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Combinatorics

On the triplex substitution – Rauzy fractal ☆

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Abstract

This Note is the continuation of our previous one. Here we study the Rauzy fractal associated with the triplex substitution $\sigma : a \mapsto ab$, $b \mapsto acb$, $c \mapsto acc$. Analyzing the combinatorial structure of the prefixes, we characterize the set of points in the Rauzy fractal in terms of Fibonacci numbers, which is quite different from the case of Pisot substitutions. Some lexicographic properties and local similarity structures are presented. ***To cite this article: B. Tan et al., C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Sur la substitution triplexe – fractal de Rauzy. On étudie le fractal de Rauzy engendré par la substitution triplexe $\sigma : a \mapsto ab$, $b \mapsto acb$, $c \mapsto acc$. Analysant la structure combinatoire des préfixes, qui est différente de celle des substitutions de Pisot, nous caractérisons les points du fractal de Rauzy en termes des nombres de Fibonacci. En particulier, des propriétés lexicographiques et de similarité locale sont données. ***Pour citer cet article : B. Tan et al., C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Version française abrégée

La plupart des fractals des Rauzy sont compacts car on projette sur un plan de contraction. Pour la substitution triplexe, dont le polynôme caractéristique est réductible (1 est un valeur propre), le fractal de Rauzy est un ensemble non borné contenu dans des droites parallèles.

La substitution triplexe $\sigma = (ab, acb, acc)$ sur l'alphabet $\mathcal{A} = \{a, b, c\}$ possède un point fixe $\xi = abacbabaccacb\cdots$. Sa matrice de substitution M est primitive. Sa valeur propre de Perron–Frobenius est $\phi + 1$, où $\phi = \frac{\sqrt{5}+1}{2}$ désigne le nombre d'or. Le vecteur propre correspondant est $E = (\phi^{-2}, \phi^{-3}, \phi^{-2})^t$. Soient Q le plan dans $\mathbb{R}^3$ orthogonal à E et π' la projection orthogonale de $\mathbb{R}^3$ sur Q. Notons $P(w) = (|w|_a, |w|_b, |w|_c)^t$, $w \in \mathcal{A}^*$, le vecteur de Parikh, et $\pi = \pi' \circ P : \mathcal{A}^* \to Q$ (c'est un homomorphisme). Soit enfin $\mathcal{P}_\xi = \{P(\xi_1 \cdots \xi_N) \mid N \geqslant 1\} \subset \mathbb{R}^3$, l'ensemble des vecteurs de Parikh des préfixes de ξ.

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Définition 1. Soit $\mathfrak{R} = \pi(\mathcal{P}_\xi)$. Son adhérence $\overline{\mathfrak{R}}$ est appelée fractal de Rauzy de σ.

Dans cette Note, nous choisissons $\boldsymbol{u} = (\phi, -1, -1)^t$ et $\boldsymbol{v} = (1, 0, -1)^t$ comme base de Q. Remarquons que $\boldsymbol{u}$ et $\boldsymbol{v}$ ne sont pas orthogonaux. Avec cette base, on peut identifier les points de Q à leurs coordonnées par rapport à $\{\boldsymbol{u}, \boldsymbol{v}\}$.

Le lemme suivant est la clé de cette note. Rappelons que $\xi[1 \dots N] = \xi_1 \cdots \xi_N$.

Lemme 1. *Pour tout $N \geqslant 1$, il existe des entiers uniques $n, q_1, p_1, \dots, q_n, p_n$ tels que*

$$\xi[1 \dots N] = [\![q_n, p_n]\!][\![q_{n-1}, p_{n-1}]\!] \cdots [\![q_1, p_1]\!] \tag{1}$$

où, pour $q \geqslant p \geqslant 0$, on pose $[\![q, p]\!] = \sigma^q(a)\sigma^{q-1}(ac) \cdots \sigma^p(ac)$ (en convenant que $[\![q, p]\!] = \sigma^q(a)$ quand $q = p$), avec

$$n \geqslant 1, \quad q_n \geqslant p_n > q_{n-1} \geqslant p_{n-1} > \cdots > q_1 \geqslant p_1 \geqslant 0. \tag{2}$$

Réciproquement, si $n, q_1, \dots, p_n$ satisfont (2)*, il existe un unique entier $N \geqslant 1$ tel que* (1) *soit satisfait.*

D'après Lemme 1 et l'additivité de π, il suffit d'étudier la projection des mots $[\![q, p]\!]$.

Dans le lemme suivant, $\pi(w) = \pi'(P(w)) = (u(w), v(w))^t,\ w \in \mathcal{A}^*$.

Lemme 2. (i) $\pi([\![q_n, p_n]\!][\![q_{n-1}, p_{n-1}]\!] \cdots [\![q_1, p_1]\!]) = (\sum_{i=1}^n (\phi^{-2p_i} + \phi^{-2q_i-1}), n)^t$.

