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Improved version of Bohr's inequality

Version améliorée de l'inégalité de Bohr

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A B S T R A C T

In this article, we prove several different improved versions of the classical Bohr's inequality. All the results are proved to be sharp.
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R É S U M É

Nous montrons ici plusieurs améliorations de l'inégalité de Bohr classique. Nous montrons également que les constantes numériques dans nos résultats sont optimales.
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1. Introduction and main results

The classical theorem of Bohr [3] (after subsequent improvements due to M. Riesz, I. Schur and F. Wiener) states that if f is a bounded analytic function on the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, with the Taylor expansion $\sum_{k=0}^{\infty} a_k z^k$, and $\|f\|_\infty := \sup_{z\in\mathbb{D}} |f(z)| < \infty$, then

$$M_f(r) := \sum_{n=0}^{\infty} |a_n| r^n \le \|f\|_\infty \quad \text{for } 0 \le r \le 1/3 \tag{1}$$

and the constant 1/3 is sharp. There are a number of articles that deal with Bohr's phenomenon. See, for example, [2,10], the recent survey on this topic by Abu-Muhanna et al. [1] and the references therein. Bombieri [4] considered the function $m(r)$ defined by $m(r) = \sup\left\{M_f(r)/\|f\|_\infty\right\}$, where the supremum is taken over all nonzero bounded analytic functions, and proved that

$$m(r) = \frac{3 - \sqrt{8(1-r^2)}}{r} \quad \text{for } 1/3 \le r \le 1/\sqrt{2}.$$

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Later Bombieri and Bourgain [5] studied the behaviour of $m(r)$ as $r \to 1$ (see also [6]) and proved the following result, which validated a question raised in [11, Remark 1] in the affirmative.

Theorem A. ([5, Theorem 1]) *If $r > 1/\sqrt{2}$, then $m(r) < 1/\sqrt{1-r^2}$. With $\alpha = 1/\sqrt{2}$, the function $\varphi_\alpha(z) = (\alpha - z)/(1-\alpha z)$ is extremal, giving $m(1/\sqrt{2}) = \sqrt{2}$.*

A lower estimate for $m(r)$ as $r \to 1$ is also obtained in [5, Theorem 2]. We are now ready to state several different improved versions of the classical Bohr inequality (1).

Theorem 1. *Suppose that $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is analytic in $\mathbb{D}$, $|f(z)| \le 1$ in $\mathbb{D}$, and S_r denotes the area of the Riemann surface of the function f^{-1} defined on the image of the subdisk $|z| < r$ under the mapping f. Then*

$$B_1(r) := \sum_{k=0}^{\infty} |a_k| r^k + \frac{16}{9}\left(\frac{S_r}{\pi}\right) \le 1 \quad \text{for } r \le \frac{1}{3} \tag{2}$$

and the numbers $1/3$ and $16/9$ cannot be improved. Moreover,

$$B_2(r) := |a_0|^2 + \sum_{k=1}^{\infty} |a_k| r^k + \frac{9}{8}\left(\frac{S_r}{\pi}\right) \le 1 \quad \text{for } r \le \frac{1}{2} \tag{3}$$

and the constants $1/2$ and $9/8$ cannot be improved.

Remark 1. Let us remark that if f is a univalent function then S_r is the area of the image of the subdisk $|z| < r$ under the mapping f. In the case of multivalent function, S_r is greater than the area of the image of the subdisk $|z| < r$. This fact could be shown by noting that

$$S_r = \int_{f(\mathbb{D}_r)} |f'(z)|^2 \, dA(w) = \int_{f(\mathbb{D}_r)} \nu_f(w) \, dA(w) \ge \int_{f(\mathbb{D}_r)} dA(w) = \text{Area}(f(\mathbb{D}_r)),$$

where $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$ and $\nu_f(w) = \sum_{f(z)=w} 1$ denotes the counting function of f.

Theorem 2. *Suppose that $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is analytic in $\mathbb{D}$ and $|f(z)| \le 1$ in $\mathbb{D}$. Then*

$$|a_0| + \sum_{k=1}^{\infty} \left(|a_k| + \frac{1}{2}|a_k|^2\right) r^k \le 1 \quad \text{for } r \le \frac{1}{3} \tag{4}$$

and the numbers $1/3$ and $1/2$ cannot be improved.

Theorem 3. *Suppose that $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is analytic in $\mathbb{D}$ and $|f(z)| \le 1$ in $\mathbb{D}$. Then*

$$\sum_{k=0}^{\infty} |a_k| r^k + |f(z) - a_0|^2 \le 1 \quad \text{for } r \le \frac{1}{3}$$

and the number $1/3$ cannot be improved.

