Available online at www.sciencedirect.com

ScienceDirect

C. R. Acad. Sci. Paris, Ser. I 347 (2009) 953–958

COMPTES RENDUS MATHEMATIQUE

Dynamical Systems/Probability Theory

Some optimal pointwise ergodic theorems with rate ☆

Christophe Cuny

Université de la Nouvelle-Calédonie, Équipe ERIM, B.P. R4, 98800 Nouméa, New Caledonia

Received 15 February 2009; accepted 29 April 2009

Available online 3 June 2009

Presented by Wendelin Werner

Abstract

Let T be a Dunford–Schwartz operator on the probability space (X, Σ, μ) and $p > 1$. For f in the range of suitable operators of $L^p(X, \Sigma, \mu)$, we obtain pointwise ergodic theorems with rate, using a method of Derriennic and Lin (2001). When T is induced by a μ-preserving transformation, these results are shown to be optimal. The proof of the optimality is inspired from a construction of Déniel (1989). ***To cite this article: C. Cuny, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Théorèmes ergodiques ponctuels avec vitesse optimale. Soit T un opérateur de Dunford–Schwartz sur l'espace de probabilité (X, Σ, μ) et $p > 1$. Pour f dans l'image d'opérateurs judicieux de $L^p(X, \Sigma, \mu)$, nous obtenons des théorèmes ergodiques ponctuels avec vitesse, par une méthode due à Derriennic et Lin (2001). Lorsque T est induit par une transformation préservant μ, nous montrons l'optimalité des résultats, la preuve étant inspirée par une construction de Déniel (1989). ***Pour citer cet article : C. Cuny, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Version française abrégée

Soit T un opérateur de Dunford–Schwartz sur l'espace de probabilité (X, Σ, μ) (i.e. T contracte chaque espace $L^r(X, \mu)$, $1 \leqslant r \leqslant \infty$). Pour $\alpha \in]0, 1[$, Derriennic et Lin [5] ont utilisé le développement en série entière $(1-t)^\alpha = 1 - \sum_{n\geqslant 1} a_n t^n$, où $a_n = a_n^{(\alpha)} > 0$ et $\sum_{n\geqslant 1} a_n = 1$, pour définir l'opérateur $(I-T)^\alpha$ sur $L^p(X, \mu)$, par

$$(I-T)^\alpha = I - \sum_{n\geqslant 1} a_n T^n.$$

Pour $f \in (I-T)^\alpha L^p(X, \mu)$, des théorèmes ergodiques ponctuels avec vitesse sont établis dans [5]. Cette approche a ensuite été développée par Zhao et Woodroofe [10], puis dans [3], en considérant des séries entières plus générales. Dans ces derniers travaux, le seul cas $p = 2$ a été traité, en raison des applications en vue. Dans cette note nous montrons que la méthode s'applique aisément à tout $p > 1$. Puis, nous montrons que les vitesses de convergence

☆ Research partially carried out at Ben-Gurion University, supported by its Center for Advanced Studies in Mathematics.

E-mail address: cuny@univ-nc.nc.

1631-073X/$ – see front matter
doi:10.1016/j.crma.2009.04.034

obtenues dans le théorème ergodique ponctuel sont, en un certain sens, optimales. Les résultats sont particulièrement précis dans le cas $p = 2$. La preuve de l'optimalité s'inspire d'une construction due à Déniel [4]. Par exemple, en corollaire nous obtenons

Théorème. *Soit* (X, Σ, μ, θ) *un système dynamique (non nécessairement ergodique). Si* $f \in L^2(X, \mu)$ *satisfait* $\sum_{n\geqslant 2} \log n(\log\log n)^{1+\varepsilon} \frac{\|f\circ\theta+\cdots+f\circ\theta^n\|_2^2}{n^2} < +\infty$, *pour* $\varepsilon > 0$, *alors*

$$\frac{1}{\sqrt{n}}\sum_{k=1}^{n} f\circ\theta^k \xrightarrow[n\to+\infty]{} 0 \quad \mu\text{-a.s.}$$

