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Complex variable approach to the analysis of a fractional differential equation in the real line

Approche par variable complexe de l'analyse d'une équation différentielle fractionnaire sur la droite réelle

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ABSTRACT

The first aim of this work is to establish a Peano-type existence theorem for an initial value problem involving a complex fractional derivative, and then, as a consequence of this theorem, to give a partial answer for the local existence of the continuous solution to the initial value problem:

$$\begin{cases} D_x^q u(x) = f\big(x, u(x)\big), \\ u(0) = b, \quad (b \neq 0). \end{cases}$$

Moreover, for some special cases of the problem, we investigate the corresponding geometric properties of the solutions.

RÉSUMÉ

L'objectif principal de ce travail est d'établir un théorème d'existence de type Peano pour un problème aux valeurs initiales faisant intervenir une dérivée fractionnaire, puis, comme conséquence, de donner une réponse partielle à l'existence locale d'une solution continue du problème aux valeurs initiales suivant :

$$\begin{cases} D_x^q u(x) = f\big(x, u(x)\big), \\ u(0) = b, \quad (b \neq 0). \end{cases}$$

De plus, nous étudions les propriétés géométriques des solutions pour quelques cas particuliers.

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1. Introduction and motivation

In the theory of ordinary differential equations, Peano's existence theorem is practical and important because one can easily check the local existence of the solution to a differential equation. The Peano-type existence theorem in the theory of fractional differential equations was first given by Laksmikantham and Vatsala [10] by using Tonelli's approach for the initial value problem

$$\begin{cases} D_x^q u(x) = f\big(x, u(x)\big), \\ u(0) = b, \end{cases} \tag{1.1}$$

where D_x^q is the well-known Riemann–Liouville fractional derivative in the real line, b is a real number, and $f \in C([0,T] \times \mathbb{R}, \mathbb{R})$. However, as indicated in [24], problem (1.1) with $b \neq 0$ is not a suitable problem. For this reason, this problem was considered with different initial data or with a modified equation by some researchers, see, for example [1], [6], [9], [15], [24]. Nevertheless, if the initial condition in (1.1) is taken as homogenous, one can see that problem (1.1) is meaningful provided that $f \in C([0,T] \times \mathbb{R}, \mathbb{R})$. In any case, problem (1.1) with the non-homogenous initial condition, that is, $b \neq 0$, can be suitable if the nonlinear function f in this problem satisfies the following two conditions:

(i) $f(x, y)$ and $x^q f(x, y)$ $(0 < q < 1)$ are continuous on $(0, T] \times \mathbb{R}$ and $[0, T] \times \mathbb{R}$, respectively,
(ii) $x^q f(x, b)\big|_{x=0} = b/\Gamma(1-q)$, where Γ is the well-known Gamma function.

It is known from the study of Delbosco and Rodino [3] that if condition (i) holds, then the equation (1.1) is equivalent to the following integral equation:

$$u(x) = \frac{1}{\Gamma(q)} \int_0^x \frac{f(\zeta, u(\zeta))}{(x-\zeta)^{1-q}} \mathrm{d}\zeta, \tag{1.2}$$

Now, if the initial condition with $b \neq 0$ in (1.1) is considered, then condition (ii) is necessary for the problem to be suitable. Otherwise

$$b = u(0) = \lim_{x \to 0^+} u(x) = \frac{1}{\Gamma(q)} \lim_{x \to 0^+} \int_0^1 \frac{(xt)^q\, f(xt, u(xt))}{t^q (1-t)^{1-q}} \mathrm{d}t \neq b,$$

which is a contradiction.

It is to be observed that the existence and uniqueness of continuous solutions to this problem without initial condition was proved in Theorem 3.5 in [3] by posing a Lipschitz-type condition in addition to (i). However, the local existence of continuous solutions to this problem only under condition (i) is still an open problem.

