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Group theory

On generalized categories of Soergel bimodules in type A_2

Sur les catégories de bimodules de Soergel généralisées de type A_2

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A R T I C L E   I N F O

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A B S T R A C T

In this note, we compute the split Grothendieck ring of a generalized category of Soergel bimodules of type A_2, where we take one generator for each reflection. We give a presentation by generators and relations of it and a parametrization of the indecomposable objects of the category, by realizing them as rings of regular functions on certain unions of graphs of group elements on a reflection faithful representation.

R É S U M É

Le but de cette note est de décrire l'anneau de Grothendieck scindé d'une catégorie de bimodules de Soergel généralisée de type A_2, où l'on prend un générateur par réflexion. On donne une présentation par générateurs et relations de cette algèbre ainsi qu'une paramétrisation des objets indécomposables de la catégorie, en les réalisant comme anneaux de fonctions régulières sur des réunions de graphes d'éléments du groupe de Coxeter sur une représentation réflexion-fidèle.

1. Introduction

Let (W, S) be a Coxeter system, V a reflection faithful representation (as defined in [5, Definition 1.5]) of W over the real numbers. Let $R = \mathcal{O}(V) = S(V^*)$ be the ring of polynomial functions on V, graded so that $\deg(V^*) = 2$. To each simple reflection $s \in S$, one associates the graded R-bimodule $B_s := R \otimes_{R^s} R(1)$ (here (1) denotes a grading shift), where $R^s \subseteq R$ is the graded subring of s-invariant functions. Soergel showed that the split Grothendieck ring $\langle \mathcal{B} \rangle$ of the Karoubian envelope $\mathcal{B}$ of the category of graded R-bimodules generated by (shifted) tensor products of the B_s over R is isomorphic to the Iwahori–Hecke algebra $\mathcal{H}(W)$ of the Coxeter system (W, S) ([5, Theorem 1.10, Remark 1.14]). The class $\langle B_s \rangle$ of B_s in $\langle \mathcal{B} \rangle$ corresponds to the Kazhdan–Lusztig generator C'_s of $\mathcal{H}(W)$, see [3]. More generally, Soergel's conjecture [5, Conjecture 1.13] (which was proven in full generality by Elias and Williamson [1]) asserts that to each element C'_w of the canonical

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Kazhdan–Lusztig basis corresponds an indecomposable bimodule in $\mathcal{B}$. The bimodule corresponding to w generalizes the (equivariant) intersection cohomology of the Schubert variety X_w (which can be defined only in the case where W is a Weyl group; see [4]).

The definition of B_s makes sense for every reflection $s \in T := \bigcup_{w \in W} wSw^{-1}$, not only for the simple reflections $s \in S$. In this note, we compute the split Grothendieck ring $\langle \mathcal{C} \rangle$ of the category $\mathcal{C}$ generated by the $B_t := R \otimes_{R^t} R(1)$, $t \in T$ in the case where W is of type A_2 (note that in Soergel's category $\mathcal{B}$, the notation B_t stands for the indecomposable bimodule associated with the group element t, which in general is *not* isomorphic to $R \otimes_{R^t} R(1)$; in this note, B_t will always denote $R \otimes_{R^t} R(1)$). This ring is a free $\mathbb{Z}[v^{\pm 1}]$-algebra $A(W)$ of rank 20, and we identify the indecomposable objects in $\mathcal{C}$: they are in one-to-one correspondence with subsets $A \subseteq W$ such that there is $t \in T$ with $tA = A$. We give a presentation by generators and relations of the resulting algebra $A(W)$, which turns out to be a quotient of the Iwahori–Hecke algebra of type $\widetilde{A_2}$. While the category $\mathcal{C}$ can be defined for an arbitrary Coxeter group W, we do not know how to compute $\langle \mathcal{C} \rangle$ for W of type A_3 or B_2, where it is not even clear to us that $\langle \mathcal{C} \rangle$ has finite rank.

It would be interesting to find a well-behaved generalization of the algebra obtained in type A_2 for finite W or in type A_n, possibly by considering split Grothendieck rings of suitable subcategories of $\mathcal{C}$ containing $\mathcal{B}$.