(ii) (*propriété lexicographique*) $u([\![q_n, p_n]\!][\![q_{n-1}, p_{n-1}]\!] \cdots [\![q_1, p_1]\!]) > u([\![q'_n, p'_n]\!][\![q'_{n-1}, p'_{n-1}]\!] \cdots [\![q'_1, p'_1]\!])$ *si et seulement s'il existe $i \in \{1, \dots, n\}$ tel que $(q_1, p_1) = (q'_1, p'_1), \dots, (q_{i-1}, p_{i-1}) = (q'_{i-1}, p'_{i-1})$ et $u([\![q_i, p_i]\!]) > u([\![q'_i, p'_i]\!])$. La dernière inégalité équivaut à ($p < p'$, ou $p = p'$ mais $q < q'$).*

Maintenant les préfixes de ξ sont découpés comme suit : $\mathcal{P}_\xi = \{\xi[1 \dots N] \mid N \geqslant 1\} = \bigcup_{n \geqslant 1} \mathcal{W}_n$, où

$$\mathcal{W}_n = \big\{[\![q_n, p_n]\!][\![q_{n-1}, p_{n-1}]\!] \cdots [\![q_1, p_1]\!] \mid q_n \geqslant p_n > q_{n-1} \geqslant p_{n-1} > \cdots > q_1 \geqslant p_1 \geqslant 0\big\}.$$

Soit $\mathfrak{R}_n = \pi(\mathcal{W}_n)$, alors $\mathfrak{R}_n = \{(u, v) \in \mathfrak{R} \mid v = n\}$ et $\mathfrak{R} = \bigcup_{n \geqslant 1} \mathfrak{R}_n$. Puis le fractal de Rauzy peut être écrit comme $\bar{\mathfrak{R}} = \bigcup_{n \geqslant 1} \bar{\mathfrak{R}}_n$.

Pour préciser la structure géométrique de $\bar{\mathfrak{R}}$, posons $\mathfrak{U}_n = \{u \mid (u, n) \in \mathfrak{R}_n\} = \{u(w) \mid w \in \mathcal{W}_n\}$. Alors $\mathfrak{R}_n = \mathfrak{U}_n \times \{n\}$, $\bar{\mathfrak{R}}_n = \bar{\mathfrak{U}}_n \times \{n\}$ et $\bar{\mathfrak{R}} = \bigcup_{n \geqslant 1} \bar{\mathfrak{U}}_n \times \{n\}$.

Il est important de remarquer que la formule (i) du Lemme 2 peut être étendue au cas où $q_k = \infty$ ou $p_k = \infty$ avec la convention $\phi^{-\infty} = 0$. Cette extension permet de décrire les points d'accumulation de $\mathfrak{U}_n$ et donne lieu à une notion de « mots au bord' ».

Théorème 2. (i) *Il existe une bijection entre $\mathcal{P}_\xi$ et $\mathfrak{R} = \bigcup(\mathfrak{U}_n \times \{n\})$, où $\mathfrak{U}_n \subset [0, \phi^2]$.*

(ii) $\mathfrak{U}_n = \bigcup_{u \in \mathfrak{U}_{n-1}} \{u + \phi^{-2\,\mathrm{ord}(u)-2} \cdot \mathfrak{U}_1\}$ *où la réunion est disjointe.*

(iii) $\mathfrak{U}_n = \{u([\![q_n, p_n]\!] \cdots [\![q_1, p_1]\!]) \mid 0 \leqslant q_j,\ p_j < \infty$ *satisfait à* (2)$\}$,

(iv) $\bar{\mathfrak{U}}_n = \{u([\![q_n, p_n]\!] \cdots [\![q_1, p_1]\!]) \mid 0 \leqslant q_j, p_j \leqslant \infty$ *satisfait à* (2)$\}$ (*voir Lemme* 2(i)).

(v) $\max(\mathfrak{U}_n) = u([\![n-1, n-1]\!] \cdots [\![0, 0]\!])$.

Remarque 1. Si on considère l'adhérence de la projection de $\mathfrak{R}$ sur le premier axe de coordonnées, on obtient l'intervalle $[0, \phi^2]$. En effet, c'est équivalent au fractal de Rauzy de la substitution (ab, abb) sur $\{a, b\}$.

1. Introduction and notations

This is a continuation of the study of the triplex substitution [5]. We study here the associated Rauzy fractal, and we shall use the terminology and notations from [5].

Rauzy fractals were first introduced in [4] in the case of the Tribonacci substitution. They can be more generally associated with Pisot substitutions. The substitutions with irreducible characteristic polynomials have been studied extensively (see the survey in [1,3] and references therein for example); while in [2], the authors considered the cases when the polynomials are reducible.

