Partial differential equations

Canard cycle transition at a slow–fast passage through a jump point

Transition de cycles canard pour un passage lent–rapide par un point de saut

Peter De Maesschalck [a], Freddy Dumortier [a], Robert Roussarie [b]

[a] *Hasselt University, Martelarenlaan 42, B-3500 Hasselt, Belgium*
[b] *Institut de mathématique de Bourgogne, UMR 5584 du CNRS, Université de Bourgogne, BP 47 870, 21078 Dijon cedex, France*

ARTICLE INFO

ABSTRACT

We introduce transitory canard cycles for slow–fast vector fields in the plane. Such cycles separate "canards without head" and "canards with head", like for example in the Van der Pol equation. We obtain optimal upper bounds on the number of periodic orbits that can appear near the cycle under whatever condition on the related slow divergence integral I, including the challenging case $I = 0$.

© 2014 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

RÉSUMÉ

On introduit des cycles canard transitoires pour les champs de vecteurs lents–rapides du plan. De tels cycles font la transition entre des « canards sans tête » et des « canards avec tête », comme par exemple dans l'équation de Van der Pol. On obtient des bornes supérieures optimales pour le nombre des orbites périodiques qui peuvent apparaître près du cycle canard transitoire, quelles que soient les conditions sur l'intégrale de divergence lente I associée, ce qui inclut le cas difficile $I = 0$.

© 2014 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

Version française abrégée

Dans cette note, on utilise la théorie géométrique des perturbations singulières pour présenter quelques résultats nouveaux sur un type particulier de cycles canard de systèmes lents–rapides dans le plan. Ces cycles canard, appelés cycles canard transitoires, peuvent être trouvés dans les systèmes lents–rapides sur les surfaces lisses, mais dans un souci de simplicité, on se restreint dans cette note au système (1), où b et c sont des paramètres de bifurcation et $\varepsilon > 0$ est le paramètre singulier, supposé petit. Quand $\varepsilon = 0$, les trajectoires sont horizontales en dehors de la courbe critique $y = F(x, c)$. Cette courbe est composée d'*arcs lents* séparés par les points de contact (x, y) où $F_x(x, c) = 0$. Pour $\varepsilon > 0$ petit, chaque trajectoire suit une orbite rapide en direction d'un arc lent attractant, puis longe typiquement cet arc en accord avec la *dynamique lente*. En arrivant à un point de contact, la trajectoire va suivre une autre orbite rapide en direction d'un nouvel arc lent attractant, et ainsi de suite. Les trajectoires peuvent aussi avoir un comportement atypique connu comme comportement de *type canard*.

Les orbites périodiques pour $\varepsilon > 0$ petit bifurquent des *cycles lents–rapides*, courbes fermées unions d'un nombre fini d'orbites rapides, d'arcs lents et de points de contact. Un cycle lent–rapide est *commun* lorsque les arcs lents sont tous, soit attractants, soit répulsifs, et est un *cycle canard* sinon.

Un cycle commun bifurque en une seule orbite périodique, nécessairement hyperbolique [3]. Les cycles canards apparaissent en famille. Dans cette note, on considère une famille $\Gamma_{b,c,s}$, avec $s \in (\mathbf{R}, 0)$, crée soit par un mécanisme de Hopf (cas I), soit par un mécanisme de saut (cas II), comme il est montré dans la Fig. 1. Le cycle canard $\Gamma_{b,c,0}$ est transitoire entre des *cycles canard sans tête* et des *cycles canard avec tête*, définis pour $s \neq 0$. Les bifurcations de $\Gamma_{b,c,s}$ en orbites périodiques sont complètement contrôlées par l'*intégrale de divergence lente* du cycle canard, qui, pour une équation de Liénard (1), est donnée par $I(\Gamma_{b,c,s}) = \sum_{j=1}^{n} \int_{u_j}^{v_j} \frac{F_x(x,c)^2}{b-x}\,dx$, où n est le nombre d'arcs lents contenus dans $\Gamma_{b,c,s}$ et où u_j, v_j sont les abscisses des extrémités de ces arcs [7]. Ces résultats ne s'appliquent pas au cycle canard transitoire $\Gamma_{b,c,0}$ car, comme on peut le voir sur la Fig. 3, la fonction $s \to I(\Gamma_{b,c,s})$ n'est pas différentiable en $s = 0$. Cependant, le théorème principal suivant montre que l'on a bien la relation attendue entre le nombre de zéros de l'intégrale de divergence lente et le nombre d'orbites périodiques bifurquant de $\Gamma_{b,c,0}$.

