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Number theory

Small values of signed harmonic sums

Petites valeurs de sommes harmoniques signées

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ABSTRACT

For every $\tau \in \mathbb{R}$ and every integer N, let $\mathfrak{m}_N(\tau)$ be the minimum of the distance of τ from the sums $\sum_{n=1}^{N} s_n/n$, where $s_1, \dots, s_n \in \{-1, +1\}$. We prove that $\mathfrak{m}_N(\tau) < \exp(-C(\log N)^2)$, for all sufficiently large positive integers N (depending on C and τ), where C is any positive constant less than $1/\log 4$.

RÉSUMÉ

Pour tout $\tau \in \mathbb{R}$ et tout entier N, soit $\mathfrak{m}_N(\tau)$ la distance minimale de τ aux sommes $\sum_{n=1}^{N} s_n/n$, où $s_1, \dots, s_n \in \{-1, +1\}$. On montre que $\mathfrak{m}_N(\tau) < \exp(-C(\log N)^2)$ pour tout entier positif N suffisamment grand (dépendant de C et τ), quelle que soit la constante positive C, inférieure à $1/\log 4$.

1. Introduction

For each positive integer n, let

$$H_n := 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

be the nth *harmonic number*. Harmonic numbers have long been an active area of research. For instance, Wolstenholme [18] proved that, for any prime number $p \geq 5$, the numerator of H_{p-1} is divisible by p^2, while Taeisinger [17, p. 3115] showed that H_n is never an integer for $n > 1$. This latter result has been generalized by Erdös [5] to sums of inverses of numbers in arithmetic progression. Also, the p-adic valuation of H_n has been studied by Boyd [3], Eswarathasan and Levine [6], Wu and Chen [19], and Sanna [15]. Moreover, harmonic numbers are special cases of *Egyptian fractions* (rational numbers that are sums of distinct unit fractions), themselves an active area of research [7, §D11].

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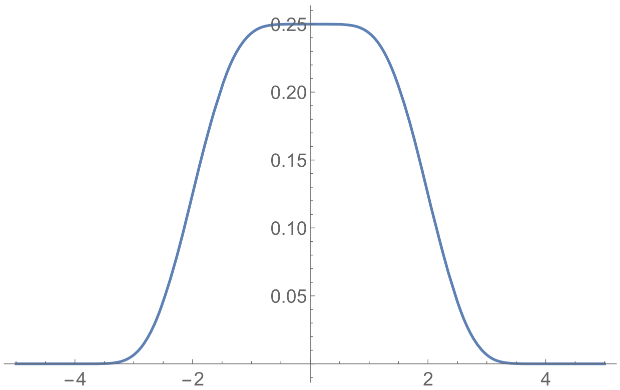

Fig. 1. The graph of the distribution function $g(x)$ of X.

Fig. 1. Le graphe de la fonction de distribution $g(x)$ de X.

It is well known that $H_n \to +\infty$ as $n \to +\infty$. More precisely,

$$H_n = \log n + \gamma + O(1/n) \tag{1.1}$$

for all positive integers n, where γ is the *Euler–Mascheroni constant*. On the other hand, the alternating-sign harmonic number

$$H'_n = 1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{(-1)^{n+1}}{n}$$

converges to $\log 2$ as $n \to +\infty$. Building on earlier work by Morrison [8,9], Schmuland [16] proved that the random harmonic series

$$X := \sum_{n=1}^{\infty} \frac{s_n}{n},$$

where $s_1, s_2, \ldots$ are independent uniformly distributed random variables in $\{-1, +1\}$, converges almost surely to a random variable with smooth density function g supported on the whole real line. Interestingly, $g(0)$ and $g(2)$ are extremely close to, but slightly smaller than, $\frac{1}{4}$ and $\frac{1}{8}$ respectively (the error being of the order of 10^{-6} and 10^{-43}, respectively). We refer to [2, p. 101] and [16] for some more information on these constants and to [4,10] for more information on the random variable X (Fig. 1).

In this paper we are interested in the set

$$\mathfrak{S}_N := \left\{ \sum_{n=1}^{N} \frac{s_n}{n} : s_1, \ldots, s_N \in \{-1, +1\} \right\}.$$

Clearly, $\mathfrak{S}_N$ is symmetric respect to the origin and

$$\max \mathfrak{S}_N = H_N \sim \log N$$

as $N \to +\infty$, by (1.1). On the other hand, the quantity

$$\mathfrak{m}_N := \min\{|\mathfrak{s}| : \mathfrak{s} \in \mathfrak{S}_N\}$$

is much more mysterious. It is not difficult to prove (see Proposition 2.6 below) that $\mathfrak{m}_N \neq 0$ for all $N \in \mathbb{N}$. In particular, estimating the least common multiple of the denominators using the Prime Number Theorem, one easily obtains the following lower bound for $\mathfrak{m}_N$,

$$\mathfrak{m}_N > \exp(-N + o(N)), \tag{1.2}$$

as $N \to +\infty$.

More generally, we shall study the function

$$\mathfrak{m}_N(\tau) := \min\{|\mathfrak{s} - \tau| : \mathfrak{s} \in \mathfrak{S}_N\}, \qquad \tau \in \mathbb{R}.$$

Using an easy argument, in Proposition 2.7 below we show that for almost every τ,

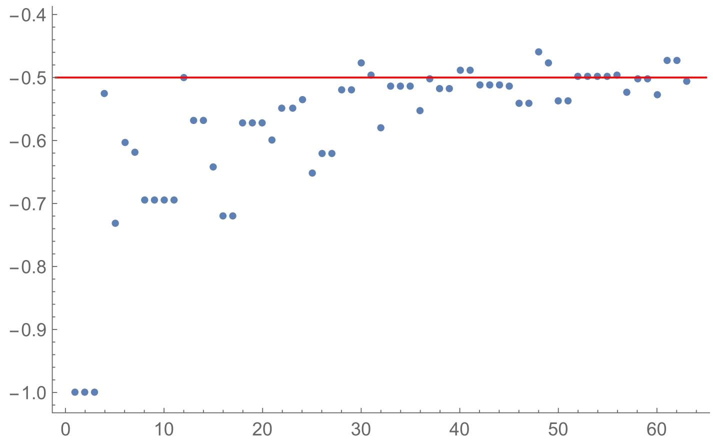

Fig. 2. The first 64 values of $\frac{\log \mathfrak{m}_N}{\log L_N}$, where $L_N = \text{lcm}\{1,\dots,N\} = e^{N+o(N)}$, plotted against the constant line $-\frac{1}{2}$.

