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A new canonical induction formula for p -permutation modules



Une nouvelle formule d'induction canonique pour modules de p -permutation

Laurence Barker, Hatice Mutlu

Department of Mathematics, Bilkent University, 06800 Bilkent, Ankara, Turkey

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ABSTRACT

Applying Robert Boltje's theory of canonical induction, we give a restriction-preserving formula expressing any p -permutation module as a $\mathbb{Z}[1/p]$ -linear combination of modules induced and inflated from projective modules associated with subquotient groups. The underlying constructions include, for any given finite group, a ring with a \mathbb{Z} -basis indexed by conjugacy classes of triples (U, K, E) where U is a subgroup, K is a p' -residue-free normal subgroup of U , and E is an indecomposable projective module of the group algebra of U/K .

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R É S U M É

En application de la théorie de l'induction canonique de Robert Boltje, nous présentons une formule stable par restriction au moyen de laquelle tout module de p -permutation est exprimé sous forme de combinaison $\mathbb{Z}[1/p]$ -linéaire des inductions des inflations des modules projectifs associés à des groupes de sous-quotients. Les constructions concernées comprennent, pour tout groupe fini, un anneau qui a une \mathbb{Z} -base indexée par les classes de conjugaison des triplets (U, K, E) avec U un sous-groupe, $O^{p'}(K) = K \trianglelefteq U$ et E un module projectif indécomposable de l'algèbre de groupe de U/K .

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1. Introduction

We shall be applying Boltje's theory of canonical induction [2] to the ring of p -permutation modules. Of course, p is a prime. We shall be considering p -permutation modules for finite groups over an algebraically closed field \mathbb{F} of characteristic p . A review of the theory of p -permutation modules can be found in Bouc–Thévenaz [6, Section 2].

E-mail addresses: barker@fen.bilkent.edu.tr (L. Barker), hatice.mutlu@bilkent.edu.tr (H. Mutlu).

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A canonical induction formula for p -permutation modules was given by Boltje [3, Section 4] and shown to be \mathbb{Z} -integral. It expresses any p -permutation module, up to isomorphism, as a \mathbb{Z} -linear combination of modules induced from a special kind of p -permutation module, namely, the 1-dimensional modules.

We shall be inducing from another special kind of p -permutation module. Let G be a finite group. We understand all $\mathbb{F}G$ -modules to be finite-dimensional. An indecomposable $\mathbb{F}G$ -module M is said to be **exprojective** provided the following equivalent conditions hold up to isomorphism: there exists a normal subgroup $K \trianglelefteq G$ such that M is inflated from a projective $\mathbb{F}G/K$ -module; there exists $K \trianglelefteq G$ such that M is a direct summand of the permutation $\mathbb{F}G$ -module $\mathbb{F}G/K$; every vertex of M acts trivially on M ; some vertex of M acts trivially on M . Generally, an $\mathbb{F}G$ -module X is called **exprojective** provided every indecomposable direct summand of X is exprojective.

The exprojective modules do already play a special role in the theory of p -permutation modules. Indeed, the parametrization of the indecomposable p -permutation modules, recalled in Section 2, characterizes any indecomposable p -permutation module as a particular direct summand of a module induced from an exprojective module.

We shall give a $\mathbb{Z}[1/p]$ -integral canonical induction formula, expressing any p -permutation $\mathbb{F}G$ -module, up to isomorphism, as a $\mathbb{Z}[1/p]$ -linear combination of modules induced from exprojective modules. More precisely, we shall be working with the Grothendieck ring for p -permutation modules $T(G)$ and we shall be introducing another commutative ring $\mathcal{T}(G)$ which, roughly speaking, has a free \mathbb{Z} -basis consisting of lifts of induced modules of indecomposable exprojective modules. We shall consider a ring epimorphism $\text{lin}_G : \mathcal{T}(G) \rightarrow T(G)$ and its \mathbb{Q} -linear extension $\text{lin}_G : \mathbb{Q}\mathcal{T}(G) \rightarrow \mathbb{Q}T(G)$. The latter is split by a \mathbb{Q} -linear map $\text{can}_G : \mathbb{Q}T(G) \rightarrow \mathbb{Q}\mathcal{T}(G)$ which, as we shall show, restricts to a $\mathbb{Z}[1/p]$ -linear map $\text{can}_G : \mathbb{Z}[1/p]T(G) \rightarrow \mathbb{Z}[1/p]\mathcal{T}(G)$.

