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Singular Hochschild cohomology via the singularity category

La cohomologie de Hochschild singulière via la catégorie des singularités

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ABSTRACT

We show that the singular Hochschild cohomology (= Tate–Hochschild cohomology) of an algebra A is isomorphic, as a graded algebra, to the Hochschild cohomology of the differential graded enhancement of the singularity category of A. The existence of such an isomorphism is suggested by recent work by Zhengfang Wang.

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RÉSUMÉ

Nous montrons que la cohomologie de Hochschild singulière (cohomologie de Tate–Hochschild) d'une algèbre A est isomorphe, en tant qu'algèbre graduée, à la cohomologie de Hochschild de l'enrichissement différentiel gradué de la catégorie des singularités de A. L'existence d'un tel isomorphisme est suggérée par des travaux récents de Zhengfang Wang.

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1. Introduction

Let k be a commutative ring. We write $\otimes$ for $\otimes_k$. Let A be a right Noetherian (non-commutative) k-algebra projective as a k-module. The *stable derived category* or *singularity category* of A is defined as the Verdier quotient

$$\mathsf{Sg}(A) = \mathcal{D}^b(\mathsf{mod}\, A)/\,\mathsf{per}(A)$$

of the bounded derived category of finitely generated (right) A-modules by the *perfect derived category* $\mathsf{per}(A)$, *i.e.* the full subcategory of complexes quasi-isomorphic to bounded complexes of finitely generated projective modules. It was introduced by Buchweitz in an unpublished manuscript [4] in 1986 and rediscovered, in its scheme-theoretic variant, by Orlov in 2003 [24]. Notice that it vanishes when A is of finite global dimension and thus measures the degree to which A is 'singular', a view confirmed by the results of [24].

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Let us suppose that the enveloping algebra $A^e = A \otimes A^{\mathrm{op}}$ is also right Noetherian. In analogy with Hochschild cohomology, in view of Buchweitz' theory, it is natural to define the *Tate–Hochschild cohomology* or *singular Hochschild cohomology* of A to be the graded algebra with components

$$HH^n_{\mathrm{sg}}(A, A) = \mathrm{Hom}_{\mathrm{Sg}(A^e)}(A, \Sigma^n A)\ ,\ n \in \mathbb{Z}\ ,$$

where Σ denotes the suspension (=shift) functor. It was studied, for example, in [10,2,23] and more recently in [29,30,28, 31,27,5]. Wang showed in [29] that, like Hochschild cohomology [11], singular Hochschild cohomology carries a structure of Gerstenhaber algebra. Now recall that the Gerstenhaber algebra structure on Hochschild cohomology is a small part of much richer higher structure on the Hochschild cochain complex $C(A, A)$ itself, namely the structure of a B_∞-algebra in the sense of Getzler–Jones [12, 5.2] given by the brace operations [1,16]. In [27], Wang improves on [29] by defining a singular Hochschild cochain complex $C_{\mathrm{sg}}(A, A)$ and endowing it with a B_∞-structure, which in particular yields the Gerstenhaber algebra structure on $HH^*_{\mathrm{sg}}(A, A)$.

Using [17], Lowen–Van den Bergh showed in [21, Theorem 4.4.1] that the Hochschild cohomology of A is isomorphic to the Hochschild cohomology of the canonical differential graded (= dg) enhancement of the (bounded or unbounded) derived category of A and that the isomorphism lifts to the B_∞-level. Together with the complete structural analogy between Hochschild and singular Hochschild cohomology described above, this suggests the question whether the singular Hochschild cohomology of A is isomorphic to the Hochschild cohomology of the canonical dg enhancement $\mathrm{Sg}_{\mathrm{dg}}(A)$ of the singularity category $\mathrm{Sg}(A)$ (note that such an enhancement exists by the construction of $\mathrm{Sg}(A)$ as a Verdier quotient [19,6]). Chen–Li–Wang show in [5] that this does hold at the level of Gerstenhaber algebras when A is the radical square zero algebra associated with a finite quiver without sources or sinks. Our main result is the following.

Theorem 1.1. *There is a canonical isomorphism of graded algebras between the singular Hochschild cohomology of A and the Hochschild cohomology of the dg singularity category* $\mathrm{Sg}_{\mathrm{dg}}(A)$.

Conjecture 1.2. *The isomorphism of the theorem lifts to an isomorphism*

$$C_{\mathrm{sg}}(A, A) \xrightarrow{\sim} C(\mathrm{Sg}_{\mathrm{dg}}(A), \mathrm{Sg}_{\mathrm{dg}}(A))$$

in the homotopy category of B_∞-algebras.

