)^{l^I(\gamma) - l^I(\gamma_o)} T_E \Theta(\gamma). \tag{19}$$

To a conjugacy class (H, Δ^+), we associate $\mathcal{L}$, the centraliser of the maximal $\boldsymbol{R}$-split torus of H, and π, the discrete series representation of $\mathcal{L}(\boldsymbol{R})$ parametrised by the intersection of Δ^+ with the compact part of $\check{\mathfrak{h}}$ (*cf.* [Vog81], Theorem 6.6.9). There is a one-to one correspondence between the conjugacy classes γ and conjugacy classes of pairs $(\mathcal{L}, \pi)$ of Levi components of cuspidal parabolic subgroups of $\mathcal{G}(\boldsymbol{R})$ and discrete series representations π of $\mathcal{L}(\boldsymbol{R})$ such that $I_{\mathcal{L}}^{\mathcal{G}}(\pi)$ has the same infinitesimal character as $\boldsymbol{C}$. Let $(\mathcal{L}_\gamma, \pi_\gamma)$ be the pair corresponding to γ. We have an equality of virtual representations

$$\Theta(\gamma) = I_{\mathcal{L}_\gamma}^{\mathcal{G}}(\pi_\gamma), \tag{20}$$

by the comparison between the Vogan-Zuckerman and Langlands constructions [Vog81], Theorem 6.6.15. Let F_γ be the irreducible representation of $\mathcal{L}_\gamma$ whose highest weight is an extremal weight of E belonging to the same Weyl chamber for $\Delta(\mathfrak{h}, \mathfrak{g})$ as the infinitesimal character of π_γ. Then

$$T_E\Theta(\gamma) = I_{\mathcal{L}_\gamma}^{\mathcal{G}}(T_{F_\gamma}\pi_\gamma), \tag{21}$$

by (20) and the effect of Jantzen-Zuckerman tensoring on induction (*cf.* [Vog81], Proposition 7.4.1). Moreover, $T_{F_\gamma}\pi_\gamma$ is again a discrete series representation of $\mathcal{L}_\gamma$ and the set $(\mathcal{L}_\gamma, T_{F_\gamma}\pi_\gamma)$ contains precisely one representative out of each conjugacy class of pairs $(\mathcal{L}, \pi)$, where $\mathcal{L}$ is a Levi component of a real parabolic subgroup of $\mathcal{G}$ and π is a discrete series representation of $\mathcal{L}(\boldsymbol{R})$ such that $I_{\mathcal{L}}^{\mathcal{G}}(\pi)$ has the same infinitesimal character as $\check{E}$.

By (19) and (21), it remains to verify

$$(-1)^{l^I(\gamma)-l^I(\gamma_o)} = (-1)^{q(\mathcal{L})+2q(\mathcal{G})+a_{\mathcal{L}}^{\mathcal{G}}} n_{T_{F_\gamma}\pi_\gamma}, \tag{22}$$

where $n_{T_{F_\gamma}\pi_\gamma}$ has the same meaning as in (16). As we explained at the beginning of the proof, this will prove not only **ii**, but also **i** since the left hand side of (22) is independent of the choice of $\mathcal{R}_\lambda$.

Fix a pair $\pi = (H, \Delta^+)$, and let $\mathcal{L}$ and $\mathfrak{h}$ be defined as above. Let $\lambda \in \check{\mathfrak{a}}_{\mathcal{L}}$ be defined by the action of the centre of $\mathcal{L}$ on $T_{F_\gamma}\pi_\gamma$. It is the projection of μ to $\check{\mathfrak{a}}_{\mathcal{L}}$, where $\mu \in \check{\mathfrak{h}}$ is a representative of the infinitesimal character of $\check{E}$ which belongs to the positive Weyl chamber for Δ^+. Let $\mathcal{R} = \mathcal{R}_\lambda$. Let A_- be the set of roots α of $\mathfrak{h}$ in $\mathfrak{n}_{\mathcal{R}}$ with $\langle \check{\alpha}, \mu \rangle < 0$, in other words, the set of elements of $-\Delta^+$ which occur in $\mathfrak{n}_{\mathcal{R}}$. For such a root α, $\vartheta(\alpha)$ again belongs to $-\Delta^+$. For our assumption on $\mathcal{R}$ implies

$$\langle \alpha - \vartheta(\alpha), \mu \rangle = 2\langle \alpha, \lambda \rangle \geq 0.$$