2. General facts

Let $x \in W$ and let R_x denote the R-bimodule R with the right operation twisted by x, that is, for $a \in R_x$, $r \in R$ we set $a \cdot r = ax(r)$. Note that the embedding $V \hookrightarrow V \times V$, $v \mapsto (v, x^{-1}v)$ induces an isomorphism of graded bimodules $R_x \cong \mathcal{O}(\mathrm{Gr}(x))$, where $\mathrm{Gr}(x) := \{(xv, v) \mid v \in V\}$ and $\mathcal{O}(-)$ denotes the $\mathbb{R}$-algebra of regular functions. We have $R_x \otimes_R R_y \cong R_{xy}$ for all $x, y \in W$. Note the following isomorphisms:

$$R \otimes_{R^t} R_t \cong R \otimes_{R^t} R \cong R_t \otimes_{R^t} R, \forall t \in T, \tag{2.1}$$

$$R_w B_t \cong B_{wtw^{-1}} R_w, \forall t \in T, w \in W. \tag{2.2}$$

The first isomorphisms are given by the maps $a \otimes b \mapsto a \otimes t(b)$ and $a \otimes b \mapsto t(a) \otimes b$, respectively. Hence $B_t \otimes_R R_t \cong B_t \cong R_t \otimes_R B_t$. The last one is given by the well-defined invertible map $R_w \otimes_{R^t} R \to R \otimes_{R^{wtw^{-1}}} R_w$, $a \otimes b \mapsto a \otimes w(b)$.

For simplicity, we will denote tensor products over R by juxtaposition. A consequence of the above isomorphisms is that for all $s, t_1, \ldots, t_k \in T$, one has isomorphisms of graded R-bimodules

$$B_s B_{t_1} B_{t_2} \cdots B_{t_k} B_s \cong B_s B_{st_1s} B_{st_2s} \cdots B_{st_ks} B_s. \tag{2.3}$$

We denote by $\mathcal{C}$ the category obtained as the Karoubi envelope of the category of (shifted) tensor products of B_t for $t \in T$. We denote by $\mathcal{C}^{\text{ext}}$ the category generated by (shifted) tensor products of B_t for $t \in T$ and R_w for $w \in W$. Note that one has inclusions as full subcategories $\mathcal{B} \subseteq \mathcal{C} \subseteq \mathcal{C}^{\text{ext}}$. By (2.2), observe also that $\mathcal{C}^{\text{ext}}$ is generated by the bimodules B_s for $s \in S$ and R_w, for $w \in W$.

The isomorphisms given in (2.3) yield a family of relations satisfied by the generators of $\mathcal{C}$, in addition to those that hold in $\mathcal{B} \subseteq \mathcal{C}$. We assume the reader to be familiar with the combinatorics of Soergel's category $\mathcal{B}$.

Given $A \subseteq W$, we write $R(A)$ for the graded R-bimodule $\mathcal{O}\left(\bigcup_{x \in A} \mathrm{Gr}(x)\right)$. Note that $\bigcup_{x \in A} \mathrm{Gr}(x)$ is a closed subscheme of $V \times V$, inducing a surjective map of graded R-bimodules $\mathcal{O}(V \times V) \cong R \otimes_{\mathbb{R}} R \twoheadrightarrow R(A)$. It implies that $R(A)$ is generated as a graded R-bimodule by any nonzero element in its degree-zero component, hence that it is indecomposable, since this component is one-dimensional. By convention we set $R(\emptyset) = 0$.

We have $B_t \cong R(\{e, t\})(1)$ (see [5, Remark 4.3]). For more on the properties of rings of regular functions on unions of graphs, we refer the reader to [6, Section 4.3].

For $w \in W$, $A \subseteq W$, we have

$$R_w \otimes_R R(A) \cong R(wA) \text{ and } R(A) \otimes_R R_w \cong R(Aw), \tag{2.4}$$

where the first map is given by $a \otimes b \mapsto \{(u, v) \mapsto a(u) b\left(w^{-1}u, v\right)\}$ and the second one by $a \otimes b \mapsto \{(u, v) \mapsto a(u, wv)\, b(wv)\}$.