In this study, we give a partial answer to this open problem by establishing a Peano-type existence theorem for the complex version of problem (1.1):

$$\begin{cases} D_z^q u(z) = f\big(z, u(z)\big), \quad z \in \mathbb{U}_R, \\ u(0) = b, \end{cases} \tag{1.3}$$

where $\mathbb{U}_R : \{z \in \mathbb{C} : |z| < R\}$ is an open disc in the complex plane $\mathbb{C}$, $b \in \mathbb{C}$ and D^q is the complex fractional derivative operator, which is a direct extension of the well-known Riemann–Liouville derivative in the real line and given in [18], [20].

The existence of a solution to problem (1.3) is investigated in the Banach space $\mathcal{B}(\mathbb{U}_R) := C(\overline{\mathbb{U}}_R) \cap \mathcal{A}$, where $\mathbb{U}_R := \{z \in \mathbb{C} : |z| < R\}$, $C(\overline{\mathbb{U}}_R)$ is the space of continuous functions on $\overline{\mathbb{U}}_R$, and $\mathcal{A}$ is the space of analytic functions on $\mathbb{U}_R$, provided that the nonlinear function f satisfies the following two conditions:

(iii) $f(z, t)$ is a multivalued function on $\overline{\mathbb{U}}_R \times \mathbb{C}$ such that $z^q f(z, t)$ is analytic on $\mathbb{U}_R \times \mathbb{C}$ and continuous on $\overline{\mathbb{U}}_R \times \mathbb{C}$;
(iv) $z^q f(z, b)\big|_{z=0} = b/\Gamma(1-q)$.

By the help of this existence theorem (see Corollary 2.6 and Example 2.7), it is possible to show the existence of continuous solutions to problem (1.1) in the following way: if the nonlinear function $f(x, t)$ satisfying conditions (i)–(ii) can be extended to the function $f(z, t)$ satisfying conditions (iii)–(iv), then, as a result of this existence theorem, one can say that there exists a solution $u \in \mathcal{B}(\mathbb{U}_R)$ of (1.3). At the same time, the real part of $u \in \mathcal{B}(\mathbb{U}_R)$ is a continuous solution to problem (1.1). From this point of view, this existence theorem contributes to the area of fractional differential equations in the real line. Moreover, in the proof of the theorem, we use a new technique related to Schwarz' Lemma for analytic functions of one variable and two variables varying on generally different closed balls in $\mathbb{C}$, which we prove in the next section. From this aspect, this existence result is a nice consequence of the analytic functions.

On the other hand, as shown by Diethelm in [5], the existence of a real analytic solution to problem (1.1) with Caputo-derivative D_x^q is a rare event when f is analytic. Of course, the same result is valid for problems (1.1) and (1.3) when f is analytic.

Furthermore, complex fractional integral and derivative operators have been widely used in complex analysis and have many applications in the univalent function theory (see, for example, [2], [8], [21]). We here show that the solutions to (1.3) for some special cases of f are univalent or starlike on $\mathbb{U}$ by using the results of Noshiro–Warschawski in [7] and Mocanu in [11]. Such investigations were previously made for the solutions to the ordinary differential equations in [14], [17], [19].

2. Preliminaries and main result

In this section, we intend to present a Peano-type existence theorem for the initial value problem (1.3) involving the fractional derivative D_z^q, and therefore the fractional integral I_z^q, which are defined as follows (see [13], [18]).

Definition 2.1. Let the function $u(z)$ be defined on a certain domain of complex plane containing the points 0 and z. Then, the fractional integral and derivative of order q $\big(0<q<1\big)$ of $u(z)$ are defined, respectively, by

$$I_z^q u(z) := \frac{1}{\Gamma(q)} \int_0^z \frac{u(\zeta)}{(z-\zeta)^{1-q}}\,\mathrm{d}\zeta, \tag{2.1}$$

and

$$D_z^q u(z) = \frac{1}{\Gamma(1-q)} \frac{\mathrm{d}}{\mathrm{d}z} \int_0^z \frac{u(\zeta)}{(z-\zeta)^{q}}\,\mathrm{d}\zeta, \tag{2.2}$$

where the integrations are along the line segment connecting points 0 and z as a rule, and with the principal value

$$(z-\zeta)^{1-q} = |z-\zeta|^{1-q}\,\mathrm{e}^{\mathrm{i}(1-q)\arg z}, \quad \arg z \in (-\pi, \pi].$$

