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Optimal Control

Nonlinear stabilization of abstract systems via a linear observability inequality and application to vibrating PDE's

Stabilisation non linéaire de systèmes abstraits via un critère d'observabilité linéaire et applications aux EDP vibrantes

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ABSTRACT

This Note is concerned with the links between nonlinear stabilization of hyperbolic systems and linear observability for the unforced corresponding linear system, for locally distributed and boundary feedbacks as well. We show that if the linear system is observable through a locally distributed (resp. boundary) observation, then any dissipative nonlinear feedback locally distributed (resp. active only on a part of the boundary) stabilize the system and we give a general energy decay formula. Our results generalize previous results by Haraux (1989) and Ammari and Tucsnak (2001) for linear feedbacks. We show by this way that for the locally distributed case, one can combine the optimal geometric conditions of Bardos et al. (1992) and the method of Alabau-Boussouira (2005) to deduce energy decay rates for nonlinear damped systems.

RÉSUMÉ

On étudie dans cette Note, le problème de la stabilisation par rétro-action non linéaire localement distribuée ou frontière d'équations abstraites, comme conséquence d'une inégalité d'observabilité pour le problème linéaire associé sans rétro-action. On montre sous des hypothèses très générales sur le feedback, notamment sans hypothèse de croissance à l'origine, que si le système conservatif linéaire est observable par une observation localement distribuée (resp. par une observation frontière), dans l'espace d'énergie naturelle (resp. dans le domaine de l'opérateur), alors tout feedback non linéaire localement distribué (resp. frontière) stabilise le système et on donne un taux de décroissance de l'énergie quasi-optimal. On donne des exemples d'application de ces résultats à des EDP. On montre ainsi, pour le cas localement distribué, que l'on peut combiner les hypothèses géométriques optimales de Bardos et al. (1992) et la méthode de Alabau-Boussouira (2005) pour déduire des résultats de stabilisation non linéaire.

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Introduction

Cette Note concerne la stabilisation des équations hyperboliques du deuxième ordre par des feedbacks non linéaires. Nous renvoyons à par ex. [7,17,18,13,12,14,1,2] pour des résultats de stabilisation dans le cas de feedbacks frictionnels.

Résultats principaux

Pour des soucis de longueur, nous renvoyons le lecteur à la version anglaise pour le détail de certaines équations et pour les hypothèses (A1) et (A2). On considère les systèmes (1) (cas localement distribué) et (5) (cas frontière).

Les résultats principaux de cette Note sont les suivants :

Théorème 1 *(Cas localement distribué). On suppose que ρ et a vérifient* (A1) *et qu'il existe $r_0 > 0$ suffisamment petit tel que la fonction H définie par* (14) *soit strictement convexe sur $[0, r_0^2]$. On suppose aussi que* (17) *ou bien* (18) *est vérifiée. De plus, soit $T > 0$ tel qu'il existe $c_T > 0$ pour lequel l'inégalité d'observabilité* (19) *soit satisfaite pour les solutions de* (12)*. Alors l'énergie des solutions de* (1) *vérifie* (20)*.*

Théorème 2 *(Cas frontière). On suppose que $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$ et que ρ et a satisfont* (A2)*. On suppose de plus qu'il existe $r_0 > 0$ suffisamment petit tel que la fonction H définie par* (14) *soit strictement convexe $[0, r_0^2]$ et que* (17) *ou bien* (18) *est vérifiée. Soit $T > 0$ tel qu'il existe $c_T > 0$ pour lequel l'inégalité d' observabilité* (22) *soit satisfaite pour les solutions de* (12)*. Alors, l'énergie des solutions de* (5) *satisfait* (20)*.*

Nous renvoyons à la partie anglaise pour une application de ces résultats à des exemples d'EDP et pour une idée de la preuve.

1. Introduction

In this Note, we show that if a second-order linear hyperbolic conservative system is observable either by a locally distributed or boundary observation, then the corresponding damped system is stabilized by very general nonlinear dissipative feedbacks located on the same region than the observation. Using the method of Alabau-Boussouira [2], we give sharp energy decay rates for the nonlinear damped system. We give applications of these results to examples of PDE's. Our results generalize Haraux's result [11] valid for the case of bounded linear observation and linear feedback operators, and Ammari–Tucsnak's [5,6] result valid for linear unbounded observation and linear unbounded feedbacks. We mention that interesting results in this direction have also been obtained by Daoulati et al. [10].

One interesting aspect of our results is that they allow to combine the optimal conditions of geometric optics of Bardos–Lebeau–Rauch [7] to the optimal or quasi-optimal energy decay rates given by Alabau-Boussouira [1,2] provided that the observation region satisfies the condition of geometric optics so that observability holds for the linear conservative system.

We consider both cases of locally distributed and boundary feedbacks.

