



Partial Differential Equations

Exponential rates of convergence by an iteration technique

Michel Chipot, Karen Yeessian

Institute of Mathematics, University of Zürich, Winterthurerstrasse 190, CH-8057 Zurich, Switzerland

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Abstract

The goal of this Note is to introduce a technique leading to a convergence of exponential type for the solution of problems set in cylinders becoming unbounded in some directions. **To cite this article:** *M. Chipot, K. Yeessian, C. R. Acad. Sci. Paris, Ser. I 346 (2008).*

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Résumé

Convergences exponentielles par une technique itérative. Le but de cette Note est la présentation d’une technique conduisant à une convergence de type exponentiel pour la solution de problèmes posés dans des cylindres dont certaines directions tendent vers l’infini. **Pour citer cet article :** *M. Chipot, K. Yeessian, C. R. Acad. Sci. Paris, Ser. I 346 (2008).*

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Version française abrégée

Supposons que pour $\ell > 0$, Ω_ℓ soit le rectangle défini par

$$\Omega_\ell = (-\ell, \ell) \times (-1, 1). \tag{1}$$

Notons $A(x)$ la matrice

$$A(x) = \begin{pmatrix} a_{11}(x) & a_{12}(x_2) \\ a_{21}(x) & a_{22}(x_2) \end{pmatrix}, \tag{2}$$

et supposons que cette matrice soit définie positive à coefficients bornés i.e. telle que

$$a_{ij} \in L^\infty(\mathbb{R} \times (-1, 1)), \quad \lambda|\xi|^2 \leq A(x)\xi \cdot \xi \quad \forall \xi \in \mathbb{R}^2, \text{ p.p. } x \in \mathbb{R} \times (-1, 1). \tag{3}$$

(Les points de \mathbb{R}^2 sont notés $x = (x_1, x_2)$, « \cdot » est le produit scalaire canonique de \mathbb{R}^2 , $|\cdot|$ la norme associée, λ une constante positive.)

Soit $f = f(x_2)$ une fonction (ou distribution) ne dépendant que de x_2 , par exemple telle que

$$f \in L^2((-1, 1)). \tag{4}$$

E-mail addresses: m.m.chipot@math.unizh.ch (M. Chipot), karen.yeessian@math.unizh.ch (K. Yeessian).

Sous les conditions ci-dessus il existe un unique u_ℓ solution de

$$\begin{cases} u_\ell \in H_0^1(\Omega_\ell), \\ \int_{\Omega_\ell} A(x) \nabla u_\ell \cdot \nabla v \, dx = \int_{\Omega_\ell} f v \, dx \quad \forall v \in H_0^1(\Omega_\ell). \end{cases} \quad (5)$$

Nous voulons établir dans cette note que lorsque ℓ tend vers plus l'infini u_ℓ converge sur tout sous rectangle Ω_{ℓ_0} , $\ell_0 < \ell$, vers la fonction u_∞ où u_∞ est la solution de

$$\begin{cases} u_\infty \in H_0^1((-1, 1)), \\ \int_{-1}^1 \partial_{x_2} u_\infty \partial_{x_2} v \, dx_2 = \int_{-1}^1 f(x_2) v \, dx_2 \quad \forall v \in H_0^1((-1, 1)), \end{cases} \quad (6)$$

ceci avec une vitesse de convergence exponentielle i.e. en $e^{-\alpha\ell}$, $\alpha > 0$.

Remarque 1. Nous ne précisons pas pour quelle norme cette convergence a lieu puisque de multiples possibilités sont offertes par notre méthode. Dans le cas où A est une matrice diagonale le résultat était déjà connu (cf. [2,4]) mais par des méthodes totalement différentes.

Remarque 2. On notera que le résultat est indépendant des conditions aux limites choisies aux extrémités de Ω_ℓ i.e. sur $\{\ell, -\ell\} \times (-1, 1)$ et qu'il n'est pas limité aux équations du second degré, voir [5,3] pour plus de détails, voir également [1,10].

Remarque 3. Une fonction f indépendante de x_1 est une fonction périodique de période arbitraire. L'indépendance de f par rapport à x_1 force u_ℓ à la limite à ne dépendre que de la section du cylindre. On peut adapter notre méthode au cas d'une fonction périodique f , comme, par exemple, dans [6,7,11], démontrant que f périodique force u_ℓ à devenir périodique à la limite ceci exponentiellement rapidement sur tout sous domaine.

Remarque 4. Le cadre adopté dans cette version abrégée est le plus simple possible pour des raisons didactiques. Nos techniques s'étendent à des situations beaucoup plus générales comme on pourra le voir plus loin, aux systèmes et aux problèmes non linéaires (cf. [5,3]).