Par contre, sur tout sytème dynamique ergodique, inversible et non atomique (X, Σ, μ, θ), *il existe* $f \in L^2(X, \mu)$ *(centrée), satisfaisant la condition* $\sum_{n\geqslant 2} \log n \log\log n \frac{\|f\circ\theta+\cdots+f\circ\theta^n\|_2^2}{n^2} < +\infty$, *mais telle que* $\limsup_{n\to+\infty} \frac{1}{\sqrt{n}} \times |\sum_{k=1}^{n} f\circ\theta^k| = +\infty$ μ*-a.s.*

1. Introduction

Let T be a Dunford–Schwartz operator on the probability space (X, Σ, μ) (i.e. T is a contraction of each $L^r(X, \mu)$, $1 \leqslant r \leqslant \infty$). For $0 < \alpha < 1$, Derriennic and Lin [5] used the power series expansion $(1-t)^\alpha = 1 - \sum_{n\geqslant 1} a_n t^n$, where $a_n = a_n^{(\alpha)} > 0$ and $\sum_{n\geqslant 1} a_n = 1$, to define the operator $(I - T)^\alpha$ on $L^p(X, \mu)$ by

$$(I-T)^\alpha = I - \sum_{n\geqslant 1} a_n T^n.$$

For $p > 1$ and $f \in (I - T)^\alpha L^p(X, \mu)$, pointwise ergodic theorems with rate were obtained in [5]. In this note we use generalized power series of operators (as in Zhao–Woodroofe [10] or [3]) to obtain more precise rates. Moreover, inspired by a construction of Déniel [4], we show the optimality of the obtained rates.

2. The results

Let b be a slowly varying function (as in Zygmund [11, p. 186]) and fix $\alpha \in]0, 1[$. We consider the series $B(z) := \sum_{n\geqslant 1} \beta_n z^n$, where $\beta_n = \frac{C}{n}\sum_{k\geqslant n} \frac{b(k)}{k^{1+\alpha}}$ (notice that $\sum_{k\geqslant n} \frac{b(k)}{k^{1+\alpha}} \sim \frac{Cb(n)}{\alpha n^\alpha}$), and C is such that $\sum_{n\geqslant 1} \beta_n = 1$. The series defining $B(z)$ is absolutely convergent on $\overline{D}$ ($D := \{z \in \mathbf{C}\colon |z| < 1\}$) and defines a continuous function on $\overline{D}$, analytic on D. Moreover, $B(z)$ is a (strict) convex combination of complex numbers of $\overline{D}$, hence $B(z) = 1 \Leftrightarrow z = 1$ and the function $A(z) := 1/(1 - B(z))$ is well defined, continuous on $\overline{D} - \{1\}$, analytic on D. So, there exists $\{\alpha_n\}_{n\in\mathbf{N}} \subset \mathbf{C}$, such that

$$A(z) = \sum_{n\geqslant 0} \alpha_n z^n \quad \forall z \in D.$$

For a power bounded operator T on a Banach space $\mathcal{B}$, the operator $B(T) := \sum_{n\geqslant 1} \beta_n T^n$ is well-defined. Define also $A_n(T) := \sum_{k=0}^{n} \alpha_k T^k$, $n \in \mathbf{N}$. For a given T we denote $B = B(T)$ and $A_n = A_n(T)$.

For $f \in \mathcal{B}$ such that $\{A_n f\}$ converges in $\mathcal{B}$, we denote by $A(f)$ its limit and say that $A(f)$ converges. The following is proved in [2] or [5] for some special cases:

Proposition 2.1. *Let* T *be a power bounded operator on a Banach space* $\mathcal{B}$. *If* $f \in \mathcal{B}$ *is such that* $A(f)$ *converges in* $\mathcal{B}$, *then* f *and* $h := A(f)$ *are in* $\overline{(I-T)\mathcal{B}}$ *and satisfy* $f = (I - B)h$.