Let \mathbb{K} be a field of characteristic zero that is sufficiently large for our purposes. To motivate further study of the algebras $\mathbb{Z}[1/p]\mathcal{T}(G)$ and $\mathbb{K}\mathcal{T}(G)$, we mention that, notwithstanding the formulas for the primitive idempotents of $\mathbb{K}T(G)$ in Boltje [4, 3.6], Bouc-Thévenaz [6, 4.12] and [1], the relationship between those idempotents and the basis $\{M_{P,E}^G : (P, E) \in_G \mathcal{P}(E)\}$ remains mysterious. In Section 4, we shall prove that $\mathbb{K}\mathcal{T}(G)$ is \mathbb{K} -semisimple as well as commutative, in other words, the primitive idempotents of $\mathbb{K}\mathcal{T}(G)$ comprise a basis for $\mathbb{K}\mathcal{T}(G)$. We shall also describe how, via lin_G , each primitive idempotent of $\mathbb{K}T(G)$ lifts to a primitive idempotent of $\mathbb{K}\mathcal{T}(G)$.

2. Exprojective modules

We shall establish some general properties of exprojective modules.

Given $H \leq G$, we write ${}_G\text{Ind}_H$ and ${}_H\text{Res}_G$ to denote the induction and restriction functors between $\mathbb{F}G$ -modules and $\mathbb{F}H$ -modules. When $H \trianglelefteq G$, we write ${}_G\text{Inf}_{G/H}$ to denote the inflation functor to $\mathbb{F}G$ -modules from $\mathbb{F}G/H$ -modules. Given a finite group L and an understood isomorphism $L \rightarrow G$, we write ${}_L\text{Iso}_G$ to denote the isogation functor to $\mathbb{F}L$ -modules from $\mathbb{F}G$ -modules, we mean to say, ${}_L\text{Iso}_G(X)$ is the $\mathbb{F}L$ -module obtained from an $\mathbb{F}G$ -module X by transport of structure via the understood isomorphism.

Let us classify the exprojective $\mathbb{F}G$ -modules up to isomorphism. We say that G is p' -**residue-free** provided $G = O^{p'}(G)$, equivalently, G is generated by the Sylow p -subgroups of G . Let $\mathcal{Q}(G)$ denote the set of pairs (K, F) , where K is a p' -residue-free normal subgroup of G and F is an indecomposable projective $\mathbb{F}G/K$ -module, two such pairs (K, F) and (K', F') being deemed the same provided $K = K'$ and $F \cong F'$. We define an indecomposable exprojective $\mathbb{F}G$ -module $M_G^{K,F} = {}_G\text{Inf}_{G/K}(F)$. By considering vertices, we obtain the following result.

Proposition 2.1. *The condition $M \cong M_G^{K,F}$ characterizes a bijective correspondence between:*

- (a) *the isomorphism classes of indecomposable exprojective $\mathbb{F}G$ -modules M ,*
- (b) *the elements (K, F) of $\mathcal{Q}(G)$.*

In particular, for a p -subgroup P of G , the condition $E \cong_{N_G(P)} \text{Inf}_{N_G(P)/P}(\bar{E})$ characterizes a bijective correspondence between, up to isomorphism, the indecomposable exprojective $\mathbb{F}N_G(P)$ -modules E with vertex P and the indecomposable projective $\mathbb{F}N_G(P)/P$ -modules \bar{E} . It follows that the well-known classification of the isomorphism classes of indecomposable p -permutation $\mathbb{F}G$ -modules, as in Bouc-Thévenaz [6, 2.9] for instance, can be expressed as in the next result. Let $\mathcal{P}(G)$ denote the set of pairs (P, E) where P is a p -subgroup of G and E is an exprojective $\mathbb{F}N_G(P)$ -module with vertex P , two such pairs (P, E) and (P', E') being deemed the same provided $P = P'$ and $E \cong E'$. We make $\mathcal{P}(G)$ become a G -set via the actions on the coordinates. We define $M_{P,E}^G$ to be the indecomposable p -permutation $\mathbb{F}G$ -module with vertex P in Green correspondence with E .

