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Number theory

On the denominators of harmonic numbers ☆

Sur les dénominateurs des nombres harmoniques

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ABSTRACT

Let H_n be the n-th harmonic number and let v_n be its denominator. It is well known that v_n is even for every integer $n \geq 2$. In this paper, we study the properties of v_n. One of our results is: the set of positive integers n such that v_n is divisible by the least common multiple of $1, 2, \cdots, \lfloor n^{1/4} \rfloor$ has density one. In particular, for any positive integer m, the set of positive integers n such that v_n is divisible by m has density one.

RÉSUMÉ

Soit H_n le n-ième nombre harmonique et notons v_n son dénominateur. Il est bien connu que v_n est pair pour tout entier $n \geq 2$. Dans ce texte, nous étudions les propriétés de v_n. Un de nos résultats montre que l'ensemble des entiers positifs n tels que v_n soit divisible par le plus petit commun multiple de $1, 2, \ldots, [n^{1/4}]$ est de densité 1. En particulier, pour tout entier positif m, l'ensemble des entiers positifs n tels que v_n soit divisible par m est de densité 1.

1. Introduction

For any positive integer n, let

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \frac{u_n}{v_n}, \quad (u_n, v_n) = 1, \ v_n > 0.$$

The number H_n is called *the n-th harmonic number*. In 1991, Eswarathasan and Levine [2] introduced I_p and J_p. For any prime number p, let J_p be the set of positive integers n such that $p \mid u_n$ and let I_p be the set of positive integers n such that $p \nmid v_n$. Here I_p and J_p are slightly different from those in [2]. In [2], Eswarathasan and Levine considered $0 \in I_p$ and $0 \in J_p$. It is clear that $J_p \subseteq I_p$.

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In 1991, Eswarathasan and Levine [2] conjectured that J_p is finite for any prime number p. In 1994, Boyd [1] confirmed that J_p is finite for $p \le 547$, except 83, 127, 397. For any set S of positive integers, let $S(x) = |S \cap [1, x]|$. In 2016, Sanna [3] proved that

$$J_p(x) \le 129\, p^{\frac{2}{3}}\, x^{0.765}.$$

Recently, Wu and Chen [5] proved that

$$J_p(x) \le 3\, x^{\frac{2}{3}+\frac{1}{25\log p}}. \tag{1.1}$$

For v_n, Shiu [4] proved that, for any primes $2 < p_1 < p_2 < \cdots < p_k$, there exists n such that the least common multiple of $1, 2, \cdots, n$ is divisible by $p_1 \cdots p_k v_n$.

For any positive integer m, let I_m be the set of positive integers n such that $m \nmid v_n$. In this paper, the following results are proved.

Theorem 1.1. *The set of positive integers n such that v_n is divisible by the least common multiple of $1, 2, \cdots, \lfloor n^{1/4} \rfloor$ has density one.*

Theorem 1.2. *For any positive integer m and any positive real number x, we have*

$$I_m(x) \le 4\; m^{\frac{1}{3}} x^{\frac{2}{3}+\frac{1}{25\log q_m}},$$

where q_m is the least prime factor of m.

From Theorem 1.1 or Theorem 1.2, we immediately have the following corollary.

Corollary 1.3. *For any positive integer m, the set of positive integers n such that $m \mid v_n$ has density one.*

2. Proofs

We always use p to denote a prime. Firstly, we give the following two lemmas.

Lemma 2.1. *For any prime p and any positive integer k, we have*

$$I_{p^k} = \{p^k n_1 + r : n_1 \in J_p \cup \{0\},\ 0 \le r \le p^k - 1\} \setminus \{0\}.$$

Proof. For any integer a, let $\nu_p(a)$ be the p-adic valuation of a. For any rational number $\alpha = \frac{a}{b}$, let $\nu_p(\alpha) = \nu_p(a) - \nu_p(b)$. It is clear that $n \in I_{p^k}$ if and only if $\nu_p(H_n) > -k$.