Finally, we also prove the following sharp inequality.

Theorem 4. *Suppose that $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is analytic in $\mathbb{D}$ and $|f(z)| \le 1$ in $\mathbb{D}$. Then*

$$|f(z)|^2 + \sum_{k=1}^{\infty} |a_k|^2 r^{2k} \le 1 \quad \text{for } r \le \sqrt{\frac{11}{27}} = 0.63828\ldots$$

and this number cannot be improved.

2. Proofs of Theorems 1, 2, 3 and 4

If f and g are analytic in $\mathbb{D}$, then g is *subordinate* to f, written $g \prec f$ or $g(z) \prec f(z)$, if there exists a function ω analytic in $\mathbb{D}$ satisfying $\omega(0) = 0$, $|\omega(z)| < 1$ and $g(z) = f(\omega(z))$ for $z \in \mathbb{D}$. If f is univalent in $\mathbb{D}$, then $g \prec f$ if and only if $g(0) = f(0)$ and $g(\mathbb{D}) \subset f(\mathbb{D})$ (see [7, p. 190 and p. 253] and [1,8]).

For the proof of Theorem 1, we need the following lemma, especially when $0 < r \leq 1/2$.

Lemma 1. *Let $|b_0| < 1$ and $0 < r \leq 1/\sqrt{2}$. If $g(z) = \sum_{k=0}^{\infty} b_k z^k$ is analytic and satisfies the inequality $|g(z)| < 1$ in $\mathbb{D}$, then the following sharp inequality holds:*

$$\sum_{k=1}^{\infty} k|b_k|^2 r^{2k} \leq r^2 \frac{(1-|b_0|^2)^2}{(1-|b_0|^2 r^2)^2}. \tag{5}$$

Proof. Let $b_0 = a$. Then, it is easy to see that the condition on g can be rewritten in terms of subordination as

$$g(z) = \sum_{k=0}^{\infty} b_k z^k \prec \varphi_a(z) = \frac{a-z}{1-\bar{a}z} = a - (1-|a|^2) \sum_{k=1}^{\infty} (\bar{a})^{k-1} z^k, \quad z \in \mathbb{D}, \tag{6}$$

where $\prec$ denotes the subordination. Note that φ_a is analytic in $\mathbb{D}$ and $|\varphi_a(z)| < 1$ for $z \in \mathbb{D}$. The subordination relation (6) gives

$$\sum_{k=1}^{\infty} k|b_k|^2 r^{2k} \leq (1-|a|^2)^2 \sum_{k=1}^{\infty} k|a|^{2(k-1)} r^{2k} = r^2 \frac{(1-|a|^2)^2}{(1-|a|^2 r^2)^2}$$

from which we arrive at the inequality (5), which proves Lemma 1. For $0 < r \leq 1/\sqrt{2}$, it is important to note here that the sequence $\{kr^{2k}\}$ is non-increasing for all $k \geq 1$, so that we were able to apply the classical Goluzin's inequality [8] (see also [7, Theorem 6.3]), which extends the classical Rogosinski inequality. □

Proof of Theorem 1. Since the left-hand side of (2) is an increasing function of r, it is enough to prove it for $r = 1/3$. Therefore, we set $r = 1/3$. Moreover, the present authors in the proof of Theorem 1 in [9] proved the following inequalities:

$$\sum_{k=1}^{\infty} |a_k| r^k \leq \begin{cases} A(r) := r\dfrac{1-|a_0|^2}{1-r|a_0|} & \text{for } |a_0| \geq r \\[2ex] B(r) := r\dfrac{\sqrt{1-|a_0|^2}}{\sqrt{1-r^2}} & \text{for } |a_0| < r. \end{cases} \tag{7}$$

Note that $|a_k| \leq 1 - |a_0|^2$ for $k \geq 1$ and, from the definition of S_r, we see that

$$\begin{aligned} \frac{S_r}{\pi} &= \frac{1}{\pi} \int\int_{|z|<r} |f'(z)|^2 \, dx\, dy = \sum_{k=1}^{\infty} k|a_k|^2 r^{2k} \\ &\leq (1-|a_0|^2)^2 \sum_{k=1}^{\infty} kr^{2k} = (1-|a_0|^2)^2 \frac{r^2}{(1-r^2)^2}. \end{aligned} \tag{8}$$