Theorem 2.2. *The condition $M \cong M_{P,E}^G$ characterizes a bijective correspondence between:*

- (a) *the isomorphism classes of indecomposable p -permutation $\mathbb{F}G$ -modules M ,*
- (b) *the G -conjugacy classes of elements $(P, E) \in \mathcal{P}(G)$.*

We now give a necessary and sufficient condition for $M_{P,E}^G$ to be exprojective.

Proposition 2.3. *Let $(P, E) \in \mathcal{P}(G)$. Let K be the normal closure of P in G . Then $M_{P,E}^G$ is exprojective if and only if $N_K(P)$ acts trivially on E . In that case, K is p' -residue-free, P is a Sylow p -subgroup of K , we have $G = N_G(P)K$, the inclusion $N_G(P) \hookrightarrow G$ induces an isomorphism $N_G(P)/N_K(P) \cong G/K$, and $M_{P,E}^G \cong M_G^{K,F}$, where F is the indecomposable projective $\mathbb{F}G/K$ -module determined, up to isomorphism, by the condition $E \cong_{N_G(P)} \text{Inf}_{N_G(P)/N_K(P)} \text{Iso}_{G/K}(F)$.*

Proof. Write $M = M_{P,E}^G$. If M is exprojective then K acts trivially on M and, perforce, $N_K(P)$ acts trivially on E .

Conversely, suppose $N_K(P)$ acts trivially on E . Then P , being a vertex of E , must be a Sylow p -subgroup of $N_K(P)$. Hence, P is a Sylow p -subgroup of K . By a Frattini argument, $G = N_G(P)K$ and we have an isomorphism $N_G(P)/N_K(P) \cong G/K$ as specified. Let $X = {}_G\text{Ind}_{N_G(P)}(E)$. The assumption on E implies that X has well-defined \mathbb{F} -submodules

$$Y = \left\{ \sum_k k \otimes_{N_G(P)} x : x \in E \right\}, \quad Y' = \left\{ \sum_k k \otimes_{N_G(P)} x_k : x_k \in E, \sum_k x_k = 0 \right\}$$

summed over a left transversal $kN_K(P) \subseteq K$. Making use of the well-definedness, an easy manipulation shows that the action of $N_G(P)$ on X stabilizes Y and Y' . Similarly, K stabilizes Y and Y' . So Y and Y' are $\mathbb{F}G$ -submodules of X . Since $|K : N_K(P)|$ is coprime to p , we have $Y \cap Y' = 0$. Since $|K : N_K(P)| = |G : N_G(P)|$, a consideration of dimensions yields $X = Y \oplus Y'$.

Fix a left transversal \mathcal{L} for $N_K(P)$ in K . For $g \in N_G(P)$ and $\ell \in \mathcal{L}$, we can write ${}^g\ell = \ell_g h_g$ with $\ell_g \in \mathcal{L}$ and $h_g \in N_K(P)$. By the assumption on E again, $h_g x = x$ for all $x \in E$. So

$$g \sum_{\ell} \ell \otimes x = \sum_{\ell} {}^g\ell \otimes gx = \sum_{\ell} \ell_g \otimes gx = \sum_{\ell} \ell \otimes gx$$

summed over $\ell \in \mathcal{L}$. We have shown that ${}_{N_G(P)}\text{Res}_G(Y) \cong E$. A similar argument involving a sum over \mathcal{L} shows that K acts trivially on Y . Therefore, $Y \cong M_{P,E}^{K,F}$. On the other hand, Y is indecomposable with vertex P and, by the Green correspondence, $Y \cong M_{P,E}^G$. \square

We shall be making use of the following closure property.

