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Harmonic analysis

Lipschitz conditions for the generalized discrete Fourier transform associated with the Jacobi operator on $[0, \pi]$

Conditions de Lipschitz pour la transformée de Fourier discrète généralisée associée à l'opérateur de Jacobi sur $[0, \pi]$

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A R T I C L E   I N F O

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Dedicated to Professor François Rouvière on the occasion of his 71st birthday

A B S T R A C T

Our aim in this paper is to prove an analog of the classical Titchmarsh theorem on the image under the discrete Fourier–Jacobi transform of a set of functions satisfying a generalized Lipschitz condition in the space $\mathbb{L}_2^{(\alpha,\beta)}$.

R É S U M É

L'objectif de cette Note est de prouver un analogue du théorème de Titchmarsh sur l'image sous la transformée de Fourier–Jacobi discrète d'un jeu de fonctions satisfaisant une condition de Lipschitz généralisée dans l'espace $\mathbb{L}_2^{(\alpha,\beta)}$.

1. Introduction

It is well known that many problems for partial differential equations are reduced to a power series expansion of the desired solution in terms of special functions or orthogonal polynomials (such as Laguerre, Hermite, Jacobi, etc., polynomials). In particular, this is associated with the separation of variables as applied to problems in mathematical physics (see, e.g., [10,12]).

In this article, we obtain an analog of one classical Titchmarsh theorem on description of the image under the discrete Fourier–Jacobi transform of a class of functions satisfying the Lipschitz condition in weighted function spaces on $[-1, 1]$. We now give the exact statement of this theorem.

Suppose that $f(x)$ is a function in the $L^2(\mathbb{R})$ space (all functions below are complex-valued), $\|.\|_{L^2(\mathbb{R})}$ is the norm of $L^2(\mathbb{R})$, and δ is an arbitrary number in the interval $(0, 1)$.

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Theorem 1.1. *[13, Theorem 85] Let $f \in L^2(\mathbb{R})$. Then the following conditions are equivalent:*

(i) $\|f(x+h) - f(x)\|_{L^2(\mathbb{R})} = O(h^\delta)$, *as* $h \to 0$, $0 < \delta < 1$,

(ii) $\displaystyle\int_{|\lambda|\geq r} |\widehat{f}(\lambda)|^2 \mathrm{d}\lambda = O(r^{-2\delta})$ *as* $r \to \infty$,

where $\widehat{f}$ stands for the Fourier transform of f.

This theorem has been generalized in the case of noncompact rank-1 Riemannian symmetric spaces [8], and was extended in [4] for the Fourier transform in the space $L^2(\mathbb{R}^n)$ using a spherical mean operator. The Titchmarsh's theorem has been generalized recently for a class of functions satisfying the Lipschitz condition for the Jacobi transform in [1] and also for the Dunkl transform in [3].

On the other hand, in [14, Theorem 2.17], M. S. Younis characterizes the set of functions in $L^2([-\pi, \pi])$ satisfying the Lipschitz condition by means of an asymptotic estimate growth of the norm of their discrete Fourier transform. More precisely, we have the following theorem.

Theorem 1.2. *[14, Theorem 2.17] Let $f \in L^2([-\pi, \pi])$. Then the following conditions are equivalent:*

(i) $\|f(x+h) - f(x)\|_{L^2([-\pi,\pi])} = O(h^\delta)$, *as* $h \to 0$, $\delta \in (0, 1)$,

(ii) $\displaystyle\sum_{|n|\geq N} |\widehat{f}(n)|^2 = O(N^{-2\delta})$ *as* $N \to \infty$,

where $\widehat{f}(n)$ stands for the n-th Fourier transform coefficient of f.

In our present paper, we investigate, among other things, the validity of Theorem 1.2 in case of functions of the wider Lipschitz class in weighted function spaces on $[-1, 1]$. For this purpose, we use a generalized translation operator, which was defined by Flensted-Jensen and Koornwinder (see [7]).

2. Preliminaries

Throughout the paper, α and β are arbitrary real numbers with $\alpha \geq \beta \geq -1/2$ and $\alpha \neq -1/2$. We put $w(x) = (1 - x)^\alpha (1+x)^\beta$ and consider problems of the approximation of functions in the Hilbert spaces $L_2([-1, 1], w(x)\mathrm{d}x)$.