In most of the previous works on the Rauzy fractals, either in the irreducible cases or in the reducible ones, one projects on the contracting plane, therefore the Rauzy fractal is compact. But for the triplex substitution, the characteristic polynomial is reducible with an eigenvalue 1, and its Rauzy fractal, as we shall see, is an unbounded set which consists of countably many bounded subsets in some parallel lines. We also show that the one dimensional projection is a bounded closed interval.

Recall that the triplex substitution $\sigma = (ab, acb, acc)$ over the three-letter alphabet $\mathcal{A} = \{a, b, c\}$ has the fixed point $\xi = abacbabaccacb\ldots$. Hereafter, we use the notation $\phi = \frac{\sqrt{5}+1}{2}$ for the golden ratio ($\phi^2 = \phi + 1$).

The substitution matrix

$$M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

of σ is primitive, and its Perron–Frobenius eigenvalue is $\phi + 1$ with the eigenvector $E = (\phi^{-2}, \phi^{-3}, \phi^{-2})^t$. Let Q denote the plane in $\mathbb{R}^3$ orthogonal to E, and π' the orthogonal projection of $\mathbb{R}^3$ onto Q. Define $\pi = \pi' \circ P : \mathcal{A}^* \to Q$. Denote $\mathcal{P}_\xi = \{P(\xi_1 \cdots \xi_N) \mid N \geqslant 1\} \subset \mathbb{R}^3$, the set of Parikh vectors of the prefixes of ξ (recall that $P(w) = (|w|_a, |w|_b, |w|_c)^t$ for $w \in \mathcal{A}^*$).

Definition 1. Let $\mathfrak{R} = \pi(\mathcal{P}_\xi)$. Its closure $\bar{\mathfrak{R}}$ is called the Rauzy fractal associated with σ.

To give an explicit formula for points in the Rauzy fractal $\bar{\mathfrak{R}}$, we choose some suitable coordinates for the plane Q. Let $\boldsymbol{u} = (\phi, -1, -1)^t$ and $\boldsymbol{v} = (1, 0, -1)^t$ be the eigenvectors of M^t corresponding to eigenvalues $2 - \phi$ and 1 respectively. Then $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal to E. Thus Q is spanned by $\boldsymbol{u}$ and $\boldsymbol{v}$, $\mathbb{R}^3 = E \oplus Q$ (here we also use E to denote the line spanned by E). We remark that the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are not orthogonal to each other.

To simplify our notations and calculations, we use this non-orthogonal basis $\{\boldsymbol{u}, \boldsymbol{v}\}$ for Q (had we taken an orthogonal basis instead, the two coordinates would have been equivalent up to an affine transformation). We identify the point in $Q \subset \mathbb{R}^3$ with its coordinates with respect to the basis $\{\boldsymbol{u}, \boldsymbol{v}\}$. Thus π' is the linear transformation from $\mathbb{R}^3$ onto (the $\boldsymbol{uv}$-plane) $Q = \mathbb{R}^2$ with matrix $\pi' = \left[\begin{smallmatrix} \phi & -1 & -1 \\ 1 & 0 & -1 \end{smallmatrix}\right]$, i.e., $\pi'((x, y, z)^t) = (\phi x - y - z, x - z)^t$. The mapping $\pi = \pi' \circ P : \mathcal{A}^* \to \mathbb{R}^2$ is a morphism which sends a word to a vector and is characterized by the following properties:

(1) $\pi(a) = (\phi, 1)^t$, $\pi(b) = (-1, 0)^t$ and $\pi(c) = (-1, -1)^t$;

(2) $\pi(w_1 w_2) = \pi(w_1) + \pi(w_2)$ for $w_1, w_2 \in \mathcal{A}^*$.

Let $u(w)$ and $v(w)$ be the coordinates of $\pi(w)$: $\pi(w) = \pi'(P(w)) = (u(w), v(w))^t$, $w \in \mathcal{A}^*$.

2. Decomposition of the prefixes of ξ

The following lemma on the decomposition of the prefixes plays a key rôle in the study of the Rauzy fractal. Set $\xi[1 \ldots N] = \xi_1 \cdots \xi_N$.

Lemma 1. *For any $N \geqslant 1$, there exist unique integers $n, q_1, p_1, \ldots, q_n, p_n$ such that*

$$\xi[1 \ldots N] = \left[\sigma^{q_n}(a)\sigma^{q_n - 1}(ac) \cdots \sigma^{p_n}(ac)\right] \cdots \left[\sigma^{q_1}(a)\sigma^{q_1 - 1}(ac) \cdots \sigma^{p_1}(ac)\right], \tag{1}$$

where

$$n \geqslant 1, \quad q_n \geqslant p_n > q_{n-1} \geqslant p_{n-1} > \cdots > q_1 \geqslant p_1 \geqslant 0, \tag{2}$$

and, by convention, $\sigma^{q_k}(a)\sigma^{q_k - 1}(ac) \cdots \sigma^{p_k}(ac) = \sigma^{q_k}(a)$ if $q_k = p_k$.

