C. R. Acad. Sci. Paris, Ser. I 350 (2012) 177–181

Contents lists available at SciVerse ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Partial Differential Equations

Lack of compactness in the 2D critical Sobolev embedding, the general case

Défaut de compacité de l'injection de Sobolev critique en deux dimensions d'espace

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ARTICLE INFO

Article history:
Received 18 November 2011
Accepted after revision 23 January 2012
Available online 11 February 2012

Presented by Haïm Brezis

ABSTRACT

This Note is devoted to the description of the lack of compactness of the Sobolev embedding of $H^1(\mathbb{R}^2)$ in the critical Orlicz space $\mathcal{L}(\mathbb{R}^2)$. It turns out that up to cores our result is expressed in terms of the concentration-type examples derived by J. Moser (1971) in [16] as in the radial setting investigated in Bahouri et al. (2011) [5]. However, the analysis we used in this work is strikingly different from the one conducted in the radial case which is based on an L^∞ estimate far away from the origin and which is no longer valid in the general frame work. The strategy we adopted to build the profile decomposition in terms of examples by Moser concentrated around cores is based on capacity arguments and relies on an extraction process of mass concentrations.

RÉSUMÉ

Cette Note est consacrée à l'étude du défaut de compacité de l'injection de Sobolev de $H^1(\mathbb{R}^2)$ dans l'espace d'Orlicz critique $\mathcal{L}(\mathbb{R}^2)$. Nous démontrons que la déscription donnée dans Bahouri et al. (2011) [5] concernant le cas radial reste valable dans le cas général (à des translations près par des coeurs de concentration). La preuve utilise des arguments de capacité ainsi qu'un processus d'extraction de concentrations.

Version française abrégée

Il est bien connu que $H^1(\mathbb{R}^2)$ s'injecte continûment dans tous les espaces $L^p(\mathbb{R}^2)$ pour $2 \leqslant p < \infty$, mais pas dans $L^\infty(\mathbb{R}^2)$. De plus, on sait d'après les inégalités de Moser–Trudinger (voir [16,18]) que H^1 s'injecte dans l'espace d'Orlicz $\mathcal{L}$ (défini ci-dessous). Plus précisément, on a

$$\|u\|_{\mathcal{L}} \leqslant \frac{1}{\sqrt{4\pi}} \|u\|_{H^1}. \qquad (1)$$

Cette injection est critique dans le sens où on ne peut pas remplacer l'espace d'Orlicz $\mathcal{L}$ par un espace plus petit du type Orlicz (voir [5,7]). Il est aussi à remarquer que cette injection est non compacte pour, au moins, deux raisons. La première

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1631-073X/$ – see front matter
doi:10.1016/j.crma.2012.01.016

est la perte de compacité à l'infini illustrée par l'exemple $u_n(x) = \varphi(x + x_n)$ où $\varphi \in \mathcal{D}\backslash\{0\}$ et $|x_n| \to \infty$. La seconde est due à un phénomène de concentration et un exemple typique est celui de P.-L. Lions [14] et J. Moser [16] :

$$f_\alpha(x) = \begin{cases} 0 & \text{si } |x| \geqslant 1, \\ -\frac{\log|x|}{\sqrt{2\alpha\pi}} & \text{si } \mathrm{e}^{-\alpha} \leqslant |x| \leqslant 1, \\ \sqrt{\frac{\alpha}{2\pi}} & \text{si } |x| \leqslant \mathrm{e}^{-\alpha}, \end{cases}$$

où $\alpha \to \infty$. Remarquons que f_α peut-être écrite $f_\alpha(x) = \sqrt{\frac{\alpha}{2\pi}}\,\mathbf{L}(\frac{-\log|x|}{\alpha})$ avec

$$\mathbf{L}(t) = \begin{cases} 0 & \text{si } t \leqslant 0, \\ t & \text{si } 0 \leqslant t \leqslant 1, \\ 1 & \text{si } t \geqslant 1. \end{cases}$$

L'objectif principal de cette Note est de démontrer que le défaut de compacité de l'injection de Sobolev $H^1 \hookrightarrow \mathcal{L}$ peut-être décrit à l'aide de l'exemple de Moser (à des translations près par des coeurs de concentration). Le résultat principal est le suivant :