If $n < p^k$, then $\nu_p(H_n) \ge -\nu_p([1, 2, \cdots, n]) > -k$. So $n \in I_{p^k}$. In the following, we assume that $n \ge p^k$. Let

$$n = p^k n_1 + r, \quad 0 \le r \le p^k - 1,\ n_1, r \in \mathbb{Z}.$$

Then $n_1 \ge 1$. Write

$$H_n = \sum_{m=1,\, p^k \nmid m}^{n} \frac{1}{m} + \frac{1}{p^k} H_{n_1} = \frac{b}{p^{k-1}a} + \frac{u_{n_1}}{p^k v_{n_1}} = \frac{pbv_{n_1} + au_{n_1}}{p^k a v_{n_1}}, \tag{2.1}$$

where $p \nmid a$ and $(u_{n_1}, v_{n_1}) = 1$.

If $n_1 \in J_p$, then $p \mid u_{n_1}$ and $p \nmid v_{n_1}$. Thus $p \mid au_{n_1} + pbv_{n_1}$ and $\nu_p(p^k a v_{n_1}) = k$. By (2.1), $\nu_p(H_n) > -k$. So $n \in I_{p^k}$.

If $n_1 \notin J_p$, then $p \nmid u_{n_1}$. Thus $p \nmid au_{n_1} + pbv_{n_1}$. It follows from (2.1) that $\nu_p(H_n) \le -k$. So $n \notin I_{p^k}$.

Now we have proved that $n \in I_{p^k}$ if and only if $n_1 \in J_p \cup \{0\}$.

This completes the proof of Lemma 2.1. □

Lemma 2.2. *For any prime power p^k and any positive number x, we have*

$$I_{p^k}(x) \le 4 (p^k)^{\frac{1}{3}-\frac{1}{25\log p}} x^{\frac{2}{3}+\frac{1}{25\log p}}.$$

Proof. If $x \le p^k$, then

$$I_{p^k}(x) \le x < 4x^{\frac{1}{3}-\frac{1}{25\log p}} x^{\frac{2}{3}+\frac{1}{25\log p}} \le 4(p^k)^{\frac{1}{3}-\frac{1}{25\log p}} x^{\frac{2}{3}+\frac{1}{25\log p}}.$$

Now we assume that $x > p^k$. By Lemma 2.1 and (1.1), we have

$$I_{p^k}(x) = |\{p^k n_1 + r \le x : n_1 \in J_p \cup \{0\},\ 0 \le r \le p^k - 1\}| - 1$$
$$\le p^k \left(J_p(\frac{x}{p^k}) + 1 \right) \le 4(p^k)^{\frac{1}{3} - \frac{1}{25 \log p}} x^{\frac{2}{3} + \frac{1}{25 \log p}}.$$

This completes the proof of Lemma 2.2. □

Proof of Theorem 1.1. Let m_n be the least common multiple of $1, 2, \cdots, \lfloor n^\theta \rfloor$, where $\lfloor n^\theta \rfloor$ denotes the greatest integer not exceeding the real number n^θ and $0 < \theta < 1$, which will be given later. Let $T = \{n : m_n \nmid v_n\}$. For any prime p and any positive number x with $p \le x^\theta$, let α_p be the integer such that $p^{\alpha_p} \le x^\theta < p^{\alpha_p + 1}$.

By the definitions of m_n and T,

$$T(x) \le \sum_{p \le x^\theta} I_{p^{\alpha_p}}(x).$$

In view of Lemma 2.2, we have

$$\sum_{p \le x^\theta} I_{p^{\alpha_p}}(x) \le 4 \sum_{p \le x^\theta} (p^{\alpha_p})^{\frac{1}{3}} x^{\frac{2}{3} + \frac{1}{25 \log p}} := S_1 + S_2,$$

where

$$S_1 = 4 \sum_{x^\delta < p \le x^\theta} (p^{\alpha_p})^{\frac{1}{3}} x^{\frac{2}{3} + \frac{1}{25 \log p}}, \quad S_2 = 4 \sum_{p \le x^\delta} (p^{\alpha_p})^{\frac{1}{3}} x^{\frac{2}{3} + \frac{1}{25 \log p}}$$

and δ is a positive constant less than θ which will be given later.