This consideration shows that $\alpha \to -\vartheta(\alpha)$ is a bijection between A_- and the set A_{++} of roots $\alpha \in \Delta^+$ which occur in $\mathfrak{n}_{\mathcal{R}}$ and satisfy $\vartheta(\alpha) \in \Delta^+$. Let A_{+-} be the set of $\alpha \in \Delta^+$ which occur in $\mathfrak{n}_{\mathcal{R}}$ and satisfy $\vartheta(\alpha) \in -\Delta^+$. By these definitions of the sets $A_?$, there is a disjoint union

$$\Delta(\mathfrak{h}, \mathfrak{n}_{\mathcal{R}}) = A_- \cup A_{++} \cup A_{+-}$$

of the set of roots of $\mathfrak{h}$ in $\mathfrak{n}_{\mathcal{R}}$.

By the definition of $E_{\mathcal{L}}^*$ and Kostant's theorem on $\mathfrak{n}$-homology,

$$n_\pi = (-1)^{\#(A_{++} \cup A_{+-})}.$$

Furthermore, $2q(\mathcal{G}) = 2q(\mathcal{L}) + a_{\mathcal{L}}^{\mathcal{G}} + \#(A_{++} \cup A_{+-} \cup A_{-})$. Using the bijection between A_{++} and A_{-} and the fact that $q(\mathcal{L})$ is an integer (since $\mathcal{L}(\boldsymbol{R})$ is cuspidal), we get

$$2q(\mathcal{G}) = a_{\mathcal{L}}^{\mathcal{G}} + \#(A_{+-}) \bmod 2.$$

Putting these things together, we see that the right hand side of (22) is equal to

$$(-1)^{\#(A_{++})+q(\mathcal{L})}. \tag{23}$$

We use [Vog81], Lemma 9.4.15 to compare this to the left hand side of (22). As the set B occurring in [Vog81], Lemma 9.4.15 we take A_{++}. The number $q(\mathcal{L})$ can be identified with the number of noncompact imaginary roots in Δ^+. Combining assertions a and c of [Vog81], Lemma 9.4.15, we get $l^I(\gamma_o) - l^I(\gamma) = \#(A_{++}) + q(\mathcal{L})$, and (23) is equal to the left hand side of (22). □

It follows from (17) that the coefficients (14) are equal to:

$$\tilde{\Phi}_{\mathcal{G},E,\mathcal{M}}(\gamma) = (-1)^{2q(\mathcal{G})+a_{\mathcal{M}}^{\mathcal{G}}} \left|D_{\mathcal{M}}^{\mathcal{G}}(\gamma)\right|^{1/2} \mathrm{char}_E(\gamma),$$

where $a_{\mathcal{M}}^{\mathcal{G}}$ has the same meaning as in (5). We arrive at the final formula:

THEOREM 21. – *Let us assume that* $K = K_\infty$, i.e., *that* $\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o \backslash \mathcal{G}(\boldsymbol{R})/K$ *is connected. The trace* (1) *of a Hecke correspondence is given by*

$$(-1)^{2q(\mathcal{G})} \sum_{\mathcal{M}\in\mathfrak{L}_c} (-1)^{a_{\mathcal{M}}^{\mathcal{G}}} \frac{w^{\mathcal{M}}}{w^{\mathcal{G}}} \sum_{\gamma\in(\mathcal{M})_{\mathbf{R}\text{-elliptic}}} \frac{\chi(\mathcal{M}_\gamma)}{\iota^{\mathcal{M}}(\gamma)} \left|D_{\mathcal{M}}^{\mathcal{G}}(\gamma)\right|^{1/2} \mathrm{char}_E(\gamma) O_\gamma(h_{\mathcal{M}}). \tag{24}$$