Lemma 2.5 *([5, Lemma 4.5 (1)]). Let $A \subseteq W$, $t \in T$ such that $tA = A$. Then $R \otimes_{R^t} R(A) \cong R(A) \oplus R(A)(-2)$.*

From now on, we assume that (W, S) is dihedral, that is, that $|S| = 2$. In that case, there holds the following Lemma, which will allow us to parametrize the indecomposable bimodules in $\mathcal{C}$ in the case where (W, S) is of type $A_2 = I_2(3)$.

Lemma 2.6 *([5]). Let $A \subseteq W$, $t, s \in T$ such that $tA = A$ and $|A \setminus (A \cap sA)| = 2$. Then*

$$R \otimes_{R^s} R(A) \cong R\left(A \cup sA\right) \oplus R\left(A \cap sA\right)(-2).$$

Proof. One can give exactly the same proof as in [5, Proposition 4.6]: in the proof there, one considers sets A of the form $A = \{y \in W \mid y \leq x\}$ where $\leq$ is the Bruhat order on W, and one uses the fact that $A\setminus(A \cap sA) = \{x, t'x\}$ for some $t' \in T$ with $t' \neq s$ (Soergel assumes $s \in S$). But, by looking at the proof, one sees that it works for any $s \in T$ and any A such that $A\setminus(A \cap sA)$ is a t'-stable subset of cardinality 2 for some reflection $t' \neq s$, which can be easily observed in our case: since $A = tA$, we have $|A| = 2k$ for some $k \geq 1$, and exactly half of the elements in A have odd length with respect to the generating set S of W (equivalently, are reflections). But since $A \cap sA$ is s-stable, the same holds for $A \cap sA$, implying that among the two elements x, y of $A\setminus(A \cap sA)$, exactly one, say x, has odd length, while y has even length. It implies that $t' := yx^{-1}$ has odd length, hence is a reflection, which shows the claim. □

3. Parametrizing the indecomposable objects in type A_2

In this section, we assume (W, S) to be of type A_2. The aim of this section is to identify the indecomposable bimodules in $\mathcal{C}$ in this case, using Lemmas 2.5 and 2.6. To this end, consider the set

$$\mathcal{X} := \{\emptyset \neq A \subseteq W \mid \exists t \in T : tA = A\}.$$

There are 19 elements belonging to $\mathcal{X}$, given by W, 3 t-stable subsets of cardinality 2 for each $t \in T = \{t_1, t_2, t_3\}$, and their complements.

Proposition 3.1. *The unshifted indecomposable bimodules in $\mathcal{C}$ are given by $R(A)$, $A \in \mathcal{X}$, and R. In particular, the $\mathbb{Z}[v^{\pm 1}]$-algebra $A(W) := \langle \mathcal{C} \rangle$ (where v acts by a grading shift (1)) has rank 20 as a free $\mathbb{Z}[v^{\pm 1}]$-module.*

Before establishing the above Proposition, we prove the following lemma.

Lemma 3.2. *Let $A \in \mathcal{X}$, $t \in T$. Then either $tA = A$, or $A\setminus(A \cap tA)$ is a t'-stable subset of cardinality 2 for some $t' \in T$.*

Proof. Assume that $tA \neq A$. By definition of $\mathcal{X}$, there is $s \in T$ such that $sA = A$. If $|A| = 2$, then since $s \neq t$ we have $A \cap tA = \emptyset$ and we are done. If $|A| = 4$, then $|A \cap tA| = 2$ since $A \neq tA$ and $|W| = 6$. But $A\setminus(A \cap tA)$ has exactly one element of even length and one of odd length, implying that it is t'-stable for some $t' \in T$. □

As an immediate Corollary of 2.5, 3.2, 2.6, we have that, for all $t \in T$ and $A \in \mathcal{X}$, the tensor product $B_t \otimes_R R(A)$ decomposes as a direct sum of $R(B)$, $B \in \mathcal{X}$. Together with the fact that $B_t \cong R(\{e, t\})(1)$ for all $t \in T$, we deduce that $\mathcal{C}$ has at most 20 unshifted indecomposable bimodules, and it remains to prove that $R(A)$ occurs as a an indecomposable bimodule of $\mathcal{C}$ for every $A \in \mathcal{X}$.