Fig. 2. Les premières 64 valeurs de $\frac{\log \mathfrak{m}_N}{\log L_N}$, où $L_N = \text{lcm}\{1,\dots,N\} = e^{N+o(N)}$, en comparaison avec la constante $-\frac{1}{2}$.

$$\mathfrak{m}_N(\tau) > \exp(-0.665\,N + o(N)), \tag{1.3}$$

as $N \to +\infty$ (notice that $0.665 < \log 2 = 0.693\dots$). This bound holds for almost every τ, but not for all of them: in fact, $\mathfrak{m}_N(\tau)$ can be arbitrary small infinitely often. Precisely, given any $f : \mathbb{N} \to \mathbb{R}_{>0}$, we can construct $\tau_f \in \mathbb{R}$ such that $\mathfrak{m}_N(\tau_f) < f(N)$ for infinitely many N (see [1, Proposition 5.9]). The bound in (1.3) is not optimal, and some minor variations of our proof are already able to produce some small improvement.

In this paper, we are mainly interested in the opposite direction where the upper bound for $\mathfrak{m}_N(\tau)$ is sought. Our main result is the following.

Theorem 1.1. *For every $\tau \in \mathbb{R}$ and for any positive constant C less than $1/\log 4$, we have*

$$\mathfrak{m}_N(\tau) < \exp\Big(-C(\log N)^2\Big), \tag{1.4}$$

for all sufficiently large N, depending on C and τ.

Notice that a sequence of signs $s_1,\dots,s_N$ realizing the minimum in the definition of $\mathfrak{m}_N(\tau)$ does not come from a "universal" infinite sequence $(s_n)_{n\ge 1}$ such that, setting $\sigma_N := \sum_{n=1}^N s_n/n$, we have $\mathfrak{m}_N(\tau) = |\sigma_N - \tau|$ for all N. Indeed, $|\sigma_N - \sigma_{N-1}| = 1/N$ and so σ_N and σ_{N-1} cannot both be less than $1/(2N)$ away from τ.

The upper and lower bounds given in the inequalities (1.2) and (1.4) are quite distant and thus they do not indicate clearly what is the real size of $\mathfrak{m}_N$. A heuristic argument suggests that the inequality $\mathfrak{m}_N > \exp\big(-\frac{1}{2}N + o(N)\big)$ is satisfied for infinitely many N, and numerical computations (cf. Fig. 2) might suggest that actually $\mathfrak{m}_N = \exp\big(-\frac{1}{2}N + o(N)\big)$. However, because of the exponential nature of the problem, we were able to compute only the first 64 values of $\mathfrak{m}_N$, which are clearly not enough to draw a solid conclusion. We shall give these values of $\mathfrak{m}_N$ in the appendix. Despite the limited amount on data at disposal, some interesting observations can be drawn from them. For example, $\mathfrak{m}_N$ is not a decreasing function of N and there are several repeated values. One can then perhaps expect that there are infinitely many values of N such that $\mathfrak{m}_N = \mathfrak{m}_{N+1}$ or even such that $\mathfrak{m}_N = \cdots = \mathfrak{m}_{N+k}$ for any fixed $k \in \mathbb{N}$.

We prove Theorem 1.1 using a probabilistic argument. More precisely, in Theorem 2.1 below we shall prove a small-scale distribution result for $X_N := \sum_{n=1}^N \frac{s_n}{n}$, where $s_1,\dots,s_N$ are independently uniformly distributed random variables in $\{-1,+1\}$. Theorem 1.1 will follow immediately from this result (cf. Corollary 2.3). Interestingly, this distribution problem for X_N will lead us to another classical number theoretic problem: that of bounding a short average of the number of divisors in a prescribed small interval. We will attack this problem in two different ways, first using Rankin's trick together with a bound for the divisor function $\sigma_s(n)$ proved in Ramanujan's lost notebook [12], and then using a more complicated arithmetic construction. Surprisingly, the two methods both lead to the same bound (1.4), albeit with different constants.

While the probabilistic approach has the advantage of showing the existence of several N-tuples of signs $s_1,\dots,s_N$ giving small values for $|\sigma_N - \tau|$, this approach does not produce any explicit instance of these N-tuples. If one is interested in exhibiting explicit sequences, then one can construct some special signed harmonic series converging to τ and estimate the absolute value of their partial sums. A natural candidate is the "greedy" sequence obtained by setting $s_{N+1} := +1$ if $\sigma_N \le \tau$, and $s_{N+1} := -1$ otherwise. It is clear that σ_N converges to τ, since at each step one chooses the sign that makes σ_N closer to τ and, more precisely, one has $|\sigma_N - \tau| \le 1/N$ for all N large enough (depending on τ). On the other hand, as observed above, σ_N cannot be always very close to τ and in fact the inequality $|\sigma_N - \tau| \ge 1/(N+1)$ is satisfied infinitely often. However, it is still possible to prove that, for any $A > 0$, one has $|\sigma_N - \tau| \ll_A N^{-A}$ for infinitely many positive integers N. In fact, we can show that, for almost all τ, one has

$$\liminf_{n\to+\infty} \frac{\log|\sigma_N - \tau|}{(\log N)^2} = -\frac{1}{\log 4}.$$

It is quite remarkable that this "greedy" algorithm and the probabilistic method developed in this paper both give a decay rate of $\exp\big(-\big(\frac{1}{\log 4} + o(1)\big)(\log N)^2\big)$. The study of this "greedy" sequences needs completely different tools from those employed here, thus we leave its study to another paper [1].

1.1. Notation

We employ the Landau–Bachmann "Big Oh" and "little oh" notations O and o, as well as the associated Vinogradov symbols $\ll$ and $\gg$, with their usual meanings. Any dependence of the implied constants is explicitly stated or indicated with subscripts. As usual, we write $\mathbb{E}[X]$ for the expected valued of a random variable X, and $\mathbb{P}[E]$ for the probability of an event E. Also, we indicate with $\mathcal{C}_c(\mathbb{R})$ the space of continuous functions with compact support on $\mathbb{R}$ and with $\mathcal{C}_c^\infty(\mathbb{R})$ the subspace of $\mathcal{C}_c(\mathbb{R})$ consisting of smooth functions. Finally, for each $\Phi \in \mathcal{C}_c(\mathbb{R})$, we let $\widehat{\Phi}$ denote its Fourier transform, here defined by

$$\widehat{\Phi}(x) := \int_{\mathbb{R}} \Phi(y)\mathrm{e}^{-2\pi \mathrm{i} xy}\,\mathrm{d}y$$

for all $x \in \mathbb{R}$.

2. The small-scale distribution of X_N

We start with stating our result on the small-scale distribution of X_N. We remind that X_N is the random variable defined by $X_N := \sum_{n=1}^N s_n/n$, where s_n are taken uniformly and independently at random in $\{-1, +1\}$.