Proposition 2.4. *Given exprojective $\mathbb{F}G$ -modules X and Y , then the $\mathbb{F}G$ -module $X \otimes_{\mathbb{F}} Y$ is exprojective.*

Proof. We may assume that X and Y are indecomposable. Then X and Y are, respectively, direct summands of permutation $\mathbb{F}G$ -modules having the form $\mathbb{F}G/K$ and $\mathbb{F}G/L$ where $K \trianglelefteq G \trianglerighteq L$. By Mackey decomposition and the Krull–Schmidt Theorem, every indecomposable direct summand of $X \otimes Y$ is a direct summand of $\mathbb{F}G/(K \cap L)$. \square

3. A canonical induction formula

Throughout, we let \mathfrak{R} be a class of finite groups that is closed under taking subgroups. We shall understand that $G \in \mathfrak{R}$. We shall abuse notation, neglecting to use distinct expressions to distinguish between a linear map and its extension to a larger coefficient ring.

Specializing some general theory in Boltje [2], we shall introduce a commutative ring $\mathcal{T}(G)$ and a ring epimorphism $\text{lin}_G : \mathcal{T}(G) \rightarrow T(G)$. We shall show that the $\mathbb{Z}[1/p]$ -linear extension $\text{lin}_G : \mathbb{Z}[1/p]\mathcal{T}(G) \rightarrow \mathbb{Z}[1/p]T(G)$ has a splitting $\text{can}_G : \mathbb{Z}[1/p]\mathcal{T}(G) \rightarrow \mathbb{Z}[1/p]T(G)$. As we shall see, can_G is the unique splitting that commutes with restriction and isogation.

To be clear about the definition of $T(G)$, the Grothendieck ring of the category of p -permutation $\mathbb{F}G$ -modules, we mention that the split short exact sequences are the distinguished sequences determining the relations on $T(G)$. The multiplication on $T(G)$ is given by tensor product over \mathbb{F} . Given a p -permutation $\mathbb{F}G$ -module X , we write $[X]$ to denote the isomorphism class of X . We understand that $[X] \in T(G)$. By Theorem 2.2,

$$T(G) = \bigoplus_{(P,E) \in {}_G\mathcal{P}(G)} \mathbb{Z}[M_{P,E}^G]$$

as a direct sum of regular \mathbb{Z} -modules, the notation indicating that the index runs over representatives of G -orbits. Let $T^{\text{ex}}(G)$ denote the \mathbb{Z} -submodule of $T(G)$ spanned by the isomorphism classes of exprojective $\mathbb{F}G$ -modules. By Proposition 2.4, $T^{\text{ex}}(G)$ is a subring of $T(G)$. By Proposition 2.1,

$$T^{\text{ex}}(G) = \bigoplus_{(K,F) \in {}_G\mathcal{Q}(G)} \mathbb{Z}[M_G^{K,F}].$$

For $H \leq G$, the induction and restriction functors ${}_G\text{Ind}_H$ and ${}_H\text{Res}_G$ give rise to induction and restriction maps ${}_G\text{ind}_H$ and ${}_H\text{res}_G$ between $T(H)$ and $T(G)$. Similarly, given $L \in \mathfrak{R}$ and an isomorphism $\theta : L \rightarrow G$, we have an evident isogation map ${}_L\text{iso}_G^\theta : T(L) \leftarrow T(G)$. In particular, given $g \in G$, we have an evident conjugation map ${}_{gH}\text{con}_H^g$. Boltje noted that, when \mathfrak{R} is the set of subgroups of a given fixed finite group, T is a Green functor in the sense of [2, 1.1c]. For arbitrary \mathfrak{R} , a class of admitted isogations must be understood, and the isogations and inclusions between groups in \mathfrak{R} must satisfy the

axioms of a category. Granted that, then T is still a Green functor in an evident sense whereby the conjugations replaced by isogations.