1.1. The locally distributed case

We consider the following second order differential equation

$$\begin{cases} \ddot{w}(t) + Aw(t) + a(\,.\,)\rho(.,\dot{w}) = 0, & t \in (0,\infty),\ x \in \Omega, \\ w(0) = w_0, \quad \dot{w}(0) = w_1, \end{cases} \tag{1}$$

where Ω is a bounded open set in $\mathbb{R}^N$, with a boundary Γ of class $\mathcal{C}^2$, $X = L^2(\Omega)$, with its usual scalar product denoted by $\langle\cdot,\cdot\rangle_X$ and the associated norm $|\cdot|_X$ and where $A : D(A) \subset X \to X$ is a densely defined self-adjoint linear operator satisfying

$$\langle Au, u\rangle_X \geqslant \alpha |u|_X^2, \quad \forall u \in D(A), \tag{2}$$

for some $\alpha > 0$. We also set $V = D(A^{1/2})$ and $|u|_V = |A^{1/2}u|_X$.

We make the following assumptions on the feedback ρ and on a:

Assumption (A1). $\rho \in \mathcal{C}(\Omega \times \mathbb{R}; \mathbb{R})$ is a continuous monotone nondecreasing function with respect to the second variable on Ω such that $\rho(.,0) = 0$ on Ω and there exists a continuous strictly increasing odd function $g \in \mathcal{C}([-1,1]; \mathbb{R})$, differentiable in a neighbourhood of 0 and satisfying $g(0) = g'(0) = 0$, with

$$\begin{cases} c_1 g(|v|) \leqslant |\rho(.,v)| \leqslant c_2 g^{-1}(|v|), & |v| \leqslant 1, \text{ a.e. on } \Omega, \\ c_1|v| \leqslant |\rho(.,v)| \leqslant c_2|v|, & |v| \geqslant 1, \text{ a.e. on } \Omega, \end{cases} \tag{3}$$

where $c_i > 0$ for $i = 1, 2$. Moreover $a \in \mathcal{C}(\overline{\Omega})$, with $a \geqslant 0$ on Ω and

$$\exists\, a_- > 0 \quad \text{such that} \quad a \geqslant a_- \text{ on } \omega. \tag{4}$$

1.2. The boundary case for the wave equation

We keep the above notation and hypotheses for Ω, Γ and X. We identify X with its dual space. Let Γ_1 be a subset of Γ of positive Lebesgue measure on Γ. We set $\Gamma_0 = \Gamma \setminus \Gamma_1$. Let $V = \{v \in H^1(\Omega), v = 0 \text{ on } \Gamma_0\}$ equipped with the scalar product $\langle u, z\rangle_V = \int_\Omega \nabla u \cdot \nabla z \, dx + \int_{\Gamma_1} \sigma u z \, d\gamma$, with $\sigma \in L^\infty(\Gamma_1)$ and $\sigma \geqslant \sigma_- > 0$ on Γ_1, where σ_- is a positive real number. We denote by $\tilde{A}$ the duality mapping from V on V' and by A the unbounded operator on X defined on $D(A) = \{u \in V, \tilde{A}u \in X\}$. Then $A = -\Delta$ and $D(A) = H^2(\Omega) \cap V$.

We consider the following second-order evolution equation:

$$\begin{cases} \ddot{w}(t) + Aw(t) + B(., \dot{w}) = 0, & t \in (0, \infty),\ x \in \Omega, \\ w(0) = w_0, \quad \dot{w}(0) = w_1, \end{cases} \tag{5}$$

where B is a continuous operator from $\overline{\Gamma_1} \times V$ to V' which further satisfies

$$\langle B(., v), z\rangle_{V',V} = \int_{\Gamma_1} a(.)\rho(., v) z \, d\gamma, \quad \forall u, z \in V. \tag{6}$$

This operator serve as the abstract formulation of the boundary feedback operator.

Assumption (A2). $\rho \in \mathcal{C}(\overline{\Gamma_1} \times \mathbb{R}; \mathbb{R})$ is a continuous monotone nondecreasing function with respect to the second variable on Ω such that $\rho(., 0) = 0$ on $\overline{\Gamma_1}$ and there exists a continuous strictly increasing odd function $g \in \mathcal{C}([-1, 1]; \mathbb{R})$, differentiable in a neighbourhood of 0 and satisfying $g(0) = g'(0) = 0$, with

$$\begin{cases} c_1 g(|v|) \leqslant |\rho(., v)| \leqslant c_2 g^{-1}(|v|), & |v| \leqslant 1, \text{ a.e. on } \Gamma_1, \\ c_1|v| \leqslant |\rho(., v)| \leqslant c_2|v|, & |v| \geqslant 1, \text{ a.e. on } \Gamma_1, \end{cases} \tag{7}$$

where $c_i > 0$ for $i = 1, 2$. Moreover $a \in \mathcal{C}(\overline{\Gamma_1})$, with

$$\exists a_- > 0 \quad \text{such that} \quad a \geqslant a_- \text{ on } \Gamma_1. \tag{8}$$

