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Harmonic Analysis

Uniqueness sets for unbounded spectra

Ensembles d'unicité pour des spectres non-bornés

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A R T I C L E   I N F O

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A B S T R A C T

For every set $S \subset \mathbb{R}$ of finite measure, we construct a system of exponentials $\{e^{i\lambda t}\}_{\lambda\in\Lambda}$ which is complete in $L^2(S)$ and such that the set of frequencies Λ has the critical density $D(\Lambda) = \mathrm{mes}(S)/2\pi$.

R É S U M É

Pour tout ensemble $S \subset \mathbb{R}$ de mesure finie nous construisons un système d'exponentielles $\{e^{i\lambda t}\}_{\lambda\in\Lambda}$ qui est total dans $L^2(S)$ et dont l'ensemble des fréquences a la densité critique, à savoir $\mathrm{mes}(S)/2\pi$.

1. Introduction

Let $\mathcal{F}$ be some space of continuous functions on the real line $\mathbb{R}$. A set $\Lambda \subset \mathbb{R}$ is called a uniqueness set for $\mathcal{F}$ if

$$f \in \mathcal{F}, \quad f|_\Lambda = 0 \quad \Rightarrow \quad f = 0.$$

In what follows we consider *discrete* uniqueness sets for the classical Paley–Wiener spaces PW_S. Given a set $S \subset \mathbb{R}$ of finite measure, the space PW_S consists of all functions f which are the (inverse) Fourier transform

$$f(x) = \int_{\mathbb{R}} e^{itx} F(t)\, \mathrm{d}t \tag{1}$$

of functions $F \in L^2(\mathbb{R})$, $F = 0$ a.e. outside S. Since the measure of S is finite, we have $F \in L^1(\mathbb{R})$, and so every function $f \in PW_S$ is continuous.

Let $S = [a, b]$ be an interval. Assume that a set $\Lambda \subset \mathbb{R}$ is uniformly discrete (u.d.):

$$\inf_{\lambda,\lambda^*\in\Lambda,\ \lambda\neq\lambda^*} |\lambda - \lambda^*| > 0,$$

and assume that Λ is regularly distributed in the sense that the uniform density $D(\Lambda)$ exists:

$$\mathrm{Card}\big(\Lambda \cap (x, x + r)\big) = rD(\Lambda) + o(r) \quad \text{uniformly on } x \text{ as } r \to \infty.$$

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Then the condition $D(\Lambda) \geqslant |S|/2\pi$ is necessary while the condition

$$D(\Lambda) > \frac{1}{2\pi}|S| \tag{2}$$

is sufficient for Λ to be a uniqueness set for PW_S, where $|S|$ denotes the measure of S. This can be shown by standard complex variable techniques. A deep result of Beurling and Malliavin [3] states that the same is true for an arbitrary set Λ provided the uniform density is replaced with a certain exterior density (Beurling–Malliavin density).

In the case of disconnected spectra S, condition (2) ceases to be sufficient already when S is the union of two intervals. This is where the arithmetics of Λ comes into the play. For example, one can easily check that $\Lambda = \mathbb{Z}$ is not a uniqueness set for PW_S, where $S = [0, \epsilon] \cup [2\pi, 2\pi + \epsilon]$, $\epsilon > 0$. On the other hand, Landau [4] discovered that certain small perturbations of $\mathbb{Z}$ produce uniqueness sets for PW_S whenever S is a finite union of intervals satisfying $|S_\pi| < 2\pi$, where S_π denotes the "projection" of S onto $[-\pi, \pi]$:

$$S_\pi := (S + 2\pi\mathbb{Z}) \cap [-\pi, \pi].$$

In [5], Theorem 3.1 we proved that every "exponentially small" perturbation of $\mathbb{Z}$ is a uniqueness for PW_S where S is an *arbitrary* bounded set satisfying $|S_\pi| < 2\pi$. This result remains true for unbounded sets S with a moderate accumulation at infinity. However, if this accumulation is too fast then the functions of the corresponding Paley–Wiener space loose analyticity, and the result is no longer true, see [5].

