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Harmonic Analysis

On Ingham-type interpolation in $\mathbb{R}^n$

Sur l'interpolation du type d'Ingham dans $\mathbb{R}^n$

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ABSTRACT

Let $\mathcal{S}$ and $\mathcal{K}$ be 0-symmetric convex bodies in $\mathbb{R}^n$. We are interested in determining conditions under which every set Λ satisfying $(\Lambda - \Lambda) \cap \mathcal{K} = \{0\}$ is a set of interpolation for the Paley–Wiener space of functions with spectrum in $\mathcal{S}$. Some sufficient and necessary conditions are given which, in particular, imply sharp asymptotic estimates for the l_p-balls.

RÉSUMÉ

Soient $\mathcal{S}$ et $\mathcal{K}$ deux ensembles convexes 0-symétriques (symétriques par rapport à 0). A quelle condition tout ensemble Λ vérifiant $(\Lambda - \Lambda) \cap \mathcal{K} = \{0\}$ est-il un ensemble d'interpolation pour les fonctions localement L^2 à spectre dans $\mathcal{S}$? On donne des conditions nécessaires et des conditions suffisantes pour qu'il en soit ainsi, et on en dérive des estimations précises pour les boules l_p quand $n \to \infty$.

Version française abrégée

On dit que $\mathcal{S}$ et $\mathcal{K}$, convexes et 0-symétriques dans $\mathbb{R}^n$, forment une paire admissible si, pour tout $\Lambda \subset \mathbb{R}^n$ verifiant $(\Lambda - \Lambda) \cap \mathcal{K} = \{0\}$, l'application $f \to \hat{f}|_\Lambda$ de $L^2(\mathcal{S})$ dans $l^2|_\Lambda$ est surjective (on dit alors que Λ est un ensemble d'interpolation pour $PW_{\mathcal{S}}$). Voici les principaux résultats :

Théorème 1. *Soient $\mathcal{A}$ et $\mathcal{B}$ deux ensembles convexes 0-symétriques dans $\mathbb{R}^n$. S'il existe $f \in L^2(\mathbb{R}^n)$ telle que*

$$\int_{\mathcal{A}} |f|^2 > \frac{99}{100} \int_{\mathbb{R}^n} |f|^2 \quad \text{et} \quad \int_{\mathcal{B}} |\hat{f}|^2 > \frac{99}{100} \int_{\mathbb{R}^n} |\hat{f}|^2,$$

alors $(\mathcal{A}, 12\mathcal{B})$ est admissible, et de même $(\mathcal{B}, 12\mathcal{A})$.

Théorème 2. *Si $\mathcal{S}, \mathcal{K}$ est une paire admissible, alors*

$$\mathrm{vol}(\mathcal{S}) \cdot \mathrm{vol}(\mathcal{K}) \geqslant (2\pi)^n.$$

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Théorème 3. *Soit B_p resp. B_q la boule unité de $\mathbb{R}^n$ en norme l_p resp. l_q. Il existe $C > 0$ tel que $(\mathcal{B}_p, Cn\mathcal{B}_q)$ soit admissible pour tout $n > 1$, $1 \leqslant p \leqslant \infty$ et $q = p/(p-1)$.*

Cela est faux pour $C < 1/e$.

1. Introduction

Let $\mathcal{S}$ be a bounded set in $\mathbb{R}^n, n \geqslant 1$. By $PW_{\mathcal{S}}$ we denote the space of all functions $f \in L^2(\mathbb{R}^n)$ such that $\hat{f}$ vanishes a.e. outside $\mathcal{S}$, where

$$\hat{f}(t) := \int_{\mathbb{R}^n} e^{it\cdot x} f(x)\,\mathrm{d}x$$

is the Fourier transform of f. Let Λ be a separated set in $\mathbb{R}^n$. Then the restriction operator

$$R : f \to f|_{\Lambda}$$

acts boundedly from $PW_{\mathcal{S}}$ into $l^2(\Lambda)$. Λ is called a set of interpolation for $PW_{\mathcal{S}}$ if R is surjective. The following result is classical:

Theorem A. *(See Ingham [2].) Let $\Lambda \subset \mathbb{R}$ satisfy the condition*:

$$\inf_{\lambda, \lambda' \in \Lambda,\, \lambda \neq \lambda'} |\lambda - \lambda'| > \pi.$$

Then Λ is a set of interpolation for $PW_{[-1,1]}$.

