



ELSEVIER

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com



Differential geometry

## Isometry group of Sasaki–Einstein metric



## Groupes d'isométries des métriques de Sasaki–Einstein

Weiyong He

Department of Mathematics, University of Oregon, Eugene, OR 97403, USA

## ARTICLE INFO

## Article history:

Received 4 June 2013

Accepted after revision 31 October 2013

Available online 20 November 2013

Presented by the Editorial Board

## ABSTRACT

In this short paper we prove a conjecture of Martelli–Sparks–Yau regarding the isometry group of a Sasaki–Einstein metric.

© 2013 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

## R É S U M É

Soit  $(M, g)$  une variété de Sasaki–Einstein et  $(X, J)$  la variété affine sous-jacente à son cône de Kähler. Nous montrons que la composante neutre du sous-groupe compact maximal du groupe des automorphismes de  $(X, J)$  coïncide avec la composante neutre du groupe des isométries holomorphes de  $(M, g)$ .

© 2013 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

Let  $(M, g)$  be a Sasaki–Einstein manifold of dimension  $2n + 1$ ; equivalently its Kähler cone is a Calabi–Yau cone with Ricci flat metric. Let  $(X, J)$  be the underlying affine variety of its Kähler cone and denote  $\text{Aut}(X, J)$  to be its automorphism group; denote  $\text{Aut}_0(X, J)$  to be the identity component of  $\text{Aut}(X, J)$ . We prove the following result in this short paper,

**Theorem 1.** *The identity component of the holomorphic isometry group of  $(M, g)$  is the identity component of a maximal compact subgroup of  $\text{Aut}(X, J)$ .*

This answers a conjecture proposed by Martelli, Sparks, and Yau [8] about the holomorphic isometry group of a Sasaki–Einstein metric; when a Sasaki–Einstein metric is *quasi-regular*, this is proved in [8] (Section 4.3). The statement itself can be viewed as a generalization of Matsushima's theorem [9] on a Kähler–Einstein metric on a Fano manifold, which asserts that the identity component of the isometry group of a Kähler–Einstein metric on a Fano manifold is the identity component of a maximal compact subgroup of its automorphism group. Unlike the Fano case, a Killing vector field of a Sasaki–Einstein metric does not have to be holomorphic; hence we can only assert the conclusion about holomorphic isometry group. A typical example is the  $(2n + 1)$  dimensional round sphere whose isometry group is  $SO(2n + 2)$ , but the holomorphic isometry group is  $U(n + 1)$ . By a general result on Sasaki manifolds (see Theorem 8.18, Corollary 8.19 in [3]), a Killing vector field of a Sasaki–Einstein metric is (real) holomorphic unless on a round sphere or a 3-Sasaki structure (its Kähler cone is a hyper-Kähler cone and this is the counterpart of hyper-Kähler structure; it is always quasi-regular). Hence, except these two special cases, the holomorphic condition in Theorem 1 can be dropped.

In this note, we shall prove Theorem 1 when the Sasaki metric  $(M, g)$  is *irregular*. Given a Sasaki metric  $(M, g)$ , its Reeb vector field  $\xi$  is a holomorphic Killing vector field of  $(X, J, \bar{g})$ , where  $\bar{g}$  is the Kähler cone metric. We fix a maximal torus  $\mathbb{T}^k \subset \text{Aut}_0(X, J)$  such that its Lie algebra  $\mathfrak{t}$  contains  $\xi$ ; we can assume that the dimension  $k$  of  $\mathbb{T}^k$  is at least two without

E-mail address: [whe@uoregon.edu](mailto:whe@uoregon.edu).

loss of generality (this is the case when  $\xi$  is irregular for example). Let  $K$  be a maximal compact subgroup of  $\text{Aut}(X, J)$  containing  $\mathbb{T}^k$ ; we denote its Lie algebra as  $\mathfrak{h} = \text{Lie}(K)$ . The starting point is that the Reeb vector field is in the center of  $\mathfrak{h}$ , as in the quasi-regular case [8].