Conversely, if $n, q_1, p_1, \ldots, q_n, p_n$ satisfy (2)*, then there exists a unique $N \geqslant 1$ such that* (1) *holds.*

Notation. For $q \geqslant p \geqslant 0$, denote $[\![q, p]\!] = \sigma^q(a)\sigma^{q-1}(ac)\sigma^{q-2}(ac) \cdots \sigma^p(ac)$ (with the convention $[\![q, p]\!] = \sigma^q(a)$ when $q = p$). Then (1) can be written as $\xi[1 \ldots N] = [\![q_n, p_n]\!][\![q_{n-1}, p_{n-1}]\!] \cdots [\![q_1, p_1]\!]$.

Here are some examples: $\xi[1 \ldots 1] = [\![0, 0]\!]$, $\xi[1 \ldots 2] = [\![1, 1]\!]$, $\xi[1 \ldots 3] = [\![1, 1]\!][\![0, 0]\!]$, $\xi[1 \ldots 18] = [\![3, 3]\!][\![2, 1]\!]$.

We first prove the following lemma which in particular gives the calculation of the $\boldsymbol{uv}$-coordinates and some "lexicographic" property for the prefixes of ξ. All integers or integer pairs m, n, q_k, p_k appearing in the lemma are naturally supposed to satisfy (2).

Lemma 2. *We have*:

(i) *For* $q \geqslant p \geqslant 0$, $u([\![q,p]\!]) = \phi^{-2p} + \phi^{-2q-1}$, $v([\![q,p]\!]) = 1$.

(ii) $\pi([\![q_n,p_n]\!]\cdots[\![q_1,p_1]\!]) = (\sum_{i=1}^{n}(\phi^{-2p_i} + \phi^{-2q_i-1}),\ n)^t$.

(iii) $u([\![q,p]\!]) > u([\![q',p']\!])$ *if and only if either* $(p < p')$ *or* $(p = p'$ *and* $q < q')$.

(iv) (*lexicographic property*) $u([\![q_n,p_n]\!]\cdots[\![q_1,p_1]\!]) > u([\![q'_n,p'_n]\!]\cdots[\![q'_1,p'_1]\!])$ *if and only if there exists* $i \in \{1,\dots,n\}$ *such that* $(q_k,p_k) = (q'_k,p'_k)$ *for* $1 \leqslant k < i$, *and* $u([\![q_i,p_i]\!]) > u([\![q'_i,p'_i]\!])$.

(v) $[\![q_n,p_n]\!]\cdots[\![q_1,p_1]\!] = [\![q'_m,p'_m]\!]\cdots[\![q'_1,p'_1]\!]$ *if and only if* $m = n$ *and* $(q_k,p_k) = (q'_k,p'_k)$ *for* $1 \leqslant k \leqslant n$.

(vi) $0 < u([\![q_n,p_n]\!]\cdots[\![q_1,p_1]\!]) < \phi^2$.

(vii) *The following statements are equivalent*:

(a) $u([\![q_n,p_n]\!]\cdots[\![q_1,p_1]\!]) = u([\![q'_n,p'_n]\!]\cdots[\![q'_1,p'_1]\!])$;
(b) $[\![q_n,p_n]\!]\cdots[\![q_1,p_1]\!] = [\![q'_n,p'_n]\!]\cdots[\![q'_1,p'_1]\!]$;
(c) $(q_n,p_n,\dots,q_1,p_1) = (q'_n,p'_n,\dots,q'_1,p'_1)$.

Proof. A direct calculation gives $\pi'\cdot M^n = \left[\begin{smallmatrix}\phi^{-2n+1} & -\phi^{-2n} & -\phi^{-2n}\\ 1 & 0 & -1\end{smallmatrix}\right]$, therefore

$$\pi(\sigma^n(a)) = (\phi^{-2n+1},1)^t,\quad \pi(\sigma^n(b)) = (-\phi^{-2n},0)^t \quad\text{and}\quad \pi(\sigma^n(c)) = (-\phi^{-2n},-1)^t.$$

Then some direct verifications remain to be made. □

Proof of Lemma 1. This is an induction on N. The cases for small N can be checked directly as in the above examples. Now let $N > 3$. Since $\xi_1\xi_2\cdots = \sigma(\xi_1)\sigma(\xi_2)\cdots$, there exists $n \in \mathbb{N}$ $(n < N)$ such that $\xi[1\dots N] = \sigma(\xi[1\dots n])w$ with $w \lhd \sigma(\xi_{n+1})$ and $|w| < |\sigma(\xi_{n+1})|$. Hence $w = \varepsilon$, $w = a$ or $w = ac$. By the induction assumption, $\xi[1\dots n] = [\![q_k,p_k]\!]\cdots[\![q_1,p_1]\!]$. We have to consider three cases according to the value of w:

Case 1. $w = \varepsilon$. In this case, $\xi[1\dots N] = [\![q_k+1,p_k+1]\!]\cdots[\![q_1+1,p_1+1]\!]$;

Case 2. $w = a$. In this case, $\xi[1\dots N] = [\![q_k+1,p_k+1]\!]\cdots[\![q_1+1,p_1+1]\!][\![0,0]\!]$;

Case 3. $w = ac$. In this case, $p_1 = 0$. Otherwise, suppose $p_1 > 0$, then $\xi[1\dots n] = \sigma(\xi[1\dots m])$ for some m, thus w is a prefix of $\sigma^2(\xi_{m+1})$ (of which ab is a prefix), but this contradicts $w = ac$.

So $p_1 = 0$, and we have $\xi[1\dots N] = [\![q_k+1,p_k+1]\!]\cdots[\![q_1+1,0]\!]$. This proves the first part of the lemma. The uniqueness is proven by Lemma 2(v).

Conversely, we show by induction on n that the word $[\![q_n,p_n]\!][\![q_{n-1},p_{n-1}]\!]\cdots[\![q_1,p_1]\!]$ is a prefix of $\sigma^{q_n+1}(a)b^{-1}$ (remark that b is the last letter of $\sigma^m(a)$ for any $m \geqslant 1$), and thus is a prefix of ξ.

For $n = 1$, since $\sigma^q(b)b^{-1} = \sigma^{q-1}(ac)\cdots\sigma^0(ac)$ (by induction), $\sigma^{q+1}(a)b^{-1} = \sigma^q(a)\sigma^q(b)b^{-1} = [\![q,0]\!]$. Now it is clear that $[\![q,p]\!]$ is a prefix of $[\![q,0]\!] = \sigma^{q+1}(a)b^{-1}$.

If $n \geqslant 2$, then $[\![q_{n-1},p_{n-1}]\!]\cdots[\![q_1,p_1]\!]$ is a prefix of $\sigma^{q_{n-1}+1}(a)b^{-1}$ by the induction assumption. Since $p_n > q_{n-1}$, $\sigma^{q_{n-1}+1}(a)b^{-1}$ is a prefix of $\sigma^{p_n}(a)b^{-1}$. Now, by taking the equality $\sigma^{p_n-1}(ac)c^{-1} = \sigma^{p_n}(a)b^{-1}$ into account, we see that $[\![q_n,p_n]\!]\cdots[\![q_1,p_1]\!]$ is a prefix of $[\![q_n,p_n-1]\!]$, and thus also is a prefix of

$$[\![q_n,0]\!] = \sigma^{q_n+1}(a)b^{-1}. \qquad \square$$

3. The Rauzy fractal of the triplex substitution

Due to Lemma 1 and the additivity of π, it suffices to study the projections of such words as $[\![q,p]\!]$. We showed in Lemma 2(ii) that $v([\![q_n,p_n]\!]\cdots[\![q_1,p_1]\!]) = n$. Thus the prefixes of ξ can be divided into countably many classes: $\mathcal{P}_\xi = \{\xi[1\dots N] \mid N \geqslant 1\} = \bigcup_{n\geqslant 1}\mathcal{W}_n$, where

$$\mathcal{W}_n = \{[\![q_n,p_n]\!][\![q_{n-1},p_{n-1}]\!]\cdots[\![q_1,p_1]\!] \mid q_n \geqslant p_n > q_{n-1} \geqslant p_{n-1} > \cdots > q_1 \geqslant p_1 \geqslant 0\}.$$

Now let $\mathfrak{R}_n = \pi(\mathcal{W}_n)$, then $\mathfrak{R}_n = \{(u,v) \in \mathfrak{R} \mid v = n\}$ and $\mathfrak{R} = \bigcup_{n\geqslant 1}\mathfrak{R}_n$. Since the sets $\mathfrak{R}_n$ $(n \geqslant 1)$ are metrically separated, the Rauzy fractal can be written as $\bar{\mathfrak{R}} = \bigcup_{n\geqslant 1}\bar{\mathfrak{R}}_n$.

Now, we analyze the structure of $\mathfrak{R}_n$ and $\bar{\mathfrak{R}}_n$ in details. For this, it suffices to study the $\boldsymbol{u}$-coordinates of $\mathfrak{R}_n$: $\mathfrak{U}_n = \{u \mid (u,n) \in \mathfrak{R}_n\} = \{u(w) \mid w \in \mathcal{W}_n\}$. Then we have:

$$\mathfrak{R}_n = \mathfrak{U}_n \times \{n\}, \quad \bar{\mathfrak{R}}_n = \bar{\mathfrak{U}}_n \times \{n\} \quad \text{and} \quad \bar{\mathfrak{R}} = \bigcup_{n \geqslant 1} \bar{\mathfrak{U}}_n \times \{n\}.$$

Let us begin with the study of $\mathfrak{R}_1$ and $\mathfrak{R}_2$. First, using the lexicographic property (Lemma 2(iv)), we describe explicitly all the points of $\mathfrak{U}_1$ and $\bar{\mathfrak{U}}_1$ from the right to the left on the $\boldsymbol{u}$-axis.

