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Ruelle operators and decay of correlations for contact Anosov flows

Opérateurs de Ruelle et décroissance des corrélations pour des flots de contact d'Anosov

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ABSTRACT

We prove strong spectral estimates for Ruelle transfer operators for arbitrary C^2 contact Anosov flows. As a consequence of this we obtain: (a) existence of a non-zero analytic continuation of the Ruelle zeta function with a pole at the entropy in a vertical strip containing the entropy in its interior; (b) a Prime Orbit Theorem with an exponentially small error; (c) exponential decay of correlations for Hölder continuous observables with respect to any Gibbs measure.

RÉSUMÉ

On prouve des estimations spectrales fortes pour l'opérateur de transfert de Ruelle relatif à des flots de contact d'Anosov arbitraires de classe C^2. Comme conséquence, on obtient les trois résultats suivants : (a) l'existence d'un prolongement analytique sans zéros de la fonction zêta de Ruelle dans une bande verticale contenant l'entropie dans son intérieur et ayant l'entropie comme ensemble de pôles ; (b) un théorème asymptotique pour le nombre de trajectoires périodiques primitives avec un reste exponentiellement petit ; (c) la décroissance exponentielle des corrélations pour des observables höldériennes par rapport à une mesure de Gibbs quelconque.

1. Introduction

Let $\phi_t : M \to M$ be a C^2 contact Anosov flow on a C^2 compact Riemann manifold M. Let $W^u_\epsilon(x)$ and $W^s_\epsilon(x)$ be the strong unstable and stable manifolds at $x \in M$, respectively. Let $\mathcal{R} = \{R_i\}_{i=1}^k$ be a (pseudo-) Markov partition for ϕ_t consisting of rectangles $R_i = [U_i, S_i]$, where U_i (resp. S_i) are (admissible) subsets of $W^u_\epsilon(z_i)$ (resp. $W^s_\epsilon(z_i)$) for some $\epsilon > 0$ and $z_i \in M$. Set $R = \bigcup_{i=1}^k R_i$ and $U = \bigcup_{i=1}^k U_i$. The first return time function $\tau : R \to [0, \infty)$ and the standard Poincaré map $\mathcal{P} : R \to R$ are naturally defined, while the *shift map* $\sigma : U \to U$ is defined by $\sigma = \pi^{(U)} \circ \mathcal{P}$, where $\pi^{(U)} : R \to U$ is the projection along the leaves of local stable manifolds. Let $\widehat{U}$ be the set of all $x \in U$ whose orbits do not have common points with the boundary of R, and for a given $\theta \in (0, 1)$, define the metric D_θ on $\widehat{U}$ by $D_\theta(x, y) = 0$ if $x = y$, $D_\theta(x, y) = 1$ if x, y belong to different U_i's and $D_\theta(x, y) = \theta^N$ if $\mathcal{P}^j(x)$ and $\mathcal{P}^j(y)$ belong to the same rectangle R_{i_j} for all $j = 0, 1, \dots, N-1$, and N is the largest integer with this property. Given a bounded function $f \in B(U)$, the *Ruelle transfer operator* $L_f : B(U) \to B(U)$

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is defined by $(L_f h)(x) = \sum_{\sigma(y)=x} e^{f(y)} h(y)$. Denote by $\mathcal{F}_\theta(\widehat{U})$ *the space of all functions* $h : \widehat{U} \to \mathbf{C}$ *with Lipschitz constants* $|h|_\theta = \sup\{\frac{|h(x)-h(y)|}{D_\theta(x,y)}:\ x \neq y,\ x, y \in \widehat{U}\} < \infty$. For such h and $b \in \mathbb{R} \setminus \{0\}$ set $\|h\|_{\theta,b} = \|h\|_0 + \frac{|h|_\theta}{|b|}$, where $\|h\|_0$ is the *standard* sup *norm* of h on $\widehat{U}$. Given a real-valued $f \in \mathcal{F}_\theta(\widehat{U})$, let $P_f \in \mathbb{R}$ be the unique number such that the topological pressure $\Pr_\sigma(f - P_f\tau)$ of $f - P_f\tau$ with respect to σ is zero (cf. [4]).

The main result in this paper is the following.

