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Differential Geometry

Comparison between two complexes on a singular space

Comparaison entre deux complexes sur un espace singulier

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ABSTRACT

The aim of this Note is twofold. In the first step we study the Witten deformation for stratified spaces X and radial Morse functions on them and prove a spectral gap theorem for the Witten Laplacian. In the second step we focus on spaces with isolated conic singularities, where we construct a geometric complex associated to the Morse function and give two comparison results.

RÉSUMÉ

Cette Note a deux buts : Dans une première partie on étend la déformation de Witten au cas d'un espace stratifié X muni de fonctions appelées fonctions de Morse radiales. On démontre le théorème du trou spectral pour le laplacien de Witten. Dans la deuxième partie, on se place dans la situation d'un espace à singularités isolées et on construit un complexe géométrique que l'on compare à celui des petites valeurs propres.

1. Introduction

Let X be a compact oriented stratified pseudomanifold of $\dim X = n$ and let g be a metric on X as in [2] (pp. 223–226). We refer the reader to [2] for more details and only recall from this definition the most relevant features, namely: X is equipped with a filtration $X = X_n \supset X_{n-1} = X_{n-2} \supset X_{n-3} \supset \cdots \supset X_1 \supset X_0$ of X by closed subsets X_j such that $X_{(j)} := X_j \setminus X_{j-1}$ are smooth manifolds of dimension j (if not empty), $X_{n-1} = X_{n-2}$ and $X_{(n)}$ is dense in X. Moreover for each $p \in X_{(j)}$, there exists an open neighbourhood (U_p, g) in X quasi-isometric to $(\mathbb{R}^j \times cL_p, g_{\mathbb{R}^j} + dr^2 + r^2 g_{L_p})$, where $g_{\mathbb{R}^j}$ is a Riemannian metric on $\mathbb{R}^j$, L_p is the link of the stratum $X_{(j)}$, $cL_p = L_p \times \mathbb{R}_{\geqslant 0}/\sim$ is the cone over L_p, r is the radial coordinate and g_{L_p} is a metric on the stratified space L_p. This type of iterated conic metrics has first been studied by Cheeger (see [5]); so-called metrics on a stratified space "associated with a tuple $\bar{c} = (c_2, \ldots, c_n)$" appear in [2] and [9] (p. 345). The metrics considered in the present article are precisely "metrics associated with $\bar{c} = (2, \ldots, 2)$" in the terminology of [2] and [9].

Definition 1. We call a continuous function $f : X \to \mathbb{R}$ a radial Morse function if the following conditions are satisfied:

(a) For each j, the restriction $f_{|X_{(j)}}$ to the stratum $X_{(j)}$ is a smooth Morse function. We denote by $\mathrm{Crit}_k(f_{|X_{(j)}})$ the set of critical points of index k;

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(b) For each point $p \in \mathrm{Crit}_k(f_{|X_{(j)}})$ there exists an open neighbourhood U_p of p in X as well as local coordinates $x_1, \dots, x_j$ on $X_{(j)} \cap U_p$ such that one can write $f = f(p) + \frac{1}{2}(-x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_j^2 - r^2)$, where r is the radial coordinate;
(c) The gradient ∇f on $X_{(n)}$ is bounded.

The Witten deformation on singular spaces has been studied by the author already in previous articles (see e.g. [8]). The Morse functions considered there were called admissible Morse functions and were inspired from the stratified Morse theory of Goresky and MacPherson [7]. In the context treated in these notes we are however able to prove stronger results, in particular the comparison theorems (Theorem 5 and Theorem 6) were out of reach in the setting of [8].