Théorème 1. *Soit (u_n) une suite de $H^1(\mathbb{R}^2)$ telle que*

$$u_n \rightharpoonup 0, \quad \limsup_{n\to\infty} \|u_n\|_{\mathcal{L}} = A_0 > 0 \quad \textit{et} \quad \lim_{R\to\infty} \limsup_{n\to\infty} \|u_n\|_{\mathcal{L}(|x|>R)} = 0. \tag{2}$$

Alors, il existe une suite $(\underline{\alpha}^{(j)})$ d'échelles, une suite $(\underline{x}^{(j)})$ de coeurs et une suite $(\psi^{(j)})$ de profils tels que les triplets $(\underline{\alpha}^{(j)}, \underline{x}^{(j)}, \psi^{(j)})$ soient deux à deux orthogonaux et, à extraction près, on a pour tout entier $\ell \geqslant 1$,

$$u_n(x) = \sum_{j=1}^{\ell} \sqrt{\frac{\alpha_n^{(j)}}{2\pi}}\,\psi^{(j)}\left(\frac{-\log|x - x_n^{(j)}|}{\alpha_n^{(j)}}\right) + \mathrm{r}_n^{(\ell)}(x), \quad \limsup_{n\to\infty} \big\|\mathrm{r}_n^{(\ell)}\big\|_{\mathcal{L}} \xrightarrow{\ell\to\infty} 0, \tag{3}$$

$$\|\nabla u_n\|_{L^2}^2 = \sum_{j=1}^{\ell} \big\|\psi^{(j)\prime}\big\|_{L^2}^2 + \big\|\nabla \mathrm{r}_n^{(\ell)}\big\|_{L^2}^2 + \circ(1), \quad n \to \infty. \tag{4}$$

La preuve de ce résultat repose sur un procédé d'extraction diagonale : l'extraction des échelles utilise le cas radial [5] à travers le réarrangement de Schwarz u_n^* de u_n tandis que celle des coeurs de concentraction se base essentiellement sur des arguments de capacité. Nous renvoyons à la version en anglais ci-dessous pour les définitions et les idées de la preuve de ce résultat et à [6] pour des détails et des compléments.

1. Introduction

It is well known that $H^1(\mathbb{R}^2)$ is continuously embedded in all Lebesgue spaces $L^p(\mathbb{R}^2)$ for $2 \leqslant p < \infty$, but not in $L^\infty(\mathbb{R}^2)$. On the other hand, it is also known (see for instance [1]) that $H^1(\mathbb{R}^2)$ embeds in $BMO(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$, where $BMO(\mathbb{R}^d)$ denotes the space of bounded mean oscillations which is the space of locally integrable functions f such that

$$\|f\|_{BMO} \stackrel{\text{def}}{=} \sup_B \frac{1}{|B|} \int_B |f - f_B|\,\mathrm{d}x < \infty \quad \text{with } f_B \stackrel{\text{def}}{=} \frac{1}{|B|} \int_B f\,\mathrm{d}x.$$

The above supremum being taken over the set of Euclidean balls B, $|\cdot|$ denoting the Lebesgue measure.

For the sake of geometric problems and the understanding of features of solutions to nonlinear partial differential equations with exponential growth, we investigate in this paper the lack of compactness of Sobolev embedding of $H^1(\mathbb{R}^2)$ in the Orlicz space $\mathcal{L}(\mathbb{R}^2)$ (defined below) which turns out not to be comparable with $BMO(\mathbb{R}^2)$ (for a proof of this fact, one can consult [5]). Remark that the lack of compactness of $\dot{H}^1(\mathbb{R}^d) \hookrightarrow BMO(\mathbb{R}^d)$ was characterized in [2] using a wavelet-based profile decomposition.

Note that in higher dimension, are available several works that highlight the role of the study of the lack of compactness in critical Sobolev embedding to the description of solutions of nonlinear partial differential equations. Among others, one can mention [3,4,10,12,13,15,19].

Let us introduce the so-called Orlicz spaces on $\mathbb{R}^d$ (for a complete presentation and more details, we refer the reader to [17]).

Definition 1. Let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a convex increasing function such that

$$\phi(0) = 0 = \lim_{s\to 0^+} \phi(s), \quad \lim_{s\to\infty} \phi(s) = \infty.$$

We say that a measurable function $u : \mathbb{R}^d \to \mathbb{C}$ belongs to L^ϕ if there exists $\lambda > 0$ such that $\int \phi(\frac{|u|}{\lambda}) < \infty$. We denote then

$$\|u\|_{L^\phi} = \inf\left\{\lambda > 0,\ \int_{\mathbb{R}^d} \phi\left(\frac{|u(x)|}{\lambda}\right) dx \leqslant \kappa\right\}, \tag{5}$$

where κ is the constant appearing in (7).