It is possible to rewrite this formula as a sum over conjugacy classes in G. Because of the presence of the factor $\left|D_{\mathcal{M}}^{\mathcal{G}}(\gamma)\right|^{1/2}$, the contribution of a M-conjugacy class γ to the summand indexed by $\mathcal{M}$ is zero unless $\mathcal{M}_\gamma = \mathcal{G}_\gamma$. It follows that M can be reconstructed from such a γ as the centraliser of the maximal $\boldsymbol{R}$-split torus in the centre of $\mathcal{G}_\gamma$. Therefore, if $g \in G$ and $\gamma \in M$ such that both γ and $g\gamma g^{-1}$ contribute to the summand indexed by $\mathcal{M}$ in (24), then $g \in N_G(\mathcal{M})$. In particular, this implies $\left|D_{\mathcal{M}}^{\mathcal{G}}(\gamma)\right|^{1/2} = \left|D_{\mathcal{M}}^{\mathcal{G}}(g\gamma g^{-1})\right|^{1/2}$, which means that γ and $g\gamma g^{-1}$ give the same contribution to (24). It also implies that the number of M-conjugacy classes of those $\tilde{\gamma} \in M$ which are G-conjugate to γ and contribute to (24) is equal to

$$\#(N_G(\mathcal{M})/MZ_G(\gamma)) = \frac{\iota^{\mathcal{M}}(\gamma)}{\iota^{\mathcal{G}}(\gamma)} \#(N_G(\mathcal{M})/M)$$

By the conjugacy of the maximal split tori of $\mathcal{M}$, we have

$$N_G(\mathcal{M})/M = N_G(\mathcal{M}) \cap N_G(\mathcal{A}_o)/N_M(\mathcal{A}_o).$$

Let $\mathfrak{L}_{\mathcal{M}}$ be the set of elements of $\mathfrak{L}$ which are conjugate to $\mathcal{M}$. It is isomorphic to $N_G(\mathcal{A}_o)/(N_G(\mathcal{M}) \cap N_G(\mathcal{A}_o))$. Using these facts, we see that the number of M-conjugacy classes of those $\tilde{\gamma} \in M$ which are G-conjugate to γ and contribute to (24) is equal to

$$\frac{\iota^{\mathcal{M}}(\gamma)\#(N_G(\mathcal{A}_o)/A_o)}{\iota^{\mathcal{G}}(\gamma)\#(N_M(\mathcal{A}_o)/A_o)\#(N_G(\mathcal{A}_o)/(N_G(\mathcal{A}_o) \cap N_G(\mathcal{M})))} = \frac{\iota^{\mathcal{M}}(\gamma) w^{\mathcal{G}}}{\iota^{\mathcal{G}}(\gamma) w^{\mathcal{M}} \#(\mathfrak{L}_{\mathcal{M}}.)}$$

This means that we may count the contribution of members of the G-conjugacy class of γ only once if we replace $\iota^{\mathcal{M}}(\gamma)$ by $\iota^{\mathcal{G}}(\gamma)$ and omit the factor $\frac{w^{\mathcal{L}}}{w^{\mathcal{G}}}$. Also, if γ contributes via $\mathcal{M}$ then

$$|D^{\mathcal{G}}_{\mathcal{M}}(\gamma)|^{1/2} O_\gamma(h_{\mathcal{M}}) = O_\gamma(h) \tag{25}$$

by an easy calculation, since $\mathcal{M}_\gamma = \mathcal{G}_\gamma$.