Ingham-type results for $\mathbb{R}^n$, $n > 1$, were first obtained by J.P. Kahane in [3,4]. In [5] (see also references therein) the authors extended Ingham's approach to $PW_{\mathcal{B}_2}$, where $\mathcal{B}_2$ is the unit ball in $\mathbb{R}^n$. Some estimates were obtained involving the first eigenvalue of Laplacian in certain domains. These estimates show, in particular, that every set $\Lambda \subset \mathbb{R}^n$ satisfying

$$\inf_{\lambda, \lambda' \in \Lambda,\, \lambda \neq \lambda'} \|\lambda - \lambda'\|_2 > Cn, \tag{1}$$

for some constant $C > 0$, is a set of interpolation for $\mathcal{B}_2$.

2. Results

Definition 1. Let $\mathcal{S}, \mathcal{K}, 0 \in \mathcal{K}$, be bounded measurable sets in $\mathbb{R}^n$. We say that the pair $(\mathcal{S}, \mathcal{K})$ is admissible if every set Λ with the property $(\Lambda - \Lambda) \cap \mathcal{K} = \{0\}$ is a set of interpolation for $PW_{\mathcal{S}}$.

Definition 2. Let $a \in (0, 1)$. We say that a function $f \in L^2(\mathbb{R}^n)$ is a-concentrated on a set $\mathcal{A}$ if

$$\int_{\mathcal{A}} |f(t)|^2\,\mathrm{d}t > a \int_{\mathbb{R}^n} |f(t)|^2\,\mathrm{d}t.$$

We show that the Ingham-type interpolation property is intimately connected with existence of functions f concentrated on $\mathcal{K}$ such that $\hat{f}$ is concentrated on $\mathcal{S}$. Below by a we denote a small absolute positive constant (one may take $a = 10^{-2}$).

Theorem 1. *Let $\mathcal{A}, \mathcal{B} \subset \mathbb{R}^n$ be 0-symmetric convex bodies. Suppose there exists $f \in L^2(\mathbb{R}^n)$ such that f is $(1-a)$-concentrated on $\mathcal{A}$ and $\hat{f}$ is $(1-a)$-concentrated on $\mathcal{B}$. Then the pairs $(\mathcal{A}, 12\mathcal{B})$ and $(\mathcal{B}, 12\mathcal{A})$ are admissible.*

The next theorem gives a necessary condition for admissibility:

Theorem 2. *Let $\mathcal{S}, \mathcal{K} \subset \mathbb{R}^n$ be convex 0-symmetric bodies. If the pair $(\mathcal{S}, \mathcal{K})$ is admissible then*

$$\mathrm{vol}(\mathcal{S}) \cdot \mathrm{vol}(\mathcal{K}) \geqslant (2\pi)^n. \tag{2}$$

As an application of Theorem 1, we have the following result in which $\mathcal{B}_p \subset \mathbb{R}^n$ denotes the unit ball in the l_p-norm:

Theorem 3. *There exists $C > 0$ such that the pair $(\mathcal{B}_p, Cn\mathcal{B}_q)$ is admissible for every n and $1 \leqslant p \leqslant \infty$, where $q = p/(p-1)$.*

This result is sharp, see Remark 1 below. In particular, it follows that the linear growth in (1) cannot be improved.

3. Proof of Theorem 1

Below we sketch the proof of Theorem 1.

It is well known that Λ is an interpolation set for $PW_{\mathcal{S}}$ if and only if there exists $C(\Lambda, \mathcal{S})$ such that the inequality

$$\int_{\mathcal{S}} \left| \sum_{j=1}^{N} c_j e^{i\lambda_j \cdot x} \right|^2 \mathrm{d}x \geqslant C(\Lambda, \mathcal{S}) \sum_{j=1}^{N} |c_j|^2, \tag{3}$$