In what follows we shall fix $d = 2$, $\phi(s) = e^{s^2} - 1$ and denote the Orlicz space L^ϕ by $\mathcal{L}$ endowed with the norm $\|\cdot\|_{\mathcal{L}}$. The 2D critical Sobolev embedding in the Orlicz space $\mathcal{L}$ can be stated as follows:

Proposition 1.

$$\|u\|_{\mathcal{L}} \leqslant \frac{1}{\sqrt{4\pi}} \|u\|_{H^1}. \tag{6}$$

Let us point out that the embedding (6) derives immediately from the following Trudinger–Moser inequality proved in [18]:

Proposition 2.

$$\sup_{\|u\|_{H^1}\leqslant 1} \int_{\mathbb{R}^2} \left(e^{4\pi|u(x)|^2} - 1\right) dx := \kappa < \infty, \tag{7}$$

and this is false for $\alpha > 4\pi$.

The embedding $H^1 \hookrightarrow \mathcal{L}$ is not compact at least for two reasons. The first reason is the lack of compactness at infinity. A typical example is given by $u_n(x) = \varphi(x + x_n)$ where $0 \neq \varphi \in \mathcal{D}$ and $|x_n| \to \infty$. The second reason is of concentration-type derived by P.-L. Lions [14] and J. Moser [16], and illustrated by the following fundamental example f_α defined for $\alpha > 0$ ($\alpha \to \infty$) by:

$$f_\alpha(x) = \begin{cases} 0 & \text{if } |x| \geqslant 1, \\ -\frac{\log|x|}{\sqrt{2\alpha\pi}} & \text{if } e^{-\alpha} \leqslant |x| \leqslant 1, \\ \sqrt{\frac{\alpha}{2\pi}} & \text{if } |x| \leqslant e^{-\alpha}, \end{cases}$$

which satisfies $\|f_\alpha\|_{\mathcal{L}} \to \frac{1}{\sqrt{4\pi}}$ as $\alpha \to \infty$. The heart of this work is to prove that the lack of compactness of the Sobolev embedding

$$H^1(\mathbb{R}^2) \hookrightarrow \mathcal{L}(\mathbb{R}^2)$$

can be reduced up to cores to the example by Moser. Before coming to the statement of the main theorem, let us introduce some definitions as in [8] and [5].

Definition 2. A scale is a sequence $\underline{\alpha} := (\alpha_n)$ of positive real numbers going to infinity, a core is a sequence $\underline{x} := (x_n)$ of points in $\mathbb{R}^2$ and a profile is a function ψ belonging to the set

$$\mathcal{P} := \left\{\psi \in L^2\left(\mathbb{R}, e^{-2s} ds\right);\ \psi' \in L^2(\mathbb{R}),\ \psi_{|]-\infty,0]} = 0\right\}.$$

Given two scales $\underline{\alpha}$, $\underline{\tilde{\alpha}}$, two cores $\underline{x}$, $\underline{\tilde{x}}$ and two profiles ψ, $\tilde{\psi}$, we shall say that the triplets $(\underline{\alpha}, \underline{x}, \psi)$ and $(\underline{\tilde{\alpha}}, \underline{\tilde{x}}, \tilde{\psi})$ are orthogonal (in short $(\underline{\alpha}, \underline{x}, \psi) \perp (\underline{\tilde{\alpha}}, \underline{\tilde{x}}, \tilde{\psi})$) if

$$\text{either} \quad \left|\log(\tilde{\alpha}_n/\alpha_n)\right| \to \infty, \tag{8}$$

or $\tilde{\alpha}_n = \alpha_n$ and

$$-\frac{\log|x_n - \tilde{x}_n|}{\alpha_n} \to a \geqslant 0 \quad \text{with } \psi \text{ or } \tilde{\psi} \text{ null for } s < a. \tag{9}$$

Our result is as follows:

Theorem 1. *Let (u_n) be a bounded sequence in $H^1(\mathbb{R}^2)$ such that*

$$u_n \rightharpoonup 0, \quad \limsup_{n\to\infty} \|u_n\|_{\mathcal{L}} = A_0 > 0 \quad \text{and} \quad \lim_{R\to\infty} \limsup_{n\to\infty} \|u_n\|_{\mathcal{L}(|x|>R)} = 0. \tag{10}$$

