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Partial differential equations

Stability of ODE blow-up for the energy critical semilinear heat equation

Stabilité de l'explosion type EDO pour l'équation de la chaleur énergie critique

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A R T I C L E I N F O

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A B S T R A C T

We consider the energy critical semilinear heat equation

$$\partial_t u = \Delta u + |u|^{\frac{4}{d-2}} u, \ \ x \in \mathbb{R}^d$$

in dimension $d \geq 3$. We propose a self-contained proof of the stability of solutions u blowing-up in finite time with type-I ODE blow-up

$$\|u\|_{L^\infty} \sim \kappa (T-t)^{\frac{d-2}{4}}, \ \ T>0, \ \ \kappa := \left(\frac{d-2}{4}\right)^{\frac{d-2}{4}}$$

which adapts to the energy critical case the proof of Fermanian, Merle, Zaag [4].

R É S U M É

Nous considérons l'équation de la chaleur énergie critique

$$\partial_t u = \Delta u + |u|^{\frac{4}{d-2}} u, \ \ x \in \mathbb{R}^d$$

en dimension $d \geq 3$. Nous proposons une preuve auto-contenue de la stabilité du régime explosif de type EDO

$$\|u\|_{L^\infty} \sim \kappa (T-t)^{\frac{d-2}{4}}, \ \ T>0, \ \ \kappa := \left(\frac{d-2}{4}\right)^{\frac{d-2}{4}}$$

qui adapte au cas énergie critique la preuve de Fermanian, Merle, Zaag [4].

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1. Introduction and main result

We consider the energy critical semilinear heat equation

$$(NLH)\begin{cases} \partial_t u = \Delta u + |u|^{p-1}u, & p = p_c := \frac{d+2}{d-2} \\ u(0,x) = u_0(x) \in \mathbb{R} \end{cases}, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^d. \tag{1.1}$$

We refer to [2,15,13] for the initial value problem and a complete introduction to this kind of models. Solutions may become unbounded in finite time T

$$\|u(t)\|_{L^\infty} \to +\infty \text{ as } t \to T,$$

an explicit example being given by the constant in space ODE blow-up solution

$$u(t,x) = \frac{\kappa_p}{(T-t)^{\frac{1}{p-1}}}, \quad \kappa_p = \left(\frac{1}{p-1}\right)^{\frac{1}{p-1}}, \quad \partial_t u = u^p. \tag{1.2}$$

Solutions blowing up with a self similar growth

$$\limsup_{t\to T} \|u(t)\|_{L^\infty}(T-t)^{\frac{1}{p-1}} < +\infty \tag{1.3}$$

are called type-I blow-up solutions and have attracted considerable attention in the past twenty years [4,6–12]. It is in particular known that in the energy subcritical range $1 < p < p_c$, any blow-up is of type I and that the set of blow-up solutions is open in any reasonable topology. We consider in this paper the energy critical case $p = p_c$, for which other blow-up dynamics have been constructed [5,14]. The result of this paper is that type-I blow-up is however still stable and described by the ODE blow-up (1.2).

Theorem 1.1 *(Stability of type-I blow-up, $p = p_c$). The set of solutions blowing-up in finite time with type-I blow-up (1.3) is open in $W^{3,\infty}(\mathbb{R}^d)$.*

Remark 1.2. The topology $W^{3,\infty}$ is not essential because of the parabolic regularizing effects. In particular, Theorem 1.1 implies the corresponding stability in $L^q(\mathbb{R}^d)$, $q \ge \frac{2d}{d-2}$, where (1.1) is also well-posed.

Theorem 1.1 is one of the key steps in the recent result of classification of the flow near the family of ground states (radially symmetric stationary solutions) [3]. Its proof is given in [4] in the energy subcritical range $p < p_c$ using Liouville classification arguments of the constant self-similar solution. We closely follow the argument that however requires sharpening a number of estimates, and the purpose of this note is to present a self-contained proof of these improvements. Section 3 follows [4]. In Section 4, a local control of a solution by a local energy, given without a proof in [4], which is Proposition 4.2 here, is more subtle due to the energy critical feature.

Notations. The heat kernel is denoted by $K_t(x) := \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|x|^2}{4t}}$. We forget the dependence in p in the notation of the constants in what follows.

2. Some known properties of type-I blow-up

A point $x \in \mathbb{R}^d$ is said to be a blow-up point for u blowing up at time T if there exists $(t_n, x_n) \to (T, x)$ such that:

$$|u(t_n, x_n)| \to +\infty \text{ as } n \to +\infty.$$

A fundamental fact is the rigidity for solutions satisfying the type-I blow-up estimate (1.3) that are global backward in time.

Proposition 2.1 *(Liouville-type theorem for type-I blow-up [11,12]). Let u be a solution to (1.1) on $(-\infty, 0] \times \mathbb{R}^d$ such that $\|u\|_{L^\infty} \le C(-t)^{\frac{1}{p-1}}$ for some constant $C > 0$, then there exists $T \ge 0$ such that $u = \pm\frac{\kappa}{(T-t)^{\frac{1}{p-1}}}$, where κ is defined in (1.2).*

We recall a precise description of type-I blow-up, with an asymptotic at a blow-up point and an ODE type characterization.

Lemma 2.2 *(Description of type-I blow-up [9,11,12]). Let u solve (1.1) with $u_0 \in W^{2,\infty}$ blowing up at $T > 0$. The three following properties are equivalent:*

(*i*) *the blow-up is of type I*;

$$(ii)\ \exists K>0,\ \ |\Delta u|\le \frac{1}{2}|u|^p+K \text{ on } \mathbb{R}^d\times[0,T); \tag{2.1}$$

$$(iii)\ \|u\|_{L^\infty}(T-t)^{\frac{1}{p-1}}\to\kappa \text{ as } t\to T. \tag{2.2}$$

Moreover, if u blows up with type I at x, then

$$(T-t)^{\frac{1}{p-1}}u(t,x+y\sqrt{T-t})\to\pm\kappa \ \text{ as } t\to T \tag{2.3}$$

in $L^2(e^{-\frac{|y|^2}{4}})$ *and in* $C^k(|y|<R)$ *for any* $R>0$ *and* $k\in\mathbb{N}$. *If* $u_n(0)\to u(0)$ *in* $W^{2,\infty}$, *for large n,* u_n *blows up at time* T_n *with* $T_n\to T$.

Some of the above results are stated in [4,9,11,12] in the case $1<p<p_c$, but are however still valid in the energy critical case. In particular, the only bounded solution to the self similar elliptic equation

$$\Delta w+|w|^{p-1}w=\frac{1}{2}\Lambda w,\ \ \Lambda:=\frac{2}{p-1}+x\cdot\nabla, \tag{2.4}$$

for $1<p\le p_c$ is $\pm\kappa$ as follows from the Pohozaev type identity [7]:

$$(d-2)(p_c-p)\int_{\mathbb{R}^d}|\nabla w|^2e^{-\frac{|y|^2}{4}}dy+\frac{p-1}{2}\int_{\mathbb{R}^d}|y|^2|\nabla w|^2e^{-\frac{|y|^2}{4}}dy=0. \tag{2.5}$$

3. Proof of Theorem 1.1

We argue by contradiction, following [4]. Assume the result is false. From Lemma 2.2 and from the Cauchy theory in $W^{2,\infty}$, the negation means the following. There exists $u_0\in W^{3,\infty}$ such that the solution to (1.1) starting from u_0 blows up at time 1 (without loss of generality) with:

$$\|u(t)\|_{L^\infty}\sim\kappa\,(1-t)^{-\frac{1}{p-1}} \text{ as } t\to 1, \tag{3.1}$$

and satisfies:

$$|\Delta u|\le\frac{1}{2}|u|^p+K \text{ on } \mathbb{R}^d\times[0,1). \tag{3.2}$$

There exists a sequence u_n of solutions to (1.1) blowing up at time T_n with:

$$T_n\to 1 \ \text{ and } \ u_n\to u \text{ in } \mathcal{C}_{loc}([0,1),W^{3,\infty}(\mathbb{R}^d)) \tag{3.3}$$

and there exists two sequences $0\le t_n<T_n$ and x_n such that:

$$|\Delta u_n|\le\frac{1}{2}|u_n|^p+2K \text{ on } \mathbb{R}^d\times[0,t_n), \tag{3.4}$$

$$|\Delta u_n(t_n,x_n)|=\frac{1}{2}|u_n(t_n,x_n)|^p+2K. \tag{3.5}$$

The strategy is the following. First we centralize the problem, showing that one can take without loss of generality $x_n=0$. Then we prove that u and u_n become singular near 0 as $(t,n)\to(1,+\infty)$. In view of Lemma 2.2, the ODE type bound (3.4) means that u_n behaves approximately as a type-I blowing-up solution until t_n. This intuition is made rigorous by proving that an appropriate renormalization of u_n near $(t_n,0)$ converges to the constant in space blow-up profile (1.2). We then show that the inequality (3.5) passes to the limit, contradicting (3.2).