To a semisimple $\gamma \in G$ we attach $\varepsilon(\gamma)$ as follows: If $\mathcal{G}_\gamma(\boldsymbol{R})$ has no Cartan subgroup which is compact modulo $\mathcal{A}_{\mathcal{G}_\gamma}(\boldsymbol{R})$, then we put $\varepsilon(\gamma) = 0$. Otherwise, we put $\varepsilon(\gamma) = (-1)^{\dim(\mathcal{A}_{\mathcal{G}_\gamma}/\mathcal{A}_{\mathcal{G}})}$. It is clear from the previous remarks that a semisimple G-conjugacy class γ contributes to (24) if and only if $\varepsilon(\gamma) \neq 0$, in which case it contributes through the centraliser of $\mathcal{A}_{\mathcal{G}_\gamma}$ in $\mathcal{G}$, with sign $\varepsilon(\gamma)$.

Putting these considerations together, we see that, under our assumption that $K = K_\infty$ (that the symmetric space $\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathcal{G}(\boldsymbol{R})/K$ is connected) the Lefschetz number (24) is equal to

$$(-1)^{2q(\mathcal{G})} \sum_{\gamma\in(\mathcal{G})_{\text{semisimple}}} \varepsilon(\gamma)\frac{\chi(\mathcal{G}_\gamma)}{\iota^{\mathcal{G}}(\gamma)} \mathrm{char}_E(\gamma) O_\gamma(h). \tag{26}$$

The sum is over semisimple $G = \mathcal{G}(\boldsymbol{Q})$-conjugacy classes in $G = \mathcal{G}(\boldsymbol{Q})$.

We will now explain what happens if we drop the assumption $K = K_\infty$. Let us first consider the case where $\mathcal{M}_{\mathcal{G}}(\boldsymbol{R})$ has a discrete series. In this case, the trace of a function $h \in C_c^\infty(\mathbb{G}_f)$ on the L_2-cohomology $H^*_{(2)}(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G}/K, \boldsymbol{E})$

$$\sum_{\mathcal{M}\in\mathfrak{L}} [K_\infty : K](-1)^{a^{\mathcal{G}}_{\mathcal{M}}} \frac{w^{\mathcal{M}}}{w^{\mathcal{G}}} \sum_{\gamma\in(\mathcal{M})_{\mathbf{R}\text{-elliptic}}\cap K\mathcal{G}(\boldsymbol{R})^o} \frac{\chi(\mathcal{M}_\gamma)}{\iota^{\mathcal{M}}(\gamma)} \Phi_{\mathcal{G},E,\mathcal{M}}(\gamma) O_\gamma(h_{\mathcal{M}}). \tag{27}$$

In other words, the differences to the case $K = K_\infty$ are (5)

- the restriction of the sum to conjugacy classes of those $\gamma \in \mathcal{G}(\mathbb{Q})$ which belong to a connected component of $\mathcal{G}(\boldsymbol{R})$ which is met by K and
- multiplication of the final result by the factor $[K_\infty : K]$.

This is so because if f is a Euler-Poincare function on $\mathcal{G}(\boldsymbol{R})$ for $(\mathfrak{m}_{\mathcal{G}}, K_\infty)$-cohomology, then the restriction of $\frac{f}{[K_\infty:K]}$ to those connected components of $\mathcal{G}(\boldsymbol{R})$ which meet K is a Euler-Poincare function for $(\mathfrak{m}_{\mathcal{G}}, K)$-cohomology. After making this change, all the considerations in Arthur's paper go through.