Following a construction in [2, 2.2], adaptation to the case of arbitrary \mathfrak{R} being straightforward, we form the G -cofixed quotient \mathbb{Z} -module

$$\mathcal{T}(G) = \left(\bigoplus_{U \leq G} T^{\text{ex}}(U) \right)_G$$

where G acts on the direct sum via the conjugation maps ${}_{gU}\text{con}_U^g$. Harnessing the Green functor structure of T , the restriction functor structure of T^{ex} and noting that $T^{\text{ex}}(G)$ is a subring of $T(G)$, we make \mathcal{T} become a Green functor much as in [2, 2.2], with the evident isogation maps. In particular, $\mathcal{T}(G)$ becomes a ring, commutative because $T(G)$ is commutative. Given $x_U \in T^{\text{ex}}(U)$, we write $[U, x_U]_G$ to denote the image of x_U in $\mathcal{T}(G)$. Any $x \in \mathcal{T}(G)$ can be expressed in the form

$$x = \sum_{U \leq G} [U, x_U]_G$$

where the notation indicates that the index runs over representatives of the G -conjugacy classes of subgroups of G . Note that x determines $[U, x_U]$ and x_G but not, in general, x_U . Let $\mathcal{R}(G)$ be the G -set of pairs (U, K, F) where $U \leq G$ and $(K, F) \in \mathcal{Q}(U)$. We have

$$\mathcal{T}(G) = \bigoplus_{U \leq G, (K, F) \in \mathcal{N}_G(U) \mathcal{Q}(U)} \mathbb{Z}[U, [M_U^{K, F}]] = \bigoplus_{(U, K, F) \in \mathcal{R}(G)} \mathbb{Z}[U, [M_U^{K, F}]].$$

We define a \mathbb{Z} -linear map $\text{lin}_G : \mathcal{T}(G) \rightarrow T(G)$ such that $\text{lin}_G[U, x_U] = {}_G\text{ind}_U(x_U)$. As noted in [2, 3.1], the family $(\text{lin}_G : G \in \mathfrak{R})$ is a morphism of Green functors $\text{lin} : \mathcal{T} \rightarrow T$. In particular, the map $\text{lin}_G : \mathcal{T}(G) \rightarrow T(G)$ is a ring homomorphism. Extending to coefficients in \mathbb{Q} , we obtain an algebra map

$$\text{lin}_G : \mathbb{Q}\mathcal{T}(G) \rightarrow \mathbb{Q}T(G).$$

Let $\pi_G : T(G) \rightarrow T^{\text{ex}}(G)$ be the \mathbb{Z} -linear epimorphism such that π_G acts as the identity on $T^{\text{ex}}(G)$ and π_G annihilates the isomorphism class of every indecomposable non-exprojective p -permutation $\mathbb{F}G$ -module. By \mathbb{Q} -linear extension again, we obtain a \mathbb{Q} -linear epimorphism $\pi_G : \mathbb{Q}T(G) \rightarrow \mathbb{Q}T^{\text{ex}}(G)$. After [2, 5.3a, 6.1a], we define a \mathbb{Q} -linear map

$$\text{can}_G : \mathbb{Q}T(G) \rightarrow \mathbb{Q}\mathcal{T}(G), \xi \mapsto \frac{1}{|G|} \sum_{U, V \leq G} |U| \text{m\"ob}(U, V) [U, {}_U\text{res}_V(\pi_V({}_V\text{res}_G(\xi)))]_G$$

where $\text{m\"ob}()$ denotes the Möbius function on the poset of subgroups of G .

Theorem 3.1. Consider the \mathbb{Q} -linear map can_G .

- (1) We have $\text{lin}_G \circ \text{can}_G = \text{id}_{\mathbb{Q}\mathcal{T}(G)}$.
- (2) For all $H \leq G$, we have ${}_H\text{res}_G \circ \text{can}_G = \text{can}_H \circ {}_H\text{res}_G$.
- (3) For all $L \in \mathfrak{R}$ and isomorphisms $\theta : L \leftarrow G$, we have ${}_L\text{iso}_G^\theta \circ \text{can}_G = \text{can}_L \circ {}_L\text{iso}_G^\theta$.
- (4) $\text{can}_G[X] = [X]$ for all exprojective $\mathbb{F}G$ -modules X .