Proof of Proposition 3.1. We have to show that for all $A \in \mathcal{X}$, $R(A)$ occurs as a direct summand of (a shift of) $B_{t_1} B_{t_2} \cdots B_{t_k}$ for some $t_i \in T$. Since $A \in \mathcal{X}$, let us fix $t \in T$ such that $tA = A$.

Assume that $|A| = 2$. If $A = \{e, t\}$, then $R(A) = B_t(-1)$ in which case we are done. If $A \neq \{e, t\}$, then A is necessarily of the form $A = \{t_1, tt_1\}$ for $t, t_1 \in T$, $t \neq t_1$ (in particular, we have $tt_1t = t_1tt_1$). Applying twice Lemma 2.6, we have first $B_{t_1} B_{t_1tt_1} \cong R(\{t_1, t_1tt_1, e, tt_1\})(2)$ and then

$$B_t R(\{t_1, t_1tt_1, e, tt_1\}) \cong R(W)(1) \oplus R(\{t_1, tt_1\})(-1),$$

which shows that $R(A)$ and $R(W)$ both appear as a summand of a shift of $B_t B_{t_1} B_{t_1tt_1}$.

Assume that $|A| = 4$. Write $A = \{tx, x, ty, y\}$. If $e \in A$, then, without loss of generality, we can assume that $x = e$ and $y = t_1 \in T$ with $t_1 \neq t$. We then have $R(A) \cong B_t B_{t_1}(-2)$ by Lemma 2.6. If $e \notin A$, then we can assume, without loss of generality, that $x = t_1 \in T$ and $y = tt_1t$. We then have by Lemma 2.6 that $R(A) \cong B_t \otimes R(\{t_1, t_1t\})(-1)$. But, as already proved, the bimodule associated with the (t_1tt_1)-stable set $\{t_1, t_1t\}$ of cardinality 2 has to appear as a summand of $B_{t_1tt_1} B_{t_1} B_t(-1)$, hence $R(A)$ appears as a summand of $B_t B_{t_1tt_1} B_{t_1} B_t(-2)$. □

By Relation (2.4), tensoring a $B_t(-1)$ with an R_w gives an indecomposable bimodule $R(A)$ with $A \in \mathcal{X}$, which by Proposition 3.1, lies in $\mathcal{C}$. As a consequence we get:

Lemma 3.3. *The unshifted indecomposable bimodules in $\mathcal{C}^{\text{ext}}$ are given by the unshifted indecomposable ones in $\mathcal{C}$ and the R_w for $w \in W\setminus\{e\}$. In particular, the split Grothendieck ring $\langle \mathcal{C}^{\text{ext}} \rangle$ has rank 25.*

4. A presentation by generators and relations

In this section, we give a presentation by generators and relations of $A(W) = \langle \mathcal{C} \rangle$ in type A_2. Let (W, S) be of type A_2, with set of reflections $T = \{t_1, t_2, t_3\}$.

Theorem 4.1. *The algebra $A(W)$ is generated as $\mathbb{Z}[v^{\pm 1}]$-algebra by C_i, $i = 1, 2, 3$ with relations*

(1) $C_i^2 = (v + v^{-1})C_i$, $\forall i = 1, 2, 3$,
(2) $C_iC_jC_i + C_j = C_i + C_jC_iC_j$, $\forall i \neq j$,
(3) $C_iC_jC_i = C_iC_kC_i$, *if* $\{i, j, k\} = \{1, 2, 3\}$,
(4) $C_iC_jC_kC_i = C_iC_kC_jC_i$, *if* $\{i, j, k\} = \{1, 2, 3\}$.

For all i, we have $C_i = \langle B_{t_i} \rangle$ *and* $\langle R(1) \rangle = v$.