Conversely, if $f \in (I - B)\overline{(I-T)\mathcal{B}}$, *then* $A(f)$ *converges.*

In the sequel, T will be a Dunford–Schwartz operator on (X, Σ, μ) and $\mathcal{B}$ will be $L^p(X, \mu)$, with $p > 1$. In order to find conditions for the convergence of $A(f)$ we need the following, proved in [3] for $\alpha = 1/2$:

Proposition 2.2. *Let* A *be as above. Then* $\alpha_n > 0$, $\forall n \geqslant 0$, *and there exist* $L_\alpha, K > 0$, *such that*

(i) $|\alpha_n - \alpha_{n+1}| = 0(\frac{1}{b(n)n^{2-\alpha}})$;
(ii) $|A(z)| \underset{1}{\sim} \frac{L_\alpha}{b(\frac{1}{|1-z|})|1-z|^\alpha}$;
(iii) $\sup_{n\geqslant 0} |\sum_{k=0}^n \alpha_k z^k| \leqslant \frac{K}{b(\frac{1}{|1-z|})|1-z|^\alpha}$, $\forall z \in \overline{D} - \{1\}$.

Proposition 2.3 below may be proved using Abel summation by parts and (i) of Proposition 2.2. For $n \geqslant 1$, write $S_n(f) := \sum_{k=1}^n T^k f$.

Proposition 2.3. *Let T be as above. Let $f \in L^p(X, \mu)$, b be a slowly varying function and $\alpha \in]0, 1[$. Then $f \in (I - B)L^p(X, \mu)$, whenever f satisfies the condition*

$$(C_p) \qquad \sum_{n\geqslant 1} \frac{\|S_n(f)\|_p}{n^{2-\alpha} b(n)} < +\infty.$$

We recall that an operator T on $L^2(X, \mu)$ is normal if $T^*T = TT^*$. The next proposition follows from Lemma 2.1 and Proposition 2.3 of [3] (see also the proof of Theorem 3.3 there) using (ii) and (iii) of Proposition 2.2:

Proposition 2.4. *Let (X, Σ, μ) be a probability space and T be a Dunford–Schwartz operator, whose restriction to $L^2(X, \mu)$ is normal. Let b be a slowly varying function and $\alpha \in]0, 1[$. Then $f \in (I - B)L^2(X, \mu)$ if and only if*

$$(C_2') \qquad \sum_{n\geqslant 1} \frac{\|S_n(f)\|_2^2}{n^{3-2\alpha} b(n)^2} < +\infty.$$

For $p > 1$, define the dual index $q := p/(p - 1)$. Our main results are the following.

Theorem 2.5. *Let T be a Dunford–Schwartz operator on (X, Σ, μ). Let b be a slowly varying function, $\alpha \in]0, 1[$, $p > 1$, and B as above. Let $f \in (I - B)L^p(X, \Sigma, \mu)$ (e.g. f satisfies (C_p)). Then*

$$\frac{\sum_{k=1}^n T^k f}{n^{1/p}(\sum_{k=1}^n \frac{b(k)^q}{k^{q\alpha}})^{1/q}} \xrightarrow[n\to+\infty]{} 0 \quad \mu\text{-a.s.} \tag{1}$$

Moreover, if (X, Σ, μ) is non-atomic and T is induced by an ergodic invertible measure preserving transformation θ, then for every positive function ψ satisfying $\lim_{n\to+\infty} \frac{(\sum_{k=1}^n \frac{b(k)^q}{k^{q\alpha}})^{1/q}}{\psi(n)} = +\infty$, there exists $f \in (I - B)L^p(X, \Sigma, \mu)$ (hence $\int_X f \, d\mu = 0$) such that

$$\limsup_{n\to+\infty} \frac{|\sum_{k=1}^n f \circ \theta^k|}{n^{1/p}\psi(n)} = +\infty \quad \mu\text{-a.s.} \tag{2}$$

It is possible to precise the rate in (1) according to the value of $q\alpha$.