1. A general result for second order elliptic equations

We denote a point $x \in \mathbb{R}^n$ also as $x = (X_1, X_2)$ where

$$X_1 = (x_1, \dots, x_p), \quad X_2 = (x_{p+1}, \dots, x_n) \quad (7)$$

i.e. we split the components of a point in \mathbb{R}^n into the p first components and the $n - p$ last ones.

Let ω_1 be an open subset of \mathbb{R}^p that we suppose to satisfy

$$\omega_1 \text{ is bounded and star-shaped with respect to } 0. \quad (8)$$

Let ω_2 be a bounded open subset of \mathbb{R}^{n-p} , then we set

$$\Omega_\ell = \ell\omega_1 \times \omega_2. \quad (9)$$

Remark 1.1. In our introduction we had $\omega_1 = \omega_2 = (-1, 1)$. ω_1 can be for instance a unit ball B_1 (for an arbitrary norm), then $\Omega_\ell = B_\ell \times \omega_2$.

We denote by

$$A(x) = \begin{pmatrix} A_{11}(X_1, X_2) & A_{12}(X_2) \\ A_{21}(X_1, X_2) & A_{22}(X_2) \end{pmatrix} = (a_{ij}(x)) \quad (10)$$

a $n \times n$ -matrix divided into four blocks such that

$$A_{11} \text{ is a } p \times p\text{-matrix,} \quad A_{22} \text{ is a } (n - p) \times (n - p)\text{-matrix.} \quad (11)$$

We assume that

$$a_{ij} \in L^\infty(\mathbb{R}^P \times \omega_2) \tag{12}$$

and that for some constants λ, Λ we have

$$\lambda|\xi|^2 \leq A(x)\xi \cdot \xi \quad \forall \xi \in \mathbb{R}^n, \text{ a.e. } x \in \mathbb{R}^P \times \omega_2, \tag{13}$$

$$|A(x)\xi| \leq \Lambda|\xi| \quad \forall \xi \in \mathbb{R}^n, \text{ a.e. } x \in \mathbb{R}^P \times \omega_2. \tag{14}$$

Then by the Lax–Milgram theorem (see [8,9]) for $f \in H^{-1}(\omega_2)$ there exists a unique u_∞ solution to

$$u_\infty \in H_0^1(\omega_2), \quad \int_{\omega_2} A_{22} \nabla_{X_2} u_\infty \cdot \nabla_{X_2} v \, dX_2 = \langle f, v \rangle \quad \forall v \in H_0^1(\omega_2). \tag{15}$$

(In the above system ∇_{X_2} stands for the gradient in X_2 , that is $(\partial_{x_{p+1}}, \dots, \partial_{x_n})$, $dX_2 = dx_{p+1} \cdots dx_n$ and $\langle \cdot, \cdot \rangle$ denotes the duality, here between $H^{-1}(\omega_2)$ and $H_0^1(\omega_2)$.)

Let us denote

$$H_{\text{lat}}^1(\Omega_\ell) = \{v \in H^1(\Omega_\ell) \mid v = 0 \text{ on } \ell\omega_1 \times \partial\omega_2\} \tag{16}$$

i.e. the set of functions in $H^1(\Omega_\ell)$ vanishing on the lateral boundary of Ω_ℓ . Then for $v \in H_{\text{lat}}^1(\Omega_\ell)$

$$v \longmapsto \int_{\ell\omega_1} \langle f, v(X_1, \cdot) \rangle \, dX_1 \tag{17}$$

defines a continuous linear form that we will yet denote by $\langle f, \cdot \rangle$.

Let V_ℓ be a closed subspace of $H_{\text{lat}}^1(\Omega_\ell)$, equipped with the $H^1(\Omega_\ell)$ topology such that

$$H_0^1(\Omega_\ell) \subset V_\ell \subset H_{\text{lat}}^1(\Omega_\ell). \tag{18}$$

(The first inclusion is only useful for our convergence result.) By the Lax–Milgram theorem there exists a unique u_ℓ solution to

$$u_\ell \in V_\ell, \quad \int_{\Omega_\ell} A \nabla u_\ell \cdot \nabla v \, dx = \langle f, v \rangle \quad \forall v \in V_\ell. \tag{19}$$

Moreover we have

Theorem 1.1. *There exists two constants $c, \alpha > 0$ independent of ℓ such that*

$$\int_{\Omega_\ell} |\nabla(u_\ell - u_\infty)|^2 \, dx \leq c e^{-\alpha\ell} |f|_{\star}^2 \tag{20}$$

($|\cdot|_{\star}$ denotes the strong dual norm in $H^{-1}(\omega_2)$).

Proof. The proof is divided into three steps.