**Proposition 0.1.** *The Reeb vector field  $\xi$  of a Sasaki–Einstein metric  $(M, g)$  is in the center of  $\mathfrak{h} = \text{Lie}(K)$ .*

**Proof.** Let  $\mathfrak{z}$  be in the center of  $\mathfrak{h}$ . We can then write  $\mathfrak{t} = \mathfrak{z} \oplus \mathfrak{t}'$ . The Reeb vector fields form a convex subset of  $\mathfrak{t}$ , called *Reeb cone* and denoted by  $\mathcal{R}$ . As in [7], we shall be mainly interested in the *normalized Reeb vector fields* that lie in a hyperplane  $\mathcal{H}$  in  $\mathfrak{t}$ ; we denote it as  $\mathcal{R}' = \mathcal{R} \cap \mathcal{H}$ . In [8] (see [5] for expository), it was proved that the volume functional  $V : \mathcal{R}' \rightarrow \mathbb{R}$  of a Sasaki structure depended only on the Reeb vector fields, and was a convex functional in  $\mathcal{R}'$ ; moreover, the Reeb vector field  $\xi$  of a Sasaki–Einstein metric has to be the (unique) critical point of the volume functional. Actually, it was proved further that the volume functional is actually proper in  $\mathcal{R}'$ , and hence such a minimizer always exists [7]. Clearly we can restrict our discussion on  $\mathfrak{z}$  and there is a unique minimizer, denoted as  $\xi_*$  of the volume functional when it is restricted to the normalized Reeb cone contained in  $\mathfrak{z}$ . It remains to show that  $\xi = \xi_*$ . When  $\xi_*$  is quasi-regular, this is proved in [8]. Hence we assume that  $\xi_*$  is irregular and hence that  $\dim \mathfrak{z} \geq 2$ . We can choose a sequence-normalized Reeb vector field  $\{\xi_n\}$  in  $\mathfrak{z}$  such that  $\xi_n \rightarrow \xi$  by a result of Rukimbira; moreover, each  $\xi_n$  can be taken as quasi-regular (see [10] or Theorem 7.1.10 [3]). Now, for any  $\zeta \in \mathfrak{t}'$ , we suppose that  $\zeta$  satisfies the normalized condition such that, for any normalized Reeb vector field  $\tilde{\xi}$ ,  $\tilde{\xi} + t\zeta$  is still a normalized Reeb vector field for a (small) real number  $t$ . We then consider the volume functional  $v(t) = V(\xi_n + t\zeta)$ . We claim that  $V(\xi_n) \leq V(\xi_n + t\zeta)$  for small  $t$ . Clearly,  $v(t)$  is a convex function of  $t$  and we only need to show that  $v'(0) = 0$ . Since  $\xi_n$  is quasi-regular and we can consider the quotient orbifold  $Z = M/\mathcal{F}_{\xi_n}$ . Then  $\mathfrak{t}'$  descends to a Lie subalgebra of  $\text{aut}_{\mathbb{R}}(Z)$ . Recall now that the variation of the volume functional  $dV$  coincides with the Futaki invariant (up to a multiplication of a constant). Now recall that the Futaki invariant  $F_{\mathbb{C}} : \text{aut}(Z) \rightarrow \mathbb{C}$  is only nontrivial on the center of  $\text{aut}(Z)$  and that, in particular, it vanishes on the complexification of  $\mathfrak{t}'$ . Hence it follows that  $dV_{\xi_n}(\zeta) = v'(0) = 0$  and the claim  $V(\xi_n) \leq V(\xi_n + t\zeta)$  is proved. By the smoothness of the volume functional on Reeb vector fields, we know that  $V(\xi_*) \leq V(\xi_* + t\zeta)$  for any normalized  $\zeta \in \mathfrak{t}'$  and small  $t$ . It follows that  $dV_{\xi_*}(\zeta) = 0$  for any  $\zeta \in \mathfrak{t}'$ . It follows that  $\xi_*$  is also a critical point of  $V$  in  $\mathcal{R}'$  (hence a minimizer of  $V$ ). By the uniqueness of the minimizer in  $\mathcal{R}'$ ,  $\xi_* = \xi$ .  $\square$

Now we suppose  $\xi \in \mathfrak{z}$  and  $\dim(\mathfrak{z}) \geq 2$ . Let  $G$  be the identity component of the isometry group of  $(M, g)$  with Lie algebra  $\mathfrak{g}$ ; clearly  $\xi$  is also in the center of  $\mathfrak{g}$ . Now we can choose a sequence of normalized Reeb vector fields  $\xi_n$  that are quasi-regular and lie in  $\mathfrak{z}$  and the center of  $\mathfrak{g}$ . When  $n$  is sufficiently large, then we have the following.