Theorem 2.6. *Let T be a Dunford–Schwartz operator on (X, Σ, μ). Let b be a slowly varying function, $\alpha \in]0, 1[$, $p > 1$, and B as above. Let $f \in (I - B)L^p(X, \Sigma, \mu)$ (e.g. f satisfies (C_p)). Then*

(i) *If $1 - \alpha > 1/p$, $\frac{\sum_{k=1}^n T^k f}{n^{1-\alpha} b(n)} \xrightarrow[n\to+\infty]{} 0$ μ-a.s.*
(ii) *If $1 - \alpha = 1/p$, $\frac{\sum_{k=1}^n T^k f}{n^{1/p}(\sum_{k=1}^n \frac{b(k)^q}{k})^{1/q}} \xrightarrow[n\to+\infty]{} 0$ μ-a.s.*
(iii) *If $1 - \alpha < 1/p$, $\frac{\sum_{k=1}^n T^k f}{n^{1/p}} \xrightarrow[n\to+\infty]{} 0$ μ-a.s.*

Remark 1. When $b \equiv 1$ we recover Theorem 3.2 of [5], hence Theorem 2.5 shows the optimality of Theorem 3.2 of [5] in the above sense. Weber [9] and Cohen–Lin [1] obtained pointwise ergodic theorems with rate, in the context of power-bounded operators in $L^p(X,\mu)$. The use of condition (C_p) in Theorem 2.6 yields in case (iii) a better rate than that of [9] or [1]; but, our rate is not as good. A similar discussion holds in case (ii) according to the chosen function b.

Theorem 2.7. *Let T be a Dunford–Schwartz operator on (X,Σ,μ), which is normal on $L^2(X,\mu)$. Let b be a slowly varying function, $\alpha\in]0,1[$ and B as above. Let $f\in L^2(X,\Sigma,\mu)$ satisfying (C_2'). Then*

(i) *If $\alpha<1/2$, $\frac{\sum_{k=1}^n T^k f}{n^{1-\alpha}b(n)}\xrightarrow[n\to+\infty]{}0$ μ-a.s.*
(ii) *If $\alpha=1/2$, $\frac{\sum_{k=1}^n T^k f}{\sqrt{n}(\sum_{k=1}^n \frac{b(k)^2}{k})^{1/2}}\xrightarrow[n\to+\infty]{}0$ μ-a.s.*
(iii) *If $\alpha>1/2$, $\frac{\sum_{k=1}^n T^k f}{\sqrt{n}}\xrightarrow[n\to+\infty]{}0$ μ-a.s.*

Remark 2. As in Theorem 2.5, the rates obtained are optimal under condition (C_2'). Our rate in (i) is essentially the same as that obtained by Gaposhkin [6] for unitary operators on $L^2(X,\mu)$.

In the case $p=2$, it is also possible to give optimal conditions on f to obtain a specific rate. For example, for T induced by a measure-preserving transformation, we have

Theorem 2.8. *Let (X,Σ,μ,θ) be a dynamical system, with μ a probability. Let b_0 be any slowly varying function with $\sum_{n\geqslant1}\frac{b_0(n)^2}{n}<+\infty$. Then for every $f\in L^2(X,\mu)$ such that $\sum_{n\geqslant1}\frac{\|S_n(f)\|_2^2}{n^2b_0(n)^2}<+\infty$, we have*

$$\frac{1}{\sqrt{n}}S_n(f)\xrightarrow[n\to+\infty]{}0\quad\mu\text{-a.s.}\tag{3}$$

Moreover, the series $\sum_{n\geqslant1}\frac{f\circ\theta^n}{\sqrt{n}}$ converges μ-a.s.

On the other hand, if θ is invertible and the system is ergodic and non-atomic, for every slowly varying function b_1 with $\sum_{n\geqslant1}\frac{b_1(n)^2}{n}=+\infty$, there exists a function $f\in L^2(X,\mu)$ such that $\sum_{n\geqslant1}\frac{\|S_n(f)\|_2^2}{n^2b_1(n)^2}<+\infty$ (hence $\int_X f\,d\mu=0$) and $\limsup|\frac{1}{\sqrt{n}}S_n(f)|=+\infty$ μ-a.s.