We recall that, under Assumption (A1) (resp. (A2)), the system (1) (resp; (5)), with initial data $(w_0, w_1) \in V \times X$ has a unique mild solution $w \in \mathcal{C}([0, T]; V) \cap \mathcal{C}^1([0, T]; X)$. Moreover for smoother initial data, that is for $(w_0, w_1) \in D(A) \times V$, w is a strong solution, i.e. $w \in \mathcal{C}([0, T]; D(A)) \cap \mathcal{C}^1([0, T]; V) \cap \mathcal{C}^2([0, T]; X)$. The energy of a solution w of (1) or (5) is defined by

$$E_w(t) = \frac{1}{2}\left(|\dot{w}|_X^2 + |w|_V^2\right). \tag{9}$$

It is well known that the energy of mild solutions of (1) satisfies the dissipation relation

$$\int_0^T \langle a(.)\rho(., \dot{w}), \dot{w}\rangle_X = E_w(0) - E_w(T) \tag{10}$$

whereas the energy of mild solutions of (5) satisfies the dissipation relation

$$\int_0^T \int_{\Gamma_1} a(.)\rho(., \dot{w})\dot{w} \, d\gamma = E_w(0) - E_w(T). \tag{11}$$

One important question is at which rate the energy of solutions of the damped equations (1) and (5) goes to 0 as time goes to ∞. We already give in [2] a direct method to derive sharp energy decay rates for system (1) (resp. (5)), provided that the set ω (resp. Γ_1) satisfies the hypotheses of the piecewise multiplier method [16] (see also [17]) (resp. the multiplier condition [15,18]). Here we are interested in an indirect method based on an observability inequality for the linear undamped system

$$\begin{cases} \ddot{\phi}(t) + A\phi(t) = 0, & t \in (0, T),\ x \in \Omega, \\ \phi(0) = \phi_0, \quad \dot{\phi}(0) = \phi_1. \end{cases} \tag{12}$$

We define the energy of a solution ϕ of (12) by

$$E_\phi(t) = \frac{1}{2}\left(|\dot{\phi}|_X^2 + |\phi|_V^2\right). \tag{13}$$

2. Main results

Before to give our main results, we need to introduce some functions and notation as in [2].

We assume that ρ and a satisfy Assumption (A1) or (A2). Moreover we assume that the function H defined by

$$H(x)=\sqrt{x}g(\sqrt{x}),\quad x\in\left[0,r_0^2\right], \tag{14}$$

is of class C^1 and is strictly convex on $[0,r_0^2]$, where $r_0>0$ is a sufficiently small number. We denote by $\widehat{H}$ the extension of H to $\mathbb{R}$ where $\widehat{H}(x)=+\infty$ for $x\in\mathbb{R}\backslash[0,r_0^2]$. We also define a function F by

$$F(y)=\begin{cases}\frac{\widehat{H}^\star(y)}{y}, & \text{if } y\in(0,+\infty),\\ 0, & \text{if } y=0,\end{cases} \tag{15}$$

where $\widehat{H}^\star$ stands for the convex conjugate function of $\widehat{H}$, i.e.: $\widehat{H}^\star(y)=\sup_{x\in\mathbb{R}}\{xy-\widehat{H}(x)\}$.

We prove in [2] that F is strictly increasing continuous and onto from $[0,+\infty)$ on $[0,r_0^2)$.

We also define a function Λ_H on $(0,r_0^2]$ by

$$\Lambda_H(x)=\frac{H(x)}{xH'(x)}. \tag{16}$$

The main results of this Note are the following:

Theorem 2.1 *(Locally distributed case). Assume that ρ and a satisfy Assumption* (A1) *and that there exists $r_0>0$ sufficiently small so that the function H defined by* (14) *is strictly convex on $[0,r_0^2]$. Assume also either that*

$$0<\liminf_{x\to 0^+}\Lambda_H(x) \tag{17}$$

or

$$\lim_{x\to 0^+}\frac{H'(x)}{\Lambda_H(x)}=0. \tag{18}$$

Moreover let $T>0$ be such that there exists $c_T>0$ for which the following observability inequality is satisfied for the linear conservative system (12)

$$c_T E_\phi(0)\leqslant\int_0^T|\sqrt{a}\dot{\phi}|_X^2\,\mathrm{d}t,\quad\forall(\phi_0,\phi_1)\in V\times X. \tag{19}$$