Lemma 3.1. *Let* u, u_n *be solutions to* (1.1), t_n *and* x_n *satisfy* (3.1), (3.2), (3.3), (3.4) *and* (3.5). *Then*

$$t_n\to 1 \tag{3.6}$$

and there exist $\hat{u}$ *and* $\hat{u}_n$ *solutions to* (1.1) *satisfying* (3.1), (3.2), (3.4) *and* (3.5) *with* $\hat{x}_n=0$. *In addition,* $\hat{u}$ *blows up with type I at* $(1,0)$, $\hat{u}_n$ *blows up at time* T_n *and*[1] $\hat{u}(t_n,0)\to+\infty$.

[1] Without loss of generality for the sign.

Proof of Lemma 3.1. Step 1 Proof of (3.6). At time t_n, u satisfies the inequality (3.2), whereas u_n does not from (3.5). As u_n converges to u in $C^{1,2}_{\text{loc}}([0,1)\times\mathbb{R}^d)$ from (3.3), this forces t_n to tend to 1.

Step 2 Centering and limit objects. Define $\hat{u}_n(t,x)=u_n(t,x+x_n)$. Then $\hat{u}_n$ is a solution satisfying (3.4), (3.5) with $\hat{x}_n=0$, and blowing up at time $T_n\to 1$ from (3.3). From parabolic regularizing effects, $(t,x)\mapsto u(t,x_n+x)$ is uniformly bounded in $C^{\frac{3}{2},3}_{\text{loc}}([0,1),\mathbb{R}^d)$, hence as $n\to+\infty$ using Arzela Ascoli theorem it converges to a function $\hat{u}$ that also solves (1.1), satisfies (3.2) and

$$\|\hat{u}(t)\|_{L^\infty}\lesssim\kappa\,(1-t)^{-\frac{1}{p-1}}. \tag{3.7}$$

As u_n converges to u in $C_{\text{loc}}([0,1),W^{3,\infty}(\mathbb{R}^d))$ from (3.3), $\hat{u}_n$ converges to $\hat{u}$ in $C^{1,2}_{\text{loc}}([0,1)\times\mathbb{R}^d)$, establishing (3.3).

Step 3 Conditions for boundedness. We claim two facts. 1) If $\hat{u}$ does not blow up at $(1,0)$, then there exists $r,C>0$ such that for all $(t,y)\in[0,t_n]\times B(0,r)$, $|\hat{u}_n(t,y)|\le C$. 2) If there exists $C>0$ such that $|\hat{u}_n(t_n,0)|\le C$, then $\hat{u}$ does not blow up at $(0,1)$.

Proof of the first fact. We reason by contradiction. If $\hat{u}$ does not blow up at $(1,0)$, there exists $r,C>0$ such that for all $(t,y)\in[0,1)\times B(0,r)$, $|\hat{u}(t,y)|\le C$. Assume that there exists $(\tilde{x}_n,\tilde{t}_n)$ such that $\tilde{x}_n\in B(0,r)$ and $|\hat{u}_n(\tilde{x}_n,\tilde{t}_n)|\to+\infty$. As $\hat{u}_n$ solves (1.1), from (3.5) one then has that:

$$\forall t\in[0,\tilde{t}_n],\ \ \partial_t|\hat{u}_n(t,\tilde{x}_n)|\le\frac{3}{2}|\hat{u}_n(t,\tilde{x}_n)|^p+2K,\ \ |\hat{u}_n(\tilde{x}_n,\tilde{t}_n)|\to+\infty.$$

This then implies that for any $M>0$, there exists $s>0$ such that for n large enough, $|\hat{u}_n(\tilde{x}_n,t)|\ge M$ on $[\max(0,\tilde{t}_n-s),\tilde{t}_n]$. But this contradicts the convergence in $C_{\text{loc}}([0,1)\times B(0,r))$ established in Step 2 to the bounded function $\hat{u}$.

Proof of the second fact. We also prove it by contradiction. Assume that $\hat{u}$ blows up at $(0,1)$ and $|\hat{u}_n(t_n,0)|\le C$. Then we claim that

$$\forall t\in[0,t_n),\ \ |\hat{u}_n(t,0)|\le\max((4K)^{\frac{1}{p}},C).$$

Indeed, as $\hat{u}_n$ is a solution to (1.1) satisfying (3.4) one has that:

$$\forall t\in[0,t_n],\ \ \partial_t|\hat{u}_n(t,0)|\ge\frac{1}{2}|\tilde{\hat{u}}_n(t,0)|^p-2K.$$

So if the bound we claim is violated at some time $0\le t_0\le\tau_n'$, then $|\hat{u}_n(t,0)|$ is non-decreasing on $[t_0,\tau_n']$, strictly greater than C, which at time t_n is a contradiction. But now as this bound is independent of n, valid on $[0,t_n)$ with $t_n\to 1$, and as $\hat{u}_n(t,0)\to\hat{u}(t,0)$ on $[0,1)$, one obtains at the limit that $\hat{u}(t,0)$ is bounded on $[0,1)$. From (2.3), this contradicts the blow up of $\hat{u}$ at $(1,0)$.

Step 4 End of the proof. It remains to prove the singular behavior near 0: that $\hat{u}$ blows up at $(1,0)$ and that $|\hat{u}_n(t_n,0)|\to+\infty$. We reason by contradiction. From Step 3 we assume that there exists $C,r>0$ such that $|\hat{u}|+|\hat{u}_n|\le C$ on $[0,1)\times B(0,r)$. A standard parabolic estimate then implies that

$$\|\hat{u}(t)\|_{W^{3,\infty}(B(0,r'))}+\|\hat{u}_n(t)\|_{W^{3,\infty}(B(0,r'))}\le C' \tag{3.8}$$

for all $t\in[\frac{1}{2},1)$ for some $0<r'\le r$. Let χ be a cut-off function, $\chi=1$ on $B(0,\frac{r'}{2})$, $\chi=0$ outside $B(0,r')$. The evolution of $\tilde{u}_n=\chi\hat{u}_n$ is given by:

$$\tilde{u}_{n,\tau}-\Delta\tilde{u}_n=\chi|\hat{u}_n|^{p-1}\hat{u}_n+\Delta\chi\hat{u}_n-2\nabla\cdot\left(\nabla\chi\hat{u}_n\right)=F_n$$

with $\|F_n\|_{W^{1,\infty}}\le C$ from (3.8). Fix $0<s\ll 1$. One has:

$$\begin{aligned}\Delta\hat{u}_n(t_n,0)\ &=\ K_s*(\Delta\tilde{u}_n(t_n-s))(0)+\textstyle\sum_1^d\int_0^s\left[\partial_{x_i}K_{s-s'}*\partial_{x_i}F(t_n-s+s')\right](0)\\ &=\ \Delta\hat{u}(t_n-s,0)+o_{n\to+\infty}(1)+o_{s\to 0}(1)\end{aligned}$$

from (3.3), the estimate on F_n and (3.8). Similarly,

$$\hat{u}_n(t_n,0)=\hat{u}(t_n,0)+o_{n\to+\infty}(1)+o_{s\to 0}(1).$$

The equality (3.5) and the two above identities imply the following asymptotics: lim-inf $|\Delta\hat{u}(t_n)|-\frac{|\hat{u}(t_n,0)|^p}{2}\ge 2K$, which is in contradiction with (3.2). Hence $\hat{u}$ blows up at $(1,0)$ with type-I blow-up from (3.7) and $|\hat{u}(t_n,0)|\to+\infty$. □