By Proposition 2.4, with $\alpha=1/2$, the convergence of $\sum_{n\geqslant1}\frac{\|S_n(f)\|_2^2}{n^2b_i(n)^2}$ ($i\in\{0,1\}$) is equivalent to the fact that $f\in(I-B)L^2(X,\mu)$ for the corresponding b_i. Theorem 2.8 then becomes a direct application of Theorem 2.5.

Remark 3. For example, take in Theorem 2.8, $b_0=\frac{1}{\sqrt{\log n(\log\log n)^{1+\varepsilon}}}$, for $\varepsilon>0$, and $b_1=\frac{1}{\sqrt{\log n\log\log n}}$. Then the condition $\sum_{n\geqslant2}\log n(\log\log n)^{1+\varepsilon}\frac{\|S_n(f)\|_2^2}{n^2}<+\infty$ is sufficient for (3), but, in general, the condition

$$\sum_{n\geqslant2}\log n\log\log n\frac{\|S_n(f)\|_2^2}{n^2}<+\infty$$

is not. Theorem 2.8 has applications in probability, see [3].

3. Proof of Theorem 2.5

Let $f\in(I-B)L^p(X,\mu)$. There exists $h\in L^p(X,\mu)$ such that $f=(I-B)h$, and we may and do assume that $h\in\overline{(I-T)L^p(X,\mu)}$, since T is mean ergodic on $L^p(X,\mu)$ and $B(1)=1$. It therefore suffices to show that, for every $h\in\overline{(I-T)L^p(X,\mu)}$

$$\frac{1}{n^{1/p}(\sum_{k=1}^n\frac{b(k)^q}{k^{q\alpha}})^{1/q}}\sum_{k=1}^n T^k\big(I-B(T)\big)h\xrightarrow[n\to+\infty]{}0\quad\mu\text{-a.s.}\tag{4}$$

For $n \geqslant 1$ write $\sum_{k=1}^{n} T^k(I-B) = C_n - D_n - E_n$, where

$$C_n = T + \sum_{m=2}^{n}\Big(\sum_{k\geqslant m}\beta_k\Big)T^m, \qquad E_n = \sum_{k=1}^{n}\Big(\sum_{m\geqslant 2n+1}\beta_{m-k}\Big)T^m$$

and

$$D_n = \sum_{m=n+1}^{2n}\sum_{k=m-n}^{m-1}\beta_k T^m = \sum_{l=1}^{n}\Big(\sum_{k=l}^{l+n-1}\beta_k\Big)T^{l+n}.$$

Hence it suffices to study separately the operator sequences $\{C_n\}$, $\{D_n\}$ and $\{E_n\}$ on $L^p(X,\mu)$. The first part of Theorem 2.5 will follow from the next propositions, which may be proved as in [5, Theorem 3.2].

Proposition 3.1. *Let T be a Dunford–Schwartz operator on a probability space (X,Σ,μ). Let $\alpha \in]0,1[$, b be any slowly varying function and B as above. Then, for every $h \in L^p(X,\mu)$*

$$\sup_{n\geqslant 1}\frac{|C_n(h)| + |E_n(h)|}{n^{1-\alpha}b(n)} < +\infty \quad \mu\text{-a.s.}$$

Proposition 3.2. *Let T be a Dunford–Schwartz operator on a probability space (X,Σ,μ). Let $\alpha \in]0,1[$, b be any slowly varying function and B as above. Then, for every $h \in L^p(X,\mu)$*

$$\sup_{n\geqslant 1}\frac{|D_n(h)|}{n^{1/p}(\sum_{k=1}^{n}\frac{b(k)^q}{k^{q\alpha}})^{1/q}} < +\infty \quad \mu\text{-a.s.} \tag{5}$$

One can see that there exists $K > 0$ such that $n^{1/p}(\sum_{k=1}^{n}\frac{b(k)^q}{k^{q\alpha}})^{1/q} \geqslant K n^{1-\alpha}b(n)$. Hence an application of Banach's principle (see e.g. [7, Theorem 7.2a, p. 64]) yields that the set of functions of $L^p(X,\mu)$ satisfying (4) is closed in $L^p(X,\mu)$. It is not difficult to check that (4) is true for $f \in (I-T)L^p(X,\mu)$, hence the first part of the theorem is proved.