Let us mention an application of Theorem 1.1 obtained in joint work with Zheng Hua. Let k be an algebraically closed field of characteristic 0 and P the power series algebra $k[[x_1, \ldots, x_n]]$.

Theorem 1.3 *([15]). Suppose that $Q \in P$ has an isolated singularity at the origin and $A = P/(Q)$. Then A is determined up to isomorphism by its dimension and the dg singularity category* $\mathrm{Sg}_{\mathrm{dg}}(A)$.

In [8, Theorem 8.1], Efimov proves a related but different reconstruction theorem: he shows that if Q is a polynomial, it is determined, up to a formal change of variables, by the differential $\mathbb{Z}/2$-graded endomorphism algebra E of the residue field in the differential $\mathbb{Z}/2$-graded singularity category together with a fixed isomorphism between H^*B and the exterior algebra $\Lambda(k^n)$.

In section 2, we generalize Theorem 1.1 to the non-Noetherian setting and prove the generalized statement. We comment on a possible lift of this proof to the B_∞-level in section 3. We prove Theorem 1.3 in section 4.

2. Generalization and proof

2.1. Generalization to the non-Noetherian case

We assume that A is an arbitrary k-algebra projective as a k-module. Its singularity category $\mathrm{Sg}(A)$ is defined as the Verdier quotient $\mathcal{H}^{-,b}(\mathrm{proj}\,A)/\mathcal{H}^b(\mathrm{proj}\,A)$ of the homotopy category of right bounded complexes of finitely generated projective A-modules by its full subcategory of bounded complexes of finitely generated projective A-modules. Notice that when A is right-Noetherian, this is equivalent to the definition given in the introduction.

The (partially) *completed singularity category* $\widehat{\mathrm{Sg}}(A)$ is defined as the Verdier quotient of the right bounded derived category $\mathcal{D}^- A$ by its full subcategory consisting of all complexes quasi-isomorphic to bounded complexes of arbitrary projective modules.

Lemma 2.2. *The canonical functor* $\mathrm{Sg}(A) \to \widehat{\mathrm{Sg}}(A)$ *is fully faithful.*

Proof. Let M be a right-bounded complex of finitely generated projective modules with bounded homology and P a bounded complex of arbitrary projective modules. Since the components of M are finitely generated, each morphism $M \to P$ in the derived category factors through a bounded complex P' with finitely generated projective components. This yields the claim. □

Since we do not assume that A^e is Noetherian, the A-bimodule A will not, in general, belong to the singularity category $\mathsf{Sg}(A^e)$. But it always belongs to the completed singularity category $\widehat{\mathsf{Sg}}(A^e)$. We define the singular Hochschild cohomology of A to be the graded algebra with components

$$HH^n_{\mathrm{sg}}(A, A) = \mathrm{Hom}_{\widehat{\mathsf{Sg}}(A^e)}(A, \Sigma^n A)\ ,\ n \in \mathbb{Z}.$$

Theorem 2.3. *Even if A^e is non-Noetherian, there is a canonical isomorphism of graded algebras between the singular Hochschild cohomology of A and the Hochschild cohomology of the dg singularity category $\mathsf{Sg}_{\mathrm{dg}}(A)$.*

Let P be a right bounded complex of projective A^e-modules. For $q \in \mathbb{Z}$, let $\sigma_{>q}P$ and $\sigma_{\leq q}P$ denote its stupid truncations:

$$\sigma_{>q}P : \ \ldots \longrightarrow 0 \longrightarrow P^{q+1} \longrightarrow P^{q+1} \longrightarrow \ldots$$
$$\sigma_{\leq q}P : \ \ldots \longrightarrow P^{q-1} \longrightarrow P^{q} \longrightarrow 0 \longrightarrow \ldots$$

so that we have a triangle

$$\sigma_{>q}P \longrightarrow P \longrightarrow \sigma_{\leq q}P \longrightarrow \Sigma\sigma_{>q}P.$$

We have a direct system

$$P = \sigma_{\leq 0}P \longrightarrow \sigma_{\leq -1}P \longrightarrow \sigma_{\leq -2}P \longrightarrow \ldots \longrightarrow P_{\leq q} \longrightarrow \ldots.$$