holds for every finite sequence $c_1, \dots, c_N$ and every $\lambda_1, \dots, \lambda_N \in \Lambda$. Suppose f satisfies the assumptions of Theorem 1. It clearly suffices to show that $(\mathcal{A}, 12\mathcal{B})$ is admissible. Let g be the inverse Fourier transform of $\hat{f}(x)\mathbf{1}_{\mathcal{B}}(x)$. Then $g \in PW_{\mathcal{B}}$ and g is $(1-\alpha)$-concentrated on $\mathcal{A}$, for some small $\alpha = \alpha(a) > 0$. We may assume that $\|g\|_2 = 1$, and so $\|g \cdot \mathbf{1}_{\mathcal{A}}\|_2^2 \geqslant 1 - \alpha$. Set

$$\eta(t) := 1, \qquad t \in 2\mathcal{A}, \qquad \eta(t) := -1, \qquad t \in \mathbb{R}^n \setminus 2\mathcal{A},$$

and denote by $h(t)$ the convolution of $\eta(t)$ and $|g(t)|^2$. It is straightforward to check that $|h(t)| \leqslant 1$ for all $t \in \mathbb{R}^n$ and that

$$h(t) \geqslant 1 - 2\alpha > 0, \quad t \in \mathcal{A}, \qquad h(t) \leqslant -1 + 2\alpha < 0, \quad t \in \mathbb{R}^n \setminus 3\mathcal{A}.$$

Next, we set $\Psi(t) := h(t)|g(t)|^2$. Since $|g(t)|^2 \in PW_{2\mathcal{B}}$, we have $\Psi \in PW_{4\mathcal{B}}$. Also, we have $\Psi(t) < 0$ for $t \in \mathbb{R}^n \setminus 3\mathcal{A}$, and

$$\hat{\Psi}(0) = \int_{\mathbb{R}^n} \Psi(t)\,\mathrm{d}t \geqslant \int_{\mathcal{A}} (1-2\alpha)\big|g(t)\big|^2\,\mathrm{d}t - \int_{\mathbb{R}^n \setminus \mathcal{A}} \big|f(t)\big|^2\,\mathrm{d}t \geqslant (1-2\alpha)(1-\alpha) - \alpha > 0.$$

Suppose that $(\Lambda - \Lambda) \cap 4\mathcal{B} = \{0\}$. Following Ingham [2], we have

$$\sup_{x \in 3\mathcal{A}} \Psi(x) \int_{3\mathcal{A}} \left| \sum_j c_j e^{i\lambda_j \cdot x} \right|^2 \mathrm{d}x \geqslant \int_{\mathbb{R}^n} \Psi(x) \left| \sum_j c_j e^{i\lambda_j \cdot x} \right|^2 \mathrm{d}x = \sum_{j,l} c_j \bar{c}_l \hat{\Psi}(\lambda_j - \lambda_l) = \hat{\Psi}(0) \sum_j |c_j|^2,$$

which proves (3) for $\mathcal{S} = 3\mathcal{A}$. Hence, every Λ with $(\Lambda - \Lambda) \cap 4\mathcal{B} = \{0\}$ is a set of interpolation for $3\mathcal{A}$. This shows that $(3\mathcal{A}, 4\mathcal{B})$ is admissible, which is equivalent to the admissibility of $(\mathcal{A}, 12\mathcal{B})$.

4. Proof of Theorem 2

By Minkowski–Hlawka theorem (see [1, ch. 6]), there is a lattice $\Lambda := T\mathbb{Z}^n$, where $T : \mathbb{R}^n \to \mathbb{R}^n$ is an invertible linear operator, such that:

$$\text{(i)} \quad \Lambda \cap \mathcal{K} = \{0\}; \qquad \text{(ii)} \quad d(\Lambda) < \operatorname{vol}(K),$$

where $d(\Lambda) = \operatorname{vol}(T[0,1]^n)$ is the volume of the fundamental parallelepiped of Λ. By assumption, Λ is an interpolation set for $PW_{\mathcal{S}}$. Hence $\mathbb{Z}^n$ is a set of interpolation for $PW_{T^{-1}\mathcal{S}}$. Since the exponential system $\{e^{ik\cdot t}\}_{k \in \mathbb{Z}^n}$ forms an orthogonal basis in $L^2[-\pi, \pi]^n$, we have $\operatorname{vol}(T^{-1}\mathcal{S}) \geqslant (2\pi)^n$. Now this and estimate (ii) above prove (2).