**Proposition 0.2.** *For  $\xi_n$ , there exists a Sasaki–Ricci soliton  $g_n$  such that its underlying Kähler cone is  $(X, J)$  and its identity component of the isometry group is still  $G$ .*

**Proof.** This is really just the local deformation of Sasaki–Ricci solitons with the Kähler cone fixed while Reeb vector fields vary. The existence of such Sasaki–Ricci solitons follows from an argument of implicit function theory (in a  $G$ -invariant way). The argument of Theorem 4.1 [7] proves such a local deformation theory in a  $\mathbb{T}$ -invariant way; since  $\xi$  and  $\xi_n$  are all in the center of  $\mathfrak{g}$ , the same argument of Theorem 4.1 still applies with the maximal torus replaced by  $G$ . In particular, the isometry group of  $(M, g_n)$  contains  $G$ . Now by a general theorem of Grove, Kratcher and Ruh [6], we know that when  $n$  is large enough, there is an inclusion, up to conjugation, of the isometry group of  $(M, g_n)$  into the isometry group  $G$  of  $(M, g)$  (see Lemma 8.2 [7], for example). It follows that the isometry group of  $(M, g_n)$  also has an identity component  $G$ , up to conjugation.

Hence we only need to prove that the identity component of isometry group  $G$  of  $(M, g_n, \xi_n)$  is the identity component of a maximal compact subgroup of  $\text{Aut}(X, J)$ , for sufficiently large  $n$ . This is a Calabi-type theorem [4] and it is proved by Tian and Zhu [11] for Kähler–Ricci solitons on Fano manifolds.

**Theorem 2 (Tian–Zhu).** *Suppose that  $(M, g, J)$  is a Kähler–Ricci soliton on a Fano manifold  $(M, J)$ . Then the identity component of the isometry group of  $(M, g)$  is a maximal compact group of the identity component of  $\text{Aut}(M, J)$ .*

By a direct adaption of Tian and Zhu's argument, we have:

**Proposition 0.3.** *For quasi-regular Sasaki–Ricci solitons  $(M, g_n, \xi_n)$ , the identity component of its isometry group is the identity component of a maximal compact subgroup of  $\text{Aut}(X, J)$ .*

**Proof.** Let  $K$  be a maximal group in  $\text{Aut}(X, J)$  such that  $\xi_n$  is in its Lie algebra  $\mathfrak{h}$  and let  $K_0$  be its identity component. Then, by Proposition 0.1,  $\xi_n$  is in  $\mathfrak{z}$ , the center of  $\mathfrak{h}$ . Since  $\xi_n$  is quasi-regular, it generates a  $U(1)$  action of  $(X, J)$  contained in  $K_0$ . Let  $Z = M/\mathcal{F}_{\xi_n}$  be the quotient orbifold and let the corresponding Kähler–Ricci soliton be  $h$ . The compact group  $K_0$ , modulo  $U(1)$ , generated by  $\xi_n$  then descends to a compact subgroup of the complex automorphism group  $\text{Aut}_0(Z)$ . By Tian

and Zhu's theorem and its proof applied to  $(Z, h)$ , we know that  $K_0$  acts isometrically on  $(Z, h)$ . It then follows that  $K_0$  acts isometrically on  $(M, g_n, \xi_n)$ . Hence  $K_0$  coincides with  $G$ , the identity component of the isometry group of  $(M, g_n, \xi_n)$ .  $\square$

**Theorem 1** is then a corollary of **Proposition 0.2** and **Proposition 0.3**. Matsushima's theorem is on Lie algebra level and does not apply directly to a finite discrete subgroup which is not contained in the identity component. Bando and Mabuchi [1] proved that a Kähler–Einstein metric on a Fano manifold is unique modulo automorphisms; in particular, a Kähler–Einstein metric must be invariant under  $\text{Aut}(M, J)$  if there is no holomorphic vector field. The following short argument uses the similar idea as in [1], but relies on the convexity of Ding's  $\mathcal{F}$ -functional, established by Berndtsson [2].