We return to the study of u and u_n introduced at the beginning of this Section to prove Theorem 1.1 by contradiction. From Lemma 3.1, keeping the notation u and u_n for $\hat{u}$ and $\hat{u}_n$ introduced there, one can assume without loss of generality that in addition to (3.1), (3.2), (3.3) and (3.4), u and u_n satisfy (3.6), and:

$$|\Delta u_n(t_n,0)| = \frac{1}{2}|u_n(t_n,0)|^p + 2K, \tag{3.9}$$

$$u_n(t_n,0) \to +\infty, \tag{3.10}$$

$$|u(t,0)| \sim \frac{\kappa}{(1-t)^{\frac{1}{p-1}}}. \tag{3.11}$$

To renormalize appropriately u_n near $(1,0)$ we do the following. Define

$$M_n(t) := \left(\frac{\kappa}{\|u_n(t)\|_{L^\infty}}\right)^{p-1}. \tag{3.12}$$

For $(\tilde{t}_n)_{n\in\mathbb{N}}$ a sequence of times, $0 \le \tilde{t}_n < T_n$, the renormalization near $(\tilde{t}_n, 0)$ is

$$v_n(\tau, y) := M_n^{\frac{1}{p-1}}(\tilde{t}_n) u_n\left(\tilde{t}_n + \tau M_n(\tilde{t}_n), M_n^{\frac{1}{2}}(\tilde{t}_n) y\right) \tag{3.13}$$

for $(\tau, y) \in \left[-\frac{\tilde{t}_n}{M_n(\tilde{t}_n)}, \frac{T_n - \tilde{t}_n}{M_n(\tilde{t}_n)}\right) \times \mathbb{R}^d$. One has the following asymptotics.

Lemma 3.2. *Assume $0 \le \tilde{t}_n \le t_n$ and $\tilde{t}_n \to 1$. Then*

$$\|u_n(\tilde{t}_n)\|_{L^\infty} \sim \frac{\kappa}{(T_n - \tilde{t}_n)^{\frac{1}{p-1}}}, \quad \text{i.e. } M_n(\tilde{t}_n) \sim (T_n - \tilde{t}_n). \tag{3.14}$$

Moreover, up to a subsequence[2]*:*

$$v_n \to \frac{\kappa}{\left[\left(\lim \frac{\|u_n(\tilde{t}_n)\|_{L^\infty}}{u_n(\tilde{t}_n,0)}\right)^{p-1} - t\right]^{\frac{1}{p-1}}} \quad \text{in } C^{1,2}_{loc}((-\infty,1)\times\mathbb{R}^d). \tag{3.15}$$

Proof of Lemma 3.2. Step 1 Upper bound for $M_n(\tilde{t}_n)$. We claim that one always has $\|u_n(\tilde{t}_n)\|_{L^\infty} \ge \frac{\kappa}{(T_n-\tilde{t}_n)^{\frac{1}{p-1}}}$, i.e.

$$M_n(\tilde{t}_n) \le (T_n - \tilde{t}_n). \tag{3.16}$$

Indeed, if it is false, then there exists $\delta > 0$ such that $\|u_n(\tilde{t}_n)\|_{L^\infty} < \frac{\kappa}{(T_n+\delta-\tilde{t}_n)^{\frac{1}{p-1}}}$. Therefore, from a parabolic comparison argument, this inequality propagates for the solutions, yielding that $-\frac{\kappa}{(T_n+\delta-t)^{\frac{1}{p-1}}} \le u_n \le \frac{\kappa}{(T_n+\delta-t)^{\frac{1}{p-1}}}$ for all times $t \ge \tilde{t}_n$. This implies that u_n stays bounded up to T_n, which is a contradiction.

Step 2 Proof of (3.15). Let $(x_n)_{n\in\mathbb{N}} \in (\mathbb{R}^d)^{\mathbb{N}}$ and define:

$$\tilde{v}_n(\tau, y) := M_n^{\frac{1}{p-1}}(\tilde{t}_n) u_n\left(\tilde{t}_n + \tau M_n(\tilde{t}_n), x_n + M_n^{\frac{1}{2}}(\tilde{t}_n) y\right). \tag{3.17}$$

From (3.13), $\tilde{v}_n$ is defined on $\left[-\frac{\tilde{t}_n}{M_n(\tilde{t}_n)}, \frac{T_n-\tilde{t}_n}{M_n(\tilde{t}_n)}\right) \times \mathbb{R}^d$. The lower bound, $-\frac{\tilde{t}_n}{M_n(\tilde{t}_n)}$, then goes to $-\infty$ from (3.16). $\tilde{v}_n$ is a solution to (1.1) satisfying:

$$\|\tilde{v}_n(0)\|_{L^\infty} \le \kappa, \tag{3.18}$$

$$\forall (\tau, y) \in \left[-\frac{\tilde{t}_n}{M_n(\tilde{t}_n)}, 0\right] \times \mathbb{R}^d, \quad |\Delta \tilde{v}_n| \le \frac{1}{2}|\tilde{v}_n|^p + 2KM_n^{\frac{p}{p-1}}(\tilde{t}_n), \tag{3.19}$$

from (3.4) and (3.13).

Precompactness of the renormalized functions. We claim that $\tilde{v}_n$ is uniformly bounded in $C^{\frac{3}{2},3}_{\text{loc}}(]-\infty,1)\times\mathbb{R}^d)$. We now prove this result. First, we claim that

$$|\tilde{v}_n| \le \max\left\{(4K)^{\frac{1}{p}} M_n^{\frac{1}{p-1}}(\tilde{t}_n), \kappa\right\}. \tag{3.20}$$

Indeed, as $\tilde{v}_n$ is a solution to (1.1) satisfying (3.19), one has that:

$$\partial_t |\tilde{v}_n| \ge \frac{1}{2}|\tilde{v}_n|^p - 2KM_n^{\frac{p}{p-1}}(\tilde{t}_n).$$

[2] With the convention that if the limit in the denominator is $+\infty$ the limit function is 0.

So if the bound we claim is violated, then $\|\tilde v_n\|_{L^\infty}$ is strictly increasing, greater than κ, which at time 0 is a contradiction to (3.18). Moreover, as $\|\tilde v_n(0)\|_{L^\infty}\le\kappa$, from a comparison argument, for $0\le t<1$, on has that $\|\tilde v_n(0)\|_{L^\infty}\le\kappa(1-t)^{-\frac{1}{p-1}}$. This and the above bound implies that for any $T<1$, $\tilde v_n$ is uniformly bounded, independently of n, in $L^\infty((-\frac{\tilde t_n}{M_n(\tilde t_n)},T]\times\mathbb{R}^d)$. From standard parabolic regularization, it is uniformly bounded in $C^{\frac32,3}((-\frac{\tilde t_n}{M_n}+1,T)\times\mathbb{R}^d)$, yielding the desired result.

Rigidity at the limit. From Step 2 and Arzela Ascoli theorem, up to a subsequence, v_n converges in $C^{1,2}_{\mathrm{loc}}((-\infty,0]\times\mathbb{R}^d)$ to a function v. The equation (1.1) passes to the limit and v also solves (1.1). (3.20) and (3.16) imply that $|v|\le\kappa$. (1.1), (3.16) and (3.19) imply that:

$$\partial_t|v|\ge\frac12|v|^p.$$

Reintegrating this differential inequality, one obtains that $|v|\le\frac{C}{|c-\tau|^{\frac{1}{p-1}}}$ for some $C,c>0$. Applying the Liouville Lemma 2.1, one has that v is constant in space. Up to a subsequence, $v(0,x_n)=\kappa\lim\frac{u_n(\tilde t_n,x_n)}{\|u_n(\tilde t_n)\|_{L^\infty}}$. The particular choice $x_n=0$, $\tilde v_n=v_n$ gives in particular the desired identity (3.15).

