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Canard-cycle transition at a fast–fast passage through a jump point

Transition de cycles canard pour un passage rapide–rapide par un point de saut

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ABSTRACT

We consider transitory canard cycles that consist of a generic breaking mechanism, i.e. a Hopf or a jump breaking mechanism, in combination with a fast–fast passage through a jump point. Such cycle separates two types of canard cycles with a different shape. We obtain upper bounds on the number of periodic orbits that can appear near the canard cycle, and this under very general conditions.

RÉSUMÉ

On considère des cycles canard transitoires comportant un mécanisme de cassure générique, de type Hopf ou bien de saut, en combinaison avec un passage de type rapide–rapide par un point de saut. De tels cycles séparent deux types de cycles canard de formes différentes. On obtient des bornes supérieures sur le nombre d'orbites périodiques qui peuvent apparaître près du cycle canard, sous certaines conditions très générales.

Version française abrégée

Un champ de vecteurs lent–rapide est une famille de champ de vecteurs, de paramètre ε, ayant une courbe de points singuliers pour $\varepsilon = 0$, appelée *courbe critique*. Les cycles limites, pour $\varepsilon > 0$ petit, bifurquent des ensembles limites périodiques définis pour $\varepsilon = 0$. Un ensemble limite périodique, homéomorphe à un cercle, défini par une combinaison de dynamiques lentes et rapides, est appelé *cycle lent–rapide*. Quand il est de type *canard* (c'est-à-dire contenant au moins un arc attractant et un arc répulsif de la courbe critique), un tel cycle lent–rapide appartient à une famille d'au moins un paramètre, la *variable de couche*. Nous considérons un cycle canard contenant une connection rapide entre deux branches A_- et A_+ de la courbe critique : cette connection rapide appartient alors à une couche remplie de connections rapides entre A_- and A_+, voir Fig. 1(a). Si Γ_Y, où Y est une valeur intérieure au domaine de la variable de couche, l'existence d'une borne finie pour le nombre des cycles limites bifurquant de Γ_Y a été obtenue dans [5]. Ce nombre (la cyclicité) est en relation avec la multiplicité de l'*intégrale de divergence lente* $I(Y)$, écrite comme fonction de la variable de couche Y. Une situation plus délicate survient lorsque l'on étudie la cyclicité d'un cycle canard Γ_{Y_0}, où Y_0 est une valeur de la variable de couche

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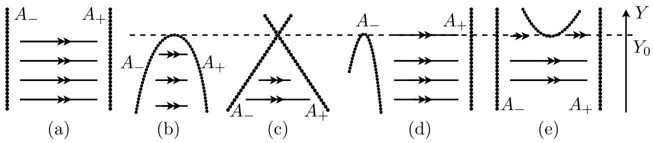

Fig. 1. Layers and their boundaries.

Fig. 1. Les couches et leurs frontières.

à la frontière du domaine (voir différentes situations typiques (b)–(e) dans la Fig. 1). La cyclicité des cas (b) et (d) a été étudiée dans [6], [7] et [1]. Dans cet article, nous étudions un cycle canard avec un *passage rapide–rapide par un point de saut*. Cette situation est montrée dans la Fig. 1(e). Il découle de [3] que la cyclicité du cycle canard est au plus de 1 lorsque $I(Y_0) \neq 0$ (sous certaines conditions très générales). Aussi, nous nous focalisons ici sur le cas $I(Y_0) = 0$. Dans la Fig. 2, nous distinguons quatre situations correspondant à la Fig. 1(e), dépendant des orientations des dynamiques lentes définies sur A_- and A_+.

Le résultat présenté ci-dessous est basé sur la multiplicité k de $\mathcal{I}_{b_0,\mu_0}(y)$ pour $y = 0$. Ce nombre k est égal à 0 ou 1 dans les cas (A) et (B) de la Fig. 2, et peut être supérieur seulement dans les cas (C) et (D). La cyclicité d'un cycle lent–rapide Γ est la borne supérieure du nombre de cycles limites de $X_{\varepsilon,b,\mu}$, qui sont voisins de Γ au sens de Hausdorff.