**Proposition 0.4.** *Let  $(M, g)$  be a Kähler–Einstein metric on a Fano manifold  $(M, J)$ . Then  $\text{Aut}(M, J) = \text{Aut}_0(M, J)G$ , where  $G$  is the isometry group of  $(M, g)$ .*

**Proof.** Suppose  $\lambda \in \text{Aut}(M, J)$ , which is a Kähler–Einstein metric on  $(M, J)$ . Note that the Kähler class of  $g$  and  $\lambda^*g$  are both in  $c_1(M, J)$ , under appropriate normalization. Suppose  $g \neq \lambda^*g$ . Recall that in the space of Kähler potentials  $\mathcal{H}$ , there exists a unique geodesic function  $\gamma(t)$  connecting  $g, \lambda^*g$  by a fundamental result of Chen. Recall that a Kähler–Einstein metric in  $c_1(M, J)$  is minimum of Ding's  $\mathcal{F}$ -functional, which is convex along geodesics in  $\mathcal{H}$ . It follows that  $\mathcal{F}$ -functional is linear (constant) along  $\gamma(t)$ . By Berndtsson's theorem [2],  $\gamma(t)$  is generated by a holomorphic vector field  $\zeta$ . In particular, there exists a one-parameter subgroup  $\sigma_t$  generated by  $\zeta$  such that  $\sigma_0 = id$ , and  $\sigma_1^*g = \lambda^*g$ . The proposition then follows.  $\square$

One may wonder whether the above Bando and Mabuchi's result for Kähler–Einstein metrics on Fano manifolds holds or not for a Sasaki–Einstein metric. We believe this might not be the case in Sasaki setting due to the possible complexity of  $\text{Aut}(X, J)$ . The main point is that in a Kähler setting, under the action of an automorphism group, the first Chern class (hence the Kähler class of the Kähler–Einstein metric, modulo scaling) is invariant. In Sasaki's setting, the Reeb vector field is also unique given a fixed Reeb cone; but we are not sure that such a Reeb cone is unique or not even within the Lie algebra  $\mathfrak{t}$  of a fixed (maximal) torus  $\mathbb{T} \subset \text{Aut}(X, J)$  (see Remark 2.9 in [7]). We ask the following problem:

**Question 3.** Let  $(M, g)$  be a Sasaki–Einstein metric and let  $G$  be the holomorphic isometry group of  $(M, g)$ . Is it true  $\text{Aut}(X, J) = \text{Aut}_0(X, J)G$ ?

## Acknowledgement

We thank Sun Song for valuable discussions on Sasaki geometry. We also thank the referee for valuable comments and suggestions. The author is partially supported by an NSF grant.

## References

- [1] S. Bando, T. Mabuchi, Uniqueness of Einstein–Kähler metrics modulo connected group actions, in: Algebraic Geometry, Sendai, 1985, in: Adv. Stud. Pure Math., vol. 10, North-Holland, Amsterdam, 1987, pp. 11–40.
- [2] B. Berndtsson, A Brunn–Minkowski-type inequality for Fano manifolds and the Bando–Mabuchi uniqueness theorem, arXiv:1103.0923 [math.DG].
- [3] C. Boyer, K. Galicki, Sasakian Geometry, Oxford Mathematical Monographs, Oxford University Press, USA, 2008, 614 p.
- [4] E. Calabi, Extremal Kähler metrics. II, in: Differential Geometry and Complex Analysis, Springer, Berlin, 1985, pp. 95–114.
- [5] A. Futaki, H. Ono, G.-F. Wang, Transverse Kähler geometry of Sasaki manifolds and toric Sasaki–Einstein manifolds, J. Differ. Geom. 83 (2009) 585–636.
- [6] K. Grove, H. Karcher, E. Ruh, Group actions and curvature, Invent. Math. 23 (1974) 31–48.
- [7] W.-Y. He, S. Sun, Frankel conjecture and Sasaki geometry, arXiv:1202.2589 [math.DG].
- [8] D. Martelli, J. Sparks, S.-T. Yau, Sasaki–Einstein Manifolds and Volume Minimisation, Commun. Math. Phys. 280 (2007) 611–673.
- [9] Y. Matsushima, Sur la structure du groupe d'homeomorphismes analytiques d'une certaine variété kählerienne, Nagoya Math. J. 11 (1957) 145–150 (in French).
- [10] P. Rukimbira, Chern–Hamilton's conjecture and  $K$ -contactness, Houston J. Math. 21 (4) (1995) 709–718.
- [11] G. Tian, X.-H. Zhu, Uniqueness of Kähler–Ricci solitons, Acta Math. 184 (2) (2000) 271–305.