Théorème 1. *Quand $I(\Gamma_{b,c,0}) \neq 0$, il y a au plus une orbite périodique proche de $\Gamma_{b,c,0}$ au sens de Hausdorff, pour $\varepsilon > 0$, assez petit. Quand $I(\Gamma_{b,c,0}) = 0$, il y a au plus deux orbites périodiques proches $\Gamma_{b,c,0}$ au sens de Hausdorff dans le cas I et au plus trois dans le cas II, pour $\varepsilon > 0$, assez petit.*

Dans le cas I, on peut montrer la présence de bifurcations d'orbites périodiques de type selle–nœud en fonction du paramètre b. Dans le cas II, si le paramètre c détruit les cycles canard de façon générique, on peut montrer la présence de bifurcations d'orbites périodiques de type catastrophe élémentaire de codimension deux, en fonction des paramètres (b, c).

La méthode pour prouver le Théorème 1 est de faire un *éclatement* du point de saut à l'origine, ce qui permet de zoomer sur la région où la transition entre les canards avec tête et les canards sans tête est cruciale. L'éclatement remplace le point de saut par une demi-sphère S munie d'une dynamique non triviale (voir [6,8] par exemple). Le cycle canard transitoire est remplacé par une famille de cycles canard secondaires s'intercalant entre les canards sans tête et les canards avec tête. L'étude de la transition au-dessus de S se subdivise en un cas intérieur et deux cas frontière. Des preuves seront données dans [2].

1. Introduction

Canard cycles are by now a well-known phenomenon in the context of slow–fast systems, and two-dimensional systems in particular. We refer the reader, for example, to [1,6,8]. In this note, we use geometric singular perturbation theory to present some new results on specific types of canard cycles for which the treatment is intrinsically delicate. These canard cycles, e.g., the so-called *transitory canard cycles*, can in general be found in slow–fast systems on 2-dimensional smooth manifolds, but for the sake of the presentation in this short article, we restrict to a formulation in the plane that allows us to present some computations in a simpler and more didactic way. Consider

$$X_{\varepsilon,b,c}\colon \left\{\dot{x} = y - F(x,c),\ \dot{y} = \varepsilon(b-x)\right\}, \tag{1}$$

where b and c are bifurcation parameters and $\varepsilon > 0$ is the singular parameter that is kept small. When $\varepsilon = 0$, the dynamics trivially points along horizontal fibers towards or away from the *critical curve* $y = F(x,c)$, depending on the sign of $F_x(x,c)$. The critical curve is composed of (c-families of) *slow arcs*. These are connected components of $y = F(x,c)$, separated by *contact points* (x,y), where $F_x(x,c) = 0$. For $\varepsilon > 0$ small, orbits follow a fast orbit towards an attracting slow arc, and will then typically drift along the slow arc either upwards or downwards. The direction of the slow dynamics is given by the sign of $b - x$, as is clear from the $\dot{y}$-equation, see also Fig. 1. At a contact point, orbits will typically revert again following another fast orbit to another stable slow arc. Orbits may also exhibit an untypical behavior, known as canard behavior. We will describe canard behavior below.

Definition 1. Given an ordered sequence of fast orbits (compact parts including α and ω limits) and compact pieces of slow arcs, oriented by the respective fast and slow dynamics. Assuming that the union is a topological circle, we call the sequence a *slow–fast cycle*. It is called a *common* slow–fast cycle when the slow parts are either all attracting or all repelling, and a *canard cycle* if it contains both attracting and repelling parts.

