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Probability Theory

Stable distributions and harmonic analysis on convex cones

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Abstract

We consider α-stable random elements in a general convex cone $\mathbb{K}$ and show how their main properties: sign and range of α, LePage representation and Lévy decomposition are related to the algebraic and metric properties of $\mathbb{K}$: distributivity laws, coincidence of the origin and the neutral elements, existence of a homogeneous norm, and separating family of semi-continuous characters. ***To cite this article: Y. Davydov et al., C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

Lois stables et analyse harmonique sur les cônes convexes. On considère les éléments aléatoires α-stables à valeurs dans un cône convexe abstrait $\mathbb{K}$. On montre que les propriétés principales de stabilité (telles que : le signe et l'intervalle du paramètre α, la représentation de LePage et la décomposition de Lévy) sont étroitement liées aux propriétés algébriques et métriques de $\mathbb{K}$ (telles que : les lois de distributivité, coïncidence de l'origine et de l'élément neutre, l'existence de la norme homogène et d'une famille de caractères semi-continus). ***Pour citer cet article : Y. Davydov et al., C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Version française abrégée

Soit $\mathbb{K}$ un cône convexe, c'est-à-dire un semigroupe topologique métrisable abélien muni d'une opération commutative et associative $+$ et d'une opération de multiplication par des nombres non-négatifs. Nous supposons qu'il existe un *élément neutre* $\mathbf{e}$ tel que $x + \mathbf{e} = x$, $\forall x \in \mathbb{K}$. Le cône est appelé *pointu* s'il a un élément unique dit *l'origine* $\mathbf{0}$ tel que $\lim_{a\downarrow 0} ax = \mathbf{0}$, $\forall x \in \mathbb{K} \setminus \{\mathbf{e}\}$. Nous supposons la validité de la première loi de distributivité : $a(x + y) = ax + ay$, mais pas nécessairement de la deuxième loi : $(a + b)x = ax + bx$ pour tous $x, y \in \mathbb{K}$, $a, b > 0$. Même si elle est normalement supposée dans la littérature sur les cône convexe, cette contrainte réduit considérablement le nombre des cas intéressants. Si la topologie est métrisable par une norme homogène en $\mathbf{0}$ telle que $d(x + h, x) \leqslant d(x, \mathbf{0}) = \|h\|$ pour tous $x, h \in \mathbb{K}$, on appelle $\mathbb{K}$ cône avec une norme *sous-invariante*.

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Un élément aléatoire ξ sur $\mathbb{K}$ est dit *strictement stable* de paramètre $\alpha \neq 0$ (noté $S\alpha S$) s'il satisfait (1). La représentation de LePage pour ξ est donnée par (2). Si $\mathbb{K}$ est un espace de Banach, on sait que (2) est vraie pour tout ξ $S\alpha S$ avec $\alpha \in (0,2)$.

Soit $\mathbb{S} = \{x\colon \|x\| = 1\}$ la sphère unitaire et $\mathbb{K}' = \mathbb{K} \setminus \{\mathbf{0}, \mathbf{e}\}$. Introduisons le processus de Poisson $\Pi_\alpha = \Pi_{\alpha,\sigma}$ sur $\mathbb{K}'$ avec la mesure d'intensité $\theta_\alpha \times \sigma$ en coordonées polaires $\mathbb{R}_+ \times \mathbb{S}$, où $\theta_\alpha\{(r,+\infty)\} = r^{-\alpha}$ si $\alpha > 0$, et $\theta_\alpha\{(0,r)\} = r^{-\alpha}$ si $\alpha < 0$, $r > 0$, σ étant une mesure σ-finie dite *mesure spectrale* sur $\mathbb{S}$.

Le Théorème 2.1 montre que le processus Π_α est une mesure aléatoire stable, c.à.d. satisfaisant (4). Puisque la série de LePage (2) s'écrit comme $\sum_{x_i \in \Pi_\alpha} x_i = \int x\, \Pi_\alpha(\mathrm{d}x)$, la convergence de la valeur principale de cette intégrale donne un élément $S\alpha S$ dans $\mathbb{K}$, voir le Théorème 2.2.