Lemma 2.4. *Let $L \in \mathcal{D}^-(A^e)$. We have a canonical isomorphism*

$$\mathrm{colim}\,\mathrm{Hom}_{\mathcal{D}A^e}(L, \sigma_{\leq q}P) \xrightarrow{\sim} \mathrm{Hom}_{\widehat{\mathsf{Sg}}(A^e)}(L, P)\ ,\ n \in \mathbb{Z}.$$

In particular, if P is a projective resolution of A over A^e, we have

$$\mathrm{colim}\,\mathrm{Hom}_{\mathcal{D}A^e}(A, \Sigma^n\sigma_{\leq q}P) \xrightarrow{\sim} \mathrm{Hom}_{\widehat{\mathsf{Sg}}(A^e)}(A, \Sigma^n A)\ ,\ n \in \mathbb{Z}.$$

Proof. Clearly, if Q is a bounded complex of projective modules, each morphism $Q \to P$ in the derived category $\mathcal{D}A^e$ factors through $\sigma_{>q}P \to P$ for some $q \ll 0$. This shows that the morphisms $P \to \sigma_{\leq q}P$ form a cofinal subcategory in the category of morphisms $P \to P'$, whose cylinder is a bounded complex of projective modules. Whence the claim. □

2.5. Proof of Theorem 2.3

We refer to [18,20,25] for foundational material on dg categories. We will follow the terminology of [20]. Let $\mathcal{M} = \mathcal{C}^{-,b}_{\mathrm{dg}}(\mathrm{proj}\,A)$ denote the dg category of right-bounded complexes of finitely generated projective A-modules with bounded homology. Let $\mathcal{S}$ denote the dg quotient of $\mathcal{M}$ by its full dg subcategory $\mathcal{P} = \mathcal{C}^{b}_{\mathrm{dg}}(\mathrm{proj}\,A)$ of bounded complexes of finitely generated projective A-modules. In the homotopy category of dg categories, we have an isomorphism between $\mathsf{Sg}_{\mathrm{dg}}(A)$ and $\mathcal{S} = \mathcal{M}/\mathcal{P}$. We view A as a dg category with one object whose endomorphism algebra is A. We have the obvious inclusion and projection dg functors

$$A \xrightarrow{\ i\ } \mathcal{M} \xrightarrow{\ p\ } \mathcal{S}.$$

For a dg category $\mathcal{A}$, denote by $\mathcal{D}\mathcal{A}$ its derived category, by $\mathcal{A}^e$ the enveloping dg category $\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}$ and by $I_{\mathcal{A}}$ the *identity bimodule*

$$I_A : (A, B) \mapsto \mathcal{A}(A, B).$$

In the case of the algebra A, the identity bimodule is the A-bimodule A. The dg functors i and p induce dg functors in the enveloping dg categories, which we will denote by the same symbols

$$A^e \xrightarrow{\ i\ } \mathcal{M}^e \xrightarrow{\ p\ } \mathcal{S}^e.$$

The restriction along i has the fully faithful left adjoint $i^* : \mathcal{D}A^e \to \mathcal{D}\mathcal{M}^e$. We claim that it takes the identity bimodule A to $I_{\mathcal{M}}$. For this, we use the bar resolution of A as a bimodule

$$\ldots \longrightarrow A \otimes A^{\otimes p} \otimes A \longrightarrow \ldots \longrightarrow A \otimes A \otimes A \longrightarrow A \otimes A.$$

Its image under i^* is the sum total dg module of the complex

$$\dots \longrightarrow \mathcal{M}(A,-)\otimes A\otimes \mathcal{M}(?,A) \longrightarrow \mathcal{M}(A,-)\otimes \mathcal{M}(A,?)$$

with pth term $\mathcal{M}(A,-)\otimes A^{\otimes p}\otimes \mathcal{M}(?,A)$. We have to show that the sum total dg module of the augmented complex

$$\dots \longrightarrow \mathcal{M}(A,-)\otimes A\otimes \mathcal{M}(?,A) \longrightarrow \mathcal{M}(A,-)\otimes \mathcal{M}(?,A) \longrightarrow \mathcal{M}(?,-) \longrightarrow 0$$

is acyclic. Denote this augmented complex by $C(?,-)$. Let P and Q in $\mathcal{M}$ be given. We have to show that $C(P,Q)$ is acyclic, *i.e.* that the sum total dg module of

$$\dots \longrightarrow Q\otimes A^{\otimes p}\otimes \mathcal{M}(P,A) \longrightarrow \dots \longrightarrow Q\otimes \mathcal{M}(P,A) \longrightarrow \mathcal{M}(P,Q) \longrightarrow 0$$

is acyclic. This is clear if $P=A$, since then $C(Q,A)$ is just the bar resolution of the right dg module Q. Thus, it also holds if P is a bounded complex of finitely generated projective modules. In the general case, we consider the filtration of P by the stupid truncations $\sigma_{\geq q}P$, $q\leq 0$. Clearly, $C(P,Q)$ is the inverse limit of the acyclic complexes $C(\sigma_{\geq q}P,Q)$ and the transition maps in this inverse system are componentwise surjective. It follows that $C(P,Q)$ is acyclic as was to be shown.