If $p > x^\delta$, then

$$x^{\frac{1}{25 \log p}} = e^{\frac{\log x}{25 \log p}} \le e^{\frac{\log x}{25 \delta \log x}} = e^{\frac{1}{25\delta}}.$$

It follows from $p^{\alpha_p} \le x^\theta$ and a Chebychev-type bound for $\pi(x)$ that

$$S_1 = 4 \sum_{x^\delta < p \le x^\theta} (p^{\alpha_p})^{\frac{1}{3}} x^{\frac{2}{3} + \frac{1}{25 \log p}} \le 4 e^{\frac{1}{25\delta}} \sum_{x^\delta < p \le x^\theta} x^{\frac{\theta}{3} + \frac{2}{3}} \ll \frac{1}{\log x} x^{\frac{4\theta}{3} + \frac{2}{3}}.$$

For S_2, by $p^{\alpha_p} \le x^\theta$ and a Chebychev-type bound for $\pi(x)$, we have

$$S_2 = 4 \sum_{p \le x^\delta} (p^{\alpha_p})^{\frac{1}{3}} x^{\frac{2}{3} + \frac{1}{25 \log p}}$$
$$\le 4 \sum_{p \le x^\delta} x^{\frac{\theta}{3} + \frac{2}{3} + \frac{1}{25 \log 2}}$$
$$\ll \frac{1}{\log x} x^{\delta + \frac{\theta}{3} + \frac{2}{3} + \frac{1}{25 \log 2}}.$$

We choose $\theta = \frac{1}{4}$ and $\delta = 0.1$. Then

$$S_1 \ll \frac{x}{\log x}, \quad S_2 \ll x^{0.91}.$$

Therefore,

$$T(x) \le \sum_{p \le x^\theta} I_{p^{\alpha_p}}(x) = S_1 + S_2 \ll \frac{x}{\log x}.$$

It follows that the set of positive integers n such that v_n is divisible by the least common multiple of $1, 2, \cdots, \lfloor n^{1/4} \rfloor$ has density one. This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. We use induction on m to prove Theorem 1.2.

By Lemma 2.2, Theorem 1.2 is true for $m = 2$. Suppose that Theorem 1.2 is true for all integers less than m $(m > 2)$.

If $x \le m$, then

$$I_m(x) \le x < 4 x^{\frac{1}{3}} x^{\frac{2}{3} + \frac{1}{25 \log q_m}} \le 4 m^{\frac{1}{3}} x^{\frac{2}{3} + \frac{1}{25 \log q_m}}.$$

In the following, we always assume that $x > m$.

If m is a prime power, then, by Lemma 2.2, Theorem 1.2 is true. Now we assume that m is not a prime power. Write m as $m = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ with

$$1 < p_1^{\alpha_1} < \cdots < p_r^{\alpha_r},$$

where $p_1, \cdots, p_r$ are distinct primes, $r \geq 2$, and let $p_1^{\alpha_1} = p^\alpha$ and $m_1 = m/p^\alpha$. Then $m_1 > p^\alpha$. It is clear that $I_m = I_{m_1} \bigcup (I_{p^\alpha} \setminus I_{m_1})$. By Lemma 2.1 and the definition of p^α, $\{1, 2, \cdots, p^\alpha - 1\} \subseteq I_{m_1}$. Hence

$$I_m(x) = I_{m_1}(x) + (I_{p^\alpha} \setminus I_{m_1})(x) \leq I_{m_1}(x) + I_{p^\alpha}(x) - (p^\alpha - 1).$$