Those four properties, taken together for all $G \in \mathfrak{R}$, determine the maps can_G .

Proof. By [2, 6.4], part (1) will follow when we have checked that, for every indecomposable non-exprojective p -permutation $\mathbb{F}G$ -module M , we have $[M] \in \sum_{K < G} \text{ind}_K(\mathbb{Q}T(K))$. By [3, 2.1, 4.7], we may assume that G is p -hypoelementary. By [3, 1.3(b)], M is induced from $N_G(P)$ where P is a vertex of M . But M is non-exprojective, so P is not normal in G . The check is complete. Parts (2), (3), (4) follow from the proof of [2, 5.3a]. \square

Parts (2) and (3) of the theorem can be interpreted as saying that $\text{can}_* : T \rightarrow \mathcal{T}$ is a morphism of restriction functors. It is not hard to check that, when \mathfrak{R} is closed under the taking of quotient groups, the functors T , T^{ex} , \mathcal{T} can be equipped with inflation maps, and the morphisms lin_* and can_* are compatible with inflation.

The latest theorem immediately yields the following corollary.

Corollary 3.2. Given a p -permutation $\mathbb{F}G$ -module X , then

$$[X] = \frac{1}{|G|} \sum_{U, V \leq G} |U| \text{m\"ob}(U, V) {}_G\text{ind}_U\text{res}_V(\pi_V({}_V\text{res}_G[X])).$$

Given p -permutation $\mathbb{F}G$ -modules M and X , with M indecomposable, we write $m_G(M, X)$ to denote the multiplicity of M as a direct summand of X . We write $\pi_G(X)$ to denote the direct summand of X , well-defined up to isomorphism, such that $[\pi_G(X)] = \pi_G[X]$.

Lemma 3.3. *Let \mathfrak{p} be a set of primes. Suppose that, for all $V \in \mathfrak{K}$, all p -permutation $\mathbb{F}V$ -modules Y , all $U \triangleleft V$ such that V/U is a cyclic \mathfrak{p} -group, and all V -fixed elements $(K, F) \in \mathcal{Q}(U)$, we have*

$$m_U(M_U^{K,F}, \pi_U({}_U\text{Res}_V(Y))) = \sum_{(J,E) \in \mathcal{Q}(V)} m_U(M_U^{K,F}, {}_U\text{Res}_V(M_V^{J,E})) m_V(M_V^{J,E}, \pi_V(Y)).$$

Then, for all $G \in \mathfrak{K}$, we have $|G|_{\mathfrak{p}'} \text{can}_G[Y] \in \mathcal{T}(G)$, where $|G|_{\mathfrak{p}'}$ denotes the \mathfrak{p}' -part of $|G|$.

Proof. This is a special case of [2, 9.4]. \square

We can now prove the $\mathbb{Z}[1/p]$ -integrality of can_G .

Theorem 3.4. *The \mathbb{Q} -linear map can_G restricts to a $\mathbb{Z}[1/p]$ -linear map $\mathbb{Z}[1/p]T(G) \rightarrow \mathbb{Z}[1/p]\mathcal{T}(G)$.*

Proof. Let \mathfrak{p} be the set of primes distinct from p . Let V, Y, U, K, F be as in the latest lemma. We must obtain the equality in the lemma. We may assume that Y is indecomposable. If Y is exprojective, then $\pi_U({}_U\text{Res}_V(Y)) \cong {}_U\text{Res}_V(Y)$ and $\pi_V(Y) \cong X$, whence the required equality is clear. So we may assume that Y is non-exprojective. Then $\pi_V(Y)$ is the zero module. It suffices to show that $M_U^{K,F}$ is not a direct summand of ${}_U\text{Res}_V(Y)$. For a contradiction, suppose otherwise. The hypothesis on $|V : U|$ implies that U contains the vertices of Y . So $Y \mid_V \text{Ind}_U(X)$ for some indecomposable p -permutation $\mathbb{F}U$ -module X . Bearing in mind that (K, F) is V -stable, a Mackey decomposition argument shows that $M_U^{K,F} \cong X$. The V -stability of (K, F) also implies that $K \triangleleft V$. So

$$Y \mid_V \text{Ind}_U \text{Inf}_{U/K}(F) \cong_V \text{Inf}_{V/K} \text{Ind}_{U/K}(F).$$