In the present Note we show that for every set S in $\mathbb{R}$ of finite measure, the Paley–Wiener space PW_S possesses a uniformly discrete uniqueness set:

Theorem 1. *Assume $S \subset \mathbb{R}$, $|S| < \infty$ and $|S_\pi| < 2\pi$. There is a u.d. set $\Lambda \subset \mathbb{R}$, $D(\Lambda) = 1$, which is a uniqueness set for PW_S.*

Corollary. *For every set $S \subset \mathbb{R}$, $|S| = 2\pi$, there is a u.d. set $\Lambda \subset \mathbb{R}$, $D(\Lambda) = 1$, which is a uniqueness set for PW_S.*

2. Proof

Here we sketch the proof of the results. Set $\mathbb{Z}_n := \mathbb{Z} \cap \{|t| \geqslant n\}$ and $\mathbb{T} := (-\pi, \pi)$.

Lemma 1. *Let w (a weight) be an L^1-function on a set $A \subset \mathbb{T}$, $|A| < 2\pi$, $w(t) \geqslant 1$. Then there is a sequence of disjoint sets $\Lambda_j \subset \mathbb{Z}$, $j \in \mathbb{N}$, such that:*

(i) *Every exponential system $\{e^{i\lambda t}\}_{\lambda \in \Lambda_j}$ is complete in $L^2(w, A)$.*
(ii) *No neighboring integers belong to different subsets Λ_j.*

Proof. Set $X := L^2(w, A)$. One can identify X^* with $L^2(1/w, A)$ and check that X^* is embedded in $L^1(A)$. So, if $g \in X^*$ is orthogonal to $E(\mathbb{Z}_n)$, then $g = 0$ a.e. Hence, $E(\mathbb{Z}_n)$ is complete in $X, n \in \mathbb{N}$.

To construct Λ_j, we run an induction process, where on m-th step one adds to $\Lambda_{j(m)}$ a "large" finite set of integers, so that there is a trigonometric polynomial with these frequencies which approximates an exponential function $e^{il(m)t}$ with an error $\epsilon(m)$. The couple (j, l) runs over $\mathbb{Z}^2$ infinitely many times and $\epsilon(m) \to 0$ as $m \to \infty$. □

Proof of Theorem 1. 1. Define a weight w to be the multiplicity function $w(t) := \operatorname{Card}\{k \in \mathbb{Z}\colon\ t + 2\pi k \in S\}$, where $t \in \mathbb{T}$. Set $A_j := \{t \in \mathbb{T}\colon\ w(t) = j\}$, $j = 1, 2, \ldots$.

2. Take any function $f \in PW_S$ and let F be the Fourier transform in (1). Then

$$F(t) = \sum_{k\in\mathbb{Z}} F_k(t + 2\pi k), \qquad F_k(t) := F(t - 2\pi k) \cdot \mathbf{1}_{\mathbb{T}}(t),$$

where the functions $\{F_k\}$ are supported by S_π.

3. Set

$$G(a, t) := \sum_{k\in\mathbb{Z}} e^{i2\pi ka} F_k(t), \quad 0 \leqslant a < 1,\ t \in \mathbb{T}. \tag{3}$$

One may check that G belongs to $L^2(1/w, S_\pi)$ for every $a \in [0, 1)$. A simple calculation shows that for each $\nu \in \mathbb{Z}$ we have

$$f(\nu + a) = \int_{\mathbb{T}} G(a, t) e^{i(\nu+a)t}\, dt. \tag{4}$$

4. By Lemma 1, there exist disjoint sets $\Lambda(n, j) \subset \mathbb{Z}$, $n = 1, 2, \ldots$, $0 \leqslant j < n$, so that each system of exponentials $E(\Lambda(n, j))$ is complete in the space $X := L^2(w, S_\pi)$. Set

$$\Lambda_n := \bigcup_{j=0}^{n}\left(\Lambda(n,j)+\frac{j}{n}\right); \qquad \Lambda := \bigcup_{n\in\mathbb{N}} \Lambda_n.$$

By Lemma 1(ii), we may assume that Λ is uniformly discrete and $D(\Lambda)=1$.