Let us prove the second part of Theorem 2.5. By Banach's principle (see [7, Theorem 7.2b, p. 64]), it suffices to show that there does not exist positive decreasing function χ on $[0,+\infty[$, with $\lim_{\lambda\to+\infty}\chi(\lambda) = 0$, such that for every $f \in L^p(X,\mu)$ we have

$$\mu\bigg(\bigg\{x \in X\colon \sup_{n\geqslant 1}\frac{|\sum_{k=1}^{n}(I-B)f\circ\theta^k|}{n^{1/p}\psi(n)} \geqslant \lambda\|f\|_p\bigg\}\bigg) \leqslant \chi(\lambda) \quad \forall\lambda > 0.$$

Hence it suffices to find $\delta > 0$, $L_m \xrightarrow[m\to+\infty]{} +\infty$, and $\{f_m\} \subset L^p(X,\mu)$ with $\sup_{m\geqslant 1}\|f_m\|_p < +\infty$, such that

$$\mu\bigg(\bigg\{x \in X\colon \sup_{n\geqslant 1}\frac{|\sum_{k=1}^{n}(I-B)f_m\circ\theta^k|}{n^{1/p}\psi(n)} \geqslant L_m\bigg\}\bigg) \geqslant \delta \quad \forall m \geqslant 1. \tag{6}$$

Using that b is slowly varying one can show that $\beta_n \sim \frac{Cb(n)}{\alpha n^{1+\alpha}}$ and that there exists $D > 0$ such that for every $n \geqslant 1$ and $l \in \{1,\dots,n\}$, $\sum_{k=l}^{l+n-1}\beta_k \geqslant D\frac{b(l)}{l^\alpha}$. Hence for every non-negative measurable function f in $L^p(X,\mu)$,

$$D_n(f) \geqslant D\sum_{l=1}^{n}\frac{b(l)}{l^\alpha}f\circ\theta^{l+n}. \tag{7}$$

The following construction is inspired by Déniel [4].

Let $n \geqslant 1$. By Rokhlin's Lemma (see e.g. [8, Lemma 4.7, p. 48]), there exists a set $Y_n \subset \Sigma$, such that the sets $\{\theta^k(Y_n)\}_{1\leqslant k\leqslant 2n}$ are disjoint and $\mu(X - \bigcup_{k=1}^{2n}\theta^k(Y_n)) < \frac{1}{2n+1}$. In particular, for every $k \in \{1,\dots,2n\}$, $\frac{1}{2n+1} \leqslant \mu(Y_n) = \mu(\theta^k(Y_n)) \leqslant \frac{1}{2n}$.

For every $n \geqslant 1$, define $u_n := \sum_{k=1}^{n}\frac{b(k)^q}{k^{q\alpha}}$ and a non-negative function f_n on X by $f_n(x) = 0$ for $x \in X - \bigcup_{k=n+1}^{2n}\theta^k(Y_n)$, and $f_n(x) = (\frac{b(k-n)}{(k-n)^\alpha})^{q/p}(\frac{n}{u_n})^{1/p}$ if $x \in \theta^k(Y_n)$, for some $k \in \{n+1,\dots,2n\}$.