Theorem 2.1. *Let C be any positive constant less than $1/\log 4$. Then, for all intervals $I \subseteq \mathbb{R}$ of length $|I| > \exp\big(-C(\log N)^2\big)$, one has*

$$\mathbb{P}[X_N \in I] = \int_I g(x)\,\mathrm{d}x + o(|I|),$$

as $N \to \infty$, where

$$g(x) := 2\int_0^{+\infty} \cos(2\pi ux) \prod_{n=1}^{\infty} \cos(2\pi u/n)\,\mathrm{d}u.$$

Remark 2.2. As shown by Schmuland [16], $g(x)$ is a smooth strictly positive function which is $O_A\big(x^{-A}\big)$ as $x \to \pm\infty$, for any $A > 0$.

Corollary 2.3. *Let C be any positive constant less than $1/\log 4$. Then, for all $\tau \in \mathbb{R}$, one has*

$$\#\left\{(s_1, \ldots, s_N) \in \{-1, +1\}^N : \left|\tau - \sum_{n=1}^N \frac{s_n}{n}\right| < \delta\right\} \sim 2^{N+1} g(\tau)\delta(1 + o_{C,\tau}(1))$$

as $N \to \infty$ and $\delta \to 0$, uniformly in $\delta \geq \exp\big(-C(\log N)^2\big)$. In particular, for all large enough N, one has $\mathfrak{m}_N(\tau) < \exp\big(-C(\log N)^2\big)$.

Proof. The result follows immediately from Theorem 2.1 and Remark 2.2. □

We now proceed to proving Theorem 2.1. For each $N \in \mathbb{N} \cup \{\infty\}$ and for any real number x, define the product

$$\varrho_N(x) := \prod_{n=1}^N \cos\Big(\frac{\pi x}{n}\Big)$$

and let $\varrho(x) := \varrho_\infty(x)$ (Fig. 3).

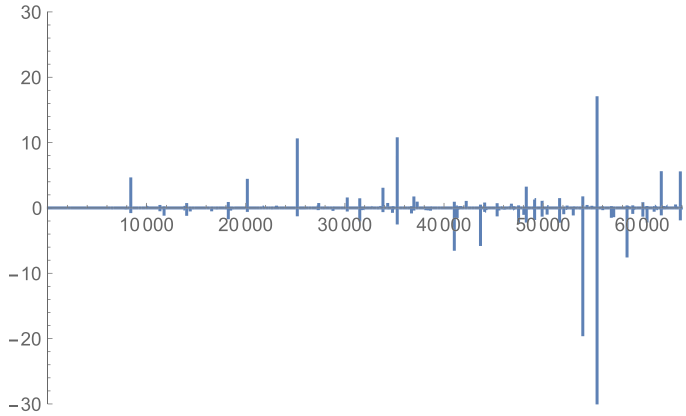

Fig. 3. Plot of $\varrho_N(x)/x$ for $N = 40$ and $x \in [0, N^3]$.

Fig. 3. Graphe de $\varrho_N(x)/x$ pour $N = 40$ et $x \in [0, N^3]$.

Lemma 2.4. *We have*

$$\mathbb{E}[\Phi(X_N)] = \int_{\mathbb{R}} \widehat{\Phi}(x)\varrho_N(2x)\,\mathrm{d}x$$

for all $\Phi \in \mathcal{C}_c^1(\mathbb{R})$.

Proof. By the definition of the expected value and by using the inverse Fourier transform, we get

$$\begin{aligned}\mathbb{E}[\Phi(X_N)] &= \frac{1}{2^N}\sum_{s_1,\ldots,s_N \in \{-1,+1\}} \Phi\left(\sum_{n=1}^{N}\frac{s_n}{n}\right) \\ &= \frac{1}{2^N}\sum_{s_1,\ldots,s_N \in \{-1,+1\}} \int_{\mathbb{R}} \widehat{\Phi}(x)\exp\left(2\pi i x\sum_{n=1}^{N}\frac{s_n}{n}\right)\mathrm{d}x \\ &= \frac{1}{2^N}\int_{\mathbb{R}} \widehat{\Phi}(x)\sum_{s_1,\ldots,s_N \in \{-1,+1\}} \exp\left(2\pi i x\sum_{n=1}^{N}\frac{s_n}{n}\right)\mathrm{d}x \\ &= \int_{\mathbb{R}} \widehat{\Phi}(x)\varrho_N(2x)\,\mathrm{d}x,\end{aligned}$$

as desired. □

In the following lemma, whose proof we postpone to Section 3, we collect some results on ϱ_N.

Lemma 2.5. *For all* $N \in \mathbb{N}$ *and* $x \in \big[0, \sqrt{N}\big]$, *we have*

$$\varrho_N(x) = \varrho(x)(1 + O(x^2/N)). \tag{2.1}$$

Moreover, there exist absolute constants $B, C, E > 0$ *such that*

$$|\varrho_N(x)| < \exp\Big(-B\exp\Big(E\sqrt{\log x}\Big)\Big) \tag{2.2}$$

for all sufficiently large positive integers N and for all $x \in \big[1, \exp\big(C(\log N)^2\big)\big]$. *In particular, C can be taken as any positive real number less than* $1/\log 4$.

We are now in a position to prove Theorem 2.1. Let C be any positive constant less than $1/\log 4$. Pick any small $\varepsilon > 0$, and set $\xi_{N,\pm\varepsilon} := \exp\big(-(1 \pm \varepsilon)C(\log N)^2\big)$ and $\xi_N := \xi_{N,0}$. For an interval $I = [a, b]$ with $b - a > 2\xi_N$, let $\Phi^+_{N,\varepsilon}$ (respectively $\Phi^-_{N,\varepsilon}$) be a smooth function $\mathbb{R} \to [0, 1]$ with support contained in $[a - \xi_{N,-\varepsilon}, b + \xi_{N,-\varepsilon}]$ (resp. I) and with $\Phi^+_{N,\varepsilon}(x) = 1$ if $x \in I$, (resp. $x \in [a + \xi_{N,-\varepsilon}, b - \xi_{N,-\varepsilon}]$). Also, suppose $\Phi^{\pm(j)}_{N,\varepsilon}(x) \ll_j (\xi_{N,-\varepsilon})^{-j}$ for all $j \ge 0$. It is not difficult to construct such functions. It follows that the Fourier transform of $\Phi^\pm_{N,\varepsilon}$ satisfies

$$\widehat{\Phi}^{\pm}_{N,\varepsilon} \ll_A (1+|x|\xi_{N,-\varepsilon})^{-A} \tag{2.3}$$

for all $A > 0$ and all $x \in \mathbb{R}$. Since

$$\mathbb{E}[\Phi^-_{N,\varepsilon}(X_N)] \le \mathbb{P}[X_N \in I] \le \mathbb{E}[\Phi^+_{N,\varepsilon}(X_N)],$$