At first, we consider the case $|a_0| \geq r = 1/3$. In this case, using (7) and (8), we have

$$\begin{aligned} B_1(r) = |a_0| + \sum_{k=1}^{\infty} |a_k| r^k + \frac{16}{9\pi} S_r &\leq |a_0| + A(1/3) + \frac{16}{9\pi} S_{1/3} \\ &\leq |a_0| + \frac{1-|a_0|^2}{3-|a_0|} + \frac{(1-|a_0|^2)^2}{4} \\ &= 1 - \frac{(1-|a_0|)^3(5-|a_0|^2)}{4(3-|a_0|)} \leq 1. \end{aligned}$$

Next we consider the case $|a_0| < r = 1/3$. Again, using (7) and (8), we deduce that

$$B_1(r) = \sum_{k=0}^{\infty} |a_k| r^k + \frac{16}{9\pi} S_r \leq |a_0| + B(1/3) + \frac{16}{9\pi} S_{1/3}$$

$$\leq |a_0| + \frac{\sqrt{1-|a_0|^2}}{\sqrt{8}} + \frac{(1-|a_0|^2)^2}{4}$$
$$\leq \frac{1}{3} + \frac{1}{\sqrt{8}} + \frac{1}{4} < 1 \quad (\text{since } |a_0| < 1/3)$$

and the desired inequality (2) follows.

To prove that the constant $16/(9\pi)$ is sharp, we consider the function $f = \varphi_a$ given by

$$\varphi_a(z) = \frac{a-z}{1-az} = a - (1-a^2)\sum_{k=1}^{\infty} a^{k-1}z^k, \quad z \in \mathbb{D},$$

where $a \in (0,1)$. For this function, straightforward calculations show that

$$\sum_{k=0}^{\infty} |a_k| r^k + \frac{\lambda}{\pi} S_r = a + r\frac{1-a^2}{1-ra} + \lambda(1-a^2)^2 \frac{r^2}{(1-a^2r^2)^2}.$$

In the case $r = 1/3$, the last expression becomes

$$a + \frac{1-a^2}{3-a} + 9\lambda \frac{(1-a^2)^2}{(9-a^2)^2} = 1 - \frac{2(1-a)^3(19+12a+a^2)}{(a^2-9)^2} + (9\lambda - 16)\frac{(1-a^2)^2}{(9-a^2)^2}$$

which is obviously bigger than 1 in case $\lambda > 16/9$ and $a \to 1$. The proof of the first part of Theorem 1 is complete.

Let us now verify the inequality (3). To do it we will use the method presented above and Lemma 1 for $r \leq 1/2$. From Lemma 1, it follows that

$$\frac{S_r}{\pi} \leq (1-|a_0|^2)^2 \frac{r^2}{(1-|a_0|^2r^2)^2}, \quad r \leq 1/2. \tag{9}$$

Let $r \leq 1/2$ and we first consider the case $|a_0| \geq 1/2$. Then, using (7) and (9), we obtain that

$$B_2(r) = |a_0|^2 + \sum_{k=1}^{\infty} |a_k| r^k + \frac{9}{8\pi} S_r \leq |a_0|^2 + A(1/2) + \frac{9}{8\pi} S_{1/2}$$
$$\leq |a_0|^2 + \frac{1-|a_0|^2}{2-|a_0|} + \frac{4(1-|a_0|^2)^2}{(4-|a_0|^2)^2}$$
$$= 1 - \frac{(1-|a_0|)^3(1+|a_0|)(7+6|a_0|+2|a_0|^2)}{2(4-|a_0|^2)^2} \leq 1.$$

Now we consider the case $|a_0| < 1/2$. In this case, using (7) and (9), we have

$$B_2(r) \leq |a_0|^2 + B(1/2) + \frac{9}{8\pi} S_{1/2}$$
$$\leq |a_0|^2 + \frac{\sqrt{1-|a_0|^2}}{\sqrt{3}} + \frac{4(1-|a_0|^2)^2}{(4-|a_0|^2)^2}$$
$$\leq \frac{1}{\sqrt{3}} + |a_0|^2 + \frac{4(1-|a_0|^2)^2}{(4-|a_0|^2)^2}$$
$$\leq \frac{1}{\sqrt{3}} + \frac{41}{100} - \frac{(1-4|a_0|^2)(256-104|a_0|^2+25|a_0|^4)}{100(|a_0|^2-4)^2}$$

which is less than 1. The sharpness of the constant 9/8 can be established as in the previous case and thus, we omit the details. The proof of the theorem is complete. □