The definitions given just above are the direct generalization of the well-known Riemann–Liouville fractional integral and derivative in the real line. Since the integrations in these definitions are over the line segment connecting points 0 and z, we can say that

$$D_x^q u(x) = D_z^q(\Re\big(u(z)\big)) \text{ on } z \in (0, R)$$

for any $R>0$ and for $u(x)$, real part of $u(z)$.

Under certain conditions, the compositional relations for these operations, that is,

$$D_z^q I_z^q u(z) = u(z) \text{ and } I_z^q D_z^q u(z) = u(z)$$

are given and proved in Lemma 3.2 and Remark 3.3 in [20].

By means of these relations, the following equivalence was obtained there, for the investigation of the existence of solutions in $\mathcal{B}^b(\mathbb{U}_R) := \{u \in \mathcal{B}(\mathbb{U}_R) : u(0) = b\}$ to problem (1.3).

Lemma 2.2. *Under conditions (iii)–(iv), problem (1.3) is equivalent to the following Volterra-type integral equation*

$$u(z) = \frac{1}{\Gamma(q)} \int_0^z \frac{f(\zeta, u(\zeta))}{(z-\zeta)^{1-q}}\,\mathrm{d}\zeta, \quad z \in \overline{\mathbb{U}}_R, \tag{2.3}$$

that is, every solution to (1.3) is also a solution to (2.3) and vice versa.

Existence theorems in the ordinary and fractional differential equations (for example, [3], [22], [23]) were proved by means of Schauder's fixed point theorem. However, making only use of this theorem is not enough to prove the local existence of desired solutions to the integral equation (2.3) for any f satisfying conditions (iii) and (iv). This is given in the following remark.

Remark 2.3. For simplicity, let us take $b=0$ in (1.3) with f satisfying conditions (iii) and (iv). Then, for the fixed positive real numbers R and r there exists a positive real number M such that

$$\sup_{(z,t)\in\overline{\mathbb{U}}_R\times\overline{\mathbb{U}}_r} \left|z^q f(z,t)\right| \leq M,$$

where $\mathbb{U}_R := \{z \in \mathbb{C} : |z| < r\}$ and $\mathbb{U}_r := \{t \in \mathbb{C} : |t| < r\}$ are the open discs. Then, for the operator T defined as (2.6), we have

$$|Tu(z)| \leq \frac{1}{\Gamma(q)} \int_0^{|z|} \frac{\left|\zeta^q f(\zeta, u(\zeta))\right|}{|\zeta|^q |z-\zeta|^{1-q}} \mathrm{d}\zeta \leq M\Gamma(1-q) \leq r \text{ on } \overline{\mathbb{U}}_{R_0}$$

for any $R_0 \leq R$, if $M\Gamma(1-q) \leq r$. That is, we can consider the existence of a solution $u \in \mathcal{B}^0(\mathbb{U}_R)$ only if f satisfies $M\Gamma(1-q) \leq r$, in addition to conditions (iii) and (iv). The same result is obtained when one investigates the local existence of continuous solutions to the equation in (1.1) (or problem (1.1)) under condition (i) (or conditions (i) and (ii)). For this reason, the existence of continuous solutions to (1.1) under these conditions is open.

As a consequence of the explanation in Remark 2.3, for the proof of the existence theorem that we have obtained, we use two Lemmas including Schwarz' Lemma for analytic functions of one variable (see [4], [16], [20]) and Schwarz' Lemma for analytic functions of two variables varying on different closed balls, which by following the way in Theorem 1.9 in [12] is obtained in the following lemma.