Remark 4.2. Note that the algebra defined above is a quotient of the Hecke algebra $\mathcal{H}(W_{\widetilde{A}_2})$ of the affine Weyl group of type $\widetilde{A}_2$, as the Kazhdan–Lusztig presentation of this Hecke algebra has three generators satisfying precisely the Relations (1) and (2) above.

Proof. We first show that the above relations are satisfied in $\mathcal{C}$: the first relation is a consequence of Lemma 2.5. The second relation holds in Soergel's category $\mathcal{B}$ in case $\{t_i, t_j\} = S$, but it generalizes here using either Lemma 2.6 or by conjugating the relation in Soergel's category by an R_s, $s \in S$ and using Relations (2.1)–(2.3). The last two relations are just a particular case of Relation (2.3). This shows that $\langle \mathcal{C} \rangle$ is a quotient of $A(W)$. Since, by Proposition 3.1, we know that $\langle \mathcal{C} \rangle$ is free of rank 20, it suffices to show that the algebra defined by the above presentation is $\mathbb{Z}[v^{\pm 1}]$-linearly spanned by a set of 20 elements. We show that every monomial in the C_i's can be expressed as a linear combination of the 20 elements 1, C_1, C_2, C_3, C_1C_2, C_2C_1, C_2C_3, C_3C_2, C_1C_3, C_3C_1, $C_1C_2C_1$, $C_iC_jC_k$ with $\{i, j, k\} = \{1, 2, 3\}$, $C_1C_2C_3C_1$, $C_2C_1C_3C_2$ and $C_3C_1C_2C_3$. To this end, it suffices to show that any word of length at most five in the C_i's is a linear combination of these 20 elements.

For words of length at most two, it is clear since $C_i^2 = (v + v^{-1})C_i$. Given a word $C_iC_jC_k$ of length three, if $|\{i, j, k\}| = 3$, then our word is an element of the above list. If two consecutive letters are the same, then we are done using again the quadratic relation. If $i = k$, $j \neq i$, then using the second and third relation, we can express our word as a linear combination of $C_1C_2C_1$ and C_1, C_2, C_3. Now consider a word $C_iC_jC_kC_\ell$ of length four. If $i = \ell$, then if $|\{i, j, k\}| = 3$, up to permuting the two middle letters using the fourth relation, our word belongs to the above list. If $|\{i, j, k\}| \neq 3$, then two consecutive letters in the word have to agree, hence we can apply a quadratic relation to express our word as a linear combination of words of length three for which we already shown the result. Assume that $i \neq \ell$. Then either a quadratic relation can be applied, or the word is $C_iC_jC_iC_\ell$ or $C_iC_\ell C_kC_\ell$, with $|\{i, j, \ell\}| = 3$ (resp. $|\{i, k, \ell\}| = 3$) or $C_iC_\ell C_iC_\ell$. Assume that the word is $C_iC_jC_iC_\ell$, the other case is symmetric. Then applying the third relation, we get $C_iC_\ell C_iC_\ell$, which, using the first two relations, can be expressed as a linear combination or words of length at most three. Now, if $C_iC_jC_kC_\ell C_m$ is a word of length five, then we can assume that $\ell = i$, otherwise we already saw that the word $C_iC_jC_kC_\ell$ is a linear combination of words of length at most three: it implies that $C_iC_jC_kC_\ell C_m$ is a linear combination of words of length at most four, for which we already know the result. If $m = i$, then the word is $C_iC_jC_kC_i^2$, which can be reduced to a linear combination of words of length four by applying the quadratic relation. Note that we can assume $\{i, j, k\} = 3$, otherwise we already saw that the word $C_iC_jC_kC_i$ can be expressed as a linear combination of words of length at most three. Assume that $m = k$, the case $m = j$ being similar after applying the relation $C_iC_jC_kC_i = C_iC_kC_jC_i$. Thanks to the third relation, we have

$$C_iC_jC_kC_iC_k = C_iC_jC_kC_jC_k,$$

and using the first two relations, the word $C_jC_kC_jC_k$ can be expressed as a linear combination of words of smaller length. Hence, $C_iC_jC_kC_iC_k$ can be expressed as a linear combination of words of length at most four, for which we have already shown the property. □