We deduce that Y is exprojective. This is a contradiction, as required. \square

Proposition 3.5. *The \mathbb{Z} -linear map $\text{lin}_G : \mathcal{T}(G) \rightarrow T(G)$ is surjective. However, the $\mathbb{Z}[1/p]$ -linear map $\text{can}_G : \mathbb{Z}[1/p]T(G) \rightarrow \mathbb{Z}[1/p]\mathcal{T}(G)$ need not restrict to a \mathbb{Z} -linear map $T(G) \rightarrow \mathcal{T}(G)$. Indeed, putting $p = 3$ and $G = \text{SL}_2(3)$, letting Y be the isomorphically unique indecomposable non-simple non-projective p -permutation $\mathbb{F}G$ -module and X the isomorphically unique 2-dimensional simple $\mathbb{F}Q_8$ -module, then the coefficient of the standard basis element $[Q_8, X]_G$ in $\text{can}_G([Y])$ is equal to $2/3$.*

Proof. Since every 1-dimensional $\mathbb{F}G$ -module is exprojective, the surjectivity of the \mathbb{Z} -linear map lin_G follows from Boltje [3, 4.7]. Routine techniques confirm the counter-example. \square

4. The \mathbb{K} -semisimplicity of the commutative algebra $\mathbb{K}\mathcal{T}(G)$

Let $\mathcal{I}(G)$ be the G -set of pairs (P, s) where P is a p -subgroup of G and s is a p' -element of $N_G(P)/P$. Let \mathbb{K} be a field of characteristic zero such that \mathbb{K} has roots of unity whose order is the p' -part of the exponent of G . Choosing and fixing an arbitrary isomorphism between a suitable torsion subgroup of $\mathbb{K} - \{0\}$ and a suitable torsion subgroup of $\mathbb{F} - \{0\}$, we can understand Brauer characters of $\mathbb{F}G$ -modules to have values in \mathbb{K} . For a p' -element $s \in G$, we define a species $\epsilon_{1,s}^G$ of $\mathbb{K}\mathcal{T}(G)$, we mean, an algebra map $\mathbb{K}\mathcal{T}(G) \rightarrow \mathbb{K}$, such that $\epsilon_{1,s}^G[M]$ is the value, at s , of the Brauer character of a p -permutation $\mathbb{F}G$ -module M . Generally, for $(P, s) \in \mathcal{I}(G)$, we define a species $\epsilon_{P,s}^G$ of $\mathbb{K}\mathcal{T}(G)$ such that $\epsilon_{P,s}^G[M] = \epsilon_{1,s}^{N_G(P)/P}[M(P)]$, where $M(P)$ denotes the P -relative Brauer quotient of M^P . The next result, well-known, can be found in Bouc–Thévenaz [6, 2.18, 2.19].

Theorem 4.1. *Given $(P, s), (P', s') \in \mathcal{I}(G)$, then $\epsilon_{P,s}^G = \epsilon_{P',s'}^G$ if and only if we have G -conjugacy $(P, s) =_G (P', s')$. The set $\{\epsilon_{P,s}^G : (P, s) \in_G \mathcal{I}(G)\}$ is the set of species of $\mathbb{K}\mathcal{T}(G)$ and it is also a basis for the dual space of $\mathbb{K}\mathcal{T}(G)$. The dual basis $\{e_{P,s}^G : (P, s) \in_G \mathcal{I}(G)\}$ is the set of primitive idempotents of $\mathbb{K}\mathcal{T}(G)$. As a direct sum of trivial algebras over \mathbb{K} , we have*

$$\mathbb{K}\mathcal{T}(G) = \bigoplus_{(P,s) \in_G \mathcal{I}(G)} \mathbb{K}e_{P,s}^G.$$