Then, there exist a sequence of scales $(\underline{\alpha}^{(j)})$, a sequence of cores $(\underline{x}^{(j)})$ and a sequence of profiles $(\psi^{(j)})$ such that the triplets $(\underline{\alpha}^{(j)}, \underline{x}^{(j)}, \psi^{(j)})$ are pairwise orthogonal and, up to a subsequence extraction, we have for all $\ell \geqslant 1$,

$$u_n(x) = \sum_{j=1}^{\ell} \sqrt{\frac{\alpha_n^{(j)}}{2\pi}}\, \psi^{(j)}\left(\frac{-\log|x - x_n^{(j)}|}{\alpha_n^{(j)}}\right) + \mathrm{r}_n^{(\ell)}(x), \quad \limsup_{n\to\infty} \big\|\mathrm{r}_n^{(\ell)}\big\|_{\mathcal{L}} \xrightarrow{\ell\to\infty} 0, \tag{11}$$

$$\|\nabla u_n\|_{L^2}^2 = \sum_{j=1}^{\ell} \big\|\psi^{(j)\prime}\big\|_{L^2}^2 + \big\|\nabla \mathrm{r}_n^{(\ell)}\big\|_{L^2}^2 + \circ(1), \quad n \to \infty. \tag{12}$$

The elements used in our orthogonal asymptotic decomposition are completely different from the ones involving in the decomposition derived by P. Gérard in [8] in the framework of Sobolev embedding in Lebesgue spaces or by S. Jaffard in [11] in the more general case of Riesz potential spaces.

2. Ideas of the proof

We just give some basic ideas of our strategy, and we refer the reader to [6] for more details and some complements. The proof of Theorem 1 is done in four steps. In the first step, according to Schwarz symmetrization, we are reduced to the study of u_n^* the symmetric decreasing rearrangement of u_n. Since $u_n^* \in H^1_{rad}(\mathbb{R}^2)$ and satisfies assumptions of Theorem 1.16 in [5], it can be written as an orthogonal asymptotic decomposition by means of elementary concentrations of type $g_n^{(j)}(x) = \sqrt{\frac{\alpha_n^{(j)}}{2\pi}}\varphi^{(j)}(\frac{-\log|x|}{\alpha_n^{(1)}})$. Then in the second step, taking advantage of a cut-off argument, we reduce to $g_n^{(1)}$ and extract the first core $(x_n^{(1)})$ and the first profile $\psi^{(1)}$ which leads to the extraction of the first element $\sqrt{\frac{\alpha_n^{(1)}}{2\pi}}\psi^{(1)}(\frac{-\log|x-x_n^{(1)}|}{\alpha_n^{(j)}})$. This step constitutes the heart of the matter and relies on capacity arguments: the basic idea is to show that the mass does not scatter and is mainly concentrated around some point that will constitute the core. In the third step, we study the remainder term. If the limit of its Orlicz norm is null we stop the process. If not, we prove that this remainder term satisfies the same properties as the sequence start which allows us to extract a second elementary concentration concentrated around a second core $(x_n^{(2)})$. Thereafter, we establish the property of orthogonality between the two first elementary concentrations and finally we prove that this process converges.

As it is mentioned above, one can reduce to studying the case of a nonnegative sequence u_n satisfying

$$u_n^*(x) = \sqrt{\frac{\alpha_n}{2\pi}}\, \varphi\left(\frac{-\log|x|}{\alpha_n}\right). \tag{13}$$

Now, the heart of the matter is to extract cores and profiles and the key point in that process relies in the following lemma.

Lemma 1. *There exist $\delta_0 > 0$ and $N \in \mathbb{N}$ such that for any $n \geqslant N$ there exists x_n such that*

$$\big|E_n \cap B\big(x_n, \mathrm{e}^{-b\alpha_n}\big)\big| \geqslant \delta_0 A_0^2 |E_n|, \tag{14}$$

where $E_n := \{x \in \mathbb{R}^2;\ |u_n(x)| \geqslant \sqrt{2\alpha_n}(1 - \frac{\varepsilon_0}{10})A_0\}$, $B(x_n, \mathrm{e}^{-b\alpha_n})$ designates the ball of center x_n and radius $\mathrm{e}^{-b\alpha_n}$ with $b = 1 - 2\varepsilon_0$ and $0 < \varepsilon_0 < \frac{1}{2}$.