Lemma 2.4. *Let R and r be fixed positive real numbers and b be a fixed complex number. Let also $\mathbb{U}_r^b := \{t \in \mathbb{C} : |t-b| \leq r\}$ and let $g : \overline{\mathbb{U}}_R \times \overline{\mathbb{U}}_r^b \longrightarrow \mathbb{C}$ be analytic on $\mathbb{U}_R \times \mathbb{U}_r^b$, continuous on $\overline{\mathbb{U}}_R \times \overline{\mathbb{U}}_r^b$ with $g(0,b) = 0$ and bounded with $|g(z,t)| \leq M$ ($M > 0$) for all $(z,t) \in \overline{\mathbb{U}}_R \times \overline{\mathbb{U}}_r^b$. Then,*

$$|g(z,t)| \leq M \max\left(\frac{|z|}{R}, \frac{|t-b|}{r}\right) \tag{2.4}$$

for all $(z,t) \in \overline{\mathbb{U}}_R \times \overline{\mathbb{U}}_r^b$.

Proof. Let $\xi = (\xi_1, \xi_2)$ be a fixed complex number on $\partial(\mathbb{U}_R) \times \partial(\mathbb{U}_r^b)$, and we define a function Φ_ξ by:

$$\Phi_\xi : \overline{\mathbb{U}}_R \to \overline{\mathbb{U}}_R \times \overline{\mathbb{U}}_r^b \ , \ \ \Phi_\xi(\eta) := \left(\frac{\xi_1}{R}\eta, \frac{(\xi_2 - b)}{R}\eta + b\right).$$

Clearly, from the hypotheses of this theorem, $g \circ \Phi_\xi$ is analytic on $\mathbb{U}_R$, continuous on $\overline{\mathbb{U}}_R$ and bounded with $\left|(g\circ\Phi_\xi)(\eta)\right| \leq M$ for all $\eta \in \overline{\mathbb{U}}_R$.

Now, let a function $\psi(\eta)$ be defined by

$$\psi(\eta) := \frac{(g\circ\Phi_\xi)(\eta)}{\eta}, \quad \left(\eta \in \overline{\mathbb{U}}_R\right).$$

Because of $(g\circ\Phi_\xi)(0) = 0$, the function ψ has a removable singularity at the point $\eta = 0$. Moreover, ψ is continuous both on $\partial(\mathbb{U}_R)$ and at $\eta = 0$, since $\psi(0) = (g\circ\Phi_\xi)'(0)$ and also ψ is an analytic function on $\mathbb{U}_R - \{0\}$. It follows from Cauchy's integral formula that $|\psi(0)| = \left|(g\circ\Phi_\xi)'(0)\right| \leq \frac{M}{R}$ is then obtained and, from the hypothesis of this theorem, that $|\psi(z)| \leq \frac{M}{R}$ on $\partial(\mathbb{U}_R)$. In view of these two results, by applying the maximum modulus theorem to the function ψ, we easily get that $\left|(g\circ\Phi_\xi)(\eta)\right| \leq \frac{M|\eta|}{R}$ on $\overline{\mathbb{U}}_R$, which immediately yields the inequality (2.4).

Furthermore, the equality (2.4) holds for all $(z,t) \in \overline{\mathbb{U}}_R \times \overline{\mathbb{U}}_r^b$, since, for any $(z,t) \in \overline{\mathbb{U}}_R \times \overline{\mathbb{U}}_r$, there exists a fixed number $\xi^* := \left(\xi_1^*, \xi_2^*\right) \in \partial(\mathbb{U}_R) \times \partial(\mathbb{U}_r^b)$ and a function $\Phi_\xi^* : \overline{\mathbb{U}}_R \to \overline{\mathbb{U}}_R \times \overline{\mathbb{U}}_r^b$ such that

$$(z,t) = \left(\frac{\xi_1^*}{R}\eta, \frac{\xi_2^*}{R}\eta\right) = \Phi_{\xi^*}(\eta).$$

This completes the proof.