Théorème principal. *Considérons un cycle canard Γ_{0,b_0,μ_0} correspondant à l'un des quatre cas de la Fig. 2. Soit $k \geqslant 0$ la multiplicité du zéro de $\mathcal{I}_{b_0,\mu_0}(y)$ à $y = 0$. Soit $\rho := \frac{\partial}{\partial y} I_{b_0,\mu_0}(0) / \frac{\partial}{\partial y} J_{b_0,\mu_0}(0)$. Alors, lorsque $k = 1$, la cyclicité de Γ_{0,b_0,μ_0} est inférieure à 4 si $\rho \notin]0,1[$ et inférieure à 5 si $\rho \in]0,1[$. Quand $k \geqslant 2$ (c'est-à-dire quand $\rho = 1$), la cyclicité de Γ_{0,b_0,μ_0} est inférieure à $k + 4$.*

Dans la preuve du théorème, nous éclatons le point de saut. Le cycle canard Γ_{0,b_0,μ_0} est alors éclaté en une famille compacte d'ensembles limites périodiques (voir Fig. 3) : chaque orbite du lieu critique de l'éclatement, connectant les deux parties de la fibre rapide à la hauteur $y = 0$ correspond à l'un des membres de cette famille. La partie sud de l'équateur E, d'une part, et la connection formée par une partie de l'équateur et la séparatrice S, d'autre part, sont des orbites particulières. Ces deux orbites particulières délimitent la région hachurée dans la Fig. 3. La détermination du nombre de points fixes de l'application de premier retour nécessite des traitements différents pour les orbites passant près de S, pour les orbites passant près de E et pour les orbites traversant le milieu M du lieu critique de l'éclatement. La plus grande partie de la multiplicité vient du passage près de E. Près de S, la cyclicité est limitée par l'influence prédominante de la branche attractante B_-. Sur M, la cyclicité est déterminée par les propriétés globales du champ de vecteurs éclaté $\{\dot{X} = -Y + X^2, \dot{Y} = -b\}$, dont l'intégration utilise les fonctions de Airy. De plus amples informations ainsi qu'une preuve détaillée seront données dans [2].

1. Introduction

In this paper, we provide theoretical results in the study of limit cycles of slow–fast vector fields on a two-dimensional smooth manifold. A slow–fast vector field is an ε-family of vector fields having a curve (or union of curves) of singular points for $\varepsilon = 0$, called the *critical curve*. Following the ideas from geometric singular perturbation theory, the dynamics for $\varepsilon > 0$ small can be analyzed by studying two limiting dynamics for $\varepsilon = 0$: the fast one and the slow one. In particular, the limit cycles for $\varepsilon > 0$ small can be detected by studying limit periodic sets defined for $\varepsilon = 0$. Limit periodic sets, homeomorphic to a circle, defined by a combination of the slow and fast limiting dynamics are called *slow–fast cycles*. When a slow–fast cycle is of canard type, i.e. when it contains both attracting and repelling parts of the critical curve, it belongs to at least one 1-parameter family parameterized by a *layer variable*. We consider a canard cycle containing a fast connection between two branches A_- and A_+ of the critical curve: the fast connection is then part of a layer filled with fast connections between A_- and A_+, and nearby canard cycles can be considered by choosing another fast connection in this layer, see Fig. 1(a). In this figure, it is assumed that the canard cycle passes from A_- to A_+ via one of the fast fibers. To visualize a closed cycle, Fig. 1(a) should be completed by additional slow–fast segments.

Given a family of canard cycles Γ_Y, where Y is some layer variable, results have been obtained in [5] to bound the number of periodic orbits near Γ_Y. The number of such limit cycles is related to the multiplicity of the slow divergence integral $I(Y)$, written as a function of the layer variable.