Using (27) in our proof of the trace formula for full cohomology, we see that for $K \subset K_\infty$ the trace of h on the full cohomology $H^*(G\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})^o\backslash\mathbb{G}/K, \boldsymbol{E})$ is given by

$$(-1)^{2q(\mathcal{G})}[K : K_\infty] \sum_{\mathcal{M}\in\mathfrak{L}_c} (-1)^{a^{\mathcal{G}}_{\mathcal{M}}} \frac{w^{\mathcal{M}}}{w^{\mathcal{G}}} \sum_{\gamma\in(\mathcal{M})_{\mathbf{R}\text{-elliptic}}\cap\mathcal{G}(\boldsymbol{R})^o K} \frac{\chi(\mathcal{M}_\gamma)}{\iota^{\mathcal{M}}(\gamma)} |D^{\mathcal{G}}_{\mathcal{M}}(\gamma)|^{1/2} \mathrm{char}_E(\gamma) O_\gamma(h_{\mathcal{M}}), \tag{28}$$

(5) It seems that the claim of Remark 3 on p. 282 of [Arthur:L2Lefschetz] is valid only if K contains the torsion points of $\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})$. Then, the restriction of the sum over conjugacy classes is unnecessary since the discrete series characters vanish outside $\mathcal{G}(\boldsymbol{R})^o\mathcal{A}_{\mathcal{G}}(\boldsymbol{R})$.

or, equivalently,

$$(29)\qquad [K_\infty : K](-1)^{2q(\mathcal{G})} \sum_{\gamma \in (\mathcal{G})_{\text{semisimple}} \cap \mathcal{G}(\boldsymbol{R})^\circ K} \varepsilon(\gamma) \frac{\chi(\mathcal{G}_\gamma)}{\iota^{\mathcal{G}}(\gamma)} \mathrm{char}_E(\gamma) O_\gamma(h).$$

In other words, the modifications that lead from the case $K = K_\infty$ to the case where K is an open subgroup of K_∞ are the same as in the case of the Lefschetz formula for L_2-cohomology.

Finally, let us sketch how our trace formula should be related to the formula of Goresky and MacPherson. As was conjectured by Harder, certain truncated Hecke operators play a crucial role in their formula. Roughly speaking, the truncation of the Hecke correspondence should correspond to distributing the contribution of $\mathcal{M}$ to (24) over the several parabolic subgroups with Levi component $\mathcal{M}$. The truncation is ambiguous for those γ for which $H_{\mathcal{M}}(\gamma)$ is singular. It seems that those conjugacy classes contribute to the Goresky-MacPherson formula via a parabolic subgroup with Levi component bigger than $\mathcal{M}$. So far, the comparison has not been carried out in detail.

Appendix A

The Hochschild-Serre spectral sequence for $(\mathfrak{g}, K)$-modules

The aim of this appendix is to prove the following version of the Hochschild-Serre spectral sequence:

PROPOSITION 1. – *Let $(\mathfrak{q}, K)$ satisfy the assumptions of* [Vog81], Def. 6.1.1. *Let $\mathfrak{n}$ be a normal divisor of $\mathfrak{q}$ whose intersection with the Lie algebra of K is zero, and let $\mathfrak{l} = \mathfrak{q}/\mathfrak{n}$. Then for every injective $(\mathfrak{q}, K)$-module I and every $n > 0$, we have $H^n(\mathfrak{n}, I) = 0$. Therefore, the Grothendieck spectral sequence implies*

$$E_2^{p,q} = H^p_{(\mathfrak{l},K)}(H^q(\mathfrak{n}, X)) \Rightarrow H^{p+q}_{(\mathfrak{q},K)}(X),$$

where the $\mathfrak{n}$-cohomology can be understood as the derived functor either in the category of $\mathfrak{n}$-modules or in the category of $(\mathfrak{q}, K)$-modules.

Some textbooks on this subject claim to give a proof of this spectral sequence. However, they base their argument on the wrong assertion that under the above mentioned assumptions, the restriction of i to $\mathfrak{n}$ is an injective $\mathfrak{n}$-module. The "proof" of this assertion rests on an invalid interchange of the functor Hom with the functor of passage to the submodule of K-finite vectors. I will first attempt to show that in general the restriction of an injective $(\mathfrak{q}, K)$-module to $\mathfrak{n}$ fails to be injective. Afterwards, I will try to prove Proposition 1.