Now, we are ready to give the Peano-type existence theorem, which is our first result.

Theorem 2.5. *Let q be fixed in $(0,1)$ and let the hypotheses (iii)–(iv) be satisfied. Furthermore, for the fixed positive real numbers R and r, let*

$$\sup_{(z,t)\in\overline{\mathbb{U}}_R\times\overline{\mathbb{U}}_r^b} \left|z^q f(z,t) - \frac{b}{\Gamma(1-q)}\right| \leq M, \quad (M > 0).$$

Then, the initial value problem (1.3) possesses at least one solution u in $\mathcal{B}^b(\mathbb{U}_{R_0})$, where

$$R_0 := \begin{cases} R & , \text{ if } M\Gamma(2-q) \le r, \\ \frac{rR}{M\Gamma(2-q)} & , \text{ if } r < M\Gamma(2-q). \end{cases} \tag{2.5}$$

Proof. As indicated in Lemma 2.2, problem (1.3) is equivalent to the integral equation (2.3). Let us now define an operator T as follows:

$$Tu(z) = \frac{1}{\Gamma(q)} \int_0^z \frac{f(\zeta, u(\zeta))}{(z-\zeta)^{1-q}} \mathrm{d}\zeta, \quad z \in \overline{\mathbb{U}}_R. \tag{2.6}$$

It is clear that $T : \mathcal{B}^b(\mathbb{U}_{R_0}) \to \mathcal{B}^b(\mathbb{U}_{R_0})$. Hence, the fixed points of the operator T in $\mathcal{B}^b(\mathbb{U}_{R_0})$ coincide with the solutions to problem (1.3). Thus, our aim is to prove the existence of the fixed points of the operator T by means of Schauder's fixed point theorem. For the proof, we first need to show the inclusion

$$T(B_r) \subseteq B_r, \tag{2.7}$$

where B_r is an appropriate closed, convex and bounded subset of $\mathcal{B}^b(\mathbb{U}_{R_0})$. The choice of B_r differs greatly depending on the value of $z^q f(z,t)$. This choice is made in the following way: let us define g as $g(z,t) := z^q f(z,t) - \frac{b}{\Gamma(1-q)}$. Since the hypotheses of Lemma 2.4 are satisfied, we have

$$\left| z^q f(z,t) - \frac{b}{\Gamma(1-q)} \right| \le M \max\left(\frac{|z|}{R}, \frac{|t-b|}{r} \right)$$

for all $(z,t) \in \overline{\mathbb{U}}_R \times \overline{\mathbb{U}}_r^b$, that is, either

(a) $\frac{|z|}{R} := \max\left(\frac{|z|}{R}, \frac{|t-b|}{r} \right)$,

or

(b) $\frac{|t-b|}{r} := \max\left(\frac{|z|}{R}, \frac{|t-b|}{r} \right)$.

For both cases above, we define a B_r such that (2.7) is satisfied. When case (a) holds, let

$$B_r = \left\{ u \in \mathcal{B}^b(\mathbb{U}_{R_0}) : \sup_{z \in \overline{\mathbb{U}}_{R_0}} |u(z) - b| \le r \right\}$$

with $r < M\Gamma(2-q)$. Then, for any $u \in B_r$, from (2.6) we have:

$$\begin{aligned} |Tu(z) - b| &= \left| \frac{1}{\Gamma(q)} \int_0^z \frac{f(\zeta, u(\zeta))}{(z-\zeta)^{1-q}} \mathrm{d}\zeta - b \right| \\ &\le \frac{1}{\Gamma(q)} \int_0^{|z|} \frac{\left| f(\zeta, u(\zeta)) - \zeta^{-q} \frac{b}{\Gamma(1-q)} \right|}{|z-\zeta|^{1-q}} |\mathrm{d}\zeta|, \\ &= \frac{1}{\Gamma(q)} \int_0^{|z|} \frac{\left| \zeta^q f(\zeta, u(\zeta)) - \frac{b}{\Gamma(1-q)} \right|}{|\zeta|^q |z-\zeta|^{1-q}} |\mathrm{d}\zeta|, \\ &\le \frac{M}{\Gamma(q)} \int_0^{|z|} \frac{\frac{|\zeta|}{R}}{|\zeta|^q |z-\zeta|^{1-q}} |\mathrm{d}\zeta|. \end{aligned}$$