By Lemma 2 and some simple calculations, the rightmost point in $\mathfrak{U}_1$ is $u([\![0,0]\!]) = \phi$. The next point on the left in $\mathfrak{U}_1$ is $u([\![1,0]\!]) = 1 + \phi^{-3}$; the next one is $u([\![2,0]\!]) = 1 + \phi^{-5}; \ldots$; the $(n+1)$th point from the right to the left is $u([\![n,0]\!]) = 1 + \phi^{-2n-1}; \ldots$; at last, we get an accumulation point 1.

As a convention, we naturally write $u([\![\infty,0]\!]) = 1$ ($[\![\infty,0]\!]$ will be called a "boundary word").

From right to left, the next point to the accumulation point 1 is $u([\![1,1]\!]) = \phi^{-1}$, then $u([\![2,1]\!]) = \phi^{-2} + \phi^{-5}, \ldots,$ and at last, we get the second accumulation point $u([\![\infty,1]\!]) = \phi^{-2}$.

Going on this way, we get the mth accumulation point $u([\![\infty, m-1]\!]) = \phi^{-2m+2}$, and then the points $u([\![m,m]\!]) = \phi^{-2m+1}$, $u([\![m+1,m]\!]) = \phi^{-2m} + \phi^{-2m+3}, \ldots,$ and then the $(m+1)$st accumulation point $u([\![\infty,m]\!]) = \phi^{-2m}$. Finally, we get the "last" accumulation point $u([\![\infty,\infty]\!]) = 0$.

For any point $u = u([\![q,p]\!])$ except the rightmost one in $\mathfrak{U}_1$, we define $I(u)$ to be the "gap interval" right to u in $\mathfrak{U}_1$, and $|I(u)|$ to be its length. More precisely, if we set $u' = \inf\{z \in \mathfrak{R}_1 \mid z > u\}$, then $I(u) = [u,u']$ and $|I(u)| = u' - u$.

Proposition 3. (1) *The set of the accumulation points of* $\mathfrak{U}_1 = \{u([\![q,p]\!]) \mid 0 \leqslant p \leqslant q < \infty\}$ *is the countable set* $\{u([\![\infty,m]\!]) \mid m = 0,1,2,\ldots,\infty\}$, *and* $\bar{\mathfrak{U}}_1 = \{u([\![q,p]\!]) \mid 0 \leqslant p \leqslant q \leqslant \infty\}$, *where* $u([\![q,p]\!]) = \phi^{-2p} + \phi^{-2q+1}$ *(with the convention* $\phi^{-\infty} = 0$*).*

Furthermore $u([\![q,p]\!]) < u([\![q',p']\!])$ *if and only if either* $(p < p')$ *or* $(p = p'$ *and* $q < q')$.

(2) *For any point* $u([\![q,p]\!])$ *in* $\mathfrak{U}_1$ *with* $q > 0$, $|I(u([\![q,p]\!]))| = \phi^{-2q}$.

Proof. Assertion (1) has already been shown. To prove assertion (2), we consider two cases:

If $q > p$, then the right point next to $u([\![q,p]\!])$ in $\mathfrak{U}_1$ is $u([\![q-1,p]\!])$, and

$$\bigl|I\bigl(u\bigl([\![q,p]\!]\bigr)\bigr)\bigr| = \bigl(\phi^{-2p} + \phi^{-2q+1}\bigr) - \bigl(\phi^{-2p} + \phi^{-2q-1}\bigr) = \phi^{-2q}.$$

If $q = p$, then the right point next to $u([\![q,p]\!])$ in $\bar{\mathfrak{U}}_1$ is $u([\![\infty,q-1]\!])$, then $|I(u([\![q,p]\!]))| = \phi^{-2q}$. □

The terminology "gap interval" can be generalized to points in $\mathfrak{U}_n$ in an obvious way.

To describe the structure of the Rauzy fractal, we use the following notations. For a prefix $w = [\![q_n,p_n]\!][\![q_{n-1},p_{n-1}]\!] \cdots [\![q_1,p_1]\!] \in \mathcal{W}_n$, we call the number q_n the *order* of w, written $\operatorname{ord}(w) = q_n$.