Soit ξ_k, $k \geqslant 1$, des éléments i.i.d. dans $\mathbb{K}$ avec une norme sous-invariante et $\{\beta_n, n \geqslant 1\}$ une suite de processus binomiaux $\beta_n = \sum_{k=1}^n \delta_{\xi_k/b_n}$, où $b_n = n^{1/\alpha} L(n)$ avec $\alpha \neq 0$ et une fonction L à variation lente à l'infini. Sous la condition de variation régulière (6) de la queue de la répartition de ξ_k, on montre que la suite β_n converge vers Π_α dans une topologie plus forte que la topologie faible impliquant pour $\alpha \in (0,1)$ la convergence des sommes correspondantes $\int x\, \beta_n(\mathrm{d}x) = b_n^{-1}(\xi_1 + \cdots + \xi_n)$ vers $\int x\, \Pi_\alpha(\mathrm{d}x)$, voir Théorème 2.3.

Dans les espaces de Banach, ce type de convergence est obtenu par l'application du centrage et de la symétrisation ce qui n'est plus généralement possible dans le cadre des semigroupes car l'opération addition n'est pas inversible en général. Un analogue du centrage dans des semigroupes est une *involution* qui est une application continue $\star\colon \mathbb{K} \mapsto \mathbb{K}$ satisfaisant $(x+y)^\star = y^\star + x^\star$, $(x^\star)^\star = x$ et $(ax)^\star = ax^\star$ pour tous $x, y \in \mathbb{K}$ et $a > 0$. Une fonction χ de $\mathbb{K}$ dans le disque unitaire du plan complexe s'appelle *un caractère* si $\chi(\mathbf{e}) = 1$, $\chi(x+y) = \chi(x)\chi(y)$ et $\chi(x^\star) = \overline{\chi(x)}$ pour tous $x, y \in \mathbb{K}$. Un semigroupe des caractères $\widetilde{\mathbb{K}}$ s'appelle *séparable*, si pour tous éléments $x \neq y$ de $\mathbb{K}$ il existe un caractère $\chi \in \widetilde{\mathbb{K}}$ tel que $\chi(x) \neq \chi(y)$. Si l'ensemble $\widetilde{\mathbb{K}}$ engendre la tribu borelienne, alors la transformée de Laplace $\mathbf{E}\chi(\xi) = \int_{\mathbb{K}} \chi(x) \mathbf{P}(\mathrm{d}x)$, $\chi \in \widetilde{\mathbb{K}}$ définit uniquement la loi de l'élément aléatoire ξ. La loi de ξ est dite *propre*, si $\mathbf{E}\chi(\xi) \neq 0$ pour tout $\chi \in \widetilde{\mathbb{K}}$. Les Théorèmes 3.1 et 3.2 montrent que la deuxième loi de distributivité a l'impact principal sur les valeurs possibles du paramètre α de la loi $S\alpha S$ propre tandis que son signe est largement dû à la relation entre les éléments $\mathbf{e}$ et $\mathbf{0}$.

Comme la loi $S\alpha S$ est infiniment divisible, sa transformée de Laplace peut s'écrire $\mathbf{E}\chi(\xi) = \exp\{-\phi(\chi)\}$, où ϕ est une fonction définie négative (la fonction de Lévy) donnée par (7). La mesure d'intégration λ (la mesure de Lévy) vit sur le semigroupe $\mathbb{K}^\sharp$ de tous les caractères sur $\widetilde{\mathbb{K}}$. Le Théorème 3.4 assure que la mesure de Lévy a une masse infinie et qu'elle est homogène d'ordre α. La loi gaussienne sur un semigroupe qui correspond à la forme quadratique dans la représentation de Lévy, peut avoir des propriétés inhabituelles comme le Théorème 3.5 et la discussion juste avant le montrent. Une généralisation de la représentation de LePage est donnée par le Théorème 3.6, montrant que l'image $\imath(\xi)$ par l'application dite *d'évaluation* est une série basée sur un processus de Poisson dans $\mathbb{K}^\sharp$. Sous certaines conditions de type réflexivité, cette représentation peut être portée par $\mathbb{K}$ donnant la série de LePage canonique (8) pour l'élément ξ lui-même. Dans ce cas λ est l'image inverse d'une mesure $\Lambda = \theta_\alpha \times \sigma$ sur $\mathbb{K}$.