• **Step 1.** The equation satisfied by $u_\ell - u_\infty$.

If $v \in H_{\text{lat}}^1(\Omega_\ell)$ then for almost every X_1 in $\ell\omega_1$ we have

$$v(X_1, \cdot) \in H_0^1(\omega_2)$$

and thus by (15)

$$\int_{\omega_2} A_{22} \nabla_{X_2} u_\infty \cdot \nabla_{X_2} v(X_1, \cdot) \, dX_2 = \langle f, v(X_1, \cdot) \rangle.$$

Integrating in X_1 we get

$$\int_{\Omega_\ell} A_{22} \nabla_{X_2} u_\infty \cdot \nabla_{X_2} v \, dx = \langle f, v \rangle \quad \forall v \in H_{\text{lat}}^1(\Omega_\ell) \tag{21}$$

(see (17) for the definition of $\langle f, v \rangle$). Now for $v \in H_0^1(\Omega_\ell)$ we have

$$\begin{aligned} \int_{\Omega_\ell} A \nabla u_\infty \cdot \nabla v \, dx &= \int_{\Omega_\ell} A_{12} \nabla_{X_2} u_\infty \cdot \nabla_{X_1} v \, dx + \int_{\Omega_\ell} A_{22} \nabla_{X_2} u_\infty \cdot \nabla_{X_2} v \, dx \\ &= \int_{\Omega_\ell} A_{22} \nabla_{X_2} u_\infty \cdot \nabla_{X_2} v \, dx = \langle f, v \rangle \end{aligned} \quad (22)$$

(since A_{12}, u_∞ are depending on X_2 only).

Combining (18), (19), (22) we get

$$\int_{\Omega_\ell} A \nabla(u_\ell - u_\infty) \cdot \nabla v \, dx = 0 \quad \forall v \in H_0^1(\Omega_\ell). \quad (23)$$

• **Step 2.** An iteration technique.

Set $0 < \ell_0 \leq \ell - 1$. In addition to (8) let us assume that there exists ρ a function of X_1 only such that

$$0 \leq \rho \leq 1, \quad \rho = 1 \quad \text{on } \ell_0 \omega_1, \quad \rho = 0 \quad \text{on } \mathbb{R}^p \setminus (\ell_0 + 1) \omega_1, \quad |\nabla_{X_1} \rho| \leq c_0 \quad (24)$$

where c_0 is a universal constant. Such a function does exist in many instances. Then we have

$$(u_\ell - u_\infty) \rho^2 \in H_0^1(\Omega_\ell)$$

and from (23) we derive

$$\begin{aligned} \int_{\Omega_\ell} A \nabla(u_\ell - u_\infty) \cdot \nabla(u_\ell - u_\infty) \rho^2 \, dx &= -2 \int_{\Omega_\ell} A \nabla(u_\ell - u_\infty) \cdot \begin{pmatrix} \nabla_{X_1} \rho \\ 0 \end{pmatrix} (u_\ell - u_\infty) \rho \, dx \\ &\leq 2 \int_{\Omega_{\ell_0+1} \setminus \Omega_{\ell_0}} |A \nabla(u_\ell - u_\infty)| |\nabla_{X_1} \rho| |u_\ell - u_\infty| \rho \, dx. \end{aligned}$$

Using (13), (14), (24) and the Cauchy–Schwarz inequality we derive

$$\begin{aligned} \lambda \int_{\Omega_\ell} |\nabla(u_\ell - u_\infty)|^2 \rho^2 \, dx &\leq 2c_0 \Lambda \int_{\Omega_{\ell_0+1} \setminus \Omega_{\ell_0}} |\nabla(u_\ell - u_\infty)| \rho |u_\ell - u_\infty| \, dx \\ &\leq 2c_0 \Lambda \left(\int_{\Omega_\ell} |\nabla(u_\ell - u_\infty)|^2 \rho^2 \, dx \right)^{\frac{1}{2}} \left(\int_{\Omega_{\ell_0+1} \setminus \Omega_{\ell_0}} (u_\ell - u_\infty)^2 \, dx \right)^{\frac{1}{2}}. \end{aligned}$$

It follows that (recall that $\rho = 1$ on Ω_{ℓ_0})

$$\int_{\Omega_{\ell_0}} |\nabla(u_\ell - u_\infty)|^2 \, dx \leq \left(2c_0 \frac{\Lambda}{\lambda} \right)^2 \int_{\Omega_{\ell_0+1} \setminus \Omega_{\ell_0}} (u_\ell - u_\infty)^2 \, dx. \quad (25)$$