Let $\mathfrak{q}$ be one of the parabolic subalgebras of $_2(\boldsymbol{C})$ with Levi component $\mathfrak{so}_2(\boldsymbol{C})$, and let $K = \mathrm{SO}_2(\boldsymbol{R})$. Let $\mathfrak{n}$ be the radical of $\mathfrak{q}$, and let n be a generator of this one-dimensional vector space. The K-action on a $(\mathfrak{q}, K)$-module V corresponds to a grading

$$(1)\qquad V = \coprod_{k=-\infty}^{\infty} V_k$$

on the underlying vector space V. Therefore, $(\mathfrak{q}, K)$-modules are just graded modules over the graded ring $\boldsymbol{C}[n]$, where $\deg(n) = 2$. The restriction from $(\mathfrak{q}, K)$-modules to $\mathfrak{n}$-modules corresponds to the functor of forgetting the grading. Since elements of the coproduct (1) have only finitely many components different from zero, there are no nonvanishing elements of V satisfying

$$(2) \qquad nv = \lambda v \qquad \text{with} \qquad \lambda \in \boldsymbol{C} - \{0\}.$$

For every $x \in \check{\mathfrak{n}}$ let $\boldsymbol{C}_x$ be the one-dimensional ungraded $\boldsymbol{C}[n]$-module on which n acts by multiplication by $x(n)$, and let $I(x)$ be the injective hull of $\boldsymbol{C}_x$. Let F be the quotient field of $\boldsymbol{C}[n]$. By a result of Matlis ([Mat86], Theorem 18.4), every injective ungraded injective $\boldsymbol{C}[n]$-module I has a canonical submodule

$$(3) \qquad I_0 \subset I$$

which has a canonical decomposition

$$I_0 = \coprod_{x \in \check{\mathfrak{n}}} I_{0,x}$$

and such that

$$(4) \qquad I/I_0 \text{ is a } F\text{-vector space.}$$

In the case where I arises from a graded $\boldsymbol{C}[n]$-module I^* by forgetting the grading, the summands $I_{0,x}$ vanish for $x \neq 0$, as (2) has no non-trivial solution. Therfore I_0 is the submodule of all $i \in I$ which are killed by some power of n. In particular, I_0 comes from a graded submodule I_0^* of I^* by forgetting the grading. But then the graded $\boldsymbol{C}[n]$-module I^*/I_0^* is free of n-torsion and also (by (4)) n-divisible. Therefore, it is a free graded $\boldsymbol{C}[n, n^{-1}]$-module, and I/I_0 is a free ungraded $\boldsymbol{C}[n, n^{-1}]$-module, in contradiction to (4) unless we have $I = I_0$. This implies that every element of an injective $\mathfrak{n}$-module which is isomorphic to the restriction to $\mathfrak{n}$ of a $(\mathfrak{q}, K)$-module is annihilated by some power of n. This means that only $(\mathfrak{q}, K)$-modules on which n acts locally nilpotently can be embedded into an injective $(\mathfrak{q}, K)$-module which remains injective after restriction to $\mathfrak{n}$. Since there are sufficiently many injective $(\mathfrak{q}, K)$-modules, and since there are $(\mathfrak{q}, K)$-modules on which n does not act locally nilpotently (for instance, the restriction to $(\mathfrak{q}, K)$ of the Harish-Chandra module of a generic principal series representation of $\mathrm{SL}_2(\boldsymbol{R})$), this implies that there are injective $(\mathfrak{q}, K)$-modules whose restrictions to $\mathfrak{n}$ are not injective as $\mathfrak{n}$-modules.

This counterexample is, of course, a θ-stable parabolic subalgebra but I am quite sure that the same situation occurs for a real parabolic subgroup of $\mathrm{GL}_3(\boldsymbol{R})$ with Levi component $\mathrm{GL}_2(\boldsymbol{R}) \times \mathrm{GL}_1(\boldsymbol{R})$. This shows that one should not attempt to prove Proposition 1 by proving that the restriction of I to $\mathfrak{n}$ remains injective. One has to be contented with proving the proposition as it is formulated. This can be done in two steps.