Proof of Theorem 2. Let $A(r)$ and $B(r)$ be defined as in (7). Furthermore, the present authors in [9] demonstrated the following inequality for the coefficients of f:

$$\sum_{k=1}^{\infty} |a_k|^2 r^k \leq \frac{r(1-|a_0|^2)^2}{1-|a_0|^2 r}. \tag{10}$$

Also, it is worth pointing out that the inequality (10) for $0 < r \leq 1/\sqrt{2}$ follows from (5) by integrating it. As remarked in the proof of earlier theorems, it suffices to prove the inequality (4) for $r = 1/3$, and thus we may set $r = 1/3$ in the proof below. At first, we consider the case $|a_0| \geq 1/3$ so that, by (7) and (10),

$$\begin{aligned}\sum_{k=0}^{\infty}|a_k|r^k+\frac{1}{2}\sum_{k=1}^{\infty}|a_k|^2r^k &\le |a_0|+A(1/3)+\frac{(1-|a_0|^2)^2}{6-2|a_0|^2}\\ &=|a_0|+\frac{1-|a_0|^2}{3-|a_0|}+\frac{(1-|a_0|^2)^2}{6-2|a_0|^2}\\ &=1-\frac{(1-|a_0|)^2}{2}\le 1 \quad (\text{since } |a_0|\le 1).\end{aligned}$$

Similarly, for the case $|a_0|<1/3$, we have, by (7) and (10),

$$\begin{aligned}\sum_{k=0}^{\infty}|a_k|r^k+\frac{1}{2}\sum_{k=1}^{\infty}|a_k|^2r^k &\le |a_0|+B(1/3)+\frac{(1-|a_0|^2)^2}{6-2|a_0|^2}\\ &\le |a_0|+\frac{\sqrt{1-|a_0|^2}}{\sqrt{8}}+\frac{(1-|a_0|^2)^2}{6-2|a_0|^2}\\ &\le \frac{1}{3}+\frac{1}{\sqrt{8}}+\frac{1}{6}<1,\end{aligned}$$

which concludes the proof of Theorem 2 since the proof of sharpness follows similarly. □

Proof of Theorem 3. Let $A(r)$ and $B(r)$ be defined as in (7). Also, we may let $r=1/3$. Accordingly, we first consider the case $|a_0|\ge 1/3$, so that

$$\begin{aligned}\sum_{k=0}^{\infty}|a_k|r^k+|f(z)-a_0|^2 &\le |a_0|+A(1/3)+A(1/3)^2\\ &=|a_0|+\frac{1-|a_0|^2}{3-|a_0|}+\frac{(1-|a_0|^2)^2}{(3-|a_0|)^2}\\ &=1-\frac{(1-|a_0|)^3(5+|a_0|)}{(3-|a_0|)^2}\le 1 \quad (\text{since } |a_0|\le 1).\end{aligned}$$

Next, we consider the case $|a_0|<1/3$ so that

$$\begin{aligned}\sum_{k=0}^{\infty}|a_k|r^k+|f(z)-a_0|^2 &\le |a_0|+B(1/3)+B(1/3)^2\\ &=|a_0|+\frac{\sqrt{1-|a_0|^2}}{\sqrt{8}}+\frac{1-|a_0|^2}{8}\\ &\le \frac{1}{3}+\frac{1}{\sqrt{8}}+\frac{1}{8}<1.\end{aligned}$$

This concludes the proof of Theorem 2 and the sharpness follows similarly. □

Proof of Theorem 4. Using (10) (see [9, Lemma 1]) and the classical inequality for $|f(z)|$, we have

$$|f(z)|^2+\sum_{k=1}^{\infty}|a_k|^2r^{2k}\le\left(\frac{r+|a_0|}{1+r|a_0|}\right)^2+\frac{r^2(1-|a_0|^2)^2}{1-|a_0|^2r^2}.$$

For $r=\sqrt{11/27}$, the last expression on the right gives

$$1-\frac{3(1-|a_0|^2)}{(9+\sqrt{33}|a_0|)^2(27-11|a_0|^2)}(135-66\sqrt{33}|a_0|+66\sqrt{33}|a_0|^3+121|a_0|^4),$$

and straightforward calculations show that this expression is less than or equal to 1 for all $|a_0|\le 1$. The example

$$f(z)=\frac{z+a}{1+az}$$

with $a=\sqrt{3/11}$ shows that $r=\sqrt{11/27}$ is sharp. This completes the proof. □

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