Let $P_n^{(\alpha,\beta)}(x)$ be the Jacobi orthogonal polynomials, $n \in \mathbb{N}_0 := \{0, 1, 2, ...\}$ (see [11] or [6]). The polynomials $P_n^{(\alpha,\beta)}(x)$, $n \in \mathbb{N}_0$, form a complete orthogonal system in the Hilbert space $L_2([-1, 1], w(x)\mathrm{d}x)$.

It is known (see [11], Ch. IV) that

$$\max_{-1\leq x\leq 1} |P_n^{(\alpha,\beta)}(x)| = P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{\alpha} = \frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+1)}.$$

The polynomials

$$R_n^{(\alpha,\beta)}(x) := \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)}$$

are called normalized Jacobi polynomials.

In what follows it is convenient to change the variable by the formula $x = \cos t$, $t \in I := [0, \pi]$. We use the notation

$$\rho(t) = w(\cos t)\sin t = 2^\gamma \left(\sin\frac{t}{2}\right)^{2\alpha+1} \left(\cos\frac{t}{2}\right)^{2\beta+1}, \quad \gamma = \alpha+\beta+1,$$

$$\varphi_n(t) = \varphi_n^{(\alpha,\beta)}(t) := R_n^{(\alpha,\beta)}(\cos t), n \in \mathbb{N}_0.$$

Let $\mathbb{L}_2^{(\alpha,\beta)}$ denote the space of square integrable functions $f(t)$ on the closed interval I with the weight function $\rho(t)$ and the norm

$$\|f\| = \sqrt{\int_0^\pi |f(t)|^2 \rho(t)\mathrm{d}t}.$$

The Jacobi differential operator is defined as

$$\mathcal{B} := \frac{d^2}{dt^2} + \left(\left(\alpha + \frac{1}{2}\right)\cot\frac{t}{2} - \left(\beta + \frac{1}{2}\right)\tan\frac{t}{2}\right)\frac{d}{dt}.$$

The function $\varphi_n(t)$ satisfies the differential equation

$$\mathcal{B}\varphi_n = -\lambda_n\varphi_n, \qquad \lambda_n = n(n+\gamma), n \in \mathbb{N}_0,$$

with the initial conditions $\varphi_n(0) = 1$ and $\varphi_n'(0) = 0$.

Lemma 2.1. *The following inequalities are valid for Jacobi functions $\varphi_n(t)$:*

(*i*) *for $t \in (0, \pi/2]$, we have*

$$|\varphi_n(t)| < 1;$$

(*ii*) *for $t \in [0, \pi/2]$, we have*

$$1 - \varphi_n(t) \le c_1\lambda_n t^2;$$

(*iii*) *For $t \in [0, 1]$ and $tn \le 2$ we have*

$$1 - \varphi_n(t) \ge c_2\lambda_n t^2.$$

Proof. See [9, Proposition 3.5. and Lemma 3.1]. □

Recall from [9], the Fourier–Jacobi series of a function $f \in \mathbb{L}_2^{(\alpha,\beta)}$ is defined by

$$f(t) = \sum_{n=1}^{\infty} a_n(f)\tilde{\varphi}_n(t), \tag{1}$$

where

$$\tilde{\varphi}_n = \frac{\varphi_n}{\|\varphi_n\|} \quad , \quad a_n(f) = \langle f, \tilde{\varphi}_n\rangle = \int_0^{\pi} f(t)\tilde{\varphi}_n(t)\rho(t)dt.$$

The sequence $\{a_n(f), n \in \mathbb{N}_0\}$ is called the discrete Fourier–Jacobi transform of f.

Let

$$S_m f(t) = \sum_{n=1}^{m-1} a_n(f)\tilde{\varphi}_n(t),$$

be a partial sums of series (1), and let

$$E_m(f) = \inf_{P_m} \|f - P_m\|,$$

denote the best approximation of $f \in \mathbb{L}_2^{(\alpha,\beta)}$ by polynomials of the form

$$P_m(t) = \sum_{n=1}^{m-1} c_n\tilde{\varphi}_n(t), c_n \in \mathbb{R}.$$

It is well known that

$$\|f\| = \sqrt{\sum_{n=1}^{\infty} |a_n(f)|^2},$$

$$E_m(f) = \|f - S_m f\| = \sqrt{\sum_{n=m}^{\infty} |a_n(f)|^2}.$$

The Jacobi generalized translation is defined by the formula

$$T_h f(t) = \int_0^{\pi} f(\theta) K(t,h,\theta)\rho(\theta)\mathrm{d}\theta, \quad 0 < t, h < \pi,$$

where $K(t,s,\theta)$ is a certain function (see [2]).