By using change of variables $\zeta = zt$ in the last inequality we obtain

$$|Tu(z) - b| \le \sup_{z \in \overline{\mathbb{U}}_{R_0}} \frac{M\Gamma(2-q)|z|}{R} \le r,$$

because of the definition of R_0 in (2.5). Hence, the inclusion in (2.7) is fulfilled.

Now, we suppose that case (b) holds and let us define

$$B_r = \left\{u \in \mathcal{B}^b(\mathbb{U}_R) : \sup_{z\in\overline{\mathbb{U}}_R} |u(z) - b| \le r\right\}$$

with $M\Gamma(2-q) \le r$. Then, in a similar way to the previous one, we get

$$|Tu(z) - b| \le \frac{M}{\Gamma(q)} \int_0^{|z|} \frac{\frac{|u(\zeta)-b|}{r}}{|\zeta|^q |z-\zeta|^{1-q}} |\mathrm{d}\zeta| \text{ for all } z \in \overline{\mathbb{U}}_R. \tag{2.8}$$

On the other hand, from the Schwarz Lemma of one variable, we can write

$$\frac{|u(z)-b|}{r} \le \frac{r|z|}{rR} = \frac{|z|}{R} \quad \text{for all } z \in \overline{\mathbb{U}}_R.$$

If we use this inequality in (2.8) and the fact that $M\Gamma(2-q) \le r$, we obtain

$$|Tu(z) - b| \le \frac{M}{\Gamma(q)} \int_0^{|z|} \frac{\frac{|\zeta|}{R}}{|\zeta|^q |z-\zeta|^{1-q}} |\mathrm{d}\zeta| \le M\Gamma(2-q) \le r,$$

which shows that (2.7) holds for the case (b).

Let us now show that T is a continuous operator on B_r (for both cases). It is supposed that $\{u_n\}_{n=1}^{\infty} \subset B_r$ is a sequence with $u_n \to u$ in $\mathcal{B}^b(\mathbb{U}_{R_0})$ as $n \to \infty$. Then, it follows that u_n converges uniformly to $u \in B_r$. By using this and the uniform continuity of $z^q f(z,t)$ on $\overline{\mathbb{U}}_R \times \overline{\mathbb{U}}_r^b$, we conclude that

$$\begin{aligned} |Tu_n(z) - Tu(z)| &= \left| \frac{1}{\Gamma(q)} \int_0^z \frac{[f(\zeta, u_n(\zeta)) - f(\zeta, u(\zeta))]}{(z-\zeta)^{1-q}} \mathrm{d}\zeta \right| \\ &\le \frac{1}{\Gamma(q)} \int_0^1 \frac{\left|(\xi z)^q f(\xi z, u_n(\xi z)) - (\xi z)^q f(\xi z, u(\xi z))\right|}{\xi^q (1-\xi)^{1-q}} \mathrm{d}\xi \to 0 \end{aligned}$$

as $n \to \infty$.