Step 3 Lower bound on M_n. We claim that $\lim-\inf\frac{M_n}{T_n-\tilde t_n}\ge1$. We prove it by contradiction using a blow-up criterion from Section 4. From (3.12), and up to a subsequence, assume that there exists $0<\delta\ll1$ and $x_n\in\mathbb{R}^d$ such that $u_n(\tilde t_n,x_n)>\frac{(1+\delta)\kappa}{(T_n-\tilde t_n)^{\frac{1}{p-1}}}$ and $\frac{u_n(\tilde t_n,x_n)}{\|u_n(\tilde t_n)\|_{L^\infty}}\to1$. Therefore the renormalized function $\tilde v_n$ defined by (3.17) blows up at $\frac{T_n-\tilde t_n}{M_n(\tilde t_n)}\ge(1+\delta)^{p-1}$. From Step 2, $v(0,\cdot)$ is uniformly bounded and converges to κ. Hence, defining the self-similar renormalization near $((1+\delta)^{p-1},0)$

$$w^{(n)}_{0,(1+\delta)^{p-1}}(t,y)=((1+\delta)^{p-1}-t)^{\frac{1}{p-1}}\tilde v_n\left(t,\sqrt{(1+\delta)^{p-1}-t}\,y\right),$$

one has that $I(w_{0,(1+\delta)^{p-1}}(0,\cdot))\to I((1+\delta)^{p-1}\kappa)>0$ where I is defined by (4.6). From (4.7), for n large enough, this implies that $\tilde v_n$ should have blown up before $(1+\delta)^{p-1}$, which yields the desired contradiction. □

To end the proof of Theorem 1.1, we now distinguish two cases for which one has to find a contradiction (which cover all possible cases up to subsequence):

$$\text{Case 1:}\quad\lim\frac{u_n(x_n,t_n)}{\|u_n(t_n)\|_{L^\infty}}>0,\tag{3.21}$$

$$\text{Case 2:}\quad\lim\frac{u_n(x_n,t_n)}{\|u_n(t_n)\|_{L^\infty}}=0.\tag{3.22}$$

Proof of Theorem 1.1 in Case 1. In this case, we can renormalize at time t_n. Let $\tilde t_n=t_n$ and define v_n and $M_n(\tilde t_n)$ by (3.13) and (3.12). (3.15) and (3.21) imply that $\Delta v_n(0,0)\to0$ and $v_n(0,0)\to v(0,0)>0$. From (3.9), v_n satisfies at the origin:

$$|\Delta v_n(0,0)|=\frac12|v_n(0,0)|^p+2KM_n^{\frac{p}{p-1}}(t_n).$$

As $M_n(t_n)\to0$ from (3.14), at the limit we get $0=\frac12v(0,0)>0$, which is a contradiction, ending the proof of Theorem 1.1 in Case 1. □

Proof of Theorem 1.1 in Case 2. Step 1 Suitable renormalization before t_n. We claim that for any $0<\kappa_0\ll1$ one can find a sequence of times $\tilde t_n$ such that $0\le\tilde t_n\le t_n$, $\tilde t_n\to1$ and such that v_n defined by (3.13) satisfies up to a subsequence:

$$v_n\to\frac{\kappa}{\left[\left(\frac{\kappa}{\kappa_0}\right)^{p-1}-1-t\right]^{\frac{1}{p-1}}}\ \text{in } C^{1,2}_{\mathrm{loc}}(]-\infty,1)\times\mathbb{R}^d).\tag{3.23}$$

We now prove this fact. On the one hand, $\frac{|u(t,0)|}{\|u(t)\|_{L^\infty}}\to1$ as $t\to1$ (from (3.11) and (2.2) as u blow up with type I at 0) and for any $0\le T<1$ u_n converges to u in $C([0,T],L^\infty(\mathbb{R}^d))$ from (3.3). As $t_n\to1$, using a diagonal argument and Lemma 3.2, up to a subsequence there exists a sequence of times $0\le t_n'\le t_n$ such that $\frac{u_n(t_n',0)}{\|u(t_n')\|_{L^\infty}}\to1$. On the other hand, from the assumption (3.22) and (3.6), $\lim\frac{|u_n(t_n,0)|}{\|u_n(t_n)\|_{L^\infty}}=0$ and $t_n\to1$. From a continuity argument, for κ_0 small enough, there exists a sequence $t_n'\le\tilde t_n\le t_n$ such that $\lim\frac{u_n(\tilde t_n,0)}{\|u_n(\tilde t_n)\|_{L^\infty}}=\left[\left(\frac{\kappa}{\kappa_0}\right)^{p-1}-1\right]^{-\frac{1}{p-1}}$. From Lemma 3.2, one obtains the desired convergence result (3.23).

Step 2 Local boundedness. Take $\tilde{t}_n$ and v_n as in Step 1. From (3.13) and (3.14) v_n blows up at time $\tau_n = \frac{T_n - \tilde{t}_n}{M_n(\tilde{t}_n)} \to 1$. Up to time $\tau'_n = \frac{t_n - \tilde{t}_n}{M_n(\tilde{t}_n)}$, $0 \le \tau'_n$, v_n satisfies:

$$|\Delta v_n| \le \frac{1}{2}|v_n|^p + 2KM_n^{\frac{p}{p-1}}(\tilde{t}_n) \tag{3.24}$$

and we recall that $M_n(\tilde{t}_n) \to 0$ from (3.14). Let $R > 0$ and $a \in B(0, R)$. Define

$$w^{(n)}_{a,\tau_n}(y,t) := (\tau_n - t)^{\frac{1}{p-1}} v_n(t, a + \sqrt{\tau_n - t}\, y).$$

Then as $v_n(-1) \to \kappa_0$ from (3.23), one has that for n large enough

$$E[w^{(n)}_{a,\tau_n}(-1, \cdot)] = O(\kappa_0^2)$$

where the energy is defined by (4.4). One can then apply the result (4.15) of Proposition 4.2: there exists $r > 0$ such that for κ_0 small enough and n large enough one has:

$$\forall t \in [0, \tau'_n], \quad \|v_n(t)\|_{W^{2,\infty}(B(0,r))} \le C. \tag{3.25}$$

Step 3 End of the proof. Let χ be a cut-off function, $\chi = 1$ on $B(0, \frac{r}{2})$ and $\chi = 0$ outside $B(0, r)$. The evolution of $\tilde{v}_n = \chi v_n$ is given by

$$\tilde{v}_{n,\tau} - \Delta \tilde{v}_n = \chi |v_n|^{p-1} v_n + \Delta\chi\, v_n - 2\nabla \cdot (\nabla \chi\, v_n) = F_n$$

with $\|F_n\|_{W^{1,\infty}} \le C$ from (3.25). Fix $0 < s \ll 1$. One has:

$$\begin{aligned} \Delta v_n(\tau'_n, 0) &= K_s * (\Delta \tilde{v}_n(\tau'_n - s))(0) + \textstyle\sum_1^d \int_0^s \left[\partial_{x_i} K_{s-s'} * \partial_{x_i} F(\tau'_n - s + s')\right](0) \\ &= o_{n\to+\infty}(1) + o_{s\to 0}(1) \end{aligned}$$

from (3.23) and the estimate on F_n. Hence $\Delta v_n(\tau'_n, 0) \to 0$ as $n \to +\infty$. On the other hand, $\lim v_n(\tau'_n, 0) = v(\tau'_n, 0) > 0$ from (3.23) and the fact that $0 \le \tau'_n \le 1$. We recall that at time τ'_n v_n satisfies:

$$|\Delta v_n(\tau'_n, 0)| = \frac{1}{2}|v_n(\tau'_n, 0)|^p + 2KM_n^{\frac{p}{p-1}}(\tilde{t}_n).$$

As $M_n^{\frac{p}{p-1}}(\tilde{t}_n) \to 0$ from (3.14) at the limit, one has $0 = \frac{1}{2}|v(\tau'_n, 0)|^p > 0$ which is a contradiction. This ends the proof of Theorem 1.1 in Case 2. $\square$

4. A local smallness result

This section is devoted to the proof of (3.25).