By the inductive hypothesis, we have

$$I_{m_1}(x) \leq 4m_1^{\frac{1}{3}} x^{\frac{2}{3} + \frac{1}{25 \log q m_1}} \leq 4m_1^{\frac{1}{3}} x^{\frac{2}{3} + \frac{1}{25 \log q m}}.$$

It follows that

$$I_m(x) \leq 4m_1^{\frac{1}{3}} x^{\frac{2}{3} + \frac{1}{25 \log q m}} + I_{p^\alpha}(x) - (p^\alpha - 1). \tag{2.2}$$

We divide into the following three cases:

Case 1: $p^\alpha \geq 8$. Then $m_1 > p^\alpha \geq 8$. By Lemma 2.2, we have

$$I_{p^\alpha}(x) \leq 4(p^\alpha)^{\frac{1}{3}} x^{\frac{2}{3} + \frac{1}{25 \log q m}}.$$

It follows from (2.2) that

$$\begin{aligned} I_m(x) &\leq 4m_1^{\frac{1}{3}} x^{\frac{2}{3} + \frac{1}{25 \log q m}} + 4(p^\alpha)^{\frac{1}{3}} x^{\frac{2}{3} + \frac{1}{25 \log q m}} \\ &= 4 \left(\frac{1}{(p^\alpha)^{\frac{1}{3}}} + \frac{1}{m_1^{\frac{1}{3}}} \right) m^{\frac{1}{3}} x^{\frac{2}{3} + \frac{1}{25 \log q m}} \\ &\leq 4m^{\frac{1}{3}} x^{\frac{2}{3} + \frac{1}{25 \log q m}}. \end{aligned}$$

Case 2: $p^\alpha < 8$, $p = 2$. Then $p^\alpha = 2$ or 4 and $x > m \geq 2 \times 3 = 6$. By Lemma 2.1 and $J_2 = \emptyset$, we have $I_4 = \{1, 2, 3\}$ and $I_2 = \{1\}$. It is clear that $I_{p^\alpha}(x) - (p^\alpha - 1) = 0$. It follows from (2.2) that

$$I_m(x) \leq 4m_1^{\frac{1}{3}} x^{\frac{2}{3} + \frac{1}{25 \log q m}} < 4m^{\frac{1}{3}} x^{\frac{2}{3} + \frac{1}{25 \log q m}}.$$

Case 3: $p^\alpha < 8$, $p \neq 2$. Then $\alpha = 1$ and $p = 3$, 5 or 7. In addition, $x > m \geq 3 \times 4 = 12$. Noting that $m^{\frac{1}{3}} - m_1{}^{\frac{1}{3}} = m_1{}^{\frac{1}{3}}(p^{\frac{\alpha}{3}} - 1) \geq 4^{\frac{1}{3}}(3^{\frac{1}{3}} - 1) > \frac{1}{2}$, by (2.2), it is enough to prove that $I_p(x) - (p-1) \leq 2x^{\frac{2}{3}}$. By Lemma 2.1, we have

$$I_p = \{pn_1 + r : n_1 \in J_p \cup \{0\},\ 0 \leq r \leq p-1\} \setminus \{0\}.$$

By [2], $J_3 = \{2, 7, 22\}$, $J_5 = \{4, 20, 24\}$ and

$$J_7 = \{6, 42, 48, 295, 299, 337, 341, 2096, 2390, 14675, 16731, 16735, 102728\}.$$

If $x \geq 7^3$, then $I_p(x) - (p-1) \leq 91 \leq 2x^{\frac{2}{3}}$. If $35 < x < 7^3$, then $I_p(x) - (p-1) \leq 21 \leq 2x^{\frac{2}{3}}$. If $12 < x \leq 35$, then $I_p(x) - (p - 1) \leq 6 \leq 2x^{\frac{2}{3}}$.

This completes the proof of Theorem 1.2. □

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