Theorem 1. *Let $\phi_t : M \to M$ be a C^2 contact Anosov flow on a C^2 compact Riemann manifold M, let $\mathcal{R} = \{R_i\}_{i=1}^{k_0}$ be a (pseudo-) Markov partition for ϕ_t as above and let $\sigma : U \to U$ be the corresponding shift map. There exists $\hat{\theta} \in (0, 1)$ such that for any $\theta \in [\hat{\theta}, 1)$, any real-valued $f \in \mathcal{F}_\theta(\widehat{U})$ and any $\epsilon > 0$ there exist constants $0 < \rho < 1$, $a_0 > 0$, $b_0 \geqslant 1$ and $C > 0$ such that if $a, b \in \mathbb{R}$ satisfy $|a| \leqslant a_0$ and $|b| \geqslant b_0$, then $\|L^m_{f-(P_f+a+\mathbf{i}b)\tau} h\|_{\theta,b} \leqslant C\rho^m\, |b|^\epsilon\, \|h\|_{\theta,b}$ for any integer $m > 0$ and any $h \in \mathcal{F}_\theta(\widehat{U})$.*

The latter implies in particular that the spectral radius of $L_{f-(P_f+a+\mathbf{i}b)\tau}$ on $\mathcal{F}_\theta(\widehat{U})$ does not exceed ρ.

The above was first proved by Dolgopyat [1] in the case of geodesic flows on compact surfaces of negative curvature, and also in the case of transitive Anosov flows on compact Riemann manifolds with C^1 jointly non-integrable local stable and unstable foliations for the Sinai–Bowen–Ruelle potential $f = \log\det(d\phi_\tau)_{|E^u}$. More general results were proved in [7] for mixing Axiom A flows on basic sets under some additional regularity assumptions, including Lipschitzness of the local stable holonomy maps.

We now state three immediate consequences of Theorem 1. First, using the procedure described in [1] one gets exponential decay of correlations with respect to any Gibbs measure.

Theorem 2. *Let $\phi_t : M \to M$ be a C^5 contact Anosov flow on a C^5 manifold M, let F be a Hölder continuous function on M and let ν_F be the Gibbs measure determined by F on M. For every $\alpha > 0$ there exist constants $C = C(\alpha) > 0$ and $c = c(\alpha) > 0$ such that*

$$\left| \int_M A(x)B\big(\phi_t(x)\big)\,d\nu_F(x) - \left(\int_M A(x)\,d\nu_F(x)\right)\left(\int_M B(x)\,d\nu_F(x)\right)\right| \leqslant Ce^{-ct}\|A\|_\alpha\|B\|_\alpha$$

for any two functions $A, B \in C^\alpha(M)$.

So far the only result concerning exponential decay of correlations for general Gibbs potentials appears to be that of Dolgopyat [1] for geodesic flows on compact surfaces. Liverani [3] proved exponential decay of correlations for C^4 contact Anosov flows, and finer results (which imply exponential decay of correlations) were established later by Tsujii and very recently by Nonnenmacher and Zworski (for C^3 and C^∞ contact Anosov flows, respectively), however all these three papers deal with the measure determined by the Riemann volume. Recently Giulietti, Liverani and Pollicott [2] derived exponential decay of correlations for contact Anosov flows with respect to the measure of maximal entropy (generated by the potential $F = 0$) under a bunching condition (which implies that the stable/unstable foliations are $\frac{2}{3}$-Hölder).

Consider the *Ruelle zeta function* $\zeta(s) = \prod_\gamma (1 - e^{-s\ell(\gamma)})^{-1}$, $s \in \mathbf{C}$, where γ runs over the set of primitive closed orbits of $\phi_t : M \to M$ and $\ell(\gamma)$ is the least period of γ. Denote by h_T the *topological entropy* of ϕ_t on M. Using Theorem 1 and an argument of Pollicott and Sharp [6], for flows $\phi_t : M \to M$ as in Theorem 1, one derives the following.

Theorem 3.