4.1. Self-similar variables

We follow the method introduced in [7–9] to study type-I blow-up locally. The results and the ideas of their proof are either contained in [8] or similar to the results there. A sharp blow-up criterion and other preliminary bounds are given by Lemma 4.1 and a condition for local boundedness is given in Proposition 4.2. For u defined on $[0, T_{u_0}) \times \mathbb{R}^d$, $a \in \mathbb{R}^d$ and $T > 0$, we define the self-similar renormalization of u at (T, a):

$$w_{a,T}(y,t) := (T - t)^{\frac{1}{p-1}} u(t, a + \sqrt{T - t}\, y) \tag{4.1}$$

for $(t, y) \in [0, \min(T_{u_0}, T)) \times \mathbb{R}^d$. Introducing the self-similar renormalized time:

$$s := -\log(T - t) \tag{4.2}$$

one sees that if u solves (1.1) then $w_{a,T}$ solves:

$$\partial_s w_{a,T} - \Delta w_{a,T} - |w_{a,T}|^{p-1} w_{a,T} + \frac{1}{2}\Lambda w_{a,T} = 0. \tag{4.3}$$

Equation (4.3) admits a natural Lyapunov functional,

$$E(w) = \int_{\mathbb{R}^d} \left(\frac{1}{2}|\nabla w(y)|^2 + \frac{1}{2(p-1)}|w(y)|^2 - \frac{1}{p+1}|w(y)|^{p+1} \right) \rho(y)\, dy, \tag{4.4}$$

where $\rho(y) := \frac{1}{(4\pi)^{\frac{d}{2}}} e^{-\frac{|y|^2}{4}}$ from the fact that for its solutions there holds:

$$\frac{d}{ds} E(w) = -\int_{\mathbb{R}^d} w_s^2 \rho \, dy \le 0. \tag{4.5}$$

Another quantity that will prove to be helpful is the following:

$$I(w) := -2E(w) + \frac{p-1}{p+1} \left(\int_{\mathbb{R}^d} w^2 \rho \, dy \right)^{\frac{p+1}{2}}. \tag{4.6}$$

Lemma 4.1 *([7,11]). Let w be a global solution to (4.3) with $E(w(0)) = E_0$, then*[3] *for $s \ge 0$:*

$$I(w(s)) \le 0, \quad E_0 \ge 0 \tag{4.7}$$

$$\int_0^{+\infty} \int_{\mathbb{R}^d} w_s^2 \rho \, dy \, ds \le E_0. \tag{4.8}$$

If moreover $E_0 := E(w(0)) \le 1$, then[4] *for any $s \ge 0$:*

$$\int_{\mathbb{R}^d} w^2 \rho \, dy \le C E_0^{\frac{2}{p+1}}, \tag{4.9}$$

$$\int_s^{s+1} \left(\int_{\mathbb{R}^d} (|\nabla w|^2 + w^2 + |w|^{p+1}) \rho \, dy \right)^2 ds \le C E_0^{\frac{p+3}{p+1}}. \tag{4.10}$$

Proof of Lemma 4.1. Step 1 Proof of (4.7). We argue by contradiction and assume that $I(w(s_0)) > 0$ for some $s_0 \ge 0$. The set $\mathcal{S} := \{s \ge s_0, \ I(s) \ge I(s_0)\}$ is closed by continuity. For any solution to (4.3), one has:

$$\frac{d}{ds} \left(\int_{\mathbb{R}^d} w^2 \rho \, dy \right) = 2 \int_{\mathbb{R}^d} w \, w_s \, \rho \, dy = -4 \, E(w) + \frac{2(p-1)}{p+1} \int_{\mathbb{R}^d} |w|^{p+1} \rho \, dy. \tag{4.11}$$

Therefore, for any $s \in \mathcal{S}$, from (4.6) and Jensen inequality this gives:

$$\frac{d}{ds} \left(\int_{\mathbb{R}^d} w^2 \rho \, dy \right) \ge -4 \, E(w(s)) + \frac{2(p-1)}{p+1} \left(\int_{\mathbb{R}^d} w^2 \rho \, dy \right)^{\frac{p+1}{2}} = I(w(s)) > 0 \tag{4.12}$$

as $I(w(s)) \ge I(w(s_0))$, which with (4.5) and (4.6) imply $\frac{d}{ds} I(w(s)) > 0$. Hence $\mathcal{S}$ is open and therefore $\mathcal{S} = [s_0, +\infty)$. From (4.12) and (4.5), there exists s_1 such that $E(w(s)) \le \frac{p-1}{2(p+1)} \left(\int_{\mathbb{R}^d} w^2 \rho \, dy \right)^{\frac{p+1}{2}}$ for all $s \ge s_1$, implying from (4.12):

$$\frac{d}{ds} \left(\int_{\mathbb{R}^d} w^2 \rho \, dy \right) \ge 2 \frac{p-1}{p+1} \left(\int_{\mathbb{R}^d} w^2 \rho \, dy \right)^{\frac{p+1}{2}}.$$

This quantity must then tend to $+\infty$ in finite time, which is a contradiction.

[3] From the definition (4.6) of I and (4.7) one has that for all $s \ge 0$, $E(w(s)) \ge 0$. Hence the right hand side in (4.8) is nonnegative.
[4] Idem for the right hand side of (4.9) and (4.10).

Step 2 End of the proof. (4.8) and (4.9) are consequences of (4.5), (4.6) and (4.7). To prove (4.10), from (4.11), (4.5), (4.9) and Hölder, one obtains:

$$\int_s^{s+1}\left(\int_{\mathbb{R}^d}|w|^{p+1}\rho\,dy\right)^2 ds \le \int_s^{s+1}\left(CE_0^2 + C\int_{\mathbb{R}^d} w_s^2\rho\,dy\int_{\mathbb{R}^d} w^2\rho\,dy\right)ds \le CE_0^{\frac{p+3}{p+1}}$$

as $E_0 \le 1$. This identity, using (4.4), (4.5) and as $E_0 \le 1$ implies (4.10). □

Proposition 4.2 *(Condition for local boundedness). Let $R>0$, $0<T_-<T_+$ and $\delta>0$. There exists $\eta>0$ and $0<r\le R$ such that, for any $T\in[T_-,T_+]$ and u solution to (1.1) on $[0,T)\times\mathbb{R}^d$ with $u_0\in W^{2,\infty}$ satisfying:*

$$\forall a\in B(0,R),\quad E(w_{a,T}(0,\cdot))\le\eta, \tag{4.13}$$

$$\forall (t,x)\in[0,T)\times\mathbb{R}^d,\quad |\Delta u(t,x)|\le\frac{1}{2}|u(t,x)|^p+\eta, \tag{4.14}$$

there holds

$$\forall t\in\left[\frac{T_-}{2},T\right),\quad \|u(t)\|_{W^{2,\infty}(B(0,r))}\le\delta. \tag{4.15}$$

The proof of Proposition 4.2 is done at the end of this subsection. We need intermediate results: Proposition 4.3 gives local smallness in self-similar variables, Lemma 4.7 and its Corollary 4.8 give local boundedness in L^∞ in original variables.

Proposition 4.3. *For any $R, s_0, \delta>0$, there exists $\eta>0$ such that for any w global solution to (4.3), with $w(0)\in W^{2,\infty}$ satisfying*

$$E(w(0))\le\eta \ \text{ and } \ \forall(s,y)\in[0,+\infty)\times\mathbb{R}^d,\ |\Delta w(s,y)|\le\frac{1}{2}|w(s,y)|^p+\eta, \tag{4.16}$$

there holds:

$$\forall(s,y)\in[s_0,+\infty)\times B(0,R),\quad |w(s,y)|\le\delta. \tag{4.17}$$

Proof of Proposition 4.3. It is a direct consequence of Lemma 4.4 and Lemma 4.5. □

Lemma 4.4. *For any $R, s_0, \eta'>0$, there exists $\eta>0$ such that for w a global solution to (4.3), with $w(0)\in W^{2,\infty}(\mathbb{R}^d)$, satisfying (4.16), there holds*

$$\forall s\in[s_0,+\infty),\quad \int_{B(0,R)}(|w|^2+|\nabla w|^2)dy\le\eta'. \tag{4.18}$$

Lemma 4.5. *For any $R,\delta>0$, $0<s_0<s_1$ there exists $\eta,\eta'>0$ and $0<r\le R$ such that for w a global solution to (4.3) with $w(0)\in W^{2,\infty}$, satisfying (4.16) and (4.18), there holds:*

$$\forall(s,y)\in[s_1,+\infty)\times B(0,r),\quad |w(s,y)|\le\delta. \tag{4.19}$$

We now prove the two above lemmas. In what follows we will often have to localize the function w. Let χ be a smooth cut-off function, $\chi=1$ on $B(0,1)$ and $\chi=0$ outside $B(0,2)$. For $R>0$ we define $\chi_R(x)=\chi\left(\frac{x}{R}\right)$ and:

$$v:=\chi_R w \tag{4.20}$$

(we will forget the dependence in R in the notations to ease writing, and will write χ instead of χ_R). From (4.3) the evolution of v is then given by:

$$v_s-\Delta v=\chi|w|^{p-1}w+\left(\left[\frac{1}{p-1}-\frac{d}{2}\right]\chi-\frac{1}{2}\nabla\chi\cdot y+\Delta\chi\right)w+\nabla\cdot\left(\left[\frac{1}{2}\chi y-2\nabla\chi\right]w\right). \tag{4.21}$$

Proof of Lemma 4.4. We will prove that (4.18) holds at time s_0, which will imply (4.18) at any time $s\in[s_0,+\infty)$ because of time invariance. We take $d\ge5$ for the sake of simplicity.