it suffices to show

$$\mathbb{E}[\Phi^{\pm}_{N,\varepsilon}(X_N)] = \int_{\mathbb{R}} \Phi^{\pm}_{N,\varepsilon}(x) g(x)\,\mathrm{d}x + o_\varepsilon(|I|),$$

because this quantity is evidently equal to $\int_I g(x)\,\mathrm{d}x + o(|I|)$. From now on, let $\Phi_{N,\varepsilon}$ be one of $\Phi^+_{N,\varepsilon}$, $\Phi^-_{N,\varepsilon}$. By Lemma 2.4, we have

$$\mathbb{E}[\Phi_{N,\varepsilon}(X_N)] = \frac{1}{2}\int_{\mathbb{R}} \widehat{\Phi_{N,\varepsilon}}(x/2)\varrho_N(x)\,\mathrm{d}x = I_1 + I_2 + I_3,$$

where I_1, I_2, I_3 are the integral supported in $|x| < N^\varepsilon$, $|x| \in [N^\varepsilon, M_N^{1+\varepsilon}]$, and $|x| > M_N^{1+\varepsilon}$, respectively, where $M_N := \xi^{-1}_{N,-\epsilon}$. Note that $M_N^{1+\varepsilon} > N^\varepsilon$ for N large enough, that $M_N^{1+\varepsilon} = \xi_N^{-(1-\varepsilon^2)} < \xi_N^{-1}$, and that $M_N^{1+\varepsilon}\xi_{N,-\varepsilon} = \xi^{-\varepsilon}_{N,-\varepsilon} = \xi_N^{-\varepsilon(1-\varepsilon)}$ goes to infinity when N goes to infinity.

Offner [10] showed that $\varrho(x)$ decays double exponentially. In particular, using also (2.1), we have

$$\begin{aligned} I_1 &= \frac{1}{2}\int_{-N^\varepsilon}^{N^\varepsilon} \widehat{\Phi_{N,\varepsilon}}(x/2)\varrho_N(x)\,\mathrm{d}x = \frac{1}{2}\int_{-N^\varepsilon}^{N^\varepsilon} \widehat{\Phi_{N,\varepsilon}}(x/2)\varrho(x)\,\mathrm{d}x + O\left(\|\widehat{\Phi_{N,\varepsilon}}\|_\infty N^{-1+3\varepsilon}\right) \\ &= \int_{\mathbb{R}} \widehat{\Phi_{N,\varepsilon}}(x)\varrho(2x)\,\mathrm{d}x + O_\varepsilon\left(\|\Phi_{N,\varepsilon}\|_1 N^{-1+3\varepsilon}\right). \end{aligned}$$

By (2.2) if N is sufficiently large, we have

$$\begin{aligned} |I_2| &\le \|\widehat{\Phi_{N,\varepsilon}}\|_\infty \int_{N^\varepsilon}^{M_N^{1+\varepsilon}} |\varrho_N(x)|\,\mathrm{d}x \\ &\le \|\Phi_{N,\varepsilon}\|_1 \int_{N^\varepsilon}^{+\infty} \exp\left(-B\exp\left(E\sqrt{\log x}\right)\right)\,\mathrm{d}x \ll_\varepsilon \|\Phi_{N,\varepsilon}\|_1 N^{-1}. \end{aligned}$$

Now, by (2.3) we easily have

$$\begin{aligned} |I_3| &\le \int_{|x|>M_N^{1+\varepsilon}} |\widehat{\Phi_{N,\varepsilon}}(x)|\,\mathrm{d}x \ll_A \int_{M_N^{1+\varepsilon}}^{+\infty} (1+x\xi_{N,-\varepsilon})^{-A}\,\mathrm{d}x \\ &\ll_A (1+M_N^{1+\varepsilon}\xi_{N,-\varepsilon})^{1-A} \ll_A \xi_N^{\varepsilon(1-\varepsilon)(A-1)} = o_\varepsilon(\xi_N) = o_\varepsilon(|I|), \end{aligned}$$

where, in the last steps, we have chosen $A = 1 + 2/\varepsilon$. Thus, collecting the above results,

$$\begin{aligned} \mathbb{E}[\Phi_{N,\varepsilon}(X_N)] &= \int_{\mathbb{R}} \widehat{\Phi_{N,\varepsilon}}(x)\varrho(2x)\,\mathrm{d}x + O_\varepsilon(\|\Phi_{N,\varepsilon}\|_1 N^{-1+3\varepsilon}) + o_\varepsilon(|I|) \\ &= \int_{\mathbb{R}} \Phi_{N,\varepsilon}(x)g(x)\,\mathrm{d}x + O_\varepsilon(\|\Phi_{N,\varepsilon}\|_1 N^{-1+3\varepsilon}) + o_\varepsilon(|I|), \end{aligned}$$

by Parseval's theorem, and the proof of Theorem 2.1 is completed, because $\|\Phi_{N,\varepsilon}\|_1 = O_\varepsilon(|I|)$.

We conclude the section with the following propositions, which prove the bounds (1.2) and (1.3).

Proposition 2.6. *We have $\mathfrak{m}_N \neq 0$ for each positive integer N. Moreover, as $N \to \infty$,*

$$\mathfrak{m}_N > \exp(-N + o(N)). \tag{2.4}$$

Proof. For each positive integer N, define $L_N := \operatorname{lcm}\{1, \dots, N\}$. Let k be the unique nonnegative integer such that $2^k \le N < 2^{k+1}$. Then, for all $n \in \{1, \dots, N\}$, we have that L_N/n is an integer that is odd if and only if $n = 2^k$. As a consequence, for all $s_1, \dots, s_N \in \{-1, +1\}$, we have that

$$\sum_{n=1}^{N} \frac{L_N}{n} s_n$$

is an odd integer and, in particular, the sum $\sigma_N := \sum_{n=1}^{N} s_n/n$ is nonzero, so that $\mathfrak{m}_N > 0$. Furthermore, $|\sigma_N| \ge 1/L_N$. Thanks to the Prime Number Theorem, we have

$$L_N = \exp(\psi(N)) = \exp(N + o(N))$$

as $N \to +\infty$, where ψ is Chebyshev's function, and (2.4) follows. □

Proposition 2.7. *For almost all $\tau \in \mathbb{R}$, as $N \to +\infty$ we have*

$$\mathfrak{m}_N(\tau) > \exp(-0.665N)\,.$$

Proof. The claim follows by the Borel–Cantelli lemma: suppose we have an upper bound $\#\mathfrak{S}_N \le \mathrm{e}^{\alpha N}$ for some $\alpha > 0$, for all large enough N. Then, for any fixed $\varepsilon > 0$,

$$\begin{aligned}\mathcal{E} &:= \{\tau \in \mathbb{R} : \mathfrak{m}_N(\tau) \le \mathrm{e}^{-(\alpha+\varepsilon)N} \text{ for infinitely many } N\} \\ &= \bigcap_{M=1}^{\infty} \bigcup_{N \ge M} \{\tau \in \mathbb{R} : \mathfrak{m}_N(\tau) \le \mathrm{e}^{-(\alpha+\varepsilon)N}\}.\end{aligned}$$