Notre cadre couvre, parmi d'autres, des modèles avec la deuxième loi de distributivité (les espaces de Banach, les nombres non-négatifs, les ensembles compacts convexes avec la sommation de Minkowski comme l'addition, les fonctions semi-continues supérieurement, les mesures aléatoires finies) aussi bien que des modèles sans la deuxième loi de distributivité (les valeurs extrémales et l'algèbre (max, +), les fonctions continues avec le produit et le changement d'échelle comme opérations, les ensembles compacts avec l'union comme addition et l'homothétie comme multiplication, les mesures aléatoires probabilistes intégrables avec convolution, les mesures aléatoires à support borné et le changement d'échelle de l'espace de phase comme multiplication). Pour les détails, voir [3].

1. Introduction and basic settings

A random vector ξ in a Banach space $\mathbb{B}$ is *strictly stable with characteristic exponent* $0 < \alpha \leqslant 2$ (notation $S\alpha S$) if for all $a, b > 0$,

$$a^{1/\alpha}\xi^{(1)} + b^{1/\alpha}\xi^{(2)} \stackrel{\mathcal{D}}{=} (a+b)^{1/\alpha}\xi, \tag{1}$$

where $\xi^{(1)}, \xi^{(2)}$ are independent realisations of ξ, and $\stackrel{\mathcal{D}}{=}$ denotes equality in distribution. Recall that $S\alpha S$ vectors with $\alpha \in (0,2)$ admit the *LePage series representation*:

$$\xi \overset{\mathcal{D}}{=} c \sum_{j=1}^{\infty} \Gamma_j^{-1/\alpha} \varepsilon_j, \tag{2}$$

where $\Gamma_1, \Gamma_2, \dots$ are the successive jumps of a homogeneous Poisson process on $\mathbb{R}_+$, and $\varepsilon_1, \varepsilon_2, \dots$ are i.i.d. random vectors on unit sphere $\mathbb{S}$ independent of the Γ's. The distribution of ε_i is defined by a finite *spectral measure* σ on $\mathbb{S}$.

Definition (1) makes sense for ξ being a random element in any space $\mathbb{K}$ where addition and multiplication by non-negative numbers is defined, i.e. in a *convex cone* $\mathbb{K}$, which is an Abelian *topological semigroup*, i.e. a topological space equipped with a commutative and associative continuous composition $+$. It is assumed that $\mathbb{K}$ possesses the *neutral element* $\mathbf{e}$ satisfying $x + \mathbf{e} = x$ for every $x \in \mathbb{K}$ and that $\mathbb{K}$ is complete. A *convex cone* is an Abelian topological semigroup $\mathbb{K}$ being a metrisable Polish (complete separable) space with a continuous operation $(x, a) \mapsto ax$ of multiplication by positive scalars for $x \in \mathbb{K}$ and $a > 0$ satisfying $a(bx) = (ab)x$, $1x = x$, $a\mathbf{e} = \mathbf{e}$ and the following *first distributivity law*:

$$a(x + y) = ax + ay, \quad a > 0,\ x, y \in \mathbb{K}. \tag{3}$$

We call $\mathbb{K}$ a *pointed cone* if there exists a *unique* $\mathbf{0} \in \mathbb{K}$ called the *origin* such that ax converges to $\mathbf{0}$ as $a \downarrow 0$ for all $x \in \mathbb{K} \setminus \{\mathbf{e}\}$. A pointed cone $\mathbb{K}$ is a *normed cone* if $\mathbb{K}$ (or $\mathbb{K} \setminus \{\mathbf{e}\}$ if $\mathbf{0} \neq \mathbf{e}$) is metrisable by a metric d which is *homogeneous* at the origin, i.e. $d(ax, \mathbf{0}) = ad(x, \mathbf{0})$ for every $a > 0$ and $x \in \mathbb{K}$. The value $\|x\| = d(x, \mathbf{0})$ is called the *norm* of x. Actually, $\|\cdot\|$ is a *gauge function* as we often do not require the triangular inequality. But if it holds, and $d(x + h, x) \leqslant d(h, \mathbf{0}) = \|h\|$ for all $x, h \in \mathbb{K}$, we call the metric (and the norm) *sub-invariant*. This guarantees the uniform continuity of the norm, i.e. closeness of $\|x\|$ and $\|x + h\|$ when $\|h\|$ is small.