2. The Witten deformation and the spectral gap theorem

Let $(\Omega_0^*(X_{(n)}), d)$ be the de Rham complex of smooth forms, compactly supported outside the singular set X_{n-2} of X. Let us denote by $\langle\ ,\ \rangle$ the L^2-metric on $\Omega_0^*(X_{(n)})$ induced from the metric g. Let $(\mathcal{C}, d_{\max}, \langle\ ,\ \rangle)$ be the maximal extension of the de Rham complex into a Hilbert complex, with $\mathcal{C}^i := \mathrm{dom}(d_{i,\max})$. The L^2-cohomology of X is defined as the cohomology of the Hilbert complex $(\mathcal{C}, d_{\max}, \langle\ ,\ \rangle)$, $H^*_{(2)}(X) := H^*((\mathcal{C}, d_{\max}, \langle\ ,\ \rangle))$. Throughout this article the language of Hilbert complexes as introduced in [3] is used. Let us however mention that the choice of the maximal extension $d_{\max}$ corresponds to the choice of an *ideal boundary condition* in the sense of Cheeger (see [4]). In particular, if X is an odd dimensional space with isolated singularities, the choice of $d_{\max}$ corresponds to the choice of the ideal boundary condition $V_a = H^{[n/2]}(L_p)$ in [4]. Recall from [5,9] that integration yields an isomorphism $H^i_{(2)}(X_{(n)}) \simeq \mathrm{Hom}(IH_i^{\underline{m}}(X), \mathbb{R})$, where by $IH_i^{\underline{m}}(X)$ we denote the intersection homology of X with lower middle perversity $\underline{m}$. Since the intersection homology of a compact stratified space is known to be finitely generated (see [6, Section 3.2]) we deduce, using the general theory of Hilbert complexes (see Theorem 2.4 in [3]), that the complex $(\mathcal{C}, d_{\max}, \langle\ ,\ \rangle)$ is a Fredholm complex and that therefore $0 \notin \mathrm{spec}_{ess}(\Delta)$, where Δ is the Laplacian associated to the Hilbert complex $(\mathcal{C}, d_{\max}, \langle\ ,\ \rangle)$.

The Witten deformation (see [10]) generalised to this situation consists in deforming the complex of L^2-forms using a radial Morse function f on X: Let us denote by $d_t := e^{-tf} d e^{tf}$ the deformed differential acting on compactly supported forms $\Omega_0^*(X_{(n)})$. The maximal extension of the complex $(\Omega_0^*(X_{(n)}), d_t, \langle\ ,\ \rangle)$, denoted by $(\mathcal{C}_t, d_{t,\max}, \langle\ ,\ \rangle)$, is also a Fredholm complex, which computes the L^2-cohomology of X. The Laplace operator associated to the Hilbert complex $(\mathcal{C}_t, d_{t,\max}, \langle\ ,\ \rangle)$, $\Delta_t := (d_{t,\max} + \delta_{t,\min})^2$, is called the Witten Laplacian. It is a non-negative operator with $0 \notin \mathrm{spec}_{ess}\Delta_t$.

Let us denote by $c_i(f_{|X_{(n)}})$ the number of critical points of f of index i in the top stratum and by

$$c_i(f) := c_i(f_{|X_{(n)}}) + \sum_{p \in X_{n-2}} m_p^i,$$

where the contribution of a critical point $p \in X_{n-2}$ of index k is given by $m_p^i := \dim IH_{i-k}^{\underline{m}}(cL_p, L_p)$. As in the smooth situation we get as main result a spectral gap theorem for the Witten Laplacian:

Theorem 2. *Let X be a stratified space as in the introduction and let $f : X \to \mathbb{R}$ be a radial Morse function on X. Then there exist constants $C_1, C_2, C_3 > 0$ and $t_0 > 0$ depending on X and f such that for any $t > t_0$, $\mathrm{spec}(\Delta_t) \cap (C_1 e^{-C_2 t}, C_3 t) = \emptyset$. Let us denote by $(\mathcal{S}_t, d_{t,\max}, \langle\ ,\ \rangle)$ the subcomplex of $(\mathcal{C}_t, d_{t,\max}, \langle\ ,\ \rangle)$ generated by eigenforms to small eigenvalues, then $\dim \mathcal{S}_t^i = c_i(f)$.*

As usual, as a corollary of the above spectral gap theorem one gets corresponding Morse inequalities for the L^2-cohomology of X.

As in the smooth situation the main step for the proof of the spectral gap theorem is to study the local model operator for the Witten Laplacian near critical points of f: Let $p \in X_{(0)}$, then the local model operator $\mathbb{D}_{t,p}$ is an operator on the infinite cone cL_p equipped with the Morse function $f = f(p) - r^2/2$. The boundary conditions near the cone point are those inherited from the Witten Laplacian. Let $\mathcal{H}^i(L_p)$ be the space of harmonic i-forms on the link, *i.e.* forms in $\ker(\Delta_{L_p})$, where Δ_{L_p} is the Laplacian associated to the Hilbert complex of L^2-forms on the stratified space L_p. Let $A_i := \{\eta_{p,l}^i \mid l = 1, \dots, \dim \mathcal{H}^i(L_p)\}$ be an orthonormal basis of $\mathcal{H}^i(L_p)$.