The Lebesgue measure of $\mathcal{E}$ is bounded by

$$|\mathcal{E}| \le \inf_M \sum_{N \ge M} 2\mathrm{e}^{-(\alpha+\varepsilon)N} \#\mathfrak{S}_N \le \inf_M \sum_{N \ge M} 2\mathrm{e}^{-\varepsilon N} = \inf_M \frac{2\mathrm{e}^{-\varepsilon M}}{1 - \mathrm{e}^{-\varepsilon}} = 0.$$

This implies that, for almost every τ, the lower bound $\mathfrak{m}_N(\tau) > \mathrm{e}^{-(\alpha+\varepsilon)N}$ holds for all N large enough. The upper bound for $\#\mathfrak{S}_N$ with $\alpha = \log 2$ is trivial, since $\#\mathfrak{S}_N \le 2^N$. The claim will follow from a slightly better estimation for this quantity. In fact, the sum

$$\frac{s_1}{1} + \frac{s_2}{2} + \frac{s_3}{3} + \frac{s_4}{4} + \frac{s_6}{6} + \frac{s_{12}}{12}$$

takes only 29 different values when $s_j \in \{\pm 1\}$. Thus, let

$$F := \{\,\{k, 2k, 3k, 4k, 6k, 12k\} : k \in D\}$$

with

$$D := \{2^{3a}3^{2b}m : m \ge 1,\ a, b \ge 0,\ 2, 3 \nmid m\}.$$

With this choice for D any natural number n can be contained in at most one 6-tuple. Indeed, the numbers in F associated with a given $k = 2^{3a}3^{2b}m$ are

$$\{2^{3a}3^{2b}m,\ 2^{3a+1}3^{2b}m,\ 2^{3a}3^{2b+1}m,\ 2^{3a+2}3^{2b}m,\ 2^{3a+1}3^{2b+1}m,\ 2^{3a+2}3^{2b+1}m\} \qquad 2, 3 \nmid m$$

and, comparing the evaluations in 2 and 3, we see that no number of this family can be produced twice. The cardinality of the union of all 6-tuples in F containing numbers $\le N$ is 6 times the number of $k \in D$ which are $\le N/12$. The number of such k can be easily seen to be

$$\frac{1 + o(1)}{(1 - 2^{-3})(1 - 3^{-2})} \frac{\varphi(6)}{6} \frac{N}{12} = \Big(\frac{1}{28} + o(1)\Big)N.$$

As said, any 6-tuple gives rise to only 29 different values, not 64, thus the inequality $\#\mathfrak{S}_N \le \mathrm{e}^{\alpha N}$ holds for any

$$\alpha > \Big(1 - \frac{6}{28}\Big)\log 2 + \frac{1}{28}\log 29 = 0.6648\dots,$$

and the result follows. □

3. The bounds for ϱ and ϱ_N

In this section, we prove Lemma 2.5. We observe that, for $0 \le x \le \sqrt{N}$, we have

$$\prod_{n=N+1}^{\infty} \cos(\pi x/n) = \prod_{n=N+1}^{\infty} \left(1 + O\left((x/n)^2\right)\right) = \exp(O(x^2/N)) = 1 + O(x^2/N),$$

which proves (2.1).

We now move to the proof of (2.2). We remark that it is sufficient to prove such inequality for $x \in \left[N, \exp\left(C(\log N)^2\right)\right]$; indeed, one can reduce to this case also when for $x < N$ since $|\varrho_N(x)| \le |\varrho_{\lfloor x \rfloor}(x)|$.

For positive integers k, N and for real $\delta, x \ge 0$, define

$$\mathcal{S}_k(N, \delta, x) := \left\{n \in \{1, \ldots, N\} : \|x/n^k\| \ge \delta\right\}, \tag{3.1}$$

where $\|y\|$ denotes the distance of $y \in \mathbb{R}$ from its nearest integer. By the following lemma, the set $\mathcal{S}_1(N, \delta, x)$ plays a crucial role in the proof of (2.2).

Lemma 3.1. *We have*

$$|\varrho_N(x)| \le \exp\left(-\frac{\pi^2 \delta^2}{2} \cdot \#\mathcal{S}_1(N, \delta, x)\right)$$

for each positive integers N and for all $x, \delta \ge 0$.

Proof. The claim follows easily from the inequality

$$|\cos(\pi x)| \le \exp\left(-\frac{\pi^2 \|x\|^2}{2}\right),$$

holding for all $x \in \mathbb{R}$, and from the definitions of $\varrho_N(x)$ and $\mathcal{S}_1(N, \delta, x)$. □

In the next two subsections, we will prove a bound for ϱ_N by giving two lower bounds for $\mathcal{S}_1(N, \delta, x)$ for some suitable values of δ. More precisely, in Section 3.2 we will complete the proof of Lemma 2.5, showing that (2.2) holds for all $x \in \left[N, \exp\left(C(\log N)^2\right)\right]$. However, before doing this, in the next subsection we give a simpler argument proving that, in the range $x \in \left[N, \exp\left(C'(\log N)^2\right)\right]$, one has $|\varrho_N(x)| < 1/x^2$. We remark that this weaker inequality would still be sufficient for our application for Theorems 1.1 and 2.1. If optimized, this argument would lead to the constant $C' = (4e)^{-2} + o(1)$.

3.1. A short average of the number of divisors in a prescribed interval

In this subsection we prove the following proposition.

Proposition 3.2. *There exists $C' > 0$ such that $|\varrho_N(x)| < 1/x^2$ for all sufficiently large positive integers N and for all $x \in \left[N, \exp\left(C'(\log N)^2\right)\right]$.*

We start with the following lemma, which shows that the size of $\mathcal{S}_1(N, \delta, x)$ is strictly related to the size of a certain divisor sum.