By assuming the first distributivity (3), *we do not require* the *second distributivity law*: $(a + b)x = ax + bx$ for all $a, b > 0$ and $x \in \mathbb{K}$, so, in general, $2x \neq x + x$. Although this law is typically imposed in the literature on convex cones [4], the second distributivity drastically reduces the number of interesting cases and the extent it is violated actually determines the range of possible α as we show below.

It can be shown that if $\mathbb{K}$ is a pointed cone, then $\mathbf{0} + \mathbf{0} = \mathbf{0}$, $\mathbf{0}$ and $\mathbf{e}$ are the only elements of $\mathbb{K}$ satisfying $ax = x$ for all $a > 0$; the second distributivity implies $\mathbf{0} = \mathbf{e}$; and if there exists $x \neq \mathbf{e}$ which possesses an inverse $(-x)$, i.e. $x + (-x) = \mathbf{e}$, then $\mathbf{e} = \mathbf{0}$.

2. Stability and LePage representation

A normed cone $\mathbb{K}$ is said to admit a *polar decomposition* if there is a bijection $x \leftrightarrow (\|x\|, x/\|x\|)$ between $\mathbb{K}' = \mathbb{K} \setminus \{\mathbf{0}, \mathbf{e}\}$ and $(0, \infty) \times \mathbb{S}$, where $\mathbb{S} = \{x\colon \|x\| = 1\}$ is the unit sphere.

Denote by $\Pi_\alpha = \Pi_{\alpha,\sigma}$ a Poisson point process in $\mathbb{K}'$ with intensity measure $\theta_\alpha \times \sigma$ in polar coordinates $\mathbb{R}_+ \times \mathbb{S}$, where $\theta_\alpha\{(r, +\infty)\} = r^{-\alpha}$ if $\alpha > 0$ or $\theta_\alpha\{(0, r)\} = r^{-\alpha}$ if $\alpha < 0$, $r > 0$ and σ is a σ-finite *spectral measure* on $\mathbb{S}$. LePage series (2) can then be written as $\sum_{x_i \in \Pi_\alpha} x_i = \int x\, \Pi_\alpha(\mathrm{d}x)$. The following theorem shows that Π_α is a stable random measure which will act as a 'building block' for construction of stable elements. It can be derived from basic facts about transformation of Poisson process and its intensity measure:

Theorem 2.1. *Poisson process $\Pi_\alpha = \Pi_\alpha(\sigma)$ with any finite spectral measure σ is a $S\alpha S$* random measure *with respect to measure addition and scaling, i.e.*

$$a^{1/\alpha} \Pi_\alpha^{(1)} + b^{1/\alpha} \Pi_\alpha^{(2)} \overset{\mathcal{D}}{=} (a + b)^{1/\alpha} \Pi_\alpha. \tag{4}$$

Equivalently, the support κ_α of Π_α is a union stable random set, [5].

Denote by A_r the ball of radius r centred in $\mathbf{0}$ if $\alpha > 0$, or $A_r = \{x\colon \|x\| \geqslant r^{-1}\}$ if $\alpha < 0$.

Theorem 2.2. *If the principal value* $\lim_{r \downarrow 0} \int_{A_r} x\, \Pi(\mathrm{d}x)$ *of the integral* $\int x\, \Pi_\alpha(\mathrm{d}x)$ *converges with probability* 1*, then for any* deterministic *$S\alpha S$ element $z \in \mathbb{K}$ and $c > 0$, the series*

$$\xi_\alpha = z + c \int x\, \Pi_\alpha(\mathrm{d}x) \tag{5}$$

defines a $S\alpha S$ random element in $\mathbb{K}$. If the norm on $\mathbb{K}$ is sub-invariant, the series (5) *converges a.s. absolutely for any* $\alpha \in (0, 1)$.

Let $\{\xi_k,\ k \geqslant 1\}$ be a sequence of i.i.d. $\mathbb{K}$-valued random elements. For every $n \geqslant 1$, the *binomial point process* is defined by $\sum_{i=1}^{n} \delta_{\xi_i}$, where δ_x is a unit mass measure concentrated on x. The scaled versions of the binomial process are defined by $\beta_n = \sum_{k=1}^{n} \delta_{\xi_k/b_n}$, where $b_n = n^{1/\alpha}L(n)$, $n \geqslant 1$ with $\alpha \neq 0$ and a slowly varying at infinity function L. Similarly to [6, Prop. 3.21], it can be shown that β_n weakly converges to Π_α as $n \to \infty$ iff the tail of distribution of ξ varies regularly as in (6) below. By the Skorohod theorem, β_n and Π_α can be coupled so that $\beta_n \to \Pi_\alpha$ a.s. vaguely, i.e. there exists ordering of support points so that points of β_n converge to the corresponding points of Π_α. Although this is insufficient to show convergence of their sums in general, this convergence holds if $\alpha \in (0, 1)$.