Below are some properties (see [9]):

(i) $T_h : \mathbb{L}_2^{(\alpha,\beta)} \to \mathbb{L}_2^{(\alpha,\beta)}$ is a continuous linear operator,
(ii) $\|T_h f\| \le \|f\|$,
(iii) $T_h(\varphi_n(t)) = \varphi_n(h)\varphi_n(t)$,
(iv) $a_n(T_h f) = \varphi_n(h) a_n(f)$,
(v) $\|T_h f - f\| \to 0, \quad h \to 0$,
(vi) $\mathcal{B}(T_h f) = T_h(\mathcal{B} f)$.

The following lemma will be needed in due course.

Lemma 2.2. *[5, p. 101] Suppose $b_n \ge 0$, and $0 < c < d$. Then*

$$\sum_{n=1}^{N} n^d b_n = O(N^c) \quad \textit{iff} \quad \sum_{n=N}^{\infty} b_n = O(N^{c-d}).$$

3. Main results

For every function $f \in \mathbb{L}_2^{(\alpha,\beta)}$ we define the differences $\Delta_h f$ with step h, $0 < h < \pi$, by the formula

$$\Delta_h f(t) = (T_h - I) f(t),$$

where I is the identity operator in $\mathbb{L}_2^{(\alpha,\beta)}$.

Let W_2^r, $r \in \mathbb{N}_0$, be the Sobolev space constructed by the Jacobi operator $\mathcal{B}$, that is:

$$W_2^r := \{f \in \mathbb{L}_2^{(\alpha,\beta)} : \mathcal{B}^j f \in \mathbb{L}_2^{(\alpha,\beta)}, j = 1, 2, ..., r\},$$

where $\mathcal{B}^0 f = f$, $\mathcal{B}^j f = \mathcal{B}(\mathcal{B}^{j-1} f)$, $j = 1, 2, ..., r$.

Lemma 3.1. *If $f \in W_2^r$, $r \in \mathbb{N}_0$, then*

$$a_n(f) = (-1)^r \frac{1}{\lambda_n^r} a_n(\mathcal{B}^r f).$$

Proof. Since $\mathcal{B}$ is self-adjoint (see [9]), we have

$$a_n(f) = \langle f, \tilde{\varphi}_n \rangle = -\frac{1}{\lambda_n}\langle f, \mathcal{B}\tilde{\varphi}_n \rangle = -\frac{1}{\lambda_n}\langle \mathcal{B} f, \tilde{\varphi}_n \rangle = -\frac{1}{\lambda_n} a_n(\mathcal{B} f).$$

This completes the proof of the lemma. □

Lemma 3.2. *If*

$$f(t) = \sum_{n=1}^{\infty} a_n(f)\tilde{\varphi}_n(t),$$

then

$$T_h f(t) = \sum_{n=1}^{\infty} \varphi_n(h) a_n(f) \tilde{\varphi}_n(t).$$

Here, the convergence of the series on the right-hand side is understood in the sense of $\mathbb{L}_2^{(\alpha,\beta)}$.

Proof. By the definition of the operator T_h,

$$T_h(\tilde{\varphi}_n(t)) = \varphi_n(h)\tilde{\varphi}_n(t).$$

Therefore, for any polynomial

$$Q_N(t)=\sum_{n=1}^{N}a_n(f)\tilde{\varphi}_n(t).$$

Since T_h is linear, we have

$$T_hQ_N(t)=\sum_{n=1}^{N}\varphi_n(h)a_n(f)\tilde{\varphi}_n(t). \tag{2}$$

Since T_h is a linear bounded operator in $\mathbb{L}_2^{(\alpha,\beta)}$ and the set of all polynomials $Q_N(t)$ is everywhere dense in $\mathbb{L}_2^{(\alpha,\beta)}$, the passage to the limit in (2) gives the required equality. □

Remark. Since

$$T_hf(t)-f(t)=\sum_{n=1}^{\infty}(\varphi_n(h)-1)a_n(f)\tilde{\varphi}_n(t),$$

the Parseval's identity gives

$$\|T_hf-f\|^2=\sum_{n=1}^{\infty}(1-\varphi_n(h))^2|a_n(f)|^2.$$

If $f\in W_2^r$, from Lemma 3.1, we have

$$\|\Delta_h(\mathcal{B}^rf)\|^2=\sum_{n=1}^{\infty}(1-\varphi_n(h))^2\lambda_n^{2r}|a_n(f)|^2. \tag{3}$$

