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Algebra

Extensions with Galois group $2^+S_4 * D_8$ in characteristic 3

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Abstract

For K a field of characteristic 3 we give explicitly the whole family of Galois extensions of K with Galois group $2^+S_4 * D_8$ and determine the discriminant of such an extension. In the case when K is the field of fractions of a formal power series ring in 3 variables, this result is interesting in the context of Abhyankar's Normal Crossings Local Conjecture. ***To cite this article: T. Crespo, Z. Hajto, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Extensions galoisiennes de groupe $2^+S_4 * D_8$ en caractéristique 3. Pour K un corps de caractéristique 3 nous donnons explicitement la famille complète d'extensions de K à groupe de Galois $2^+S_4 * D_8$ et déterminons le discriminant d'une telle extension. Dans le cas où K est le corps de fractions d'un anneau de séries de puissances formelles en 3 variables, ce résultat est intéressant dans le contexte de la Conjecture Locale de Croisements Normaux d'Abhyankar. ***Pour citer cet article : T. Crespo, Z. Hajto, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

1. Introduction

In this Note we give explicitly the whole family of Galois extensions of a field K of characteristic 3 with Galois group $2^+S_4 * D_8$ and determine the discriminant of such an extension. This result improves the one obtained in [3] by dropping the condition assumed there that the fields considered contain the field $\mathbb{F}_9$ of nine elements. In the case when K is the field of fractions of the formal power series ring in 3 variables over a field k of characteristic 3, the explicit determination of its $2^+S_4 * D_8$-coverings and their discriminant is interesting in the context of Abhyankar's Normal Crossings Local Conjecture (see [2,4] as well as the Introduction in [3]).

2. Preliminaries

We denote by 2^+S_n the double cover of the symmetric group S_n in which transpositions lift to involutions and products of two disjoint transpositions lift to elements of order 4 and by D_8 the dihedral group of order 8, which is a

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double cover of the Klein group V_4. We denote by $2^+S_4 * D_8$ the central product of the groups 2^+S_4 and D_8. Let K be a field of characteristic different from 2 and let $\tilde{L}|K$ be a Galois extension with Galois group the group $2^+S_4 * D_8$. Then if L is the field fixed by the center of $2^+S_4 * D_8$, we have $\mathrm{Gal}(L|K) \simeq S_4 \times V_4$ and for L_1, L_2 the fixed subfields of L by V_4 and S_4, respectively, we have $\mathrm{Gal}(L_1|K) \simeq S_4$ and $\mathrm{Gal}(L_2|K) \simeq V_4$. Therefore we obtain the whole family of Galois extensions with Galois group $2^+S_4 * D_8$ of a field K by constructing the whole family of $2^+S_4 * D_8$-extensions containing a given arbitrary S_4-extension of the field K. Let us now be given a polynomial $f(X) \in K[X]$ of degree 4 with Galois group S_4 and splitting field L_1 over K. We want to determine when L_1 is embeddable in a Galois extension of K with Galois group $2^+S_4 * D_8$. This fact is equivalent to the existence of a Galois extension $L_2|K$ with Galois group V_4, disjoint from L_1, and such that, if L is the compositum of L_1 and L_2, the Galois embedding problem

$$2^+S_4 * D_8 \to S_4 \times V_4 \simeq \mathrm{Gal}(L|K) \tag{1}$$

is solvable. We recall that a solution to this embedding problem is a quadratic extension $\tilde{L}$ of the field L, which is a Galois extension of K with Galois group $2^+S_4 * D_8$ and such that the restriction epimorphism between the Galois groups $\mathrm{Gal}(\tilde{L}|K) \twoheadrightarrow \mathrm{Gal}(L|K)$ agrees with the given epimorphism $2^+S_4 * D_8 \twoheadrightarrow S_4 \times V_4$. If $\tilde{L} = L(\sqrt{\gamma})$ is a solution, then the general solution is $L(\sqrt{r\gamma})$, $r \in K^*$. Given a Galois extension $L_1|K$ with Galois group S_4, in order to obtain all $2^+S_4 * D_8$-extensions of K containing L_1, we have to determine all V_4-extensions L_2 of K, disjoint from L_1, and such that the embedding problem (1) is solvable.

Let $E = K[X]/(f(X))$, for $f(X)$ the polynomial of degree 4 realizing L_1 and let d be the discriminant of the polynomial $f(X)$. Let $L_2 = K(\sqrt{a}, \sqrt{b})$. The obstruction to the solvability of the embedding problem (1) is equal to $\mathrm{w}(Q_E).(2,d).(a,b) \in H^2(G_K, \{\pm 1\})$, where Q_E denotes the trace form of the extension $E|K$ and $(\cdot,\cdot)$ a Hilbert symbol (see [3]).

