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Local smoothing effects for the water-wave problem with surface tension

Hans Christianson, Vera Mikyoung Hur, Gigliola Staffilani

Massachusetts Institute of Technology, Department of Mathematics 77, Massachusetts Avenue, Cambridge, MA 02139-4307, USA

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Abstract

The water-wave problem with a one-dimensional free surface of infinite depth is considered, based on the formulation as a second-order nonlinear dispersive equation. The local smoothing effects are established under the influence of surface tension, stating that on average in time solutions acquire locally $1/4$ derivative of smoothness as compared to the initial state. The analysis combines energy methods with techniques of Fourier integral operators. ***To cite this article: H. Christianson et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Effets de lissage locaux pour le problème des ondes avec tension superficielle. Nous considérons le problème des ondes avec une surface libre unidimensionnelle, de profondeur infinie, en utilisant sa formulation comme une équation non linéaire dispersive du second ordre. Nous mettons en évidence un effet de lissage local sous l'influence de la tension superficielle : en moyenne au fil du temps, les solutions acquièrent localement $1/4$ de dérivée en plus de la régularité de l'état initial. L'analyse combine des méthodes d'énergie avec des techniques d'opérateurs Fourier intégraux. ***Pour citer cet article : H. Christianson et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

1. Introduction

The *water-wave problem* in its simplest form concerns the two-dimensional dynamics of an incompressible inviscid liquid of infinite depth and the wave motion on its one-dimensional surface layer, under the influence of gravity and surface tension. The *moving* interface is given as a nonself-intersecting parametrized curve. The liquid occupies the domain below the interface, where the liquid motion is described by the Euler equations under gravity. The flow beneath the interface is required to be irrotational. The *kinematic* and *dynamic* boundary conditions hold at the moving interface, stating respectively that the normal component of velocity is continuous along the interface and that the jump in pressure across the interface is proportional to its mean curvature. The flow is assumed to be almost at rest at great depths, and the interface is taken to be asymptotically flat.

E-mail addresses: hans@math.mit.edu (H. Christianson), verahur@math.mit.edu (V.M. Hur), gigliola@math.mit.edu (G. Staffilani).

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Provided with the initial wave profile and the initial state of fluid current, the water-wave problem naturally poses as an *initial value problem*. Early mathematical results for its local well-posedness date back to [16,17] and they include [7,10,23,24]. Following the works by Wu [21,22] in recent years there has been considerable progress in the study of local well-posedness for, more generally, a class of the Euler equations with free boundary; we refer to [1,2,4,6,14, 15,18], and references therein. This progress is a consequence of the development of several different approaches to obtaining high energy expressions in the nonlinear problem and showing local existence by establishing bounds for these expressions. While the so-called *energy method* successfully yields local well-posedness, nonetheless it does not provide any further information about solutions, other than that they remain as smooth as their initial states.

On the other hand, the *dispersion relation* of the water-wave problem

$$c(k) = \left(\frac{S}{2}|k| + \frac{g}{|k|}\right)^{1/2} \frac{k}{|k|} \tag{1}$$

provides a useful guiding principle in the *linear* dynamics. Here, $c(k)$ is the speed of the simple harmonic oscillation with the wave length $2\pi/k$; $S \geqslant 0$ is the coefficient of surface tension and $g \geqslant 0$ is the gravitational constant of acceleration. Indeed, in the presence of the effects of surface tension, i.e. $S > 0$, the formula suggests a "regularizing" effect by the process of broadening out the wave profile. The dispersive property of gravity waves, i.e. $S = 0$ and $g > 0$, in contrast does not induce a regularizing effect. Taking this further, one can prove the local smoothing effect for the linear water-wave problem with surface tension (see (5) below). A natural question then is whether the *nonlinear* problem will inherit from the linear one a similar smoothing effect, which is the subject of investigation here.

2. The main result

Our treatment of the water-wave problem (with surface tension) is based on the formulation of the problem as a second-order in time nonlinear dispersive equation as

$$\partial_t^2 u - \frac{S}{2} H\partial_\alpha^3 u + gH\partial_\alpha u = -2u\partial_t\partial_\alpha u - u^2\partial_\alpha^2 u + R(u, \partial_t u). \tag{2}$$

Here, u is related to the tangential velocity at the interface; $t \in \mathbb{R}_+$ is the temporal variable and $\alpha \in \mathbb{R}$ is the arclength parametrization of the interface, which serves as the spatial variable; ∂ means partial differentiation. The Hilbert transform H may be defined via the Fourier transform as $\widehat{Hf}(\xi) = -i\,\mathrm{sgn}(\xi)\hat{f}(\xi)$. The remainder R is of lower order compared to $2u\partial_t\partial_\alpha u$ and $u^2\partial_\alpha^2 u$ in the sense that

$$\big\|R(u, \partial_t u)\big\|_{H^s} \leqslant C\big(\|u\|_{H^{s+1}}, \|\partial_t u\|_{H^s}\big)$$

for $s \geqslant 1$. Here and elsewhere, H^s means the Sobolev space of order s in the variable $\alpha \in \mathbb{R}$.