Step 1 An estimate for Δw. First one notices that the results of Lemma 4.1 apply. From (4.16) and (4.3), there exists a constant $C>0$ such that:

$$|w|^{2p} \le C(|w|^{p-1}w+\Delta w)^2 + C\eta^2 \le C|w_s|^2 + C|y|^2|\nabla w|^2 + Cw^2 + C\eta^2.$$

We integrate this in time, using (4.8), (4.9), (4.10) and (4.16), yielding for $s\ge 0$:

$$\int_s^{s+1}\int_{B(0,2R)} |w|^{2p}\, dy\, ds \le C\eta + C\eta^{\frac{p+3}{p+1}} + C\eta^{\frac{2}{p+1}} + C\eta^2 \le C\eta^{\frac{2}{p+1}}. \tag{4.22}$$

Injecting the above estimate in (4.16), using (4.9) and (4.10), we obtain for $s\ge 0$:

$$\begin{aligned} &\int_s^{s+1} \|w\|^2_{H^2(B(0,2R))}\, ds \le \int_s^{s+1}\int_{B(0,2R)} (|\Delta w|^2 + |\nabla w|^2 + w^2)\, dy\, ds \\ \le\ & \int_s^{s+1}\int_{B(0,2R)} C(|w|^{2p} + |\nabla w|^2 + w^2)\, dy\, ds + C\eta^2 \le C\eta^{\frac{2}{p+1}}. \end{aligned} \tag{4.23}$$

Step 2 Localization. We localize at scale R and define v by (4.20). From (4.20), (4.10) and (4.9), one obtains that there exists $\tilde s_0 \in [\max(0, s_0-1), s_0]$ such that:

$$\|v(\tilde s_0)\|^2_{H^1(\mathbb{R}^d)} \lesssim \int_{B(0,2R)} (w(\tilde s_0)^2 + |\nabla w(\tilde s_0)|^2)\, dy \le C\eta^{\frac{2}{p+1}} + C\eta^{\frac{p+3}{p+1}} \le C\eta^{\frac{2}{p+1}}. \tag{4.24}$$

We apply Duhamel's formula to (4.21) to find that $v(s_0)$ is given by:

$$\begin{aligned} v(s_0) \ =\ & \int_{\tilde s_0}^{s_0} K_{s_0-s} * \left\{\chi|w|^{p-1}w + \left(\left[\frac{1}{p-1} - \frac{d}{2}\right]\chi - \frac12 \nabla\chi.y + \Delta\chi\right) w\right\} ds \\ & + \int_{\tilde s_0}^{s_0} \nabla\cdot K_{s_0-s} * \left(\left[\frac12\chi y - 2\nabla\chi\right] w\right) ds + K_{s_0-\tilde s_0} * v(\tilde s_0). \end{aligned} \tag{4.25}$$

We now estimate the $\dot H^1$ norm of each term in the previous identity, using (4.24), (4.10), (A.2), Young and Hölder inequalities:

$$\|K_{s_0-\tilde s_0} * v(\tilde s_0)\|_{\dot H^1(\mathbb{R}^d)} \le \|v(\tilde s_0)\|_{\dot H^1(\mathbb{R}^d)} \le C\eta^{\frac{1}{p+1}}, \tag{4.26}$$

$$\begin{aligned} & \left\| \int_{\tilde s_0}^{s_0} K_{s_0-s} * \{([\tfrac{1}{p-1} - \tfrac{d}{2}]\chi - \tfrac{\nabla\chi\cdot y}{2} + \Delta\chi)w\} + \nabla\cdot K_{s_0-s} * ([\tfrac{\chi y}{2} - 2\nabla\chi]w) \right\|_{\dot H^1} \\ \le\ & C\int_{\tilde s_0}^{s_0} \|w\|_{H^1(B(0,2R))} ds + C\int_{\tilde s_0}^{s_0} \frac{1}{|s_0-s|^{\frac12}} \|w\|_{H^1(B(0,2R))} ds \\ \le\ & C\eta^{\frac{p+3}{4(p+1)}} + C\left(\int_{\tilde s_0}^{s_0} \frac{ds}{|\tilde s_1 - s|^{\frac12\times\frac43}}\right)^{\frac34} \left(\int_{\tilde s_0}^{s_0} \|w\|^4_{H^1(B(0,2R))} ds\right)^{\frac14} \le C\eta^{\frac{p+3}{4(p+1)}}. \end{aligned} \tag{4.27}$$

For the non-linear term in (4.25), one first compute from (4.20) that:

$$\nabla(\chi|w|^{p-1}w) = p\chi|w|^{p-1}\nabla w + \nabla\chi|w|^{p-1}w. \tag{4.28}$$

For the first term in the previous identity, using Sobolev embedding, one obtains:

$$\begin{aligned} \||w|^{p-1}\nabla w\|_{L^{\frac{2d}{d-2+(d-4)(p-1)}}(B(0,2R))} &\le C\|w\|^{p-1}_{L^{\frac{2d}{d-4}}(B(0,2R))} \|\nabla w\|_{L^{\frac{2d}{d-2}}(B(0,2R))} \\ &\le C\|w\|^p_{H^2(B(0,2R))}. \end{aligned}$$

Therefore, from (4.23) this force term satisfies:

$$\int_{\tilde s_0}^{s_0} \||w|^{p-1}\nabla w\|^{\frac2p}_{L^{\frac{2d}{d-2+(d-4)(p-1)}}(B(0,2R))} ds \le \int_{\tilde s_0}^{s_0} \|w\|^2_{H^2(B(0,2R))} ds \le C\eta^{\frac{2}{p+1}}.$$

We let (q,r) be the Lebesgue conjugated exponents of $\frac2p$ and $\frac{2d}{(d-2)+(d-4)(p-1)}$:

$$q = \frac{2}{2-p} > 2, \quad r = \frac{2d}{d+2-(d-4)(p-1)} > 2.$$

They satisfy the Strichartz relation $\frac2q + \frac dr = \frac d2$. Therefore, using (A.3), one obtains:

$$\left\| \int_{\tilde s_0}^{s_0} K_{s_0-s} * (p\chi|w(s)|^{p-1}\nabla w(s))\, ds \right\|_{L^2} \le C\left(\int_{\tilde s_0}^{s_0} \||w|^{p-1}\nabla w\|^{\frac2p}_{L^{\frac{2d}{d-2+(d-4)(p-1)}}(B(0,2R))} ds\right)^{\frac p2} \le C\eta^{\frac{p}{(p+1)}}.$$

For the second term in (4.28) using (4.22), (A.2) and Hölder, one has:

$$\left\| \int_{\tilde{s}_0}^{s_0} K_{s_0-s} * (\nabla\chi |w|^{p-1} w)\, \mathrm{d}s \right\|_{L^2} \leq C \int_{\tilde{s}_0}^{s_0} \|w\|^p_{L^{2p}(B(0,2R))} \leq C\eta^{\frac{1}{p+1}}.$$

The two above estimates and the identity (4.28) imply the following bound:

$$\left\| \int_{\tilde{s}_0}^{s_0} K_{s_0-s} * (\chi |w|^{p-1} w)\, \mathrm{d}s \right\|_{\dot{H}^1} \leq C\eta^{\frac{1}{p+1}}.$$

We come back to (4.25) where we found estimates for each term in the right-hand side in (4.26), (4.27) and the above identity, yielding $\|v(s_0)\|_{\dot{H}^1} \leq C\eta^{\frac{1}{p+1}}$. From (4.20), as v is compactly supported in $B(0,2R)$, the above estimate implies the desired estimate (4.18) at time s_0. □

To prove Lemma 4.5, we need the following parabolic regularization result. Its proof uses standard parabolic tools and we do not give it here.

