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Differential geometry

The hypoelliptic Laplacian, analytic torsion and Cheeger–Müller Theorem

Laplacien hypoelliptique, torsion analytique et théorème de Cheeger–Müller

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ABSTRACT

The purpose of this Note is to prove a formula relating the hypoelliptic Ray–Singer metric and the Milnor metric on the determinant of the cohomology of a compact Riemannian manifold by a Witten-like deformation of the hypoelliptic Laplacian in de Rham theory.

RÉSUMÉ

L'objet de cette Note est de démontrer une formule reliant les métriques de Ray–Singer hypoelliptique et de Milnor sur le déterminant de la cohomologie d'une variété riemannienne compacte par une déformation à la Witten du laplacien hypoelliptique en théorie de de Rham.

1. Introduction

Let X be a compact manifold of dimension n. Let (F, ∇^F) be a complex flat vector bundle on X. Let $H^{\cdot}(X, F)$ be the cohomology of the sheaf of locally flat sections of F. If E is a vector space of dimension m, set $\det E = \Lambda^m E$. We define the determinant of cohomology of F by $\lambda := \bigotimes_{i=0}^{n} (\det H^i(X, F))^{(-1)^i}$.

Let g^{TX} be a Riemannian metric on X, and let g^F be a Hermitian metric on F. When g^F is flat, it was conjectured by Ray and Singer [9], and was proved by Cheeger [5] and Müller [8] that the elliptic Ray–Singer metric on λ (which is obtained via elliptic Hodge theory) and the Reidemeister metric on λ (which is obtained combinatorially) coincide.

In [3], using the Witten deformation [11], Bismut and Zhang extended this result to an arbitrary g^F. Let $f : X \to \mathbf{R}$ be a Morse function with a Morse–Smale gradient field Z. With the Thom–Smale complex $(C^{\cdot}(W^u, F), \partial)$ associated with the flow $\dot{x} = -Z$, we can associate another metric $\|\cdot\|^{M,2}_{\lambda,Z}$ on λ, called the Milnor metric. Bismut and Zhang gave a formula relating the elliptic Ray–Singer metric and the Milnor metric. In this way, they gave a new proof of the Cheeger–Müller Theorem.

In [1,2], Bismut and Lebeau constructed a hypoelliptic deformation of the elliptic Hodge theory. Let $\mathcal{X}^*$ be the total space of the cotangent bundle T^*X. The hypoelliptic Laplacian, which depends on a parameter $b > 0$, is an operator acting on $\mathcal{X}^*$. When $b \to 0$, in the proper sense, it deforms the elliptic Hodge Laplacian on X. The authors also constructed the hypoelliptic Ray–Singer metric on λ, which is proved to be independent of $b > 0$ and to coincide with the elliptic Ray–Singer metric.

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In this Note, we announce a direct proof, detailed in [10], of a formula relating the hypoelliptic Ray–Singer metric and the Milnor metric on λ. Combining this with the result of Bismut–Zhang [3], we obtain the result of Bismut–Lebeau [2]. In our proof we do not use the fact that the hypoelliptic Laplacian deforms the elliptic Laplacian.

2. Hypoelliptic Laplacian and hypoelliptic Ray–Singer metric

We use the notation in the Introduction. Let $\pi : \mathcal{X}^* \to X$ be the canonical projection. We denote by $p \in C^\infty(\mathcal{X}^*, \pi^*(T^*X))$ the tautological section. Let $(\Omega^\cdot(\mathcal{X}^*, \pi^*F), d^{\mathcal{X}^*})$ be the obvious de Rham complex.

Set $\mathcal{H}(x,p) = |p|^2/2$. For $b > 0$, take $\mathcal{H}_b = \mathcal{H}/b^2$. In [1, (2.98)], to the function $\mathcal{H}_b$, Bismut associated a non-degenerated Hermitian form $\mathfrak{h}^{\Omega^\cdot(\mathcal{X}^*,\pi^*F)}_{\mathcal{H}_b}$ of signature (∞,∞) on $\Omega^\cdot_c(\mathcal{X}^*, \pi^*F)$, the space of compactly supported forms on $\mathcal{X}^*$. Let $A^2_{\phi,\mathcal{H}_b}$ be the Hodge Laplacian associated with this Hermitian form $\mathfrak{h}^{\Omega^\cdot(\mathcal{X}^*,\pi^*F)}_{\mathcal{H}_b}$. Then $A^2_{\phi,\mathcal{H}_b}$ is an $\mathfrak{h}^{\Omega^\cdot(\mathcal{X}^*,\pi^*F)}_{\mathcal{H}_b}$ formally self-adjoint operator acting on $\Omega^\cdot(\mathcal{X}^*, \pi^*F)$.