Lemma 3.3. *For any $0 < \delta < \frac{1}{2}$, $x \in \mathbb{R}$ and $N \in \mathbb{N}$ we have*

$$\frac{N}{2} - \sum_{x-\delta N < m < x+\delta N} \sum_{\substack{n \mid m \\ N/2 \le n \le N}} 1 < \#\mathcal{S}_1(N, \delta, x) < N - \sum_{x-\frac{\delta}{2}N < m < x+\frac{\delta}{2}N} \sum_{\substack{n \mid m \\ N/2 \le n \le N}} 1.$$

Proof. First we observe that

$$\tfrac{N}{2} - \#\left\{n \in \mathbb{Z} \cap [\tfrac{N}{2}, N] : \|\tfrac{x}{n}\| < \delta\right\} < \#\mathcal{S}_1(N, \delta, x) < N - \#\left\{n \in \mathbb{Z} \cap [\tfrac{N}{2}, N] : \|\tfrac{x}{n}\| < \delta\right\}.$$

Now,

$$\begin{aligned}\#\{n\in\mathbb{Z}\cap[\tfrac{N}{2},N]:\|\tfrac{x}{n}\|<\delta\} &= \#\{n\in\mathbb{Z}\cap[\tfrac{N}{2},N]:\exists\ell\in\mathbb{Z}\ \ell-\delta<x/n<\ell+\delta\}\\ &= \#\{n\in\mathbb{Z}\cap[\tfrac{N}{2},N]:\exists\ell\in\mathbb{Z}\ x-\delta n<\ell n<x+\delta n\}\\ &< \#\{n\in\mathbb{Z}\cap[\tfrac{N}{2},N]:\exists\ell\in\mathbb{Z}\ x-\delta N<\ell n<x+\delta N\}\\ &= \sum_{x-\delta N<m<x+\delta N}\ \sum_{\substack{n\,|\,m\\ N/2\le n\le N}} 1\end{aligned}$$

and the lower bound for $\#\mathcal{S}_1(N,\delta,x)$ follows. Similarly one obtains the upper bound. □

We take $\delta=\frac{4\sqrt{\log x}}{\pi}N^{-\frac12}$ and assume $x\in[N,\mathrm{e}^{N/8}]$ so that $0<\delta<\frac12$ and $\delta N<2x$. In particular, by Lemmas 3.1 and 3.3 we obtain $|\varrho_N(x)|<1/x^2$ whenever the inequality

$$D(x,N)<N/4 \tag{3.2}$$

is satisfied, where

$$D(x,N):=\sum_{x-\frac{4}{\pi}\sqrt{N\log x}<m<x+\frac{4}{\pi}\sqrt{N\log x}}\ \sum_{\substack{n\,|\,m\\ N/2\le n\le N}} 1.$$

Now, we take $w\in(0,\frac12)$ and use Rankin's trick to bound the inner sum:

$$\begin{aligned}D(x,N) &< \frac{9}{\pi}\sqrt{N\log x}\cdot\max_{m\le 2x}\sum_{\substack{n\,|\,m\\ N/2\le n\le N}}1 \le \frac{9}{\pi}\sqrt{N\log x}\cdot\max_{m\le 2x}\sum_{\substack{n\,|\,m\\ N/2\le n\le N}}\left(\frac{N}{n}\right)^w\\ &< \frac{9}{\pi}N^{\frac12+w}\sqrt{\log x}\cdot\max_{m\le 2x}\sigma_{-w}(m),\end{aligned} \tag{3.3}$$

where, for any $s\in\mathbb{R}$, $\sigma_s(m)$ is defined as the sum of the s-th powers of the divisors of m. In his lost notebook [12], Ramanujan studied the large values of $\sigma_{-s}(n)$ for any $s\in[0,1]$. We state his result in a slightly weaker form in the following Lemma.

Lemma 3.4. *For each fixed $\varepsilon>0$ there exists $C_1>0$ such that*

$$\sigma_{-s}(m)<\exp\left(C_1\frac{(\log m)^{1-s}}{\log\log m}\right),$$

for all integers $m\ge 3$ and for all $s\in[\varepsilon,1-\varepsilon]$.

Proof. This is a consequence of [12, pp. 380–382] (see the remark before (383) on how to make the inequalities unconditional). See also [14, Ch. 3, §3, 1b]. □

Applying the bound given in this lemma in (3.3), we obtain

$$D(x,N)<\frac{9}{\pi}N^{\frac12+w}\sqrt{\log x}\cdot\exp\left(C_1\frac{(\log 2x)^{1-w}}{\log\log 2x}\right)$$

for some $C_1>0$ and any $\frac14<w<\frac12$, $N\in\mathbb{N}$ and $x\in[N,\mathrm{e}^{N/8}]$. Picking $w=\frac12-1/\log\log 2x$, so that $\frac14<w<\frac12$ for sufficiently large N, this inequality becomes

$$D(x,N)<\frac{9}{\pi}N\sqrt{\log x}\cdot\exp\left(-\frac{\log N-C_1\mathrm{e}\,(\log 2x)^{\frac12}}{\log\log 2x}\right).$$

If $x<\exp\left(C'(\log N)^2\right)$, with $C':=(2C_1\mathrm{e})^{-2}$, then this is $o(N)$ and so (3.2) holds for N large enough. In particular, we obtain $|\varrho_N(x)|<1/x^2$ for $x\in[N,\exp(C'(\log N)^2)]$, and the proof of Proposition 3.2 is completed.

3.2. An arithmetic construction

Here we complete the proof of Lemma 2.5. More specifically, we show the following proposition.

Proposition 3.5. *For every positive $C < 1/\log 4$, there exists a positive constant E depending on C, such that*

$$|\varrho_N(x)| \le \exp\left(-\frac{\pi^2}{400^2}\exp\left(E\sqrt{\log x}\right)\right)$$

for all $x \in [N, \exp(C(\log N)^2)]$, for all sufficiently large N.

We start by giving a lower bound for $\#\mathcal{S}_k(N,\delta,x)$. We remind that $\mathcal{S}_k$ was defined in (3.1).

Lemma 3.6. *For all $a > 0$, $\delta \in (0, 1/2)$ and $x \in [\mathrm{e}^{ak^2}, N^k]$, we have*

$$\#\mathcal{S}_k(N,\delta,x) \ge \big((1/2-\delta)(2^{-1/k} - (3/2)\mathrm{e}^{-a}) - (2/3)^k\big)x^{1/k}$$

when k is large enough (depending on a), and $N > \mathrm{e}^{ak}$.