(a) *The Ruelle zeta function $\zeta(s)$ of the flow $\phi_t : M \to M$ has an analytic and non-vanishing continuation in a half-plane* $\mathrm{Re}(s) > c_0$ *for some $c_0 < h_T$ except for a simple pole at $s = h_T$.*
(b) *There exists $c \in (0, h_T)$ such that $\pi(\lambda) = \#\{\gamma:\ \ell(\gamma) \leqslant \lambda\} = \mathrm{li}(e^{h_T\lambda}) + O(e^{c\lambda})$ as $\lambda \to \infty$, where $\mathrm{li}(x) = \int_2^x \frac{du}{\log u} \sim \frac{x}{\log x}$ as $x \to \infty$.*

Part (b) above proves a conjecture made recently by Giulietti, Liverani and Pollicott [2]. Parts (a) and (b) were first established by Pollicott and Sharp [6] for geodesic flows on compact surfaces of negative curvature (using [1]), and then similar results were proved in [7] for mixing Axiom A flows on basic sets satisfying certain additional assumptions (as mentioned above). Recently, using different methods, it was proved in [2] that: (i) for volume preserving three dimensional Anosov flows (a) holds, and moreover, in the case of C^∞ flows, the Ruelle zeta function $\zeta(s)$ is meromorphic in $\mathbf{C}$ and $\zeta(s) \neq 0$ for $\mathrm{Re}(s) > 0$; (ii) (b) holds for geodesic flows on $\frac{1}{9}$-pinched compact Riemann manifolds of negative curvature.

Strong spectral estimates for Ruelle transfer operators as the ones described in Theorem 1 lead to a variety of deep results of various kinds – see, e.g., [7,5] for more information.

2. Idea of the proof of Theorem 1

In the proof of the main result we use the general framework of the method of Dolgopyat [1] and its development in [7], however some significant new ideas have been implemented.

Let $f \in \mathcal{F}_\theta(\widehat{U})$ for some sufficiently large $\theta \in (0,1)$ so that $\tau \in \mathcal{F}_\theta(\widehat{U})$, and let $g = f - P_f\tau$, where P_f is as in Section 1. Then f and g can be regarded as functions on R, constant on stable leaves in any R_i. Let $\mu = \mu_g$ be the Gibbs measure on R determined by g; then $\mu(\widehat{R}) = 1$. As in [1], we deal with functions $f^{(a)}(u) = f(u) - (P_f + a)\tau(u) + \ln h_a(u) - \ln h_a(\sigma(u)) - \ln\lambda_a$ and the corresponding normalized operators $L_{ab} = L_{f^{(a)} - \mathbf{i} b\tau}$ on $\mathcal{F}_\theta(\widehat{U})$, where $\lambda_a > 0$ is the largest eigenvalue of $L_{f-(P_f+a)\tau}$, and h_a is a normalized corresponding positive eigenfunction. Then $L_{f^{(a)}}1 = 1$.

We approximate the pseudo-Markov partition $\mathcal{R} = \{R_i\}_{i=1}^k$ by a true Markov partition $\{\widetilde{R}_i\}_{i=1}^{k_0}$, where each $\widetilde{R}_i$ is contained in a submanifold D_i of M of codimension one. The shift along the flow determines a bijection $\tilde{\Psi} : R \to \widetilde{R} = \bigcup_{i=1}^{k_0} \widetilde{R}_i$. The Poincaré map $\widetilde{\mathcal{P}} : \widetilde{R} = \bigcup_{i=1}^{k_0} \widetilde{R}_i \to \widetilde{R}$ is essentially Lipschitz. We use cylinders in R below – see e.g. Section 3 in [7] for the definition and some general facts about them.

Step 1. We choose a compact set P' of Lyapunov regular points in M, constants $r_0, \hat{\rho}, \hat{\rho}_1 \in (0,1)$ and integers $n_1, q_0 > 1$ such that the radius of Lyapunov regularity at any $x \in P'$ is $\geqslant r_0$ and:

(i) *For any unstable leaf V in R, any cylinder $\mathcal{C}_V$ in V of length $\geqslant n_1$ and any sub-cylinder $\mathcal{C}'_V$ of $\mathcal{C}_V$ of co-length* 1 *such that $\mathcal{C}'_V \cap P' \neq \emptyset$ we have* $\hat{\rho}\operatorname{diam}(\tilde{\Psi}(\mathcal{C}_V)) \leqslant \operatorname{diam}(\tilde{\Psi}(\mathcal{C}'_V))$.
(ii) *For any unstable leaf V in R, any cylinder $\mathcal{C}_V$ in V of length $\geqslant n_1$ and any sub-cylinder $\mathcal{C}'$ of $\mathcal{C}$ of co-length q_0 with $\mathcal{C}' \cap P' \neq \emptyset$ we have* $\operatorname{diam}(\tilde{\Psi}(\mathcal{C}')) \leqslant \hat{\rho}_1 \operatorname{diam}(\tilde{\Psi}(\mathcal{C}))$.