From now on, we assume that K is a field of characteristic 3. We write $f(X) = X^4 + s_2X^2 - s_3X + s_4$. By computation of the trace form Q_E, we obtain that the solvability of the embedding problem (1) is equivalent to

$$\bigl(-ds_2, -\bigl(s_2^2 - s_4\bigr)s_2\bigr) = (a,b). \tag{2}$$

3. Main results

Theorem 3.1. *Let K be a field of characteristic 3, $f(X) = X^4 + s_2X^2 - s_3X + s_4 \in K[X]$, with Galois group S_4 and L_1 the splitting field of $f(X)$ over K. Let $d = s_4^3 + s_2^2 s_4^2 + s_2^4 s_4 - s_2^3 s_3^2$ the discriminant of the polynomial $f(X)$. The family of elements a, b in K such that $(a,b) = (-ds_2, -ms_2)$, where $m := s_2^2 - s_4$, can be given in terms of an arbitrary invertible matrix $P = (p_{ij})_{1 \leqslant i,j, \leqslant 3} \in \mathrm{GL}(3,K)$ as $a = -dA$, $b = -s_2 mF$, where*

$$A = s_2 p_{11}^2 + m p_{21}^2 + dms_2 p_{31}^2,$$
$$F = dmP_{13}^2 + ds_2 P_{23}^2 + P_{33}^2, \quad \text{with } P_{ij} = \begin{vmatrix} p_{ii} & p_{ij} \\ p_{ji} & p_{jj} \end{vmatrix}.$$

Let $L_2 = K(\sqrt{a}, \sqrt{b})$ and assume that $L_2|K$ has Galois group V_4 and $L_1 \cap L_2 = K$ (i.e. that the elements a, b, ab, da, db, dab are not squares in K). Let $L = L_1 \cdot L_2$. For x a root of the polynomial $f(X)$, take $y = a_0 + a_1x + a_2x^2 + a_3x^3$, with

$$a_0 = -s_2 a_2,$$
$$a_1 = -dm(ns_2 p_{11} P_{23} + p_{21} P_{33} + mnp_{21} P_{13} + ds_2 p_{31} P_{23}) + m\sqrt{a}\,(dP_{13} - nP_{33}),$$
$$a_2 = ds_2(p_{11} P_{33} + s_2^2 s_3 p_{11} P_{23} + ms_2 s_3 p_{21} P_{13} - dmp_{31} P_{13}) + s_2\sqrt{a}\,(s_2 s_3 P_{33} - dP_{23}),$$
$$a_3 = -dms_2(s_2 p_{11} P_{23} + mp_{21} P_{13}) - ms_2\sqrt{a}\,P_{33},$$

where $n = s_2^2 + s_4$. Then $L(\sqrt{ry})$, $r \in K^*$, is the general solution to the embedding problem

$$2^+S_4 * D_8 \to S_4 \times V_4 \simeq \mathrm{Gal}(L|K).$$

Proof. By [5], 3.2, the equality of Hilbert symbols (2) is equivalent to the K-equivalence of quadratic forms

$$\langle -ds_2, -ms_2, -dm \rangle \sim \langle a, b, -ab \rangle. \tag{3}$$

The family of quadratic forms K-equivalent to $R := \langle -ds_2, -ms_2, -dm\rangle$ is given by $P^T RP$, for P running over $\mathrm{GL}(3, K)$. By diagonalizing $P^t RP$, we obtain $\langle -dA, -s_2mF, -dAs_2mF\rangle$, with A and F as in the statement. Let $a = -dA$, $b = -s_2mF$. Now, we have $(a, b) = 1 \in H^2(G_{K(\sqrt{a})}, \{\pm 1\})$ and, as $a \notin K^2$ and $L_1 \cap K(\sqrt{a}) = K$, the extension $L_1(\sqrt{a})|K(\sqrt{a})$ has Galois group S_4 and the Galois embedding problem $2^+S_4 \to S_4 \simeq \mathrm{Gal}(L_1(\sqrt{a})|K(\sqrt{a}))$ is solvable. Then, by Abhyankar's Embedding Criterion (see [1,3]), $L_1(\sqrt{a})$ is the splitting field of a polynomial of the form $Y^4 + c_3Y + c_4 \in K(\sqrt{a})[Y]$, so there exists elements $a_0, a_1, a_2, a_3 \in K(\sqrt{a})$ such that the irreducible polynomial over $K(\sqrt{a})$ of the element $y = a_0 + a_1x + a_2x^2 + a_3x^3$ has such a form. By computation, this is equivalent to the conditions $a_0 = -a_2s_2$ and $Q(a_1, a_2, a_3) := s_2a_1^2 + (s_2^2 - s_4)a_2^2 + s_2^3a_3^2 + (s_2^2 + s_4)a_1a_3 + 2s_2s_3a_2a_3 = 0$. Moreover, by Abhyankar's Polynomial Theorem (see [1,3]), the splitting field of the polynomial $\mathrm{Irr}(\sqrt{y}, K(\sqrt{a}))$, that is the field $L_1(\sqrt{a})(\sqrt{y})$, is a solution to the Galois embedding problem $2^+S_4 \to S_4 \simeq \mathrm{Gal}(L_1(\sqrt{a})|K(\sqrt{a}))$. Our aim now is to compute explicitly such elements a_i. Diagonalizing Q, we obtain $\langle s_2, m, s_2md\rangle$ and from (3) we get $\langle s_2, m, s_2md\rangle \sim \langle A, s_2mAF, s_2mFd\rangle$ and the basis change matrix can be written down explicitly in terms of the matrix P. Now the vector $(0, d, \sqrt{a}) \in K(\sqrt{a})^3$ anihilates the quadratic form $\langle A, s_2mAF, s_2mFd\rangle$ and from it we obtain the values for $a_1, a_2, a_3 \in K(\sqrt{a})$ such that $Q(a_1, a_2, a_3) = 0$.