Proposition 3. *Let $p \in X_{(0)}$. There exists a constant $c > 0$ such that $\mathrm{spec}(\mathbb{D}_{t,p}) \subset \{0\} \cup [ct, \infty)$ and*

$$\ker\big(\mathbb{D}_{t,p}^{(i)}\big) = \mathrm{span}\big\{ (t/\pi)^{i/2-n/4} \sqrt{\mathrm{vol}\big(S^{2(i-1)-n+1}\big)} \big(r^{-n+1+2(i-1)} e^{-tr^2/2} \eta_{p,l}^{i-1} \wedge dr\big) \bigm| \eta_{p,l}^{i-1} \in A_{i-1} \big\} \tag{1}$$

for $i \geqslant n/2 + 1$ and 0 else. Moreover, if p is an isolated singularity, $\mathbb{D}_{t,p}$ has discrete spectrum.

With other words, the above proposition shows that, for $p \in X_{(0)}$, the kernel of the local model operator satisfies $\ker(\mathbb{D}_{t,p}^{(i)}) \simeq \mathrm{Hom}(IH_i^{\underline{m}}(cL_p, L_p), \mathbb{R})$. For $p \in \mathrm{Crit}_k(f_{X_{(j)}})$ one gets the local spectral gap theorem, by writing the local model operator as $\mathbb{D}_{t,p} = \mathbb{D}_{t,p}^{\parallel} + \mathbb{D}_{t,p}^{\perp}$, where $\mathbb{D}_{t,p}^{\perp}$ is the operator studied above and $\mathbb{D}_{t,p}^{\parallel}$ is the well-known model Witten Laplacian for a smooth critical Morse point of index k on $\mathbb{R}^j$ [10].

3. The geometric complex and the comparison theorems

From now on we will assume that X is a space with isolated singularities, $\mathrm{Sing}(X) = X_{(0)}$. We moreover restrict to radial Morse functions which satisfy the following additional conditions: (i) Let $U_p \simeq c_2 L_p := L_p \times [0,2)/\sim$ be a neighbourhood of $p \in \mathrm{Sing}(X)$, then $\mathrm{Crit}(f_{|U_p \setminus \{p\}}) \subset L_p \times \{1\}$ and the direction normal to the link is stable. (ii) $L_p \times \{1\}$ is an invariant set for the negative gradient flow. (iii) The pair (f, g) is Morse–Smale.

It is not difficult to see that such a function always exists on X. One can then prove that X has a decomposition into unstable cells $X = \bigcup_{p \in \mathrm{Crit}(f)} W^u(p)$. Let $p \in \mathrm{Sing}(X)$. For this special choice of Morse function we have moreover that the link L_p has a decomposition into the unstable cells of critical points of f in $\mathrm{Crit}(f) \cap (L_p \times \{1\})$. Note that this decomposition is in the same time the unstable cell decomposition of L_p with respect to the Morse function $h_p := f_{|L_p \times \{1\}} : L_p \times \{1\} \simeq L_p \to \mathbb{R}$. Therefore we can refine the unstable cell decomposition of X into

$$X = \Bigg(\bigcup_{p \in \mathrm{Crit}(f_{|X \setminus \mathrm{Sing}(X)})} W^u(p) \Bigg) \cup \Bigg(\bigcup_{p \in \mathrm{Sing}(X)} \Bigg(\bigcup_{q \in \mathrm{Crit}(h_p)} c W^u(q) \Bigg) \Bigg).$$

Let us choose orientations on all cells of this decomposition. We can then define the complex (T_*, ∂_*) generated by all cells of the decomposition, where the boundary operator ∂ is given by the geometric boundary.

Let us denote by $(C^u_{L_p,*}, \partial_{L_p,*})$ the Morse–Thom–Smale complex of unstable cells for the smooth Morse function h_p on the link L_p. Note that, by condition (ii) obviously $(C^u_{L_p,*}, \partial_{L_p,*})$ is a subcomplex of (T_*, ∂_*).