Lemma 4.6 *(Parabolic regularization). Let $R, M > 0$, $0 < s_0 \leq 1$ and w be a global solution to (4.3) satisfying:*

$$\forall (s,y) \in [0,+\infty) \times \mathbb{R}^d, \quad \|w(s,y)\|_{H^2(B(0,R))} \leq M. \tag{4.29}$$

Then there exists $0 < r \leq R$, a constant $C = C(R, s_0)$ and $\alpha > 1$ such that:

$$\forall (s,y) \in [s_0,+\infty) \times B(0,r), \quad |w(s,y)| \leq C(M + M^\alpha). \tag{4.30}$$

Proof of Lemma 4.5. Without loss of generality we take $\eta' = \eta$, $s_0 = 0$, localize at scale $\frac{R}{2}$ by defining v by (4.20). The assumption (4.18) implies that for $s \geq 0$:

$$\int_{\mathbb{R}^d} (|v(s)|^2 + |\nabla v(s)|^2)\, \mathrm{d}y \leq C\eta. \tag{4.31}$$

We claim that for all $s \geq \frac{s_1}{2}$,

$$\|v\|_{H^2} \leq C\eta.$$

This will give the desired result (4.19) by applying Lemma 4.6 from (4.20). We now prove the above bound. By time invariance, we just have to prove it at time $\frac{s_1}{2}$.

Step 1 First estimate on v_s. Since w is a global solution starting in $W^{2,\infty}(\mathbb{R}^d)$ with $E(w(0)) \leq \eta$, from (4.8), one obtains:

$$\int_0^{+\infty} \int_{\mathbb{R}^d} |v_s|^2\, \mathrm{d}y\, \mathrm{d}s \leq C\eta. \tag{4.32}$$

Step 2 Second estimate on v_s. Let $u = v_s$. From (4.3) and (4.20), the evolution of u is given by:

$$u_s - \Delta u = p|w|^{p-1}u + \left(\left[\frac{1}{p-1} - \frac{d}{2} \right] \chi - \frac{1}{2} \nabla\chi \cdot y + \Delta\chi \right) w_s + \nabla \cdot \left(\left[\frac{1}{2}\chi y - 2\nabla\chi \right] w_s \right). \tag{4.33}$$

We first state a non-linear estimate. Using Sobolev embedding, Hölder inequality and (4.18), one obtains:

$$\int_{\mathbb{R}^d} |u|^2 |w|^{p-1} \mathrm{d}y \leq \|u\|^2_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} \|w\|^{p-1}_{L^{\frac{2d}{d-2}}(B(0,R))} \leq C\eta^{\frac{p-1}{2}} \int_{\mathbb{R}^d} |\nabla u|^2 \mathrm{d}y.$$

We now perform an energy estimate. We multiply (4.33) by u and integrate in space using Young inequality for any $\kappa > 0$ and the above inequality:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\left[\int_{\mathbb{R}^d}|u|^2\mathrm{d}y\right] &= -\int_{\mathbb{R}^d}|\nabla u|^2\mathrm{d}y+\int_{\mathbb{R}^d}\left(\left[\frac{1}{p-1}-\frac{d}{2}\right]\chi-\frac{1}{2}\nabla\chi\cdot y+\Delta\chi\right)w_s u\,\mathrm{d}y\\ &\quad+\int\left(\left[\frac{1}{2}\chi y-2\nabla\chi\right]w_s\right)\cdot\nabla u\,\mathrm{d}y+\int_{\mathbb{R}^d}u^2|w|^{2(p-1)}\mathrm{d}y\\ &\le -\int_{\mathbb{R}^d}|\nabla u|^2\mathrm{d}y+C\int_{B(0,R)}(w_s^2+u^2)\,\mathrm{d}y+\frac{C}{\kappa}\int_{B(0,R)}w_s^2\mathrm{d}y\\ &\quad+C\kappa\int_{\mathbb{R}^d}|\nabla u|^2\mathrm{d}y+C\eta^{\frac{p-1}{2}}\int_{\mathbb{R}^d}|\nabla u|^2\mathrm{d}y\\ &\le -\int_{\mathbb{R}^d}|\nabla u|^2\mathrm{d}y+C(\kappa)\int_{B(0,R)}w_s^2\,\mathrm{d}y\end{aligned}$$

if κ and η have been chosen small enough. Now because of the integrability (4.32), there exists at least one $\tilde{s}\in[\max(0,\frac{s_1}{2}-1),\frac{s_1}{2}]$ such that:

$$\int_{\mathbb{R}^d}|v_s(\tilde{s})|^2\mathrm{d}y\le C(s_1)\eta.$$

One then obtains from the two previous inequalities and (4.8):

$$\int_{\mathbb{R}^d}|v_s(s)|^2\mathrm{d}y\le\int_{\mathbb{R}^d}|v_s(\tilde{s})|^2\mathrm{d}y+C\int_{\tilde{s}}^{\frac{s_1}{2}}\int_{B(0,R)}w_s^2\mathrm{d}y\,\mathrm{d}s\le C\eta. \tag{4.34}$$

Step 3 Estimate on Δv. Applying Sobolev embedding and Hölder inequality, using the fact that $\left(\frac{\frac{2d}{d-4}}{2}\right)'=\frac{d}{4}=\frac{\frac{2d}{d-2}}{2(p-1)}$, one gets that for any $s\ge 0$:

$$\begin{aligned}&\int_{\mathbb{R}^d}v^2|w|^{2(p-1)}\mathrm{d}y\le\|v^2\|_{L^{\frac{\frac{2d}{d-4}}{2}}(\mathbb{R}^d)}\||w|^{2(p-1)}\|_{L^{\frac{\frac{2d}{d-2}}{2(p-1)}}(B(0,R))}\\ &=\|v\|^2_{L^{\frac{2d}{d-4}}(\mathbb{R}^d)}\|w\|^{2(p-1)}_{L^{\frac{2d}{d-2}}(B(0,R))}\le C\|v\|^2_{\dot{H}^2(\mathbb{R}^d)}\|w\|^{2(p-1)}_{H^1(B(0,R))}\\ &\le C\eta^{p-1}\int_{\mathbb{R}^d}|\Delta v|^2\mathrm{d}y,\end{aligned} \tag{4.35}$$

where we injected the estimate (4.18). We inject the above estimate in (4.21), using (4.20), yielding for all $s\ge 0$:

$$\begin{aligned}\int_{\mathbb{R}^d}|\Delta v|^2\mathrm{d}y &\le C\left(\int_{\mathbb{R}^d}(|v_s|^2+|w|^2+|\nabla w|^2+v^2|w|^{2(p-1)})\,\mathrm{d}y\right)\\ &\le C\int_{\mathbb{R}^d}|v_s|^2\mathrm{d}y+C\eta+C\eta^{p-1}\int_{\mathbb{R}^d}|\Delta v|^2\mathrm{d}y,\end{aligned}$$

where we used (4.29). Injecting (4.34), for η small enough:

$$\int_{\mathbb{R}^d}\left|\Delta v\left(\frac{s_1}{2}\right)\right|^2\mathrm{d}y\le C\int_{\mathbb{R}^d}\left|v_s\left(\frac{s_1}{2}\right)\right|^2\mathrm{d}y+C\eta\le C\eta. \tag{4.36}$$

Step 4 Conclusion. From (4.31) and (4.36) we infer $\|v(\frac{s_1}{2})\|_{\dot{H}^2}\le C\eta$, which is exactly the bound we had to prove. □

We now go from boundedness in L^∞ in self-similar variables provided by Proposition 4.3 to boundedness in L^∞ in original variables.

Lemma 4.7 *([9]). Let $0\le a\le\frac{1}{p-1}$ and $R,\epsilon_0>0$. Let $0<\epsilon\le\epsilon_0$ and u be a solution to (1.1) on $[-1,0)\times\mathbb{R}^d$ satisfying*

$$\forall(t,x)\in[-1,0)\times B(0,R),\quad |u(t,x)|\le\frac{\epsilon}{|t|^{\frac{1}{p-1}-a}}. \tag{4.37}$$

For ϵ_0 small enough, the following holds for all $(t,x)\in[-1,0)\times B\left(0,\frac{R}{2}\right)$.

$$\text{If }\ \frac{1}{p-1}-a<\frac{1}{2},\quad |u(t,x)|\le C(a)\epsilon, \tag{4.38}$$

$$\text{If }\ \frac{1}{p-1}-a=\frac{1}{2},\quad |u(t,x)|\le C\epsilon(1+|ln(t)|), \tag{4.39}$$

$$\text{If }\ \frac{1}{p-1}-a>\frac{1}{2},\quad |u(t,x)|\le\frac{C(a)\epsilon}{|t|^{\frac{1}{p-1}-a-\frac{1}{2}}}. \tag{4.40}$$

Corollary 4.8. *Let $R>0$ and $0<T_-<T_+$. There exists $\epsilon_0>0$, $0<r\le R$ and $C>0$ such that the following holds. For any $0<\epsilon<\epsilon_0$, $T\in[T_-,T_+]$ and u solution to (1.1) on $[0,T)\times\mathbb{R}^d$ satisfying*

$$\forall (t,x)\in[0,T)\times B(0,R),\quad |u(t,x)|\le \frac{\epsilon}{(T-t)^{\frac{1}{p-1}}}, \tag{4.41}$$

one has:

$$\forall (t,x)\in[0,T)\times B(0,r),\quad |u(t,x)|\le C\epsilon. \tag{4.42}$$

To prove Lemma 4.7, we need two technical Lemmas taken from [9], whose proof can be found there.