Step 2. We choose compact subsets $P_2 \subset P_1$ of P' with $\mu(P_2) > 0$, a constant $\hat{\delta}_0 \in (0,1)$ and a large integer $N > 1$. Using a strong non-integrability property of contact Anosov flows (see Lemma 4.3 in [8] for details), we choose a particular point $\hat{z}_0 \in S_1$ and construct continuous maps $v_i^{(l)} : U_1 \to U_1$ ($\ell = 1, \dots, \ell_0$, $i = 1,2$) with $\sigma^N(v_i^{(l)}(x)) = x$ for all $x \in U_1$ and the following property: for any cylinder $\mathcal{C}$ in $W_R^u(\hat{z}_0)$ of sufficiently large length with $\mathcal{C} \cap P_2 \neq \emptyset$ there exist sub-cylinders $\mathcal{D}$ and $\mathcal{D}'$ of $\mathcal{C}$ of co-length q_0 with $\mathcal{D} \cap P_2 \neq \emptyset$, $\mathcal{D}' \cap P_2 \neq \emptyset$ and $\ell = 1, \dots, \ell_0$ such that for any points $x \in \mathcal{D} \cap P_2$ and $z \in \mathcal{D}' \cap P_2$ we have $d(\tilde{\Psi}(x), \tilde{\Psi}(z)) \geqslant \frac{\hat{\rho}}{2}\operatorname{diam}(\tilde{\Psi}(\mathcal{C}))$ and $|\varphi_\ell(v_1^{(l)}(\pi^{(U)}(x)) - \varphi_\ell(v_2^{(l)}(\pi^{(U)}(z))))| \geqslant \hat{\delta}_0 \operatorname{diam}(\tilde{\Psi}(\mathcal{C}))$, where $\varphi_\ell(y) = \tau_N(v_1^{(l)}(y)) - \tau_N(v_2^{(l)}(y))$. The point $\hat{z}_0$ is chosen so that for $K_0 = \pi^{(U)}(P_2 \cap W_R^u(\hat{z}_0))$ we have $\nu(K_0) > 0$, where ν is the *Gibbs measure* determined by g on U.

Step 3. We choose a small constant $\epsilon_1 > 0$ and a large integer $q_1 > q_0$, and for a given b, consider the maximal cylinders $\mathcal{C}_1, \dots, \mathcal{C}_{m_0}$ in $W_R^u(\hat{z}_0)$ with $\mathcal{C}_m \cap P_2 \neq \emptyset$ and $\operatorname{diam}(\tilde{\Psi}(\mathcal{C}_m)) \leqslant \epsilon_1/|b|$. Let $\mathcal{D}_1, \dots, \mathcal{D}_{j_0}$ be the list of all their sub-cylinders of co-length q_1 that intersect P_2. Then $\mathcal{D}'_j = \pi^{(U)}(\mathcal{D}_j)$ are cylinders in U_1 that cover K_0. Set $X_{i,j}^{(\ell)} = v_i^{(l)}(\mathcal{D}'_j) \subset U_1$ and consider the characteristic function $\omega_{i,j}^{(\ell)} = \chi_{\widehat{X}_{i,j}^{(\ell)}} : \widehat{U} \to [0,1]$. Let J be a subset of $\Xi(b) = \{(i,j,\ell)\colon\ 1 \leqslant i \leqslant 2,\ 1 \leqslant j \leqslant j_0,\ 1 \leqslant \ell \leqslant \ell_0\}$ so that for every $j = 1, \dots, j_0$ there exists at most one pair (i,ℓ) with $(i,j,\ell) \in J$, and for any $m = 1, \dots, m_0$ there exists $(i,j,\ell) \in J$ such that $\mathcal{D}_j \subset \mathcal{C}_m$. Choosing appropriately a constant $\mu_0 \in (0, 1/4)$, we define $\omega_J = 1 - \mu_0 \sum_{(i,s,\ell) \in J} \omega_{i,s}^{(\ell)}$, and the *contraction operator* $\mathcal{N}_J : \mathcal{F}_\theta(\widehat{U}) \to \mathcal{F}_\theta(\widehat{U})$ by $(\mathcal{N}_J h) = L_{f^{(a)}}^N(\omega_J \cdot h)$.