If there is i $(1 \leqslant i \leqslant n)$ such that either

$$q_n = p_n = \cdots = q_i = \infty > p_i > q_{i-1} \geqslant p_{i-1} > \cdots > q_1 \geqslant p_1 \geqslant 0$$

or

$$q_n = p_n = \cdots = q_i = p_i = \infty > q_{i-1} \geqslant p_{i-1} > \cdots > q_1 \geqslant p_1 \geqslant 0,$$

we call $[\![q_n,p_n]\!][\![q_{n-1},p_{n-1}]\!] \cdots [\![q_1,p_1]\!]$ a *boundary word* of $\mathcal{W}_n$. The set of boundary words of $\mathcal{W}_n$ is denoted by $\partial\mathcal{W}_n$. For $w = [\![q_n,p_n]\!][\![q_{n-1},p_{n-1}]\!] \cdots [\![q_1,p_1]\!] \in \partial\mathcal{W}_n$, just as for words in $\mathcal{W}_n$, we define:

$$u\bigl([\![q_n,p_n]\!][\![q_{n-1},p_{n-1}]\!] \cdots [\![q_1,p_1]\!]\bigr) = \sum_{i=1}^{n} \bigl(\phi^{-2p_i} + \phi^{-2q_i-1}\bigr), \quad \text{with the convention } \phi^{-\infty} = 0. \tag{3}$$

With these notations, the preceding proposition says that the set of accumulation points of $\mathfrak{U}_1$ is $u(\partial\mathcal{W}_1)$ and $|I(w)| = \phi^{-2\operatorname{ord}(w)}$ for $w \in \mathcal{W}_1 \setminus \{[\![0,0]\!]\}$.

For $u \in \mathbb{R}$, we say that the set $u + Z := \{u + z \mid z \in Z\}$ is *attached to* u (*attachment* in short), and two attachments are said to be *separated* if their closed convex hulls are disjoint.

Theorem 4. (1) *At every point* $u([\![q,p]\!])$ *in* $\mathfrak{U}_1$, *we attach a set* $\phi^{-2q-2} \cdot \mathfrak{U}_1$, *then we get the set* $\mathfrak{U}_2$. *More precisely,* $\mathfrak{U}_2 = \bigcup_{u \in \mathfrak{U}_1} \{u + \phi^{-2\operatorname{ord}(u)-2} \cdot \mathfrak{U}_1\}$. *Moreover, the attachments are separated from each other.*

(2) *The set of accumulation points of* $\mathfrak{U}_2$ *is* $u(\partial\mathcal{W}_2)$. *Any point in* $\mathfrak{U}_1$ *is an accumulation point of* $\mathfrak{U}_2$.

(3) *The rightmost point in* $\mathfrak{U}_2$ *is* $u([\![1,1]\!][\![0,0]\!]) = \phi + \phi^{-1}$. *For other point in* $\mathfrak{U}_2$, $|I(u(w))| = \phi^{-2\,\mathrm{ord}(w)}$.

Proof. (1) The set $u(\{[\![q',p']\!] \mid q' \geqslant p' \geqslant q+1\})$ is $\phi^{-2q-2} \cdot \mathfrak{U}_1$ (which is just the part of $\mathfrak{U}_1$ laying between 0 and $u([\![q+1,q+1]\!]))$. The relation $\mathcal{W}_2 = \bigcup_{[\![q,p]\!]\in\mathcal{W}_1}\{[\![q',p']\!][\![q,p]\!] \mid q' \geqslant p' \geqslant q+1\}$ then proves the first statement. Since $\mathfrak{U}_1$ is of diameter ϕ, $\phi^{-2q-2} \cdot \mathfrak{U}_1$ attached at $u = u([\![q,p]\!])$ is of diameter ϕ^{-2q-1}. By Lemma 2(ii) the gap interval $I(u)$ ($u \in \mathfrak{U}_1$) is of length ϕ^{-2q}, so the attachments are separated.

(2) Since 0 is an accumulation point of $\mathfrak{U}_1$, any point of $\bar{\mathfrak{U}}_1$ is an accumulation point of $\mathfrak{U}_2$, and this kind of point is of form $u([\![\infty,\infty]\!]w)$ with $w \in \mathcal{W}_1 \bigcup \partial\mathcal{W}_1$. Secondly, the attachment $u([\![q,p]\!]) + \phi^{-2q-2} \cdot \mathfrak{U}_1$ has accumulation points of form $u([\![\infty,q']\!][\![q,p]\!])$ with $q' > p'$. In total, the set of accumulation points of $\mathfrak{U}_2$ is $u(\partial\mathcal{W}_2)$.

(3) Obviously, the rightmost point of $\mathfrak{U}_2$ is $u([\![1,1]\!][\![0,0]\!]) = \phi + \phi^{-1}$. Take any point $u = u([\![q',p']\!][\![q,p]\!])$ other than the rightmost one. By (1), the point u is in the attachment $u([\![q,p]\!]) + \phi^{-2q-2} \cdot \mathfrak{R}_1$.