Lemma 4.9 *([9]). Define for $0<\alpha<1$ and $0<\theta<h<1$ the integral $I(h)=\int_h^1 (s-h)^{-\alpha}s^{\theta}\mathrm{d}s$. It satisfies:*

$$\text{If } \alpha+\theta>1,\quad I(h)\le\left(\frac{1}{1-\alpha}+\frac{1}{\alpha+\theta-1}\right)h^{1-\alpha-\theta}, \tag{4.43}$$

$$\text{If } \alpha+\theta=1,\quad I(h)\le\frac{1}{1-\alpha}+|\log(h)|, \tag{4.44}$$

$$\text{If } \alpha+\theta<1,\quad I(h)\le\frac{1}{1-\alpha-\theta}. \tag{4.45}$$

Lemma 4.10 *([9]). If y, r and q are continuous functions defined on $[t_0,t_1]$ with*

$$y(t)\le y_0+\int_{t_0}^{t} y(s)r(s)\,\mathrm{d}s+\int_{t_0}^{t} q(s)\,\mathrm{d}s$$

for $t_0\le t\le t_1$, then for all $t_0\le t\le t_1$:

$$y(t)\le \mathrm{e}^{\int_{t_0}^t r(\tau)\,\mathrm{d}\tau}\left[y_0+\int_{t_0}^{t} q(\tau)\,\mathrm{e}^{-\int_{t_0}^{\tau} r(\sigma)\,\mathrm{d}\sigma}\mathrm{d}\tau\right]. \tag{4.46}$$

Proof of Lemma 4.7. We only treat the case (i), as the proof is the same for the other cases. We first localize the problem, with χ a smooth cut-off function, with $\chi=1$ on $B\left(0,\frac{R}{2}\right)$, $\chi=0$ outside $B(0,R)$ and $|\chi|\le 1$. We define

$$v:=\chi u \tag{4.47}$$

whose evolution, from (1.1), is given by:

$$v_t=\Delta v+|u|^{p-1}v+\Delta\chi u-2\nabla\cdot(\nabla\chi u). \tag{4.48}$$

We apply Duhamel's formula to (4.48) to find that for $t\in[-1,0)$:

$$v(t)=K_{t+1}*v(-1)+\int_{-1}^{t} K_{t-s}*(|u|^{p-1}v+\Delta\chi u-2\nabla\cdot(\nabla\chi u))\,\mathrm{d}s. \tag{4.49}$$

From (4.37) and (4.47), one has for free evolution term:

$$\|K_{t+1}*v(-1)\|_{L^\infty}\le\epsilon. \tag{4.50}$$

We now find an upper bound for the other terms in the previous equation.

Step 1 Case (i). For the linear terms, as $\frac{1}{p-1}-a+\frac12<1$, from (4.45) one has:

$$\begin{aligned}\|\textstyle\int_{-1}^t K_{t-s}*(\Delta\chi u-2\nabla\cdot(\nabla\chi u))\mathrm{d}s\|_{L^\infty} &\le C\int_{-1}^t \frac{1}{(t-s)^{\frac12}}\|u\|_{L^\infty(B(0,R))}\\ &\le C\epsilon\int_{-1}^t\frac{1}{(t-s)^{\frac12}}\frac{1}{|s|^{\frac{1}{p-1}-a}}\le C(a)\epsilon.\end{aligned} \tag{4.51}$$

For the nonlinear term, as $\frac{1}{p-1}-a<\frac12<\frac{1}{2(p-1)}=\frac{d-2}{8}$ because $d\ge 7$, we compute, using (4.37):

$$\begin{aligned}\| \textstyle\int_{-1}^{t} K_{t-s} * (\chi |u|^{p-1} v)\, \mathrm{d}s \|_{L^\infty} &\le \textstyle\int_{-1}^{t} \|u\|_{L^\infty(B(0,R))}^{p-1} \|v\|_{L^\infty} \mathrm{d}s \\ &\le \epsilon^{p-1} \textstyle\int_{-1}^{t} \frac{1}{|s|^{\frac{1}{2}}} \|v\|_{L^\infty} \mathrm{d}s.\end{aligned} \tag{4.52}$$

Gathering (4.50), (4.51) and (4.52), from (4.49), one has:

$$\|v(t)\|_{L^\infty} \le C(a)\epsilon + \epsilon^{p-1} \int_{-1}^{t} \frac{1}{|s|^{\frac{1}{2}}} \|v\|_{L^\infty}.$$

Applying (4.46) one obtains:

$$\|v(t)\|_{L^\infty} \le C(a), \epsilon, \mathrm{e}^{\int_{-1}^{t} |s|^{-\frac{1}{2}} \mathrm{d}s} \le C(a)\epsilon$$

which from (4.47) implies the bound (4.38) we had to prove. □

We can now end the proof of Proposition 4.2.

Proof of Proposition 4.2. For any $a \in B(0, R)$, from (4.1), (4.13) and (4.14), $w_{a,T}$ satisfies $E(w_{a,T}(0, \cdot)) \le \eta$ and:

$$|\Delta w_{a,T}| \le \frac{1}{2} |w_{a,T}|^p + \eta T_{+}^{\frac{p}{p-1}}.$$

Applying Proposition 4.3 to $w_{a,T}$, one obtains that for any $\eta' > 0$ if η is small enough:

$$\forall s \ge s\left(\frac{T_-}{4}\right), \quad |w_{a,T}(s, 0)| \le \eta'.$$

In original variables, this means:

$$\forall (t, x) \in B(0, R) \times [\frac{T_-}{4}, T), \quad |u(t, x)| \le \frac{\eta'}{(T-t)^{\frac{1}{p-1}}}.$$

Applying Corollary 4.8 for η' small enough, there exists $r > 0$ such that

$$\forall (t, x) \in B(0, R) \times [\frac{T_-}{4}, T), \quad |u(t, x)| \le C\eta'.$$

Then, a standard parabolic estimate propagates this bound for higher derivatives, yielding the result (4.15). □

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Appendix A. Parabolic estimates

We recall here some parabolic estimates. We refer to the proof of Theorem 8.18 in [1] for a proof of the Strichartz-type estimate. Let $d \ge 2$. We say that a couple of real numbers (q, r) is admissible if they satisfy:

$$q, r \ge 2, \ (q, r, d) \neq (2, +\infty, 2) \text{ and } \frac{2}{q} + \frac{d}{r} = \frac{d}{2}. \tag{A.1}$$

For any exponent $p \ge 1$, we denote by $p' = \frac{p-1}{p}$ its Lebesgue conjugated exponent.

Lemma 4.11 *(Strichartz type estimates for solutions to the heat equation). Let $d \ge 2$ be an integer. The two following inequalities hold. For any $t > 0$,*

$$\forall j \in \mathbb{N}, \ \forall q \in [1, +\infty], \quad \|\nabla^j K_t\|_{L^q} \le \frac{C(d, j)}{t^{\frac{d}{2q'} + \frac{j}{2}}} \ \textit{where } \frac{1}{q} + \frac{1}{q'} = 1. \tag{A.2}$$

For any (q_1, r_1), (q_2, r_2) satisfying (A.1), there exists a constant $C = C(d, q_1, q_2)$ such that for any source term $f \in L^{q_2'}([0, +\infty), L^{r_2'}(\mathbb{R}^d))$:

$$\left\| t \mapsto \int_0^t K_{t-t'} * f(t') \mathrm{d}t' \right\|_{L^{q_1}([0,+\infty), L^{r_1}(\mathbb{R}^d))} \le C \|f\|_{L^{q_2'}([0,+\infty), L^{r_2'}(\mathbb{R}^d))}. \tag{A.3}$$

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