Remark 4.3. It can be observed by straightforward computations that any tensor product of the generators of $\mathcal{C}^{\text{ext}}$, in case W is of type A_2, is isomorphic to a bimodule of the form

$$R_wB_{s_1}\cdots B_{s_k}R_{w'} \tag{4.4}$$

with $w, w' \in W$ and $s_1, \cdots s_k$ simple reflections. Since the indecomposable objects in $\mathcal{B}$ are fully understood and since tensoring with R_w defines an invertible functor, the classification of indecomposables in $\mathcal{C}$ that we made in the previous section can also be derived from this fact (but the proof with rings of regular functions appears as more conceptual to us). In Section 5, we show that this property is not fulfilled in the category $\mathcal{C}^{\text{ext}}$ attached to other Coxeter groups, by providing explicit counterexamples.

5. Some remarks about other Coxeter groups

The Coxeter group of type A_2 is the smallest Coxeter group with a braid relation, and in many cases the situation in type A_2 gives a hint of a more general situation. In this section, we give a few examples observed in bigger groups, which show that the strategy used in type A_2 is too naive to work in higher rank and even for other dihedral groups. We actually do not know what the correct generalization of the algebra $A(W)$ should be. In type A_2, one can think of the added generator B_t, where t is the non-simple reflection, as being associated with the highest root of W viewed as a Weyl group, but we are not even able to describe the Grothendieck ring of the category generated by the $B_s, s \in S$ and B_t, where t is the reflection associated with the highest root, in other cases.

5.1. Type B_2

In Soergel's category $\mathcal{B}$, when W is a dihedral group, then every unshifted indecomposable bimodule is isomorphic to an $R(A)$ (see [5, Section 4]). We have seen above that this stays true in $\mathcal{C}$ in the case where W is of type A_2. We give a counterexample of this fact for the category $\mathcal{C}$ in the case where W is of type B_2. Let W be such a Coxeter group with $S = \{s, t\}$. Consider the bimodule $B := B_{tst} B_s B_t$. Note that $B_s B_t \cong R(\{e, s, t, st\})(2)$ is a Soergel bimodule, but we cannot apply Lemma 2.6 to decompose B. As a Soergel bimodule, $B_s B_t$ possesses a standard and a costandard filtration (as defined in [5, Section 5]). There is a short exact sequence

$$0 \longrightarrow R_{tst}(-1) \longrightarrow B_{tst} \longrightarrow R(1) \longrightarrow 0.$$

Tensoring this short exact sequence by $B_s B_t$, we see, using the isomorphism theorems, that B has a filtration where (a shifted copy of) every R_x, $x \in W$, appears exactly once as subquotient (in fact, using twisted filtrations of $B_s B_t$ as in [2], it is not difficult to show that B has both a standard and a costandard filtration).

Lemma 5.1. *Let $w \in W$. Then $R_w B \cong B$.*

Proof. It suffices to show that $R_q B \cong B$ for all $q \in S$. We have

$$R_s B_{tst} B_s B_t \cong B_{tst} R_s B_s B_t \cong B_{tst} B_s B_t \cong B,$$

where the first isomorphism follows from Relation (2.2) and the second one from Relation (2.1). Similarly, we have

$$R_t B_{tst} B_s B_t \cong B_s R_t B_s B_t \cong B_s B_{tst} B_t \cong B_{tst} B_s B_t,$$

where the first isomorphism follows from Relation (2.2), the second one from both (2.1) and (2.2), and the last one uses the fact that $B_{tst} B_s \cong B_s B_{tst}$ because s and tst commute (which follows, for instance, from Lemma 2.6). □

It follows that if B was decomposable, then tensoring on the left by any R_w would permute the various summands, i.e. if B_1 is a summand, then $R_w B_1$ is also a summand. Using this property (for various w) together with the fact that a direct summand of a bimodule with a standard filtration also inherits a standard filtration, it is easy to check that B is indecomposable. Now, the only bimodule of the form $R(A)$ that has a filtration where each R_x for all $x \in W$ appears as a subquotient is $R(W)$, and comparing the graded dimensions of (any shift of) $R(W)$ and B, one sees that they do not coincide. This shows that there is an indecomposable bimodule in $\mathcal{C}$ which is not isomorphic to a ring of regular functions on a union of twisted diagonals.