For ξ an $(n-i-1)$-form on L_p, denote

$$\tau_p(\xi) := \sum_{q \in \mathrm{Crit}_i(h_p)} \Bigg(\int_{W^s_{L_p}(q)} \xi \Bigg) \cdot \big[W^u_{L_p}(q) \big], \tag{2}$$

where $W^{s/u}_{L_p}(q)$ denotes the stable/unstable manifold w.r.t. $h_p : L_p \to \mathbb{R}$. Recall that, in view of condition (i), one has that $W^u_{L_p}(q) = W^u(q)$. Using smooth Morse theory for the Morse function $h_p : L_p \to \mathbb{R}$ on the link L_p one can check that, for ξ a closed $(n-i-1)$-form and ζ an $(n-i-2)$-form

$$\partial \tau_p(\xi) = 0 \quad \text{and} \quad \tau_p(\xi + d\zeta) = \tau_p(\xi) \pm \partial \tau_p(\zeta). \tag{3}$$

Moreover, the current $\pm\tau_p(\xi)$ is homologous to the regular current ξ. Choose representatives $\{\xi^i_{p,l}\}_l$ for a basis of $H^{n-1-i}(L_p)$. Denote $\tau^i_{p,l} := \tau_p(\xi^i_{p,l})$ and $B^i_p := \{\tau^i_{p,l}\}_l$.

Definition 4. We denote by (C^u_*, ∂_*) the following subcomplex of (T_*, ∂_*):

$$C^u_i := \begin{cases} \bigoplus_{x \in \mathrm{Crit}_i(f)} \mathbb{R} \cdot [W^u(x)] \oplus \bigoplus_{p \in \mathrm{Sing}(X),\, \tau^{i-1}_{p,l} \in B^{i-1}_p} \mathbb{R} \cdot [c\tau^{i-1}_{p,l}] & \text{if } i \geqslant n/2 + 1, \\ \bigoplus_{x \in \mathrm{Crit}_i(f)} \mathbb{R} \cdot [W^u(x)] & \text{if } i < n/2 + 1. \end{cases} \tag{4}$$

In view of (3), we have that $\partial C^u_i \subset C^u_{i-1}$. Note moreover that the complex defined above is obviously independent of the choice of the basis of the cohomology $H^{n-1-i}(L_p) \simeq \mathcal{H}^{n-1-i}(L_p) \simeq \mathcal{H}^i(L_p)$, and moreover, in view of (3), is independent of the chosen representatives $\{\xi^i_{p,l}\}_l$. For the comparison result, Theorem 5, we will from now on choose $\xi^i_{p,l} := \tilde{*}\eta^i_{p,l}$, where $\{\eta^i_{p,l}\}_l$ is the ONB-basis of harmonic i-forms on the link used in (1) and $\tilde{*}$ denotes the Hodge-$*$-operator on the link L_p. Note that this means, that we have identified (via smooth Morse theory on the link) the cohomology of the link with the cohomology of the (smooth) Morse–Thom–Smale complex $(C^u_{L_p,*}, \partial_{L_p,*})$ and that we equip the later with the metric induced from the restriction of the L^2-metric to the harmonic forms $\mathcal{H}^*(L_p)$.

The complex (C^u_*, ∂_*) is a subcomplex of the complex of all allowed cells for the middle lower perversity, $(T^{all}_*, \partial_*) \subset (T_*, \partial_*)$. One can show that (C^u_*, ∂_*) computes the intersection homology $IH^{\underline{m}}_*(X)$. The (densely defined) map $P_\infty : (\mathcal{C}, d_{\max}) \to \mathrm{Hom}((C^u_*, \partial_*), \mathbb{R})$ given by integration is a map of chain complexes, inducing the canonical isomorphism between the intersection cohomology with middle lower perversity and the L^2-cohomology of the space X.