Bismut also introduced an operator $\mathfrak{A}'^{,2}_{\phi_b,\mathcal{H}}$ conjugated to $A^2_{\phi,\mathcal{H}_b}$. Let ∇^V be the fiberwise derivation along the fibres of T^*X, and let Δ^V be the fiberwise Laplacian. Let Y be the generator of geodesic flow on $\mathcal{X}^*$. When $F = \mathbf{C}$, the restriction of $2\mathfrak{A}'^{,2}_{\phi_b,\mathcal{H}}$ to $C^\infty(\mathcal{X}^*)$ is given by $\frac{1}{2b^2}(-\Delta^V + |p|^2 - n) - \frac{1}{b}\nabla_Y$. More generally, on forms of higher degree, $2\mathfrak{A}'^{,2}_{\phi_b,\mathcal{H}}$ coincides with this expression up to terms of order 0. The operator $\mathfrak{A}'^{,2}_{\phi_b,\mathcal{H}}$ is neither elliptic nor classically self-adjoint. By [6], it is hypoelliptic. In [1,2], the authors proved that many consequences of classical Hodge theory still hold true for the hypoelliptic theory. They constructed a non-degenerated Hermitian form $\|\cdot\|^{RS,2}_{\lambda,b}$ on λ via the analytic torsion of $\mathfrak{A}'^{,2}_{\phi_b,\mathcal{H}}$, and proved in [2, Theorem 9.0.1] that for any $b > 0$, $\|\cdot\|^{RS,2}_{\lambda,b}$ is a metric and equals to the elliptic Ray–Singer metric.

3. Our main result

Let ∇^{TX} be the Levi-Civita connection on (TX, g^{TX}). Let $\psi(TX, \nabla^{TX})$ be the Mathai–Quillen current [7] defined on the total space of the tangent bundle TX. By [3, Remark 3.8], $Z^*\psi(TX, \nabla^{TX})$ is a well-defined current of degree $n-1$ on X with values in $o(TX)$, the orientation bundle on X. Set $\theta(\nabla^F, g^F) = \mathrm{Tr}[(g^F)^{-1}\nabla^F g^F] \in \Omega^1(X)$. The main result of [10] is as follows.

Theorem 3.1. *For $b > 0$, we have*:

$$\log\left(\frac{\|\cdot\|^{RS,2}_{\lambda,b}}{\|\cdot\|^{M,2}_{\lambda,Z}}\right) = -\int_X \theta\big(\nabla^F, g^F\big) Z^*\psi\big(TX, \nabla^{TX}\big). \tag{1}$$

Proof. By [2, Theorem 6.7.1], $\|\cdot\|^{RS,2}_{\lambda,b}$ is independent of $b > 0$, and we know the explicit variation of $\|\cdot\|^{RS,2}_{\lambda,b}$ as a function of the metrics g^{TX} and g^F. By the same arguments as in [3, Section VII.b], to establish Theorem 3.1, we need only to obtain it for a particular quadruplet (b, g^{TX}, g^F, Z).

Let $\{x_\alpha\}_{1\leqslant\alpha\leqslant l}$ be the critical points of f. For $1 \leqslant \alpha \leqslant l$, we denote by n_α the Morse index at x_α. We suppose that on a neighborhood U_α of x_α, we have local coordinates $(y^1, \dots, y^n)$ such that for $y \in U_\alpha$,

$$f(y) = f(x_\alpha) - \frac{1}{2}\sum_{i=1}^{n_\alpha}\big(y^i\big)^2 + \frac{1}{2}\sum_{i=n_\alpha+1}^{n}\big(y^i\big)^2, \qquad g^{TX}_y = \sum_{i=1}^{n}\big(dy^i\big)^2, \qquad \nabla^F g^F_y = 0, \tag{2}$$

and that ∇f, the gradient of f with respect to g^{TX}, is Morse–Smale. In [10], we establish (1) for the quadruplet $(b_0, g^{TX}, e^{-2T_0 f}g^F, \nabla f)$ in the case where $b_0 > 0$ is small enough and $T_0 > 0$ is big enough.