Theorem 2.3. *If* $\mathbb{K}$ *possesses a sub-invariant norm and*

$$\lim_{n\to\infty} n\mathbf{P}\left\{\frac{\xi}{\|\xi\|} \in A,\ \|\xi\| > rb_n\right\} = \sigma(A)r^{-\alpha} \tag{6}$$

holds with $\alpha \in (0, 1)$*, then* β_n *converges to* Π_α *in a stronger than weak topology implying weak convergence of* $\int x\beta_n(\mathrm{d}x) = b_n^{-1}(\xi_1 + \cdots + \xi_n)$ *to* $\int x\Pi_\alpha(\mathrm{d}x)$*. In particular, if* ξ *is* $S\alpha S$ *with* $\alpha \in (0, 1)$ *satisfying* (6)*, i.e. possesses a spectral measure* σ*, then* ξ *admits LePage representation* (5) *with* $z = \mathbf{e}$.

In Banach spaces, the weak convergence of $b_n^{-1}(\xi_1 + \cdots + \xi_n)$ for $\alpha \in [1, 2]$ is proved by the *centring* of the ξ_is and the *symmetrisation* procedure. However, on semigroups centring is generally impossible, since the addition is not necessarily invertible. Hence, $\alpha \geqslant 1$ is impossible to treat without further insight into structure of $\mathbb{K}$ which we address in the following section.

3. Harmonic analysis on semigroups

The analogue of symmetrisation in semigroups is the *involution*, i.e. a continuous map $\star : \mathbb{K} \mapsto \mathbb{K}$ satisfying $(x + y)^\star = y^\star + x^\star$, $(x^\star)^\star = x$ and $(ax)^\star = ax^\star$ for all $x, y \in \mathbb{K}$ and $a > 0$. A function χ that maps $\mathbb{K}$ into the unit disk $\mathbb{D}$ in the complex plane is called a *character* if $\chi(\mathbf{e}) = 1$, $\chi(x + y) = \chi(x)\chi(y)$ and $\chi(x^\star) = \overline{\chi(x)}$ for all $x, y \in \mathbb{K}$. A sub-semigroup $\widetilde{\mathbb{K}}$ of characters is called *separating*, if, for any two distinct elements $x, y \in \mathbb{K}$, there exists $\chi \in \widetilde{\mathbb{K}}$ such that $\chi(x) \neq \chi(y)$. it can be shown the following fact:

Theorem 3.1. *If the characters from a separating sub-semigroup* $\widetilde{\mathbb{K}}$ *generate the Borel* σ*-algebra on* $\mathbb{K}$*, then the distribution of a random element* ξ *in* $\mathbb{K}$ *is uniquely determined by its* Laplace transform $\mathbf{E}\chi(\xi) = \int_{\mathbb{K}} \chi(x)\mathbf{P}(\mathrm{d}x)$, $\chi \in \widetilde{\mathbb{K}}$.

This generalisation of the classical result for continuous characters, see, e.g., [2, § IX.5.7] is important for our framework since many cones, unlike Banach spaces, do not possess continuous characters, e.g. $\mathbb{R}_+$ with maximum operation as addition has only semicontinuous characters: $\chi'_a(x) = \mathbb{1}_{[0,a]}(x)$ and $\chi''_a(x) = \mathbb{1}_{[0,a)}(x)$. Certain relaxation of continuity is the notion of *series-continuity*, i.e. $\chi(\sum_k x_k) = \prod_k \chi(x_k)$ for any convergent series in $\mathbb{K}$. In the example of $(\mathbb{R}_+, \max)$ above, $\{\chi'_a(x),\ a \geqslant 0\}$ is a series-continuous separating family of characters. Surely, it generates the Borel σ-algebra in $\mathbb{R}_+$ and the Laplace transform, i.e. the c.d.f., uniquely determines the distribution.