Let $p \in \mathrm{Crit}_k(f_{|X \setminus \mathrm{Sing}(X)})$. Then $\dim \ker(\mathbb{D}_{t,p}) = 1$ and a generator is $\omega_p = \sqrt{\frac{t}{\pi}}^{\,n/2} e^{-tr^2/2} \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_k$, where $x_1, \ldots, x_n$ are local Morse coordinates. For $p \in \mathrm{Sing}(X)$ let $\{\omega^i_{p,l}\}$ be the ONB-basis of $\ker(\mathbb{D}^i_{t,p})$ defined in (1). Let χ be a cut-off function and define the following forms on X

$$\big\{ \Phi_p(t) := \chi \omega_p(t) \;\big|\; p \in \mathrm{Crit}(f_{|X \setminus \mathrm{Sing}(X)}) \big\}$$
$$\cup \big\{ \Phi^i_{p,l}(t) := \chi \omega^i_{p,l}(t) \;\big|\; p \in \mathrm{Sing}(X),\ l = 1, \ldots, \dim \mathcal{H}^{i-1}(L_p),\ i \geqslant n/2 + 1 \big\}.$$

Multiplication by e^{tf} induces an isomorphism from $(\mathcal{S}_t, d_{t,\max}, \langle\ ,\ \rangle)$ into a subcomplex $(\mathcal{S}, d_{\max}, \langle\ ,\ \rangle_t) \subset (\mathcal{C}, d_{\max}, \langle\ ,\ \rangle_t)$, where $\langle\ ,\ \rangle_t$ denotes the twisted (by e^{-2tf}) L^2-metric. We denote by $P_t^{[0,1]} : \mathcal{C}_t \to \mathcal{S}_t$ the orthogonal projection. Define e_t : $\mathrm{Hom}((C^u_*, \partial_*), \mathbb{R}) \to (\mathcal{S}, d_{\max}, \langle\ ,\ \rangle_t)$, $[W^u(p)]^* \to e^{tf} P_t^{[0,1]} \Phi_p(t)$, $[c\tau_{p,l}]^* \mapsto e^{tf} P_t^{[0,1]} \Phi_{p,l}(t)$. (Note that, with the particular choice for the ξ's done above, the metric on the geometric complex has been fixed such that e_t is, up to a term of order $\mathcal{O}(e^{-ct})$, an isometry.) Let us denote by $P_{\infty,t} : (\mathcal{S}, d_{\max}, \langle\ ,\ \rangle_t) \to \mathrm{Hom}((C^u_*, \partial_*), \mathbb{R})$ the linear map given by integration. We denote by $\mathcal{F} \in \mathrm{End}(\mathrm{Hom}(C^u_k, \mathbb{R}))$ the homomorphism which acts on $[W^u(p)]^*$ and on $[c\tau^i_{p,l}]^*$ by multiplication with $f(p)$. With $\mathcal{I} \in \mathrm{End}(\mathrm{Hom}(C^u_i, \mathbb{R}))$ we denote multiplication by i.

Theorem 5. *There exists $c > 0$ such that for $t \to \infty$, we have $P_{\infty,t} \circ e_t = e^{t\mathcal{F}} (\frac{\pi}{t})^{\mathcal{I}/2 - n/4} (\mathcal{A} + \mathcal{O}(e^{-ct}))$, where $\mathcal{A}$ is the diagonal matrix with entries*

$$\mathcal{A}_\sigma = \begin{cases} 1 & \text{for } \sigma = [W^u(p)]^*,\ p \in \mathrm{Crit}(f_{|X \setminus \mathrm{Sing}(X)}), \\ 1/\sqrt{\mathrm{vol}(S^{2(i-1)-n+1})} & \text{for } \sigma = [c\tau^{i-1}_{p,l}]^*,\ p \in \mathrm{Sing}(X), l \in \{1, \ldots, \dim \mathcal{H}^{i-1}(L_p)\}. \end{cases} \tag{5}$$

In particular for large t the linear map of vector spaces $P_{\infty,t}$ is an isomorphism.

Theorem 6. *With the notations as in Theorem* 5 *we get*

$$e_t^{-1} \circ d \circ e_t = \sqrt{\frac{t}{\pi}} \big(\mathcal{A}^{-1} + O\big(e^{-ct}\big)\big) e^{-t\mathcal{F}} \partial^* e^{t\mathcal{F}} \big(\mathcal{A} + O\big(e^{-ct}\big)\big). \tag{6}$$

To prove Theorem 5 and Theorem 6 one generalises the proof of Theorems 6.11 and 6.12 in [1].

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