One obvious advantage of (2) is that its dispersive character is more pronounced. Indeed, the left side of (2) has symbol $-\tau^2 + \frac{S}{2}|\xi|^3 + g|\xi|$, where τ and ξ are the Fourier variables corresponding to t and α, respectively. Another more subtle advantage is that it suggests a natural expression for nonlinear energy.

Our main result concerns a local smoothing effect for the water-wave problem with surface tension.

Theorem. *Let $S > 0$ and $g \geqslant 0$ be held fixed. For $s > 2 + 1/2$ the initial value problem of* (2) *with the initial conditions $u(0, \alpha) = u_0(\alpha)$ and $\partial_t u(0, \alpha) = u_1(\alpha)$, where $(u_0, u_1) \in H^s(\mathbb{R}) \times H^{s-3/2}(\mathbb{R})$, is locally well-posed on the interval $t \in [0, T_0]$ for some $T_0 > 0$ and $(u(t), \partial_t u(t)) \in C([0, T]; H^s(\mathbb{R}) \times H^{s-3/2}(\mathbb{R}))$.*

Moreover, if $s \geqslant s_0 > 1$ is sufficiently large, then for $0 < T < T_0$ sufficiently small, the inequality

$$\int_0^T \int_{-\infty}^{\infty} \big|\langle\alpha\rangle^{-\rho} D_\alpha^{s+1/4} u(t, \alpha)\big|^2 \,\mathrm{d}\alpha\,\mathrm{d}t \leqslant C \tag{3}$$

holds, where $\rho \geqslant 3$ and $C > 0$ depends only on T and the Sobolev norms of the initial data. Here, $\langle\alpha\rangle = (1 + \alpha^2)^{1/2}$ is to describe the weighted Sobolev spaces and $D_\alpha = -\mathrm{i}\partial_\alpha$.

Kato in [11] first deduced a local smoothing result for the Korteweg–de Vries equation. The local smoothing effect of the kind in (3) is a common property of a general class of dispersive equations. It has been studied perhaps most extensively for the Schrödinger equation in the constant-coefficient setting [5,19,20] as well as in the variable-coefficient setting [3,8,9,13], to mention only a few of the results.

3. Ideas of the proof

Our proof of (3) is motivated by the local smoothing effect for the linear dispersive part of (2). It is standard by techniques of oscillatory integrals ([12], for instance) to show that when $S > 0$ the solution to the initial value problem of the linear homogeneous equation

$$\partial_t^2 u - \frac{S}{2} H \partial_\alpha^3 u + g H \partial_\alpha u = 0, \quad u(0,\alpha) = u_0(\alpha) \quad \text{and} \quad \partial_t u(0,\alpha) = u_1(\alpha) \tag{4}$$

possesses the local smoothing estimate

$$\sup_{\alpha\in\mathbb{R}} \left(\int_{-\infty}^{\infty} \left| D_\alpha^{1/4} u(t,\alpha) \right|^2 \mathrm{d}t \right)^{1/2} \leqslant C \left(\|u_0\|_{L^2_\alpha(\mathbb{R})} + \|u_1\|_{H_\alpha^{-3/2}(\mathbb{R})} \right). \tag{5}$$

Moreover, the solution to the corresponding inhomogeneous problem

$$\partial_t^2 v - \frac{S}{2} H \partial_\alpha^3 v + g H \partial_\alpha v = R(t,\alpha), \qquad v(0,\alpha) = 0 = \partial_t v(0,\alpha)$$

exhibits the estimate

$$\sup_{\alpha\in\mathbb{R}} \left(\int_{-\infty}^{\infty} \left| D_\alpha^{2} v(t,\alpha) \right|^2 \mathrm{d}t \right)^{1/2} \leqslant C \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \left| R(t,\alpha) \right|^2 \mathrm{d}t \right)^{1/2} \mathrm{d}\alpha.$$

The main difficulty of the proof is that the smoothing effect of the linear part of (2) is too weak to control the nonlinearity. In the application to our setting in (2), the above results say that the smoothing effect of the linear part of (2) can treat up to 2 derivatives in the inhomogeneous nonlinear terms. However, the worst nonlinear term $u\partial_t\partial_\alpha u$ in (2) contains $2 + 1/2$ derivatives (∂_t is comparable to $\partial_\alpha^{3/2}$). In other words, the water-wave problem under surface tension is *strongly nonlinear but only weakly dispersive*.