Let $\mathcal{J}(G)$ be the G -set of pairs (L, t) where L is a p' -residue-free normal subgroup of G and t is a p' -element of G/L . We define a species $\epsilon_G^{L,t}$ of $\mathbb{K}\mathcal{T}^{\text{ex}}(G)$ such that, given an indecomposable exprojective $\mathbb{F}G$ -module M , then $\epsilon_G^{L,t}[M] = 0$ unless M

is the inflation of an $\mathbb{F}G/L$ -module \overline{M} , in which case, $\epsilon_G^{L,t}$ is the value, at t , of the Brauer character of \overline{M} . It is easy to show that, given a p -subgroup $P \leq G$ and a p' -element $s \in N_G(P)/P$, then $\epsilon_{P,s}^G[M] = \epsilon_G^{L,t}[M]$ for all exprojective $\mathbb{F}G$ -modules M if and only if L is the normal closure of P in G and t is conjugate to the image of s in G/L . Hence, via the latest theorem, we obtain the following lemma.

Lemma 4.2. *Given $(L, t), (L', t') \in \mathcal{J}(G)$, then $\epsilon_G^{L,t} = \epsilon_G^{L',t'}$ if and only if $L = L'$ and $t =_{G/L} t'$, in other words, $(L, t) =_G (L', t')$. The set $\{\epsilon_G^{L,t} : (L, t) \in_G \mathcal{J}(G)\}$ is the set of species of $\mathbb{K}\mathcal{T}^{\text{ex}}(G)$ and it is also a basis for the dual space of $\mathbb{K}\mathcal{T}^{\text{ex}}(G)$.*

Let $\mathcal{K}(G)$ be the G -set of triples (V, L, t) where $V \leq G$ and $(L, t) \in \mathcal{J}(V)$. Given $(L, t) \in \mathcal{J}(G)$, we define a species $\epsilon_{G,L,t}^G$ of $\mathbb{K}\mathcal{T}(G)$ such that, for $x \in \mathcal{T}(G)$ expressed as a sum as in Section 3,

$$\epsilon_{G,L,t}^G(x) = \epsilon_G^{L,t}(x_G).$$

Generally, for $(V, L, t) \in \mathcal{K}(G)$, we define a species $\epsilon_{V,L,t}^G$ of $\mathbb{K}\mathcal{T}(G)$ such that

$$\epsilon_{V,L,t}^G(x) = \epsilon_{V,L,t}^V(v \text{res}_G(x)).$$

Using Lemma 4.2, a straightforward adaptation of the argument in [6, 2.18] gives the next result. This result also follows from Boltje–Raggi-Cárdenas–Valero-Elizondo [5, 7.5].

Theorem 4.3. *Given $(V, L, t), (V', L', t') \in \mathcal{K}(G)$, then $\epsilon_{V,L,t}^G = \epsilon_{V',L',t'}^G$ if and only if $(V, L, t) =_G (V', L', t')$. The set $\{\epsilon_{V,L,t}^G : (V, L, t) \in_G \mathcal{K}(G)\}$ is the set of species of $\mathbb{K}\mathcal{T}(G)$ and it is also a basis for the dual space of $\mathbb{K}\mathcal{T}(G)$. The dual basis $\{e_{V,L,t}^G : (V, L, t) \in_G \mathcal{K}(G)\}$ is the set of primitive idempotents of $\mathbb{K}\mathcal{T}(G)$. As a direct sum of trivial algebras over \mathbb{K} , we have*

$$\mathbb{K}\mathcal{T}(G) = \bigoplus_{(V,L,t) \in_G \mathcal{K}(G)} \mathbb{K}e_{V,L,t}^G.$$

We have the following easy corollary on lifts of the primitive idempotents $e_{P,s}^G$.

Corollary 4.4. *Given $(P, s) \in \mathcal{I}(G)$, then $e_{(P,s),P,s}^G$ is the unique primitive idempotent e of $\mathbb{K}\mathcal{T}(G)$ such that $\text{lin}_G(e) = e_{P,s}^G$.*

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