The key step in our proof is to relate the Milnor metric $\|\cdot\|^{M,2}_{\lambda,\nabla f}$ with the asymptotic limit of the hypoelliptic Ray–Singer metric associated with $(b, g^{TX}, e^{-2Tf}g^F)$, when $b \to 0$, $T \to \infty$ and $b^2T = b_0^2 T_0$. This is obtained by establishing a canonical isomorphism between the Thom–Smale complex $(C^\cdot(W^u, F), \partial)$ and a subcomplex of $(\Omega^\cdot(\mathcal{X}^*, \pi^*F), d^{\mathcal{X}^*})$ induced by the small eigenvalues of the hypoelliptic Laplacian associated with $(b, g^{TX}, e^{-2Tf}g^F)$. We will explain this step in more detail in the following sections.

The proof of Theorem 3.1 is completed by the standard arguments of [3] and by refining the hypoelliptic local index techniques developed in [2], which we remark for the right side of (1). □

4. A Witten-like deformation of the hypoelliptic Laplacian

For $b > 0, T \geqslant 0$, let $L_{b,T}$ be the operator $2\mathfrak{A}'^{,2}_{\phi_b,\mathcal{H}}$ when g^F is replaced by $e^{-2Tf}g^F$. Take $\mathcal{H}_{b,T} = \mathcal{H}_b + Tf$. By [1, Remark 2.37], $L_{b,T}$ coincides with $2\mathfrak{A}'^{,2}_{\phi_b,\mathcal{H}_{1,T}}$ which is conjugated to $2A^2_{\phi,\mathcal{H}_{b,T}}$. When $F = \mathbf{C}$, the restriction of $L_{b,T}$ to $C^\infty(\mathcal{X}^*)$ is given by:

$$\frac{1}{2b^2}\left(-\Delta^V+|p|^2-n\right)-\frac{1}{b}\left(\nabla_Y-T\nabla^V_{\nabla f}\right). \tag{3}$$

Let $\mathcal{S}^\cdot(\mathcal{X}^*,\pi^*F)$ be the subspace of $\Omega^\cdot(\mathcal{X}^*,\pi^*F)$ consisting of elements with rapid decay along the fibers T^*X, and with derivatives of any orders with rapid decay. We consider $L_{b,T}$ as an operator with domain $\mathcal{S}^\cdot(\mathcal{X}^*,\pi^*F)$. By [2, Theorem 15.5.1], it is closable, and its closure is a maximal operator with compact resolvent. We still denote by $L_{b,T}$ the corresponding closure.

For $c>0$ and $\lambda_1>0$, set $\mathcal{W}_{c,\lambda_1}=\{\lambda_1+\sigma+i\tau\in\mathbf{C}\colon \sigma\geqslant c|\tau|^{1/8}\}$. Let $\Omega^{\cdot,[0,1]}_{b,T}(\mathcal{X}^*,\pi^*F)$ be the direct sum of the characteristic spaces of $2A^2_{\phi,\mathcal{H}_{b,T}}$ for the eigenvalues in $[0,1]$. We suppose that (f,g^{TX},g^F) satisfy the conditions in (2). Inspired by [4], we show in [10] the following result:

Theorem 4.1. *Given $\kappa>0$, $M>0$ such that $M>\kappa$, there exist $c_0>0$, $c_1>0$, $c_2>0$, $\lambda_1>c_0$, $b_0>0$, $T_0>0$, such that for $0<b\leqslant b_0$, $T\geqslant T_0$, $\kappa\leqslant b^2T\leqslant M$, we have $\operatorname{Sp}(2b^2L_{b,T})\subset[0,c_0e^{-c_1T}]\cup\mathcal{W}_{c_2,\lambda_1}$. Moreover, if ∇f is Morse–Smale, then for $b>0$ small enough and $T>0$ big enough such that $\kappa\leqslant b^2T\leqslant M$, we have a canonical isomorphism of complex $(\Omega^{\cdot,[0,1]}_{b,T}(\mathcal{X}^*,\pi^*F),d^{\mathcal{X}^*})\simeq(C^\cdot(W^u,F),\partial)$.*