To overcome this difficulty, we view (2) as

$$\partial_t^2 u - \frac{S}{2} H \partial_\alpha^3 u + g H \partial_\alpha u + 2u\partial_t\partial_\alpha u + u^2 \partial_\alpha^2 u = R(u, \partial_t u).$$

That means, we view $2u\partial_t\partial_\alpha u$ and $u^2\partial_\alpha^2 u$ as "linear" components of the equation, but with variable coefficients which happen to depend on the solution itself. In effect, we reduce the size of nonlinearity at the expense of making the linear part more complicated. We then make a serious effort to establish the local smoothing effect for, more generally, the variable-coefficient linear equation

$$\partial_t^2 u - \frac{S}{2} H \partial_\alpha^3 u + g H \partial_\alpha u + 2V(t,\alpha)\partial_\alpha\partial_t u + V^2(t,\alpha)\partial_\alpha^2 u = R(t,\alpha). \tag{6}$$

Our approach to establishing the local smoothing effect for (6) is based on the construction of an approximate solution ("parametrix"). For the sake of exposition, we present the sketch of the proof for the homogeneous equation ($R = 0$) and for the initial data $u(0,\alpha) = u_0(\alpha)$ and $\partial_t u(0,\alpha) = u_1(\alpha)$ localized in high frequencies.

The ansatz is

$$w(t,\alpha) = \frac{1}{2\pi} \iint \mathrm{e}^{-\mathrm{i}\beta\xi} \left(\mathrm{e}^{\mathrm{i}\varphi^+(t,\alpha,\xi)} f^+(\beta) + \mathrm{e}^{\mathrm{i}\varphi^-(t,\alpha,\xi)} f^-(\beta) \right) \mathrm{d}\beta\, \mathrm{d}\xi,$$

where the *phase functions* φ satisfy $\varphi^\pm(0,\alpha,\xi) = \alpha\xi$.

Applying the homogeneous equation of (6) to our ansatz, we obtain a non-linear equation for $\varphi^\pm$, commonly referred to as the *Hamilton–Jacobi* equation. The usual way to solving the Hamilton–Jacobi equation is through the technique of generating functions. The equation is, however, neither homogeneous nor polyhomogeneous, and as such solutions are on a time scale $t \sim |\xi|^{1/2}$. We thus construct the parametrix for $|\xi| \sim 2^j$ and $t \sim 2^{-j/2}$ with $\varphi^\pm(t,\alpha,\xi) = \alpha\xi + t(\pm|\xi|^{3/2} + \vartheta^\pm(t,\alpha,\xi))$, where $\vartheta^\pm(0,\alpha,\xi)$ behave like classical symbols.

The oscillatory integral ansatz w satisfies the local smoothing estimate (3) for short (frequency-localized) time scale. The proof uses the change of variables and L^2-mapping properties of Fourier integral operators, analogous to the standard proof for (5). The error arising in approximating by w is, on the frequency localized time scale $t \sim 2^{-j/2}$, of the order $|\xi|^1$, and hence it is controlled by $\|u_0\|_{H^1_\alpha} + \|u_1\|_{L_\alpha^{-1/2}}$. This $|\xi|^1$-order error, incidentally, is of an oscillatory-integral form with the same phase functions as those of w, and thus it enjoys a $1/4$ derivative gain of smoothness. In consequence, the error in approximating by w^0 is controlled by $\|u_0\|_{H_\alpha^{3/4}} + \|u_1\|_{H_\alpha^{-3/4}}$.

In order to construct the parametrix on a fixed time scale, we "glue" together roughly $2^{j/2}$ parametrices in each dyadic frequency band. The gluing procedure requires fine control over propagation of singularities for short time scales. It remains to show that the "glued" parametrix is a good approximation to the actual solution u to (6). For this, we combine the energy estimate for the linear problem (6) with the improved error estimate to show that

$$\left\| \langle\alpha\rangle^{-\rho}(u-w)\right\|_{L_t^2([0,T])H_\alpha^{s+3/2}(\mathbb{R})} \leqslant C\left(\|u_0\|_{H_\alpha^{s+5/4}(\mathbb{R})} + \|u_1\|_{H_\alpha^{s-1/4}(\mathbb{R})}\right).$$

By virtue of the smoothing estimate (3) for w, in all, it follows that

$$\begin{aligned}\left\| \langle\alpha\rangle^{-\rho}u\right\|_{L_t^2([0,T])H_\alpha^{s+3/2}(\mathbb{R})} &\leqslant \left\| \langle\alpha\rangle^{-\rho}(u-w)\right\|_{L_t^2([0,T])H_\alpha^{3/2}(\mathbb{R})} + \left\| \langle\alpha\rangle^{-\rho}w\right\|_{L_t^2([0,T])H^{s+3/2}(\mathbb{R})} \\ &\leqslant C\left(\|u_0\|_{H_\alpha^{s+5/4}(\mathbb{R})} + \|u_1\|_{H_\alpha^{s-1/4}(\mathbb{R})}\right).\end{aligned}$$

This asserts (3) for the linearized problem (6).

To prove (3) for the nonlinear problem, we employ a nonlinear energy estimate for (2) to establish its local well-posedness for sufficiently regular initial data. Substituting the coefficient in (6) by the solution then completes the proof.

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