Definition 3.1. Let $\delta\in(0,1)$, $r\in\mathbb{N}_0$. A function $f\in W_2^r$ is said to be in the $(\delta,r,2)$-Jacobi–Lipschitz class, denoted by $Lip(\delta,r,2)$, if

$$\|\Delta_h(\mathcal{B}^rf)\|=O(h^\delta)\quad\text{as}\quad h\to 0.$$

Theorem 3.1. *Let $f\in W_2^r$. The following two conditions are equivalent:*

(*a*) $f\in Lip(\delta,r,2)$,
(*b*) $\sum_{n\geq N}\lambda_n^{2r}|a_n(f)|^2=O(N^{-2\delta})$, *as* $N\to\infty$.

Proof. $(a)\Rightarrow(b)$ Let $f\in Lip(\delta,r,2)$. Then we have

$$\|\Delta_h(\mathcal{B}^rf)\|=O(h^\delta)\quad\text{as}\quad h\to 0,$$

It follows from (3) that

$$\sum_{n=1}^{\infty}(1-\varphi_n(h))^2\lambda_n^{2r}|a_n(f)|^2=O(h^{2\delta}).$$

If $0\leq n\leq\frac{1}{h}$, then $nh\leq 2$, and from the third inequality of Lemma 2.1, we obtain

$$1-\varphi_n(h)\geq c_2\lambda_nh^2.$$

Therefore,

$$\sum_{n=1}^{[\frac{1}{h}]}\lambda_n^2h^4\lambda_n^{2r}|a_n(f)|^2=O(h^{2\delta}),$$

and, by $\lambda_n\geq n^2$,

$$\sum_{n=1}^{[\frac{1}{h}]} n^4\lambda_n^{2r}|a_n(f)|^2 = O(h^{2\delta-4}).$$

Putting $N = \frac{1}{h}$, we may write this inequality in the following form:

$$\sum_{n=1}^{N} n^4\lambda_n^{2r}|a_n(f)|^2 = O(N^{4-2\delta}).$$

From Lemma 2.2, we have

$$\sum_{N}^{\infty} \lambda_n^{2r}|a_n(f)|^2 = O(N^{4-2\delta-4}) = O(N^{-2\delta}).$$

Thus, the first implication is proved.

$(b) \Rightarrow (a)$. Suppose now that

$$\sum_{n\geq N} \lambda_n^{2r}|a_n(f)|^2 = O(N^{-2\delta}), \quad as \quad N \to \infty.$$

It follows from Lemma 2.2 that

$$\sum_{n=1}^{N} n^4\lambda_n^{2r}|a_n(f)|^2 = O(N^{4-2\delta}).$$

According (3), we write

$$\|\Delta_h(\mathcal{B}^r f)\|^2 \leq \sum_{n=1}^{N}(1-\varphi_n(h))^2\lambda_n^{2r}|a_n(f)|^2 + \sum_{n\geq N}(1-\varphi_n(h))^2\lambda_n^{2r}|a_n(f)|^2.$$

Note that

$$\lambda_n \leq n^2(1+\frac{\gamma}{n}) \leq n^2(1+\gamma), n = 1, 2, ... \tag{4}$$

It follows from (4) and the second inequality in Lemma 2.1 that

$$\sum_{n=1}^{N}(1-\varphi_n(h))^2\lambda_n^{2r}|a_n(f)|^2 \leq c_1h^4\sum_{n=1}^{N}\lambda_n^2\lambda_n^{2r}|a_n(f)|^2 \leq c_1(1+\gamma)^2h^4\sum_{n=1}^{N}n^4\lambda_n^{2r}|a_n(f)|^2 = O(N^{4-2\delta-4}) = O(N^{-2\delta}).$$

On the other hand, it follows from the first inequality of Lemma 2.1 that

$$\sum_{n\geq N}(1-\varphi_n(h))^2\lambda_n^{2r}|a_n(f)|^2 \leq 4\sum_{n\geq N}\lambda_n^{2r}|a_n(f)|^2 = O(N^{-2\delta}).$$

Consequently,

$$\|\Delta_h(\mathcal{B}^r f)\| = O(h^\delta),$$

and this ends the proof of the Theorem. □

We conclude this work by the following immediate consequence.

Corollary 3.1. *Let $f \in W_2^r$, and let*

$$f \in Lip(\delta, r, 2).$$

Then

$$\sum_{n\geq N}|a_n(f)|^2 = O(N^{-2\delta-4r}), \quad as \quad N \to \infty.$$

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