Now let us prove that $T(B_r)$ is an equicontinuous set of $\mathcal{B}^b(\mathbb{U}_{R_0})$. For any $u \in B_r$, the function $z^q f(z, u(z))$ is uniformly continuous on $\overline{\mathbb{U}}_R$. Therefore, for given $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that

$$\left|z_1^q f(z_1, u(z_1)) - z_2^q f(z_2, u(z_2))\right| < \frac{\epsilon}{\Gamma(1-q)},$$

for all $z_1, z_2 \in \overline{\mathbb{U}}_R$ satisfying $|z_1 - z_2| < \delta$. From here, one can conclude that

$$\begin{aligned} &\left|Tu(z_1) - Tu(z_2)\right| \\ &\le \frac{1}{\Gamma(q)} \int_0^1 \frac{\left|(\xi z_1)^q f(\xi z_1, u(\xi z_1)) - (\xi z_2)^q f(\xi z_2, u(\xi z_2))\right|}{\xi^q (1-\xi)^{1-q}} \mathrm{d}\xi \\ &< \Gamma(1-q) \frac{\epsilon}{\Gamma(1-q)} = \epsilon, \end{aligned}$$

since $|\xi z_1 - \xi z_2| < \delta$. So, the desired result is obtained.

Therefore, as a consequence of Schauder's fixed point theorem, it can be said that the operator T has at least one fixed point in $\mathcal{B}^b(\mathbb{U}_{R_0})$ for an R_0 given in (2.5), which is also a solution to problem (1.3). This completes the proof.

Now, to give a corollary of Theorem 2.5 dealing with the existence of the solution for the initial value problem (1.1) with $b \neq 0$, we suppose that the nonlinear function $f(x,t)$ in (1.1) satisfies conditions (i) and (ii). Moreover, we extend the function $f(x,y)$ to the function $f(z,t)$ by writing z and t instead of x and y, respectively, for $(z,t) \in \overline{\mathbb{U}}_R \times \mathbb{C}$. So, we obtain a new function $f(z,t)$. Then, the corollary is as follows.

Corollary 2.6. *If the new function $f(z,t)$ fulfills conditions (iii)–(iv), and*

$$f(x,y) = \Re\big(f(z,t)\big)$$

holds, then problem (1.1) with $b \neq 0$ has at least one continuous solution on the interval $[0, R_0]$, where R_0 is as in (2.5), which is the real part of the solution to the initial value problem (1.3) under conditions (iii)–(iv).

Proof. If conditions of this corollary are considered, as a result of the previous theorem, for an R_0 given in (2.5) one can say that there exists a solution $u(z)$ in $\mathcal{B}^b(\mathbb{U}_{R_0})$ to the initial value problem:

$$\begin{cases} D_z^q u(z) = f\big(z, u(z)\big), \\ u(0) = b, \quad (b \in \mathbb{R} - \{0\}). \end{cases} \tag{2.9}$$

Thus, one can easily deduce from above that $\Re\big(u(z)\big)$ satisfies (2.9) on $z \in [0, R]$.

Moreover, from the assumptions, one can write:

$$x^q f(x, y) = z^q \Re\big(f(z,t)\big) \text{ for all } z \in [0, R].$$

Therefore, it is obtained that $\Re\big(u(z)\big)$ satisfies the initial value problem

$$\begin{cases} D_x^q \big(\Re\big(u(z)\big)\big) = f\big(x, \Re\big(u(z)\big)\big), \\ u(0) = b, \quad (b \in \mathbb{R} - \{0\}), \end{cases}$$

which implies that $\Re\big(u(z)\big)$ is a continuous solution to (1.1).

Example 2.7. If we take $f(x, y) = \frac{x^{-q}}{\Gamma(1-q)}\left(y + \frac{q}{1-q}x\right)$ in (1.1), then it is clear that $f(z,t) = \frac{z^{-q}}{\Gamma(1-q)}\left(t + \frac{q}{1-q}z\right)$ fulfills conditions (iii)–(iv). As a result of Theorem 2.5, we can say that there exist a solution to (1.3) in $\mathcal{B}^b(\mathbb{U}_{R_0})$, and it is $u(z) = b + z$. Moreover, the real part of $u(z)$, i.e. $u(x) = b + x$ satisfies problem (1.1).