We want to see now that $L(\sqrt{y})|K$ is a Galois extension with Galois group $2^+S_4 * D_8$. By the assumption $L_1 \cap L_2 = K$, we have $\mathrm{Gal}(L(\sqrt{y})|L_2) \simeq 2^+S_4$. We consider now the behaviour of y under the action of $\mathrm{Gal}(L_2|K)$. Let r, s, t be the non-trivial elements of $\mathrm{Gal}(L_2|K)$ fixing respectively $\sqrt{ab}$, $\sqrt{b}$, $\sqrt{a}$. By computation we obtain $y^s y = d^2h^2b$, where $h = ms_2p_{31}x^3 - (p_{21} + s_2^2s_3p_{31})x^2 + (mnp_{31} - p_{11})x + s_2^3s_3p_{31} + s_2p_{21}$. Now $y \in K(\sqrt{a})(x)$, so $y^t = y$ and $y^r = y^s$, so $L(\sqrt{y})$ is Galois over K. Now we have $(dh\sqrt{b})^s = dh\sqrt{b}$ and $(dh\sqrt{b})^r = -dh\sqrt{b}$, so $\mathrm{Gal}(L(\sqrt{y})|L_1) \simeq D_8$, with $L(\sqrt{y})|L_1(\sqrt{ab})$ cyclic, hence $\mathrm{Gal}(L(\sqrt{y})|K) \simeq 2^+S_4 * D_8$. □

Proposition 3.2. *Let the fields K and L and the elements s_2, s_3, s_4, d, a, b, m, p_{ij} and y be as in Theorem* 3.1. *We have*

$$\mathrm{disc}\big(L(\sqrt{y})|K\big) = d^{144}a^{96}b^{120}D^{12}$$

where

$$\begin{aligned} D = {} & s_4p_{11}^4 - s_2s_3p_{11}^3p_{21} + ms_2p_{11}^2p_{21}^2 - ms_3p_{11}p_{21}^3 + (m^2 - s_2s_3^2)p_{21}^4 \\ & + dp_{31}\big(p_{11}^3 + ms_2^2p_{11}^2p_{31} + s_3p_{21}^3 + m^2s_2p_{31}p_{21}^2\big) - d^2p_{31}^3(s_2s_3p_{21} + mp_{11}) + d^3p_{31}^4. \end{aligned}$$

Proof. We have $\mathrm{disc}(L(\sqrt{y})|K) = \mathrm{disc}(L|K)^2 \cdot N_{L|K}(y)$ and $\mathrm{disc}(L|K) = (dab)^{48}$. Now

$$N_{L|K}(y) = \big(N_{L_1(\sqrt{a})|K}(y)\big)^2$$

and

$$N_{L_1(\sqrt{a})|K}(y) = N_{L_1|K}\big(N_{L_1(\sqrt{a})|L_1}(y)\big) = N_{L_1|K}\big(d^2h^2b\big) = d^{48}b^{24}N_{L_1|K}(h)^2,$$

for h as in the proof of Theorem 3.1. By computation, we obtain $N_{L_1|K}(h) = D^6$, for D as in the statement. □

4. Example

Let $K = k((Z_1, Z_2, Z_3))$ be the field of fractions of the formal power series ring in 3 variables over a field k of characteristic 3. We consider the family of polynomials $f_l(X) = X^4 + Z_1X^2 + Z_2X + Z_3^l \in K[X]$, where l is a positive integer number, i.e. we are taking $s_2 = Z_1$, $s_3 = -Z_2$, $s_4 = Z_3^l$. We can check that the polynomial f_l has Galois group S_4 over K, for all $l \in \mathbb{N}$, and let L_1 be the splitting field of f over K. We consider the extension $L_2|K$ generated by the elements $\sqrt{-ds_2}$, $\sqrt{-ms_2}$, $\sqrt{-dm}$. We can check that the elements $-ds_2$, $-ms_2$, $-dm$, $-s_2$, $-dms_2$, $-m$ are not squares in K and so, $L_2|K$ has Galois group V_4 and is disjoint with $L_1|K$. Let $L = L_1 \cdot L_2$. Let y be the element given by Theorem 3.1 for the matrix P, such that $p_{12} = p_{23} = p_{31} = 1$ and the other entries are equal to zero. Then we have $\mathrm{Gal}(L(\sqrt{y})|K) \simeq 2^+S_4 * D_8$, with $L(\sqrt{y})|L_1(\sqrt{-ds_2})$ cyclic. By applying Proposition 3.2, we see that the discriminant locus remains unchanged when going from L to $L(\sqrt{y})$.

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