Proof. Let $b > 1$ be a parameter that will be chosen later. If ℓ and n are integers such that

$$1 \le \ell \le b^k - \tfrac{1}{2} \quad \text{and} \quad \left(\frac{x}{\ell+1/2}\right)^{1/k} < n \le \left(\frac{x}{\ell+\delta}\right)^{1/k},$$

then it follows easily that $n \in \mathcal{S}_k(N,\delta,x)$. As a consequence,

$$\#\mathcal{S}_k(N,\delta,x) \ge \sum_{1\le \ell \le b^k - \frac{1}{2}} \left(\frac{x}{\ell+\delta}\right)^{1/k} - b^k. \tag{3.4}$$

For $0 \le s < t \le 1$, we have the lower bounds

$$\frac{1}{(1+s)^{1/k}} - \frac{1}{(1+t)^{1/k}} = \frac{1}{k}\int_s^t \frac{\mathrm{d}y}{(1+y)^{1+1/k}} \ge \frac{t-s}{k(1+t)^{1+1/k}}.$$

Applying these inequalities in (3.4) with $s = \delta/\ell$ and $t = 1/(2\ell)$, we get

$$\#\mathcal{S}_k(N,\delta,x) \ge \left(\frac{1}{2}-\delta\right)\frac{1}{k}\sum_{1\le \ell \le b^k-\frac{1}{2}} \frac{x^{1/k}}{(\ell+1/2)^{1+1/k}} - b^k.$$

Since

$$\sum_{1\le \ell\le b^k-\frac{1}{2}} \frac{1}{(\ell+\frac{1}{2})^{1+1/k}} \ge \int_1^{b^k-\frac{1}{2}} \frac{\mathrm{d}y}{(y+1/2)^{1+1/k}} = k((3/2)^{-1/k} - b^{-1}) \ge k(2^{-1/k} - b^{-1}),$$

this bound show that

$$\#\mathcal{S}_k(N,\delta,x) \ge (1/2-\delta)\big(2^{-1/k} - b^{-1}\big)x^{1/k} - b^k.$$

From the assumption $x \ge \mathrm{e}^{ak^2}$ we get the claim setting $b := 2\mathrm{e}^a/3$. $\square$

Now we state a well-known identity (see, e.g., [13, Ch. 1, Problem 5]).

Lemma 3.7. *For all integers $m \ge 0$, the identity*

$$\sum_{j=0}^{m} (-1)^j \binom{m}{j} \frac{1}{x+j} = \frac{m!}{x(x+1)\cdots(x+m)}$$

holds in $\mathbb{Q}(x)$.

Proof. By induction on m. $\square$

The next lemma is a simple inequality that will be useful later.

Lemma 3.8. *We have*

$$0 \le \frac{1}{n^k} - \frac{1}{n(n+1)\cdots(n+k-1)} < \frac{k^2}{2n^{k+1}},$$

for all positive integers n and k.

Proof. Since $1 + x \le \mathrm{e}^x$ for all real number x, we have

$$0 \le 1 - \prod_{j=0}^{k-1}\left(1 + \frac{j}{n}\right)^{-1} \le 1 - \exp\left(-\sum_{j=0}^{k-1}\frac{j}{n}\right) < 1 - \mathrm{e}^{-k^2/(2n)} < \frac{k^2}{2n},$$

and dividing everything by n^k, we get the desired claim. $\square$

Next, using Lemmas 3.7 and 3.8, we deduce a bound for $\#\mathcal{S}_1$ from the bound for $\#\mathcal{S}_k$ given by Lemma 3.6. Next, we use the previous lemmas to deduce a bound for $\mathcal{S}_1$ from Lemma 3.6.

Lemma 3.9. *For all $\delta \in (0, 2^{-k}/20]$ and $x \in [4^{k^2}, N^k/(k-1)!]$, we have*

$$\#\mathcal{S}_1(N, \delta, x) \ge \frac{x^{1/k}}{200}$$

when k is large enough and $N \ge k4^k$.

The assumption $N \ge k4^k$ is an easy way to ensure that $4^{k^2} < N^k/(k-1)!$.

Proof. We set $\delta = d \cdot 2^{-(k+1)}$ for some d that we fix later. First, we have

$$\begin{aligned}\#\mathcal{S}_1(N, \delta, x) &\ge \frac{1}{k} \cdot \#\big\{n \in \{1, \dots, N\} : \exists j \in \{0, \dots, k-1\} \text{ with } \|x/(n+j)\| \ge \delta\big\} \\ &= \frac{1}{k} \cdot \big(N - \#\mathcal{T}_k(N, \delta, x)\big),\end{aligned} \tag{3.5}$$

where

$$\mathcal{T}_k(N, \delta, x) := \big\{n \in \{1, \dots, N\} : \|x/(n+j)\| < \delta \text{ for all } j \in \{0, \dots, k-1\}\big\}.$$

If $n \in \mathcal{T}_k(N, \delta, x)$, then for all $j \in \{0, \dots, k-1\}$ there exists an integer ℓ_j such that

$$\left|\frac{x}{n+j} - \ell_j\right| < \delta.$$

Therefore, setting

$$\ell := \sum_{j=0}^{k-1}(-1)^j\binom{k-1}{j}\ell_j$$

and using Lemma 3.7, we obtain

$$\left|\frac{x(k-1)!}{n(n+1)\cdots(n+k-1)} - \ell\right| \le \sum_{j=0}^{k-1}\binom{k-1}{j}\left|\frac{x}{n+j} - \ell_j\right| < 2^{k-1}\delta = d/4.$$

Furthermore, assuming $n \ge \eta k (x/d)^{1/(k+1)}$ for some $\eta > 0$, thanks to Lemma 3.6 we have that

$$\begin{aligned}\left|\frac{x(k-1)!}{n^k} - \ell\right| &\le \frac{d}{4} + \left|\frac{x(k-1)!}{n^k} - \frac{x(k-1)!}{n(n+1)\cdots(n+k-1)}\right| \le \frac{d}{4} + \frac{xk^2(k-1)!}{2n^{k+1}} \\ &\le \frac{d}{4} + \frac{xdk^2k^k}{2\mathrm{e}^k\eta^{k+1}k^{k+1}x} = \frac{d}{4}\left(1 + \frac{2k}{\eta(\eta \mathrm{e})^k}\right).\end{aligned}$$

Choosing $\eta > \mathrm{e}^{-1}$ this quantity becomes $\le 3d/10$ if k is large enough (depending on the choice of η). Choosing $d < 5/3$, we ensure that this quantity is strictly smaller than $1/2$. Therefore, under these hypotheses

$$\left\| \frac{x(k-1)!}{n^k} \right\| \le \frac{3d}{10}.$$

Summarizing, we have proved that for all $n \in \mathcal{T}_k(N,\delta,x)$, but at most $\eta k (x/d)^{1/(k+1)}$ exceptions, it holds $n \notin \mathcal{S}_k(N, 3d/10, x(k-1)!)$. As a consequence,

$$\#\mathcal{T}_k(N,\delta,x) - \eta k (x/d)^{1/(k+1)} \le N - \#\mathcal{S}_k(N, 3d/10, x(k-1)!).$$