Now fix a projective resolution P of A as a bimodule. Denote by $\sigma_{\leq q}P$ its stupid truncations, $q\leq 0$. We have a direct system

$$P \longrightarrow \sigma_{\leq -1}P \longrightarrow \sigma_{\leq -2}P \longrightarrow \dots \longrightarrow P_{\leq q} \longrightarrow \dots .$$

By Lemma 2.4, we have a canonical isomorphism

$$\operatorname{colim}\operatorname{Hom}_{\mathcal{D}A^e}(A,\Sigma^n\sigma_{\leq q}P) \xrightarrow{\sim} \operatorname{Hom}_{\widehat{\mathsf{Sg}}(A^e)}(A,\Sigma^n A).$$

Since i^* is fully faithful and $i^*(A)=I_{\mathcal{M}}$, we have

$$\operatorname{colim}\operatorname{Hom}_{\mathcal{D}A^e}(A,\Sigma^n\sigma_{\leq q}P) = \operatorname{colim}\operatorname{Hom}_{\mathcal{D}\mathcal{M}^e}(I_{\mathcal{M}},\Sigma^n i^*(\sigma_{\leq q}P)).$$

Since $p:\mathcal{M}\to\mathcal{S}$ is a localization, we have $p^*(I_{\mathcal{M}})=I_{\mathcal{S}}$. Thus, we get a map

$$\operatorname{colim}\operatorname{Hom}_{\mathcal{D}\mathcal{M}^e}(I_{\mathcal{M}},\Sigma^n i^*(\sigma_{\leq q}P)) \longrightarrow \operatorname{colim}\operatorname{Hom}_{\mathcal{D}\mathcal{S}^e}(I_{\mathcal{S}},\Sigma^n (pi)^*(\sigma_{\leq q}P)).$$

We claim that it is a bijection for all n. For this, we first reinterpret the left-hand side. Since $i^*:\mathcal{D}A^e\to\mathcal{D}\mathcal{M}^e$ is fully faithful, by Lemma 2.4, it is isomorphic to

$$\operatorname{Hom}_{\mathcal{D}(\mathcal{M}^e)/\mathcal{N}}(I_{\mathcal{M}},\Sigma^n I_{\mathcal{M}}),$$

where $\mathcal{N}$ is the image under i^* of the full subcategory of $\mathcal{D}A^e$ formed by the complexes quasi-isomorphic to bounded complexes of arbitrary projective A^e-modules. Let us now consider the right-hand side. The cones over the morphisms $i^*(P)\to i^*(\sigma_{\leq q}P)$ are finite extensions of shifts of arbitrary coproducts of objects $Y(P',P'')$, where P' and P'' are finitely generated projective A-modules. The functor $p^*:\mathcal{D}\mathcal{M}^e\to\mathcal{D}\mathcal{S}^e$ commutes with arbitrary coproducts and vanishes on the $Y(P',P'')$. Thus the images under p^* of the morphisms $i^*(P)\to i^*(\sigma_{\leq q}P)$ are all invertible so that the right-hand side is isomorphic to

$$\operatorname{Hom}_{\mathcal{D}\mathcal{S}^e}(I_{\mathcal{S}},\Sigma^n I_{\mathcal{S}}) = \operatorname{Hom}_{\mathcal{D}\mathcal{S}^e}(p^*(I_{\mathcal{M}}),\Sigma^n p^*(I_{\mathcal{M}})).$$

Now notice that we have a Morita morphism of dg categories

$$\mathcal{S}^e \xrightarrow{\sim} \frac{\mathcal{M}\otimes\mathcal{M}^{\mathrm{op}}}{\mathcal{P}\otimes\mathcal{M}^{\mathrm{op}}+\mathcal{M}\otimes\mathcal{P}^{\mathrm{op}}}.$$