The uniqueness of the desired solution to problem (1.3) was previously proved in Theorem 3.8 in [20]. In the following, we give two results related to the geometric properties of the unique analytic solution on the unit disc $\mathbb{U}$ for (1.3) with $u(0) = 0$ and the function f satisfying certain conditions, by using the Noshiro–Warschawski theorem in [7] and a result of Mocanu in [11].

Theorem 2.8. *Let $f(z,t) := z^{-q}h(z)$ with $h \in \mathcal{B}^0(\mathbb{U})$ and $u(0) = 0$ in (1.3). Then the initial value problem (1.3) has a unique univalent solution, if h satisfies the following two conditions:*

(i) $h'(0) = \frac{1}{\Gamma(2-q)}$,
(ii) $\Re\big(e^{i\beta}h'(z)\big) > 0$ *for some real β and for all $z \in \mathbb{U}$.*

Proof. By virtue of Theorem 3.8 in [20], one can say that there exists a unique solution u in $\mathcal{B}^0(\mathbb{U})$ to the problem, and this solution is given by

$$u\,(z) = \frac{1}{\Gamma\,(q)} \int_0^z \frac{\zeta^{-q}h(\zeta)}{(z-\zeta)^{1-q}} \mathrm{d}\zeta. \tag{2.10}$$

If we first apply the change of variables $\zeta = zt$ in the right-hand side of (2.10), and then, if we differentiate both sides of the resulting equation, we obtain

$$u'\,(z) = \frac{1}{\Gamma\,(q)} \int_0^1 t^{1-q}(1-t)^{q-1}h'(zt)\,\mathrm{d}t \quad (\forall z \in \mathbb{U}).$$

Using condition (ii) in the above equality, it is easily seen that $\Re\big(e^{i\beta}u'(z)\big) > 0$ for some real β and for all $z \in \mathbb{U}$. Moreover, if we use condition (i) in the equality after taking $z \to 0$, then we have $u'(0) = 1$. Therefore, from the Noshiro–Warschawski theorem, the solution u to the considered problem is univalent on $\mathbb{U}$.

Theorem 2.9. *Let $f(z,t) := z^{-q}h(z)$ with $h \in \mathcal{B}^0(\mathbb{U})$ and $h'(0) = \frac{1}{\Gamma(2-q)}$ in (1.3). Moreover, if there exists an M with $0 < M \le \frac{\sqrt{20}}{5}$ such that*

$$\sup_{z\in\overline{\mathbb{U}}} \left|\Gamma(1-q)h(z) - \frac{z}{1-q}\right| \le M, \tag{2.11}$$

then (1.3) has a unique starlike solution.

Proof. As a result of Theorem 3.8 in [20] and the hypotheses, it can be said that there exists a unique solution $u \in \mathcal{B}^0(\mathbb{U})$ to the problem, which is given by (2.10). From (2.10), one can write, for all $z \in \overline{\mathbb{U}}$:

$$|u(z) - z| = \frac{1}{\Gamma(q)} \left| \int_0^z \frac{\left[\zeta^{-q} h(\zeta) - \frac{\zeta^{1-q}}{\Gamma(2-q)}\right]}{(z-\zeta)^{1-q}} \mathrm{d}\zeta \right|$$
$$\leq \sup_{z \in \overline{\mathbb{U}}} \left| \Gamma(1-q) h(z) - \frac{z}{1-q} \right|.$$

Then, utilizing (2.11) in the last inequality, it is easily obtained that $\sup_{z \in \overline{\mathbb{U}}} |u(z) - z| \leq M$ where $0 < M \leq \frac{\sqrt{20}}{5}$. Finally, from the Cauchy integral formula, we have

$$\left|u'(z) - 1\right| = \frac{1}{2\pi} \left| \int_{|\zeta|=1} \frac{u(\zeta) - \zeta}{(\zeta - z)^2} \mathrm{d}\zeta \right| \leq M$$

for all $z \in \overline{\mathbb{U}}$.

Therefore, if the result of Mocanu in [11] is considered in the last inequality, it is seen that u is starlike on $\mathbb{U}$.

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