It also shows that B cannot be isomorphic to a (shift of a) bimodule of the form $R_w B' R_{w'}$ (see Remark 4.3), where $B' \in \mathcal{B}$ is an indecomposable Soergel bimodule and $w, w' \in W$, since by [5, Section 4] every indecomposable bimodule in $\mathcal{B}$ (and hence using Relations (2.4) every twist of it by R_w) is isomorphic to $R(A)$ for some $A \subseteq W$.

5.2. Type A_3

Let W be of type A_3, with $S = \{s, t, u\}$ such that $su = us$.

It is well known that already in $\mathcal{B}$, there are indecomposable bimodules that are not isomorphic to $R(A)$ for some $A \subseteq W$. For instance, the indecomposable Soergel bimodule B associated with the group element $tsut$ has two shifted copies of R_e in its standard filtration (equivalently the Kazhdan–Luzstig polynomial $h_{e,tust}$ is not a monomial), while an indecomposable Soergel bimodule of the form $R(A)$ has exactly one shifted copy of each R_x, $x \in A$ appearing in its standard filtration.

Consider the subcategory $\mathcal{C}' \subseteq \mathcal{C}$ generated by B_s, B_t, B_u, and B_{sts}. The subcategory $\mathcal{C}_1 \subseteq \mathcal{C}'$ generated by B_s, B_t and B_{sts} is equivalent to the category of type A_2 that we described above, but the subcategory $\mathcal{C}_2$ generated by B_{sts}, B_t and B_u is not: for instance, we have $B_t B_u B_t \ncong B_t B_{sts} B_t$. Note that using Relations (2.1) and (2.2), one sees that every subcategory of $\mathcal{C}_2$ generated by two of the generators is equivalent to a Soergel category of type A_2. It follows that $\langle \mathcal{C}_2 \rangle$ is again a quotient of the affine Hecke algebra of type $\widetilde{A_2}$, but it is not even clear to us whether the algebra $\langle \mathcal{C}_2 \rangle$ has finite rank or not (equivalently if there are finitely many indecomposables in $\mathcal{C}_2$ up to shifts).

As in type B_2, we give an example of an indecomposable bimodule B in $\mathcal{C}$ that cannot be isomorphic to a (shift of a) bimodule of the form $R_w B' R_{w'}$, where B' is an indecomposable bimodule in $\mathcal{B}$. Consider $B := B_s R_t B_u \cong B_s B_{tut} R_t$. This element has a filtration with subquotients given by (shifts of) R_t, R_{st}, R_{tu} and R_{stu}, each appearing exactly once. Now, the only indecomposable Soergel bimodules $B' \in \mathcal{B}$ with exactly four subquotients in their (twisted) standard filtrations are the B' associated with the group elements $x = s_1 s_2$, $s_i \in S$, $s_1 \neq s_2$ (and up to shifts the subquotients are $R, R_{s_1}, R_{s_2}, R_{s_1 s_2}$; the subquotients are independent of the filtration, but the shifts may differ). Set $A := \{t, st, tu, stu\}$. Assume that $B \cong R_w B' R_{w'} \cong R_{\widetilde{w}} \underbrace{R_{w'^{-1}} B' R_{w'}}_{=:B''}$ for $B' \in \mathcal{B}$ associated with such a group element x. Then B'' has a filtration with four subquotients given by (possibly shifted) R_y for $y = 1, t_1, t_2, t_1t_2$, where t_i are reflections ($t_i := w'^{-1} s_i w'$). It implies that $\widetilde{w}\{1, t_1, t_2, t_1t_2\} = A$; in particular, that $\widetilde{w} \in A$. Checking with the four possible values of $\widetilde{w}$, we see that $\widetilde{w}^{-1}A$ is never of the form $\{1, t_1, t_2, t_1t_2\}$, where t_i are reflections, which concludes.

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