Proof. The main difficulty is that when $T\to\infty$, $b\to0$ and $b^2T\in[\kappa,M]$, from (3), we cannot deduce an obvious localization for characteristic forms of $L_{b,T}$ associated with small eigenvalues. For $\epsilon>0$, set $L_{\epsilon,b,T}=e^{-\epsilon bT\nabla_p f}L_{b,T}e^{\epsilon bT\nabla_p f}$. When $F=\mathbf{C}$, by (3), the restriction of $L_{\epsilon,b,T}$ to $C^\infty(\mathcal{X}^*)$ is given by:

$$\frac{1}{2b^2}\left(-\Delta^V+\langle p|1-2\epsilon b^2T\nabla^{TX}\nabla f|p\rangle-n\right)+\epsilon T^2(1-\epsilon/2)|df|^2-\frac{1}{b}\left(\nabla_Y-T(1-\epsilon)\nabla^V_{\nabla f}\right). \tag{4}$$

For $\epsilon>0$ small enough, when $b^2T\in[\kappa,M]$, the term $\langle p|1-2\epsilon b^2T\nabla^{TX}\nabla f|p\rangle$ is positive. When $T\to\infty$, the term $\epsilon T^2(1-\epsilon/2)|df|^2$ becomes very large outside a neighborhood of the critical points. Although the harmonic oscillator $-\Delta^V+\langle p|1-2\epsilon b^2T\nabla^{TX}\nabla f|p\rangle-n$ acquires negative eigenvalues, this phenomenon is compensated by $\epsilon T^2(1-\epsilon/2)|df|^2$.

Now the difficulty is concentrated over the neighborhood of the critical points of f. We deduce Theorem 4.1 by some detailed and explicit calculations, which will be explained in the next section. □

5. The model operator near the critical points

Let $X=\mathbf{R}$ equipped with the trivial metric and with its canonical orientation. Take $F=\mathbf{R}$ also equipped with the trivial metric. Let $H^{\cdot,+}(\mathbf{R})$ (resp. $H^{\cdot,-}(\mathbf{R})$) be the cohomology (resp. with compact support) of $\mathbf{R}$. Then we have the canonical isomorphisms $\lambda^\pm=\det(H^{\cdot,\pm}(\mathbf{R}))=\mathbf{R}$. Set $f^\pm(y)=\pm y^2/2$.

The Witten Laplacian [11] and [3, Section V.b] is then given by $-\frac{\partial^2}{\partial y^2}+T^2y^2\mp T\pm 2T\,dy\,i_{\partial/\partial y}$. Then $H^{\cdot,\pm}(\mathbf{R})$ can be identified with the kernel of this operator. By a direct computation, the elliptic Ray–Singer metric coincides with the canonical metric on $\lambda^\pm=\mathbf{R}$.

Let $L^\pm_{b,T}$ be the operator $L_{b,T}$ which we considered before in this special case. Our operator $L^\pm_{b,T}$ is the model of the operator $L_{b,T}$ of Section 4 near the critical points of f. We have the identity:

$$L^\pm_{b,T}=\frac{1}{2b^2}\left(-\frac{\partial^2}{\partial p^2}+p^2-1+2\,dp\,i_{\frac{\partial}{\partial p}}\right)-\frac{1}{b}\left(p\frac{\partial}{\partial y}\mp Ty\frac{\partial}{\partial p}\right)\pm T(dy-dp)i_{\frac{\partial}{\partial y}+\frac{\partial}{\partial p}}. \tag{5}$$

In [10], we show $\operatorname{Sp}L^\pm_{b,T}=\frac{1+\sqrt{1\mp4b^2T}}{2b^2}\mathbf{N}\mp\frac{-1+\sqrt{1\mp b^2T}}{2b^2}\mathbf{N}$. We calculate explicitly the kernel of $L^\pm_{b,T}$, and we identify this kernel with $H^{\cdot,\pm}(\mathbf{R})$. We show that the hypoelliptic Ray–Singer metric is still equal to the canonical metric on $\lambda^\pm=\mathbf{R}$. This fact plays an important role in our proof of Theorem 3.1.

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