Then, the energy of the solution of (1) *satisfies*

$$E_w(t)\leqslant 2\beta F\left(\frac{1}{\psi_r^{-1}(\frac{t-T}{T_0})}\right),\quad \textit{for } t \textit{ sufficiently large}, \tag{20}$$

where $F(\frac{1}{\psi_r^{-1}(\frac{t-T}{T_0})})\to 0$ as $t\to\infty$, where β is a positive constant which depends on $E_w(0)$, c_T, and T, and where

$$\psi_r(x)=\frac{1}{H'(r_0^2)}+\int_{\frac{1}{x}}^{H'(r_0^2)}\frac{1}{v^2(1-\Lambda_H((H')^{-1}(v)))}\,\mathrm{d}v,\quad x\geqslant\frac{1}{H'(r_0^2)}. \tag{21}$$

Theorem 2.2 *(Boundary case). We assume that $\overline{\Gamma_0}\cap\overline{\Gamma_1}=\emptyset$ and that ρ and a satisfy Assumption* (A2). *We further assume that there exists $r_0>0$ sufficiently small so that the function H defined by* (14) *is strictly convex on $[0,r_0^2]$ and that either* (17) *or* (18) *holds. Moreover let $T>0$ be such that there exists $c_T>0$ for which the following observability inequality is satisfied for the linear conservative system* (12)

$$c_T E_\phi(0)\leqslant\int_0^T\int_{\Gamma_1}|\sqrt{a}\dot{\phi}|^2\,\mathrm{d}t,\quad\forall(\phi_0,\phi_1)\in D(A)\times V. \tag{22}$$

Then, the energy of the solution of (5) *satisfies* (20).

Remark.

- The function Λ has been introduced for the first time in [3], in which the above energy decay formula has also been simplified by the first author.

An idea of the proof. We use the methods of [2] to prove that under the assumptions of Theorem 2.1 (resp. Theorem 2.2), the sequence $(E_w(kT))_{k\in\mathbb{N}}$ satisfies a nonlinear discrete inequality, linking two subsequent terms of this sequence.

We then prove, following the proof for the continuous case given in [2] and by comparison with an appropriate Euler scheme, that the energy decays as announced in Theorem 2.1 (resp. Theorem 2.2).

3. Examples of PDEs

We illustrate our results on two classical examples, namely the wave equation with a locally distributed, respectively boundary feedback. We refer to [4] for other examples.

3.1. The wave equation with locally distributed damping

We first consider the geometrical situation considered by Lebeau [14,7]. More precisely, (Ω, g) is assumed to be a $\mathcal{C}^\infty$ Riemannian compact and connex manifold, with a boundary of class ∞, whereas $-A$ is the Laplacian on Ω for the metrics g, and $a \in \mathcal{C}^\infty(\overline{\Omega}; [0,\infty))$, and where ρ and a satisfy Assumption (A1).

Thanks to Theorem 0 in [14] and to [7], and applying Theorem 2.1, we deduce that

Theorem 3.1. *Assume that ρ and a satisfy Assumption* (A1) *and that there exists $r_0 > 0$ sufficiently small so that the function H defined by* (14) *is strictly convex on* $[0, r_0^2]$. *Assume also either that* (17) *or* (18) *hold. Assume moreover that the geodesics of $\overline{\Omega}$ have no contact of infinite order with Γ and that there exists a time $T_- > 0$ such that every generalized geodesics of Ω of length larger than T_- meets w. Then, the energy of the solution of* (1) *satisfies* (20).

3.2. The wave equation with boundary damping

We consider (5) with $\sigma(x) = m(x)\cdot\nu(x)l(x)$ on Γ_1 with $l \in L^\infty(\Gamma_1)$, $l \geqslant l_-$ on Γ_1, where l_- is a positive real number. Then, we have the following result:

Theorem 3.2. *We assume that $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$ and that ρ and a satisfy Assumption* (A2). *We assume the multiplier geometric condition for Γ_1, that is we assume that there exists $x_0 \in \mathbb{R}^N$, such that*

$$m\cdot\nu \leqslant 0 \quad \text{on } \Gamma_0,$$
$$m\cdot\nu > 0 \quad \text{on } \Gamma_1,$$

where $m(x) = x - x_0$ for $x \in \mathbb{R}^N$.

We further assume that there exists $r_0 > 0$ sufficiently small so that the function H defined by (14) *is strictly convex on* $[0, r_0^2]$ *and that either* (17) *or* (18) *holds. Then, the energy of the solution of* (5) *satisfies* (20).

Remarks.

- Burq [8] and Burq and Gérard [9] proved an observability inequality under more general geometrical conditions than the ones given by the multiplier method, for a wave equation as in (5), but with Dirichlet boundary conditions on all Γ instead of mixed Dirichlet–Robin boundary conditions. It would be interesting to know if their method can provide more general geometrical conditions than the ones stated in Theorem 3.2, by means of microlocal analysis and defect measures for wave equation with mixed Dirichlet–Robin type boundary conditions.
- We refer to [2,3] for examples of nonlinear feedbacks.

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