5. Now we prove that Λ is a uniqueness set for PW_S. Assume $f(x)=0, x\in\Lambda$. Given n and j, we use (4) for every $\nu\in\Lambda(n,j)$. Due to completeness of $E(\Lambda(n,j))$ in X, we get: $G(j/n,t)=0$ a.e. Since G is continuous with respect to a for almost all $t\in S_\pi$, we see that $G(a,t)=0$ on $(0,1)$. From (3) we conclude that $F_k(t)=0$ a.e. on $\mathbb{T}$ for every k. Hence, $f=0$ which proves Theorem 1. □

Proof of Corollary. The only case which is not covered by Theorem 1 is when $|S|=|S_\pi|=2\pi$. Then one can check that $\Lambda=\mathbb{Z}$ is a uniqueness set for PW_S. □

3. Remarks

1. The results above hold also for the spaces $\hat{L}^p_S$, $p>1$, of all functions f defined in (1), where $F\in L^p(\mathbb{R})$ and $F=0$ a.e. outside S.

2. Let B^q_S denote the Bernstein's type spaces of all continuous functions $f\in L^q(\mathbb{R})$ whose spectrum, in the distributional sense, lies in S (cf. Section 6 in [7]). The following result shows a sharp contrast between these spaces and the Paley–Wiener spaces:

For every $q>2$ there is an (unbounded) closed set $S\subset\mathbb{R}$ of Lebesgue measure zero such that no u.d. set Λ is a uniqueness set for B^q_S.

For $q=\infty$ this follows directly from Theorem 3 in [7] (see also Theorem 3.1 in [6]). The latter results can be extended to the case $q>2$.

3. Given two sets S, Q in $\mathbb{R}$ of finite measure, it is well known that there is no non-trivial L^2-function f supported by S so that $\hat{f}$ is supported by Q (see [1,2]). By slightly changing the proof of Theorem 1, one can obtain the following extension of this result:

Suppose $S, Q\subset\mathbb{R}$, $|S|, |Q|<\infty$. There is a u.d. set $\Lambda\subset\mathbb{R}$ such that $\Lambda\cap Q=\emptyset$, $D(\Lambda)=|S|/2\pi$ and Λ is a uniqueness set for PW_S.

4. One may extend the results (and proofs) of this Note to functions of several variables. For example, the following is true:

For every set $S\subset\mathbb{R}^n$ of finite Lebesgue measure there is a u.d. set $\Lambda\subset\mathbb{R}^n$, $D(\Lambda)=|S|/(2\pi)^n$, which is a uniqueness set for the space PW_S.

5. A set $\Lambda\subset\mathbb{R}$ is called a sampling set for the space PW_S if there is a constant $C>0$ such that

$$\int_{\mathbb{R}} \left|f(x)\right|^2 dx \leqslant C\sum_{\lambda\in\Lambda}\left|f(\lambda)\right|^2, \quad \text{for every } f\in PW_S.$$

It is well known that when S is a *bounded* set, then the space PW_S possesses a u.d. sampling set Λ. The following open question seems to be interesting: Is it true that PW_S possesses a u.d. sampling set for every set S of finite measure?

References

[1] W.O. Amrein, A.M. Berthier, On support properties of L^p-functions and their Fourier transforms, J. Funct. Anal. 24 (3) (1977) 258–267.
[2] M. Benedicks, On Fourier transforms of functions supported on sets of finite Lebesgue measure, J. Math. Anal. Appl. 106 (1) (1985) 180–183.
[3] A. Beurling, P. Malliavin, On the closure of characters and the zeros of entire functions, Acta Math. 118 (1967) 79–93.
[4] H.J. Landau, A sparse regular sequence of exponentials closed on large sets, Bull. Amer. Math. Soc. 70 (1964) 566–569.
[5] A. Olevskii, A. Ulanovskii, Universal sampling and interpolation of band-limited signals, Geom. Funct. Anal. 18 (3) (2008) 1029–1052.
[6] A. Olevskii, A. Ulanovskii, Interpolation in Bernstein and Paley–Wiener spaces, J. Funct. Anal. 256 (10) (2009) 3257–3278.
[7] A. Olevskii, A. Ulanovskii, Approximation of discrete functions and size of spectrum, St. Petersburg Math. J. 21 (2010) 1015–1025.