Step 4. As in [1], the main result follows if we show that, given $f \in \mathcal{F}_\theta(\widehat{U})$, there exist constants $C > 0$ and $\rho \in (0,1)$ such that for small $|a|$ and large $|b|$ and any $h \in \mathcal{F}_\theta(\widehat{U})$ with $\|h\|_{\theta,b} \leqslant 1$ we have $\int_U |L_{ab}^{mN} h|^2 \,d\nu \leqslant C\rho^m$ for every positive integer m. To prove this, one constructs for given a and b, a sequence of subsets $J_1, J_2, \dots, J_m, \dots$ of $\Xi(b)$ – see Step 5 below. Given h with $\|h\|_{\theta,b} \leqslant 1$, set $h^{(m)} = L_{ab}^{Nm} h$, $H^{(0)} = 1$ and $H^{(m)} = \mathcal{N}_{J_m}(H^{(m-1)})$; then $|h^{(m)}| \leqslant H^{(m)}$ for all m. Using $\nu(K_0) > 0$ and the strong mixing properties of $\mathcal{P} : R \to R$ (which is a Kolmogorov automorphism), we derive that for some large integer $p_0 \geqslant 1$, if the sequence $H^{(m)}$ is defined as above, then $\int_U (H^{(pp_0)})^2 \,d\nu \leqslant C_1 \rho_1^p$ $(p \geqslant 1)$ for some constants $C_1 > 0$ and $\rho_1 \in (0,1)$, independent of a, b and h. Thus, $\int_U |L_{ab}^{mN} h|^2 \,d\nu = \int_U |h^{(m)}|^2 \,d\nu \leqslant \int_U (H^{(m)})^2 \,d\nu \leqslant C\rho^m$ for some constants $C > 0$ and $\rho \in (0,1)$, and this proves the theorem.

Step 5. Here we sketch the induction process to construct the sequence $\{J_m\} \subset \Xi(b)$. It is enough to do one step. For $u, u' \in \widehat{U}$ let $Y(u,u')$ be the smallest cylinder containing u, u'. Fix $\theta_2 \in (\theta, 1)$ sufficiently close to 1 and a sufficiently large $E > 0$. Consider pairs of functions (h, H) such that $h, H \in \mathcal{F}_\theta(\widehat{U})$, $H > 0$, $\frac{|H(u) - H(u')|}{H(u')} \leqslant E \frac{D_\theta(u,u')}{\operatorname{diam}_\theta(\mathcal{C}_m)}$ whenever $u, u' \in \widehat{U}$ and there exists an integer $p \geqslant 0$ with $\sigma^p(Y(u,u')) \subset \mathcal{C}_m$ for some $m \leqslant m_0$, and the following two properties hold: (i) $|h| \leqslant H$ on $\widehat{U}$; (ii) for any $u, u' \in \widehat{U}_1$ contained in some cylinder $\mathcal{C}$ in U_1 with $\mathcal{C} \cap K_0 \neq \emptyset$, any integer $p \geqslant 0$ and any points $v, v' \in \widehat{U}_1$ such that $\sigma^p(v) = v_i^{(l)}(u)$, $\sigma^p(v') = v_i^{(l)}(u')$ for some $i = 1,2$ and $\ell = 1, \dots, \ell_0$, and the cylinder $Y(v,v')$ has length $\geqslant N$, we have $|h(v) - h(v')| \leqslant E|b|\theta_2^{p+N} H(v') \operatorname{diam}(\tilde{\Psi}(\pi_{\hat{z}_0}(\mathcal{C})))$.

Then we have to find some subset J of $\Xi(b)$ (of the required type) such that for $h^{(1)} = L_{ab}^N h$ and $H^{(1)} = \mathcal{N}_J(H)$ the analogues of (i) and (ii) hold. This is the central and most difficult part of the proof. Here we use the fact that $K_0 \subset \pi^{(U)}(P_2)$, the properties of the set P_2 and those of the cylinders $\mathcal{D}_j$ and $\mathcal{C}_m$, and also Steps 1, 2 and 3.

A detailed proof of Theorem 1 is given in [8].

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