A more delicate situation arises when one studies the cyclicity of a canard cycle Γ_{Y_0}, where Y_0 is a value of the layer at the boundary of the layer domain. We refer to Fig. 1(b)–(e) for some of the typical situations. The cyclicity of the contact point in Fig. 1(b) has been studied in [6] and [7]. The cyclicity of the transcritical contact point in Fig. 1(c) has not yet been studied. Fig. 1(d) has been studied recently: at the critical value of the layer variable, the slow–fast cycle contains a *slow–fast passage through a jump point*, see [1]. In this paper, we are interested in studying canard cycles with a *fast–fast passage through a jump point*, as shown in Fig. 1(e). Common to the cases shown in Fig. 1(d)–(e), the slow divergence integral $I(Y)$ is only piecewise smooth at the critical value Y_0, and for this reason the results from [5] are not valid here. From [3] we know that when $I(Y_0) \neq 0$, the cyclicity of the canard cycle is at most 1 (under very weak conditions). We will focus on the case $I(Y_0) = 0$. In Fig. 2, we distinguish four situations corresponding to Fig. 1(e), depending on the orientations of the slow dynamics in A_- and A_+.

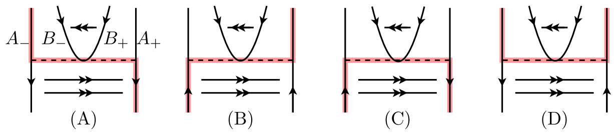

Fig. 2. Different orientations of the slow dynamics.

Fig. 2. Les différentes orientations de la dynamique lente.

2. Results

Given any of the four situations in Fig. 2, we assume that the slow–fast path continues on both sides. Along the side of A_+, we assume there is an attracting sequence (fast orbits and attracting slow arcs). Along the side of A_-, we assume the same after reversing time. We assume that both continuations meet each other in a so-called canard breaking mechanism (see [4]). This can be a Hopf breaking mechanism (an attracting and repelling branch meet each other at a singular contact point of Hopf type, similar to the one encountered in the well-known van der Pol system), or a jump breaking mechanism (attracting and repelling branches both end at a regular contact point of jump type and both jump points are joined by a fast orbit). Treating orbits along attracting sequences and creating cycles through either of the two breaking mechanisms has been studied very well by now, allowing us to focus in this paper on the peculiarity of the specific canard cycle we consider, namely the presence of the jump point inside the layer in Fig. 1(e). We will therefore consider a coordinate system where this jump point is put at the origin:

$$X_{\varepsilon,b,\mu}\colon \quad \begin{cases} \dot{x} = -y + x^2 \\ \dot{y} = -\varepsilon(b + xg(x, y, \mu, \varepsilon)). \end{cases}$$

We assume that $b > 0$, thereby excluding the case of a singular contact point. The function g is smooth and may depend on additional parameters $\mu \in \mathbf{R}^p$. This parameter may contain a breaking parameter for the aforementioned canard breaking mechanism. We will use the variable y in this coordinate system as a layer variable to parameterize the slow–fast cycles $\Gamma_{y,b,\mu}$. In fact, we will consider not only slow–fast cycles, but slow–fast segments connecting a well-chosen Poincaré section near the canard breaking mechanism with itself. Some of those segments will be closed, some not. Let the layer at height y intersect $A_\pm$ at a point $p_\pm(y, b, \mu)$. We compute the slow divergence integral along the critical branches from $p_+(y, b, \mu)$ onwards along the attracting sequence up to the canard breaking mechanism, and denote it $J_{b,\mu}(y) < 0$. Similarly, we compute (in reverse time) the slow divergence integral from $p_-(y, b, \mu)$ up to the canard breaking mechanism and denote it $I_{b,\mu}(y) < 0$. The slow divergence integral of the segments $\Gamma_{y,b,\mu}$ is given by $J_{b,\mu}(y) - I_{b,\mu}(y)$ when $y \leqslant 0$ and by $J_{b,\mu}(u) - I_{b,\mu}(y) + K_{b,\mu}(y)$ when $y > 0$, where K is some relevant part of the slow divergence integral along $y = x^2$ up to the jump point. Though the slow divergence integral has a piecewise smooth nature, we will show that the number of periodic orbits near Γ_{0,b_0,μ_0} can be bounded merely by imposing conditions on the function:

$$\mathcal{I}_{b,\mu}(y) = J_{b,\mu}(y) - I_{b,\mu}(y), \quad y \sim 0.$$

Lemma 1. *Consider the four cases* (A)–(D) *in* Fig. 2*. Then* $\frac{\partial}{\partial y} I_{b,\mu}(0) > 0$ *in cases* (A) *and* (D) *and* $\frac{\partial}{\partial y} I_{b,\mu}(0) < 0$ *in cases* (B) *and* (C)*. And* $\frac{\partial}{\partial y} J_{b,\mu}(0) > 0$ *in cases* (B) *and* (D) *and* $\frac{\partial}{\partial y} J_{b,\mu}(0) < 0$ *in cases* (A) *and* (C)*.*

Let us now announce the main result, which is based on the multiplicity k of $\mathcal{I}_{b_0,\mu_0}(y)$ at $y = 0$. From Lemma 1, it is clear that in cases (A) and (B), k is either 0 or 1. Higher multiplicities are possible in the cases (C) and (D).

Theorem 1. *Let us consider a slow–fast cycle* $\Gamma = \Gamma_{0,b_0,\mu_0}$ *as depicted in one of the four cases of* Fig. 2*. Let* $k \geqslant 0$ *be the multiplicity of the zero of* $\mathcal{I}_{b_0,\mu_0}(y)$ *at* $y = 0$*. Then for* (ε, b, μ) *close to* $(0, b_0, \mu_0)$ *an upper bound for the number of limit cycles of* $X_{\varepsilon,b,\mu}$ *that are Hausdorff close to* Γ_{0,b_0,μ_0} *is given as follows. When* $k = 0$*, the cyclicity of* Γ *is at most* 1*. When* $k = 1$ *the cyclicity of* Γ *is at most* 4 *if* $\rho := \frac{\partial}{\partial y} I_{b_0,\mu_0}(0) / \frac{\partial}{\partial y} J_{b_0,\mu_0}(0) \notin]0, 1[$ *and at most* 5 *if* $\rho \in]0, 1[$*. When* $k \geqslant 2$ *(i.e. when* $\rho = 1$*) the cyclicity of* Γ *is at most* $k + 4$*.*

In the proof[1] of the theorem, we blow up the jump point, see Fig. 3. The slow–fast cycle Γ_{0,b_0,μ_0} blows up to a compact family of limit periodic sets: each orbit on the blow-up locus connecting both parts of the fast fiber at height $y = 0$ corresponds to one member of this family. Particular such orbits are the southern part of the equator E and a connection formed by a part of the equator and a separatrix S. Both particular orbits delimit the shaded area in Fig. 3. Setting up bifurcation diagrams for the number of fixed points for the first return map needs a different treatment for orbits passing near S, for orbits passing near E and for orbits passing in the middle M of the blow-up locus. The high multiplicity comes from the passage near E, and not from the passage near S or through M. Near S, the cyclicity is limited due to the

[1] This is a succinct version of a text file deposited for five years in the French Academy of Sciences' Archives. Copy is available upon request.

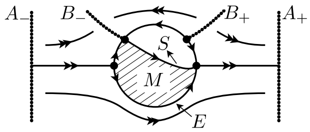

Fig. 3. Dynamics after blow-up.

Fig. 3. Dynamique après éclatement.

influence of the attracting branch B_-. On M the cyclicity is determined by global properties of the blow-up vector field $\{\dot{X} = -Y + X^2, \dot{Y} = -b\}$ whose orbits can be integrated exactly in terms of Airy functions. More information and a detailed proof will be given in [2].

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