Since $u_\ell - u_\infty$ vanishes on the lateral boundary of Ω_ℓ there exists a constant c_p independent of ℓ such that (see [2])

$$\int_{\Omega_{\ell_0+1} \setminus \Omega_{\ell_0}} (u_\ell - u_\infty)^2 \, dx \leq c_p^2 \int_{\Omega_{\ell_0+1} \setminus \Omega_{\ell_0}} |\nabla_{X_2}(u_\ell - u_\infty)|^2 \, dx. \quad (26)$$

Combining this Poincaré inequality with (25) we get

$$\int_{\Omega_{\ell_0}} |\nabla(u_\ell - u_\infty)|^2 \, dx \leq C \int_{\Omega_{\ell_0+1} \setminus \Omega_{\ell_0}} |\nabla(u_\ell - u_\infty)|^2 \, dx \quad (27)$$

which is also

$$\int_{\Omega_{\ell_0}} |\nabla(u_\ell - u_\infty)|^2 dx \leq \frac{C}{1+C} \int_{\Omega_{\ell_0+1}} |\nabla(u_\ell - u_\infty)|^2 dx$$

where $C = (2c_0c_p \frac{A}{\lambda})^2$. Iterating this formula starting from $\frac{\ell}{2}$ we obtain

$$\int_{\Omega_{\frac{\ell}{2}}} |\nabla(u_\ell - u_\infty)|^2 dx \leq \left(\frac{C}{1+C}\right)^{[\frac{\ell}{2}]} \int_{\Omega_{\frac{\ell}{2}+[\frac{\ell}{2}]}} |\nabla(u_\ell - u_\infty)|^2 dx$$

where $[\frac{\ell}{2}]$ denotes the integer part of $\frac{\ell}{2}$. Since $\frac{\ell}{2} - 1 < [\frac{\ell}{2}] \leq \frac{\ell}{2}$, it comes

$$\int_{\Omega_{\frac{\ell}{2}}} |\nabla(u_\ell - u_\infty)|^2 dx \leq e^{(-\frac{\ell}{2}+1)\ln(\frac{1+C}{C})} \int_{\Omega_\ell} |\nabla(u_\ell - u_\infty)|^2 dx = c_1 e^{-\alpha_0 \ell} \int_{\Omega_\ell} |\nabla(u_\ell - u_\infty)|^2 dx \tag{28}$$

where $c_1 = \frac{1+C}{C}$ and $\alpha_0 = \frac{1}{2} \ln(\frac{1+C}{C})$.

• **Step 3.** Evaluation of the last integral.

Taking $v = u_\ell$ in (19) we get

$$\begin{aligned} \int_{\Omega_\ell} A \nabla u_\ell \cdot \nabla u_\ell dx &= \langle f, u_\ell \rangle = \int_{\ell\omega_1} \langle f, u_\ell(X_1, \cdot) \rangle dX_1 \\ &\leq \int_{\ell\omega_1} |f|_\star |\nabla_{X_2} u_\ell(X_1, \cdot)|_{L^2(\omega_2)} dX_1 \leq |\ell\omega_1|^{\frac{1}{2}} \left(\int_{\Omega_\ell} |\nabla u_\ell|^2 dx \right)^{\frac{1}{2}} |f|_\star \end{aligned}$$

($|\cdot|$ denotes the measure of sets), from which we derive

$$\int_{\Omega_\ell} |\nabla u_\ell|^2 dx \leq \frac{|\ell\omega_1|}{\lambda^2} |f|_\star^2 = \frac{|\omega_1| |f|_\star^2}{\lambda^2} \ell^p.$$

Similarly taking $v = u_\infty$ in (15) we get

$$\int_{\omega_2} |\nabla_{X_2} u_\infty|^2 dx \leq \frac{|f|_\star^2}{\lambda^2}$$

and thus

$$\int_{\Omega_\ell} |\nabla(u_\ell - u_\infty)|^2 dx \leq 2 \int_{\Omega_\ell} (|\nabla u_\ell|^2 + |\nabla u_\infty|^2) dx \leq \frac{4|\omega_1| |f|_\star^2}{\lambda^2} \ell^p.$$

The estimate (20) follows then from (28) where α can be chosen as any constant smaller than α_0 and c large enough. \square

Remark 1.2. In the case of a diagonal matrix A such result was already known (see for instance [2,4]).

Remark 1.3. A function f independent of X_1 is a periodic function – for any period. The independence of f from X_1 forces u_ℓ at the limit to depend only on X_2 . We can adapt our method in the case of a periodic f , in the spirit of [6,7,11], showing that f periodic forces u_ℓ to become periodic at the limit exponentially quickly.

The method is not restricted to second order elliptic equations. It extends to many other situations, to more general domains, to nonlinear problems and systems (see [5,3,2,10,11]). Note also that our convergence technique applies to other norms and is not restricted to the H^1 -norms.

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