5. Admissible l_p-balls

Lemma 1. *Set*

$$S(x) := \prod_{j=1}^{n} \left(\frac{\sin x_j}{x_j} \right)^2, \qquad x = (x_1, \dots, x_n),\ n > 1.$$

Then $S \in PW_{2\mathcal{B}_\infty}$, and there exists $C > 0$ such that $S(x)$ is $(1-\epsilon)$-concentrated on $(C\sqrt{n/\epsilon})\mathcal{B}_2$ for every $0 < \epsilon < 1$.

Proof. Set

$$\beta := \int_{\mathbb{R}} \left(\frac{\sin t}{t} \right)^4 \mathrm{d}t, \qquad \gamma := \int_{\mathbb{R}} \frac{\sin^4 t}{t^2}\,\mathrm{d}t.$$

Then $\|S\|_2^2 = \beta^n$ and

$$\int_{\|x\|_2 > \sqrt{n\gamma/\epsilon\beta}} S^2(x)\,\mathrm{d}x < \frac{\epsilon\beta}{n\gamma} \int_{\mathbb{R}^n} \|x\|_2^2 S^2(x)\,\mathrm{d}x = \epsilon\beta^n = \epsilon\|S\|_2^2,$$

which shows that $S(x)$ is $(1-\epsilon)$-concentrated on $(\sqrt{n\gamma/\epsilon\beta})\mathcal{B}_2$. $\square$

Lemma 1 and Theorem 1 show that the pairs $(\mathcal{B}_2, C\sqrt{n}\mathcal{B}_\infty)$ and $(\mathcal{B}_\infty, C\sqrt{n}\mathcal{B}_2)$ are admissible, for some constant $C>0$. Clearly, if a pair $(\mathcal{S},\mathcal{K})$ is admissible and $\mathcal{S}\subset\mathcal{S}^*, \mathcal{K}\subset\mathcal{K}^*$, then $(\mathcal{S}^*,\mathcal{K}^*)$ is admissible, too. Since $1\leqslant p<p^*\leqslant\infty$ implies $n^{1/p}\mathcal{B}_p\subset n^{1/p^*}\mathcal{B}_{p^*}$, we obtain Theorem 3 in a slightly stronger form:

Theorem 3*. *There is a constant $C>0$ such that for every $n>1$ the pair $(n^{1/p}\mathcal{B}_p, Cn^{1/p^*}\mathcal{B}_{p^*})$ is admissible provided $p, p^*\geqslant 1$ and at least one of the numbers p, p^* does not exceed 2.*

6. Remarks

1. When $q=p/(p-1)$, the Blashke–Santaló inequality gives

$$\mathrm{vol}(\mathcal{B}_p)\cdot\mathrm{vol}(Cn\mathcal{B}_q)\leqslant(Cn)^n\big(\mathrm{vol}(\mathcal{B}_2)\big)^2=C^n\big(2\pi e+o(1)\big)^n.$$

By Theorem 2, we see that Theorem 3 ceases to be true for $C<e^{-1}$.

2. One can show that Theorem 3* remains true for all $p, p^*\geqslant 1$ provided one replaces C by $C\cdot\min\{p,p^*\}$.

3. Theorems 1 and 2 admit extension to the case when $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$ and $\mathcal{K}$ are arbitrary (not necessarily convex) bounded sets in $\mathbb{R}^n$.

The conclusion of Theorem 2 remains true, the only change in the proof is that one uses a theorem of Hlawka (see [1], Corollary on p. 182) instead of Minkowski–Hlawka theorem.

The proof of Theorem 1 (with obvious minor changes) yields the following variant of Theorem 1: *Under the assumptions on f, the pairs $(\mathcal{A}+\mathcal{A}-\mathcal{A}, \mathcal{B}+\mathcal{B}-\mathcal{B}-\mathcal{B})$ and $(\mathcal{B}+\mathcal{B}-\mathcal{B}, \mathcal{A}+\mathcal{A}-\mathcal{A}-\mathcal{A})$ are admissible.*

4. We conjecture that a more general version of Theorem 3 holds: *There exists $C>0$ such that every pair (K, CnK^o) is admissible, where $K\subset\mathbb{R}^n$ is a convex 0-symmetric body and $\mathcal{K}^o$ is its polar body.*

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