The functor p^* induces the quotient functor

$$\frac{\mathcal{D}(\mathcal{M}\otimes\mathcal{M}^{\mathrm{op}})}{\mathcal{N}} \longrightarrow \frac{\mathcal{D}(\mathcal{M}\otimes\mathcal{M}^{\mathrm{op}})}{\mathcal{D}(\mathcal{P}\otimes\mathcal{M}^{\mathrm{op}}+\mathcal{M}\otimes\mathcal{P}^{\mathrm{op}})} = \mathcal{D}(\mathcal{S}^e).$$

It suffices to show that p^* induces bijections in the morphism spaces with target $I_{\mathcal{M}}$

$$\operatorname{Hom}_{\mathcal{D}(\mathcal{M}^e)/\mathcal{N}}(?,I_{\mathcal{M}}) \longrightarrow \operatorname{Hom}_{\mathcal{D}(\mathcal{S}^e)}(p^*(?),p^*(I_{\mathcal{M}})).$$

For this, it suffices to show that $I_{\mathcal{M}}$ is right orthogonal in $\mathcal{D}(\mathcal{M}^e)/\mathcal{N}$ on the images under the Yoneda functor of the objects in $\mathcal{P}\otimes\mathcal{M}^{\mathrm{op}}+\mathcal{M}\otimes\mathcal{P}^{\mathrm{op}}$. To show that $I_{\mathcal{M}}$ is right orthogonal on $Y(\mathcal{M}\otimes\mathcal{P}^{\mathrm{op}})$, it suffices to show that it is right orthogonal to an object $Y(M,A)$, $M\in\mathcal{M}$. Now a morphism in $\mathcal{D}(\mathcal{M}^e)/\mathcal{N}$ is given by a diagram of $\mathcal{D}(\mathcal{M}^e)$ representing a left fraction

$$Y(M,A) \longrightarrow I'_{\mathcal{M}} \longleftarrow I_{\mathcal{M}}$$

where the cone over $I_{\mathcal{M}}\to I'_{\mathcal{M}}$ lies in $\mathcal{N}$. For each object X of $\mathcal{D}\mathcal{M}^e$, we have canonical isomorphisms

$$\mathsf{Hom}_{\mathcal{DM}^e}(Y(M,A),X) = H^0(X(M,A)) = \mathsf{Hom}_{\mathcal{DM}}(Y(M),X(?,A)).$$

Thus, the given fraction corresponds to a diagram in $\mathcal{D}(\mathcal{M})$ of the form

$$Y(M) \longrightarrow I'_{\mathcal{M}}(?,A) \longleftarrow I_{\mathcal{M}}(?,A) = \mathcal{M}(?,A)\ ,$$

where the cone over $I_{\mathcal{M}}(?,A) \to I'_{\mathcal{M}}(?,A)$ is the image under $i^* : \mathcal{DA} \to \mathcal{DM}$ of a bounded complex with projective components. Thus, we may assume that $I'_{\mathcal{M}}(?,A)$ is a finite extension of shifts of arbitrary coproducts of objects $Y(Q)$, where Q is a finitely generated projective A-module. Since M has finitely generated components, the given morphism $Y(M) \to I'_{\mathcal{M}}(?,A)$ must then factor through $Y(Q)$ for an object Q of $\mathcal{P}$. This means that the given morphism $Y(M,A) \to I'_{\mathcal{M}}$ factors through $Y(Q,A)$, which lies in $\mathcal{N}$. Thus, the given fraction represents the zero morphism of $\mathcal{D}(\mathcal{M}^e)/\mathcal{N}$, as was to be shown. The case of an object in $Y(\mathcal{P} \otimes \mathcal{M}^{\mathrm{op}})$ is analogous. In summary, we have shown that the canonical map

$$\operatorname{colim} \mathsf{Hom}_{\mathcal{DM}^e}(I_{\mathcal{M}}, \Sigma^n i^*(\sigma_{\leq q} P)) \longrightarrow \operatorname{colim} \mathsf{Hom}_{\mathcal{DS}^e}(I_{\mathcal{S}}, \Sigma^n (pi)^*(\sigma_{\leq q} P))$$

is bijective. As we have already observed, the direct system $(pi)^*(\sigma_{\leq q} P)$ is constant in $\mathcal{D}(\mathcal{S}^e)$. Moreover, we know that $i^*(P) = I_{\mathcal{M}}$ and $p^*(I_{\mathcal{M}}) = I_{\mathcal{S}}$. Thus the right-hand side is isomorphic to

$$\mathsf{Hom}_{\mathcal{DS}^e}(I_{\mathcal{S}}, \Sigma^n I_{\mathcal{S}})\ ,$$

which is the nth component of the Hochschild cohomology of the dg category $\mathsf{Sg}_{\mathrm{dg}}(A)$.