Then $\{f_n\}$ is bounded in $L^p(X,\mu)$. Indeed, we have

$$\|f_n\|_p^p = \sum_{k=n+1}^{2n} \|f_n \mathbf{1}_{\theta^k(Y_n)}\|_p^p = \frac{n}{u_n} \sum_{k=n+1}^{2n} \left(\frac{b(k-n)}{(k-n)^\alpha}\right)^q \mu(Y_n) \leqslant 1/2.$$

Let $0 \leqslant j \leqslant n-1$ and take $x \in \theta^j(Y_n)$. Let $y \in Y_n$, such that $x = \theta^j(y)$. We have

$$\begin{aligned} D_{n-j}(f)(x) &\geqslant D \sum_{l=1}^{n-j} \frac{b(l)}{l^\alpha} f_n\big(\theta^{l+n-j}(x)\big) \geqslant D \sum_{l=1}^{n-j} \frac{b(l)}{l^\alpha} f_n\big(\theta^{n+l}(y)\big) \\ &= D\left(\frac{n}{u_n}\right)^{1/p} \sum_{l=1}^{n-j} \frac{b(l)}{l^\alpha} \left(\frac{b(l)}{l^\alpha}\right)^{q/p} = D\left(\frac{n}{u_n}\right)^{1/p} u_{n-j}. \end{aligned}$$

Using that b is slowly varying, one can see that there exists $K > 0$ such that for every $n \geqslant 1$, and every $j \leqslant \frac{n}{2}$, $u_{n-j} \geqslant K u_n$.

Hence, noticing that $C_{n-j}(f_n)(x) = 0$ and $E_{n-j}(f_n)(x) \geqslant 0$, we obtain, for every $0 \leqslant j \leqslant n/2$ and $x \in \theta^j(Y_n)$,

$$\left|\sum_{k=1}^{n-j} (I-B)(f_n)\big(\theta^k(x)\big)\right| \geqslant \frac{D}{K^{1/p}} \frac{u_{n-j}^{1/q}}{\psi(n-j)} (n-j)^{1/p} \psi(n-j).$$

So, on the set $\bigcup_{0 \leqslant j \leqslant n/2} \theta^j(Y_n)$ whose measure is greater than $\frac{n/2}{2n+1} \sim \frac{1}{4}$,

$$\sup_{r \geqslant 1} \frac{|\sum_{k=1}^r (I-B)(f_n) \circ \theta^k|}{r^{1/p}\psi(r)} \geqslant \frac{D}{K^{1/p}} \inf_{s \geqslant n/2} \frac{u_s^{1/q}}{\psi(s)} \xrightarrow[n \to +\infty]{} +\infty,$$

which proves (6).

References

[1] G. Cohen, M. Lin, Extensions of the Menchoff–Rademacher theorem with applications to ergodic theory, Israel J. Math. 148 (2005) 41–86.
[2] C. Cuny, On the a.s. convergence of the one-sided ergodic Hilbert transform, Ergodic Theory Dynam. Systems, in press.
[3] C. Cuny, Pointwise ergodic theorems with rate and application to limit theorems for stationary processes, preprint.
[4] Y. Déniel, On the a.s. Cesaro-α convergence for stationary or orthogonal random variables, J. Theoret. Probab. 2 (4) (1989) 475–485.
[5] Y. Derriennic, M. Lin, Fractional Poisson equations and ergodic theorems for fractional coboundaries, Israel J. Math. 123 (2001) 93–130.
[6] V.F. Gaposhkin, The dependence of the rate of convergence in the strong law of large numbers for stationary processes on the rate of diminution of the correlation function, Teor. Veroyatnost. i Primenen. 26 (4) (1981) 720–733.
[7] U. Krengel, Ergodic Theorems, de Gruyter Studies in Mathematics, vol. 6, Walter de Gruyter & Co., Berlin, 1985.
[8] K. Petersen, Ergodic Theory, Cambridge Studies in Advanced Mathematics, vol. 2, Cambridge University Press, Cambridge, 1983.
[9] M. Weber, Uniform bounds under increment conditions, Trans. Amer. Math. Soc. 358 (2) (2006) 911–936.
[10] O. Zhao, M. Woodroofe, Law of the iterated logarithm for stationary processes, Ann. Probab. 36 (2008) 127–142.
[11] A. Zygmund, Trigonometric Series, corrected 2nd ed., Cambridge University Press, 1968.