Hence, recalling (3.5) and thanks to Lemma 3.6, since by hypothesis $x \ge \mathrm{e}^{ak^2}$ for some $a \ge 1$, we obtain

$$\begin{aligned}\#\mathcal{S}_1(N,\delta,x) &\ge \frac{1}{k}\#\mathcal{S}_k(N, 3d/10, x(k-1)!) - \eta (x/d)^{1/(k+1)} \\ &\ge \left(\left(\frac{1}{2} - \frac{3d}{10}\right)(2^{-1/k} - \frac{3}{2}\mathrm{e}^{-a}) - (2/3)^k\right)\frac{(k-1)!^{1/k}}{k} x^{1/k} - \eta (x/d)^{1/(k+1)}.\end{aligned}$$

Collecting $x^{1/k}$ and using the inequality $k! \ge (k/\mathrm{e})^k$, we get

$$\#\mathcal{S}_1(N,\delta,x) \ge x^{1/k}\left(\left(\left(\frac{1}{2} - \frac{3d}{10}\right)(2^{-1/k} - \frac{3}{2}\mathrm{e}^{-a}) - (2/3)^k\right)\frac{(1/k)^{1/k}}{\mathrm{e}} - \frac{\eta}{d^{1/(k+1)}x^{1/(k^2+k)}}\right),$$

and recalling the assumption $x \ge \mathrm{e}^{ak^2}$, we obtain

$$\#\mathcal{S}_1(N,\delta,x) \ge x^{1/k}\left(\left(\left(\frac{1}{2} - \frac{3d}{10}\right)(2^{-1/k} - \frac{3}{2}\mathrm{e}^{-a}) - (2/3)^k\right)\frac{(1/k)^{1/k}}{\mathrm{e}} - \frac{\eta}{d^{1/(k+1)}\mathrm{e}^{ak/(k+1)}}\right).$$

For k large enough, this quantity is positive as soon as

$$\left(\frac{1}{2} - \frac{3d}{10}\right)(1 - \frac{3}{2}\mathrm{e}^{-a}) > \frac{\eta \mathrm{e}}{\mathrm{e}^a}.$$

If η is very close to e^{-1} and d is very small, this inequality is satisfied by any a with $(1-(3/2)\mathrm{e}^{-a}) > 2\mathrm{e}^{-a}$, i.e. $a > \log(7/2)$. We set $a = \log 4$, allowing the choice $\eta = 0.4$ and $d = 0.1$, when k is large. An explicit computation shows that with these values for the parameters the lower bound is larger than $x^{1/k}/200$ as soon as k is larger than 400. □

We are now ready to prove Proposition 3.5. Let C be any positive constant, $C < 1/\log 4$, and pick any C' with $C < C' < 1/\log 4$. We take $\delta := 2^{-k}/20$, and $k := \left\lfloor \sqrt{C' \log x} \right\rfloor$ for every x in the given range. Then x is in the interval $[4^{k^2}, N^k/(k-1)!]$. In fact, the inequality $4^{k^2} \le x$ is evident, and

$$x \le \frac{N^k}{(k-1)!} \Leftarrow x \le \left(\frac{\mathrm{e}N}{k}\right)^k \iff \log x \le k \log\left(\frac{\mathrm{e}N}{k}\right).$$

Since $\sqrt{C' \log x} - 1 \le k = \left\lfloor \sqrt{C' \log x} \right\rfloor \le \sqrt{C' \log x}$, the last inequality is implied by

$$\frac{\log x}{\sqrt{C' \log x} - 1} + \log(\sqrt{\log x}) \le \log(\mathrm{e}N/\sqrt{C'}).$$

As a function of x, this can be written as

$$\sqrt{\frac{\log x}{C'}} + \log(\sqrt{\log x}) \le \log N + O_{C'}(1).$$

We are assuming that $\log N \le \log x \le C(\log N)^2$, hence this is implied by

$$\sqrt{C/C'}\log N + \log\log N \le \log N + O_{C,C'}(1)$$

which is true as soon as N is large enough. This proves that we can apply Lemma 3.9, getting

$$\begin{aligned}\delta^2 \cdot \#\mathcal{S}_1(N,\delta,x) &\ge \frac{1/200}{400} \cdot 4^{-k} x^{1/k} = \frac{2}{400^2}\exp\left(\frac{\log x}{k} - k\log 4\right) \\ &\ge \frac{2}{400^2}\exp(E\sqrt{\log x}),\end{aligned}$$

where $E := \frac{1}{\sqrt{C'}} - \sqrt{C'}\log 4$. Hence, applying Lemma 3.1, we get

$$|\varrho_N(x)| \le \exp\Big(-\frac{\pi^2\delta^2}{2}\cdot \#\mathcal{S}_1(N,\delta,x)\Big) \le \exp\Big(-\frac{\pi^2}{400^2}\exp(E\sqrt{\log x})\Big),$$

which is the claim.

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Appendix A

The time needed for the computation of $\mathfrak{m}_N$ with a direct exhaustive computation grows exponentially with N and becomes unpractical already for $N \approx 30$. Thus, for computing $\mathfrak{m}_N$ for larger N we used the following idea. Let

$$A := \left\{\sum_{n=1}^{R}\frac{s_n}{n} : s_1,\dots,s_R \in \{-1,+1\}\right\} \quad B := \left\{\sum_{n=R+1}^{N}\frac{s_n}{n} : s_{R+1},\dots,s_N \in \{-1,+1\}\right\},$$

for any intermediate parameter $R \in [1,N]$. Then $\mathfrak{m}_N = \min\{|a-b| : a \in A,\ b \in B\}$. The algorithm producing this minimal distance is very fast if one preorders the lists A and B. In this way, we were able to compute all $\mathfrak{m}_N$ with $N \le 64$; see the table below. The need for a large quantity of RAM for storing the lists prevents us from computing significantly larger values of N. For the computations, we have used PARI/GP [11].

N	$\mathfrak{m}_N L_N$	N	$\mathfrak{m}_N L_N$	N	$\mathfrak{m}_N L_N$	N	$\mathfrak{m}_N L_N$
1	1	17	97	33	902339	49	421936433719
2	1	18	97	34	7850449	50	175378178867
3	1	19	3767	35	7850449	51	8643193037
4	1	20	3767	36	7850449	52	8643193037
5	7	21	3767	37	10683197	53	461784703049
6	3	22	2285	38	68185267	54	461784703049
7	11	23	24319	39	37728713	55	461784703049
8	13	24	24319	40	37728713	56	461784703049
9	11	25	71559	41	740674333	57	514553001783
10	11	26	4261	42	740674333	58	116096731427
11	23	27	13703	43	1774907231	59	2810673355099
12	23	28	13703	44	1774907231	60	2810673355099
13	607	29	872843	45	1774907231	61	4723651835663
14	251	30	872843	46	1699239271	62	136420009515743
15	251	31	17424097	47	3103390393	63	136420009515743
16	125	32	13828799	48	3103390393	64	23093515509397

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