3. Remark on a possible lift to the B_∞-level

The above proof produces in fact isomorphisms in the derived category of k-modules

$$\begin{aligned}\operatorname{colim} \mathsf{RHom}_{A^e}(A, \sigma_{\leq q} P) &\to \operatorname{colim} \mathsf{RHom}_{\mathcal{M}^e}(I_{\mathcal{M}}, i^*\sigma_{\leq q} P)\\ &\to \operatorname{colim} \mathsf{RHom}_{\mathcal{S}^e}(I_{\mathcal{S}}, p^* i^* \sigma_{\leq q} P)\\ &= \mathsf{RHom}_{\mathcal{S}^e}(I_{\mathcal{S}}, I_{\mathcal{S}}).\end{aligned}$$

If we choose for P the bar resolution of A, then $\sigma_{\leq -q} P$ is canonically isomorphic to $\Sigma^q \Omega^q A$ so that the first complex carries a canonical B_∞-structure constructed by Wang. As explained in the introduction, it is classical that the last complex carries a canonical B_∞-structure. It turns out that when one makes the second complex explicit, it can be chosen identical to the first complex (essentially because i^* is fully faithful and $i^* A = I_{\mathcal{M}}$). Only the interpretation changes. Thus, the problem is to construct a compatible B_∞-structure on the third complex.

4. Proof of Theorem 1.3

By the Weierstrass preparation theorem, we may assume that Q is a polynomial. Put $R = k[x_1, \ldots, x_n]/(Q)$ so that A is isomorphic to the completion $\widehat{R}$. By Theorem 3.2.7 of [14], in sufficiently high even degrees, the Hochschild cohomology of R is isomorphic to

$$T = k[x_1, \ldots, x_n]/(Q, \partial_1 Q, \ldots, \partial_n Q)$$

as an R-module. Since $R \otimes R$ is Noetherian and Gorenstein (cf. Theorem 1.6 of [26]), by Theorem 6.3.4 of [4] the singular Hochschild cohomology of R coincides with Hochschild cohomology in sufficiently high degrees. By Theorem 1.1, the Hochschild cohomology of $\mathsf{Sg}_{\mathrm{dg}}(R)$ is isomorphic to the singular Hochschild cohomology of R and thus isomorphic to T in high even degrees. Since R is a hypersurface, the dg category $\mathsf{Sg}_{\mathrm{dg}}(R)$ is isomorphic, in the homotopy category of dg categories, to the underlying differential $\mathbb{Z}$-graded category of the differential $\mathbb{Z}/2$-graded category of matrix factorizations of Q, cf. [9], [24] and Theorem 2.49 of [3]. Thus, it is 2-periodic and so is its Hochschild cohomology. It follows that the zeroth Hochschild cohomology of $\mathsf{Sg}_{\mathrm{dg}}(R)$ is isomorphic to T as an algebra. The completion functor $? \otimes_R \widehat{R}$ yields an embedding $\mathsf{Sg}(R) \to \mathsf{Sg}(A)$ through which $\mathsf{Sg}(A)$ identifies with the idempotent completion of the triangulated category $\mathsf{Sg}(R)$, cf. Theorem 5.7 of [7]. Therefore, the corresponding dg functor $\mathsf{Sg}_{\mathrm{dg}}(R) \to \mathsf{Sg}_{\mathrm{dg}}(A)$ induces an equivalence in the derived categories and an isomorphism in Hochschild cohomology. So we find an isomorphism

$$HH^0(\mathsf{Sg}_{\mathrm{dg}}(A), \mathsf{Sg}_{\mathrm{dg}}(A)) \xrightarrow{\sim} T.$$

Since Q has an isolated singularity at the origin, we have an isomorphism

$$T \xrightarrow{\sim} k[[x_1, \ldots, x_n]]/(Q, \partial_1 Q, \ldots, \partial_n Q)$$

with the Tyurina algebra of $A = P/(Q)$. Now, by the Mather–Yau theorem [22], more precisely by its formal version [13, Prop. 2.1], in a fixed dimension, the Tyurina algebra determines A up to isomorphism.

Notice that the Hochschild cohomology of the dg category of matrix factorizations considered as a differential $\mathbb{Z}/2$-graded category is different: as shown by Dyckerhoff [7], it is isomorphic to the Milnor algebra $P/(\partial_1 Q, \dots, \partial_n Q)$ in even degree and vanishes in odd degree.

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