Sketch of proof. We argue by contradiction. Hence, up to a subsequence extraction, we have for any $\delta > 0$, $n \in \mathbb{N}$ and $x \in \mathbb{R}^2$

$$\big|E_n \cap B\big(x, \mathrm{e}^{-b\alpha_n}\big)\big| \leqslant \delta A_0^2 |E_n|. \tag{15}$$

In particular, property (15) occurs for any ball centered at a point in $\mathbf{T}_n := (\mathrm{e}^{-b\alpha_n}\mathbb{Z}) \times (\mathrm{e}^{-b\alpha_n}\mathbb{Z})$. Since the balls $B(x, \mathrm{e}^{-b\alpha_n})$ constitute a covering of $\mathbb{R}^2$ when the points x describe $\mathbf{T}_n$ and each point of $\mathbb{R}^2$ belongs at most to four balls among $\mathcal{B}_n := \{B(x, \mathrm{e}^{-b\alpha_n}),\ x \in \mathbf{T}_n\}$, we deduce that

$$\|\nabla u_n\|_{L^2(\mathbb{R}^2)}^2 \geqslant \frac{1}{4} \sum_{\mathbf{B}\in\mathcal{B}_n} \|\nabla u_n\|_{L^2(\mathbf{B})}^2. \tag{16}$$

Now, the idea is to get a contradiction by proving that for δ small enough the sum $\frac{1}{4}\sum_{\mathbf{B}\in\mathcal{B}_n} \|\nabla u_n\|_{L^2(\mathbf{B})}^2$ exceeds the energy of u_n. For this purpose, making use of capacity arguments we estimate the energy of u_n on the set $\tilde{\mathcal{B}}_n$ of balls $\mathbf{B} \in \mathcal{B}_n$

satisfying $|E_n \cap \mathbf{B}| \geqslant \mathrm{e}^{-10\alpha_n}$. To do so, we take advantage of the fact that the values of $|u_n|$ on $\mathbf{B}$ vary at least from $\sqrt{2\alpha_n}(1-\frac{\varepsilon_0}{10})A_0$ on $E_n \cap \mathbf{B}$ to $\sqrt{2\alpha_n}(1-\frac{\varepsilon_0}{2})A_0$ on a set of Lebesgue measure $\geqslant \frac{|\mathbf{B}|}{2}$, for all $n \geqslant N_{\varepsilon_0}$, where N_{ε_0} is an integer big enough which only depends on ε_0. It follows by standard capacity arguments [9] that

$$\|\nabla u_n\|^2_{L^2(\mathbf{B})} \geqslant C\left(\left(\frac{\varepsilon_0}{2}-\frac{\varepsilon_0}{10}\right)\sqrt{2\alpha_n}A_0\right)^2 \frac{1}{\log \frac{\mathrm{e}^{-\alpha_n(1-2\varepsilon_0)}}{\sqrt{|E_n \cap \mathbf{B}|}}} \geqslant C\varepsilon_0^2 A_0^2.$$

Hence,

$$4\|\nabla u_n\|^2_{L^2(\mathbb{R}^2)} \geqslant \#(\tilde{\mathcal{B}}_n) C\varepsilon_0^2 A_0^2. \tag{17}$$

But by (15), the covering of $\mathbb{R}^2$ by $\mathcal{B}_n$ and the fact that mass concentrated in E_n is mainly due to balls of $\tilde{\mathcal{B}}_n$, we necessarily have

$$\#(\tilde{\mathcal{B}}_n) \geqslant \frac{1}{2\delta A_0^2}$$

which yields a contradiction for δ small enough in view of (17). □

To extract the profile, we first consider $\psi_n(y,\theta) = \sqrt{\frac{2\pi}{\alpha_n}}v_n(\alpha_n y, \theta)$ where

$$v_n(s,\theta) = (\tau_{-x_n}u_n)\big(\mathrm{e}^{-s}\cos\theta, \mathrm{e}^{-s}\sin\theta\big)$$

and x_n is given by Lemma 1. Using energy estimate, we deduce that

$$\partial_\theta \psi_n \to 0, \quad \partial_y \psi_n \rightharpoonup g \quad \text{in } L^2(y,\theta), \quad \text{as } n \to \infty,$$

where g only depends of y and is null for $y \leqslant 0$. Finally, we prove that

$$\frac{1}{2\pi}\int_0^{2\pi} \psi_n(y,\theta)\,\mathrm{d}\theta \to \psi(y) = \int_0^y g(\tau)\,\mathrm{d}\tau,$$

and that there exists an absolute constant C so that $\|\psi'\|_{L^2} \geqslant CA_0$.

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