The interest goes to the characterization of periodic orbits near a specific canard cycle Γ_{b_0,c_0}, for $\varepsilon > 0$ small enough, and for values of (b,c) close enough to some (b_0,c_0). The cyclicity of Γ_{b_0,c_0}, i.e. the upper bound for the number of such periodic orbits, is 1 when the slow–fast cycle is common, see [3]. In this paper, we distinguish two canard mechanisms: the jump mechanism and the Hopf mechanism, see also [5]. Both mechanisms are illustrated in Fig. 1. In the top picture, the slow curve is cubic-shaped, it contains $q = (0,0)$ and $p = (1,-1)$ as local extremes, and by choosing $b = 1 + O(\varepsilon)$, $b \sim 1$, the slow dynamics points in a way that a passage from the attracting arc to the repelling arc is possible. In the bottom picture, the relevant part to discuss now are the two local extremes $p_1 = (1,-1)$ and p_3 that are located on the same
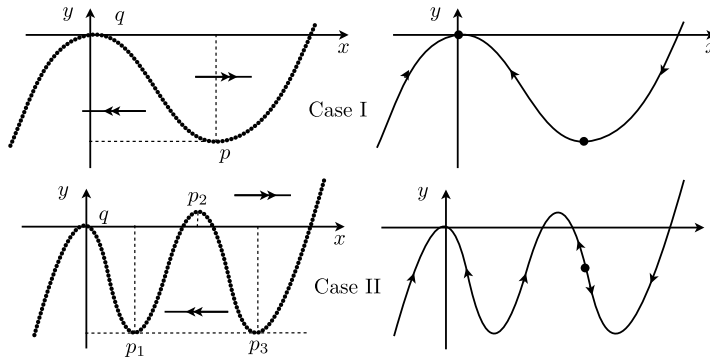
Fig. 1. Fast and slow dynamics. I: Hopf mechanism. II: Jump mechanism.
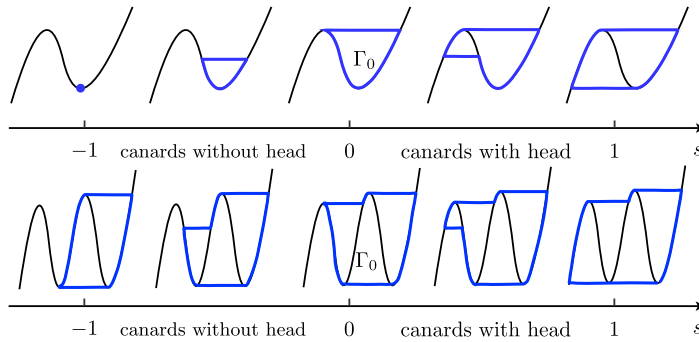**Fig. 1.** Dynamique rapide et lente. I : Mécanisme de Hopf. II : Mécanisme de saut.

Fig. 2. Canards cycles.
Fig. 2. Cycles canard.

height. By choosing the parameter b somewhere in between the x-coordinates of p_1 and p_3, the slow dynamics points in a way that a passage from the attracting arc (to the right of p_3) towards the repelling arc (between p and q) is possible, via the fast orbit in between. In both situations, the slow–fast cycles out of which limit cycles may be born are shown in Fig. 2. The family of slow–fast cycles is parameterized by s: it is the height of this left-most point when the left-most point is to the right of q, and it is the negative height of this point when it is to the left of q. The cycle at $s = 0$ is called *transitory*, because it separates canards with head from canards without head.

2. Results

Characterizing periodic orbits near a given limit periodic set such as a slow–fast cycle is typically done by dealing with the first return map towards some transverse section. It is well known that the critical points of the first return map can be related to zeros of the divergence integral of orbits near the l.p.s. In a slow–fast context, divergence integrals are essentially given by the so-called *slow divergence integral*, which in the Liénard setting, expressed in (1), is explicitly given by $I(\Gamma_{b,c,s}) = \sum_{j=1}^{n} \int_{u_j}^{v_j} \frac{F_x(x,c)^2}{b-x} \, dx$, where n is the number of involved slow arcs and where u_j and v_j are the x-coordinates of the end points of these arcs. It can be easily seen that a term in this sum is negative when it involves an attracting arc and positive otherwise. It is clear that the slow divergence integral of common slow–fast cycles is nonzero. In [3], it is shown in full generality that when $I(\Gamma_{b,c,s}) \neq 0$, there is at most *one* nearby periodic orbit. Looking at the s-family of canard cycles in Fig. 2, and considering a non-transitory canard cycle, it is shown in [7] that when $I(\Gamma_{b,c,s}) = 0$, the maximum number of limit cycles is given by the "order of zero" of the slow divergence integral function $s \mapsto I(\Gamma_{b,c,s})$. *In other words, the bifurcation diagram of the slow divergence integral as a function of s gives information on the bifurcation diagram of limit cycles.* These results specifically exclude transitory canards at $s = 0$, because of the piecewise regular nature of the bifurcation diagram of I at $s = 0$, see Fig. 3. The main theorem we state about transitory canard cycles shows that the relationship between the behavior of I and the number of limit cycles is as expected: $1 +$ the "number of zeros" of I is seen as an upper bound for the number of limit cycles:

Theorem 1. *When $I(\Gamma_{b,c,0}) \neq 0$ for the transitory canard cycle $\Gamma_{b,c,0}$, there is at most one periodic orbit Hausdorff close to $\Gamma_{b,c,0}$ for $\varepsilon > 0$ small enough. When $I(\Gamma_{b,c,0}) = 0$, there are at most two periodic orbits Hausdorff close to $\Gamma_{b,c,0}$ in case I and at most three in case II.*
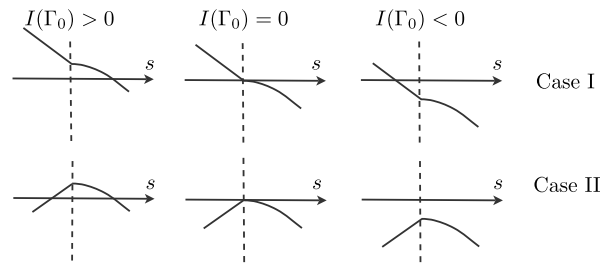
Fig. 3. Behavior of the slow divergence integral near $s = 0$.

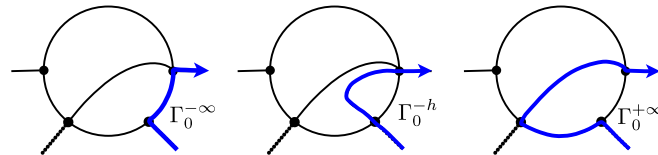**Fig. 3.** Comportement de l'intégrale de divergence lente autour de $s = 0$.

Fig. 4. Blow-up locus with different secondary canard cycles replacing the transitory cycle.

Fig. 4. Lieu d'éclatement avec différents cycles canard secondaires remplaçant le cycle transitoire.

In case I, we can furthermore use the presence of the parameter b to actually show the presence of a saddle node bifurcation of limit cycles, Hausdorff close to $\Gamma_{b,c,0}$. In case II, we assume that parameter c generically breaks the fast connection between the two local extremes p_1 and p_3. Under these circumstances, we can prove the presence of a degenerate limit cycle of order 3 near $\Gamma_{b,c,0}$, unfolding in an elementary catastrophe of codimension 2 using bifurcation parameters (b, c).

The method of proving Theorem 1 is by blowing up the jump point at the origin, zooming into the region where the change between canards with head and canards without head is crucial. Blow up replaces the jump point by a sphere and reveals a nontrivial interior dynamics (see [6,8]). We find that the transitory canard cycle is blown up into a full family of secondary canard cycles, continuously deforming from canards with head to headless canards, see Fig. 4. For the proof,[1] the interesting region on the blow-up locus is subdivided into three parts: two are neighborhoods of respectively $\Gamma_0^{-\infty}$ and $\Gamma_0^{+\infty}$, while the third one contains the Γ_0^{-h}, with h restricted to a (large) compact interval. A detailed elaboration of this transitory case, as well as an elaboration of a second transition outlined in [4], will be given in the monograph in preparation [2].

References

[1] E. Benoit, Équations différentielles: relation entrée–sortie, C. R. Acad. Sci. Paris, Ser. I 293 (5) (1981) 293–296.

[2] P. De Maesschalck, F. Dumortier, R. Roussarie, Canard Cycles from Birth to Transition, in preparation.

[3] P. De Maesschalck, F. Dumortier, R. Roussarie, Cyclicity of common slow–fast cycles, Indag. Math. (N.S.) 22 (3–4) (2011) 165–206.

[4] P. De Maesschalck, F. Dumortier, R. Roussarie, Canard-cycle transition at a fast–fast passage through a jump point, C. R. Acad. Sci. Paris, Ser. I 352 (1) (2014) 27–30.

[5] F. Dumortier, Slow divergence integral and balanced canard solutions, Qual. Theory Dyn. Syst. 10 (1) (2011) 65–85.

[6] F. Dumortier, R. Roussarie, Canard cycles and center manifolds, Mem. Amer. Math. Soc. 121 (577) (1996) x+100. With an appendix by Li Chengzhi.

[7] F. Dumortier, R. Roussarie, Multiple canard cycles in generalized Liénard equations, J. Differential Equations 174 (1) (2001) 1–29.

[8] M. Krupa, P. Szmolyan, Relaxation oscillation and canard explosion, J. Differential Equations 174 (2) (2001) 312–368.

---

[1] This is the succinct version of a text on file for five years in the Archives of the French Academy of Sciences. Copy is available upon request.