LEMMA 1. – *The restriction of an injective* $(\mathfrak{q}, K)$*-module to* $(\mathfrak{n} + \mathfrak{k}, K)$ *is injective.*

Proof. – Indeed, by [Vog81], Def. 6.1.5., Lemma 6.1.6 this restriction functor has an exact left-adjoint functor. □

LEMMA 2. – *If I is an injective $(\mathfrak{n}+\mathfrak{k}, K)$-module, then $H^i(\mathfrak{n}, I) = 0$ for $i > 0$.*

Proof. – The restriction from $(\mathfrak{n}+\mathfrak{k}, K)$ to $\mathfrak{n}$ has a right adjoint $\mathcal{I} = \mathcal{I}_{\mathfrak{n}}^{(\mathfrak{n}+\mathfrak{k},K)}$, which can be described explicitly as follows:

$$\mathcal{I}V = \{f{:}K \to V | \text{the right translates } f(\cdot k) \text{ are contained in a finite-dimensional subspace of the space of functions from } K \text{ to } V.\}$$

The actions of K and $\mathfrak{n}$ on V are given by

$$(kf)(l) = f(kl) \qquad k, l \in K$$
$$(nf)(l) = \big((\mathrm{Ad}(l))(n)\big)\big(f(l)\big).$$

One checks this by verifying that every morphism in the category of $\mathfrak{n}$-modules from a $(\mathfrak{n}+\mathfrak{k}, K)$-module to V factors over the canonical morphism $\mathcal{I}V \to V$, $f \to f(\mathbf{1})$.

If V is just a complex vector space, let $\mathcal{J}V$ be the value at V of the right adjoint functor to the forgetful functor from the category of $(\mathfrak{k}, K)$-modules to the category of vector spaces. It can be described by the same formulas as $\mathcal{I}V$, forgetting to course the formula for the action of $\mathfrak{n}$. It is clear that $\mathcal{J}$ is exact.

Let $C_{\mathfrak{n}}^*$ be the standard complex for computing $\mathfrak{n}$-cohomology, $C_{\mathfrak{n}}^i(V) = \mathrm{Hom}(\Lambda^i\mathfrak{n}, V)$ with the usual differential ([Vog81], (3.1.1)). It is easy to see that

$$C^*(\mathcal{I}(V)) \xrightarrow{j} \mathcal{J}(C^*(\mathfrak{n}, V))$$
$$\big((jf)(l)\big)(\nu_1 \wedge \ldots \wedge \nu_m) = \big(f(l)\big)\big(\mathrm{Ad}(l)^{-1}(\nu_1) \wedge \ldots \wedge \mathrm{Ad}(l)^{-1}(\nu_m)\big)$$

is an isomorphism of cochain complexes.

Therefore, $\mathcal{I}(V)$ has vanishing higher $\mathfrak{n}$-cohomology if the same is true for V. But every injective $(\mathfrak{n}+\mathfrak{k}, K)$-module I can be embedded into, and hence by its injectivity is a direct summand of, $\mathcal{I}(J)$ for some injective $\mathfrak{n}$-module J. The assertion follows. □

It is clear that the proposition to be proved in this appendix is a consequence of the two lemmata. Indeed, we have seen that the $\mathfrak{n}$-cohomology does not change if it is calculated as a derived functor on the category of $(\mathfrak{q}, K)$-modules. The Grothendieck spectral sequence is applicable since $H^0(\mathfrak{n}, I)$ is an injective $(\mathfrak{l}, K)$-module if I is an injective $(\mathfrak{q}, K)$-module. This is so because $H^0(\mathfrak{n}, \cdot)$ has an exact left adjoint functor which lets $\mathfrak{n}$ act trivially on an $(\mathfrak{l}, K)$-module.

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J. FRANKE
Mathematische Institut der Universität Bonn,
Wegelerstrasse 10 53115 Bonn.