If $(q',p') \neq (q+1,q+1)$, then the point u is not the rightmost point of the attachment, thus $|I(u)| = |I(u([\![q',p']\!]))| = \phi^{-2q'}$, where the last equality is given by Lemma 2(ii).

If $(q',p') = (q+1,q+1)$, the point u is the rightmost point of the attachment, and $|I(u)|$ is just $|I(u([\![q,p]\!]))|$ minus the diameter of $u([\![q,p]\!]) + \phi^{-2q-2} \cdot \mathfrak{U}_1$. So $|I(u)| = \phi^{-2q} - \phi^{-2q-1} = \phi^{-2q'}$.

Combining these two cases, we show that $|I(u(w))| = \phi^{-2\,\mathrm{ord}(w)}$. □

Recall that the Rauzy fractal $\bar{\mathfrak{R}} = \bigcup_{n\geqslant 1}\bar{\mathfrak{U}}_n \times \{n\}$. We have:

Theorem 5. *We have* $\bar{\mathfrak{U}}_n = \{u(w) \mid w \in \mathcal{W}_n \cup \partial\mathcal{W}_n\}$. *Moreover we have for* $n \geqslant 2$,

(1) *At every point* $u(w)$ *in* $\mathfrak{U}_{n-1}$, *where* $w \in \mathcal{W}_{n-1}$, *we attach a set* $\phi^{-2\,\mathrm{ord}(w)-2} \cdot \mathfrak{U}_1$, *then we get the set* $\mathfrak{U}_n$, *that is* $\mathfrak{U}_n = \bigcup_{u\in\mathfrak{U}_{n-1}}\{u + \phi^{-2\,\mathrm{ord}(u)-2} \cdot \mathfrak{U}_1\}$. *Moreover, the attachments are separated.*

(2) *The set of accumulation points of* $\mathfrak{U}_n$ *is the countable set* $u(\partial\mathcal{W}_n)$. *In particular, any point of* $\bar{\mathfrak{U}}_{n-1}$ *is an accumulation point of* $\mathfrak{U}_n$.

(3) $\max(\mathfrak{U}_n) = u([\![n-1,n-1]\!]\cdots[\![0,0]\!])$. *For other points in* $\mathfrak{U}_n$, *we have* $|I(u(w))| = \phi^{-2\,\mathrm{ord}(w)}$.

Proof. Similar to Theorem 4, just consider $\mathcal{W}_n = \bigcup_{w\in\mathcal{W}_{n-1}}\{[\![q,p]\!]w \mid q \geqslant p \geqslant \mathrm{ord}(w)+1\}$. □

Let us end this Note by a remark about the *one dimensional projection*.

Remark 6. Projecting $\mathfrak{R}$ to the first coordinate we get the set $\bigcup\mathfrak{U}_n$, i.e., $u(\mathfrak{R}) = \bigcup\mathfrak{U}_n$. Taking the closure again, we get the interval $[0,\phi^2]$. This is not surprising, because it is just the Rauzy fractal (up to scaling) of the substitution (ab, abb) over a two-letter alphabet $\{a,b\}$ (see [6]).

In fact, $u(a) = \phi$ and $u(b) = u(c) = -1$. The image under the letter-to-letter projection $a \mapsto a$, $b \mapsto b$, $c \mapsto b$ of ξ is just the fixed point of the invertible substitution (ab, abb). And $(\phi,-1)^t$ is the contracting eigenvector of the matrix of (ab, abb) (the matrix is symmetric). These facts verify our conclusion.

References

[1] P. Arnoux, S. Ito, Pisot substitutions and Rauzy fractals, in: Journées Montoises d'Informatique Théorique, Marne-la-Vallée, 2000, Bull. Belg. Math. Soc. Simon Stevin 8 (2001) 181–207.

[2] H. Ei, S. Ito, H. Rao, Atomic surfaces, tilings and coincidences II. Reducible case, Ann. Inst. Fourier (Grenoble) 56 (7) (2006) 2285–2313.

[3] N. Pytheas Fogg, Substitutions in Dynamics, Arithmetics and Combinatorics, Lecture Notes in Mathematics, vol. 1794, Springer-Verlag, Berlin, 2002.

[4] G. Rauzy, Nombres algébriques et substitutions, Bull. Soc. Math. France 110 (2) (1982) 147–178.

[5] B. Tan, Z.-X. Wen, Y. Zhang, On the triplex substitution – combinatorial properties C. R. Acad. Sci. Paris, Ser. I 346 (2008), in press, doi:10.1016/j.crma.2008.06.013.

[6] Z.-Y. Wen, T. Janssen, F.M. Dekking, Fibonacci chain as a periodic chain with discommensurations, J. Phys. A 27 (5) (1994) 1691–1702.