Examples show that in some cases the LePage series converges and so defines a $S\alpha S$ random element for any $\alpha \neq 0$. The range of possible values for the characteristic exponent α is a certain characteristic of a cone reflecting how 'strongly' the second distributivity is violated. Note that a random element ξ is said to be *proper* if $\mathbf{E}\chi(\xi) \neq 0$ for all $\chi \in \widetilde{\mathbb{K}}$.

Theorem 3.2. *Assume the second distributivity law holds and all* $\chi \in \widetilde{\mathbb{K}}$ *are continuous. Then every proper* $S\alpha S$ *random element* ξ *has parameter* $\alpha \in (0, 2]$*. If the involution is identical then* $\alpha \in (0, 1]$.

Theorem 3.3. *Assume that the characters have semicontinuous modulus and continuous argument and that* $\chi(sx) \to 1$ *for all* $\chi \in \widetilde{\mathbb{K}}$ *and* $x \in \mathbb{K}'$*, where* $s \to 0$ *if* $\mathbf{e} = \mathbf{0}$ *or* $s \to \infty$ *if* $\mathbf{e} \neq \mathbf{0}$*. Then for every proper* $S\alpha S$ *random element* ξ*, its characteristic exponent* $\alpha > 0$ *if and only if* $\mathbf{e} = \mathbf{0}$*, while otherwise* $\alpha < 0$.

Since each $S\alpha S$ ξ is infinitely divisible, $\mathbf{E}\chi(\xi)=\exp\{-\phi(\chi)\}$, where ϕ is negative-definite function (Lévy exponent) given by

$$\phi(\chi)=\mathrm{i}\ell(\chi)+q(\chi)+\int\limits_{\mathbb{K}^\sharp\setminus\{\mathbf{1}\}}\big(1-\rho(\chi)+\mathrm{i}L(\chi,\rho)\big)\lambda(\mathrm{d}\rho),\quad \chi\in\widetilde{\mathbb{K}},\tag{7}$$

where $\mathbb{K}^\sharp$ is the set of all characters on $\widetilde{\mathbb{K}}$ (the second dual semigroup). In (7) above, $\ell:\widetilde{\mathbb{K}}\to\mathbb{R}$ is a $\star$-additive function, $q:\widetilde{\mathbb{K}}\mapsto\mathbb{R}$ is *quadratic form*, $L(\chi,\rho)$ is the *Lévy function*, and λ is the *Lévy measure*, see, e.g., [1, Ch. 4] for details. If $\widetilde{\mathbb{K}}$ is separating, $\mathbb{K}^\sharp$ is always 'richer' than $\mathbb{K}$ in a sense that the *evaluation map* $\imath:\mathbb{K}\mapsto\mathbb{K}^\sharp$ is injection defined by $\imath(x)=\rho_x$, where $\rho_x(\chi)=\chi(x)$ are characters on $\widetilde{\mathbb{K}}$. The multiplication can be uplifted to $\widetilde{\mathbb{K}}$ and $\mathbb{K}^\sharp$ by setting $(s\circ\chi)(x)=\chi(sx)$ and $(s\circ\rho)(\chi)=\chi(s\circ\chi)$.

Theorem 3.4. *If all the characters have semicontinuous modulus and continuous argument, then for a proper $S\alpha S$ random element one has $\phi(s\circ\chi)=s^\alpha\phi(\chi)$ and $\lambda(s\circ B)=s^{-\alpha}\lambda(B)$ for some $\alpha\neq 0$ and any subset B of $\mathbb{K}^\sharp$. Moreover, λ has infinite total mass outside of any neighbourhood of character* **1**.

In Banach spaces, the quadratic form q corresponds to the Gaussian component of the stable law. In general semigroups Bochner's theorem does not hold: a probability law with Laplace transform $\mathbf{E}\chi(\xi)=\mathrm{e}^{-q(\chi)}$, $\forall\chi\in\widetilde{\mathbb{K}}$ may not exist. If it does, we call the corresponding element ξ *Gaussian* although it may exhibit rather unusual properties. In particular, it may not be $S\alpha S$ or it may be $S\alpha S$ with any α, $\xi+\xi^\star$ is deterministic, ξ is deterministic if the involution is identical, $\chi(\xi)$ is deterministic for every real $\chi\in\widetilde{\mathbb{K}}$, and for every $\chi(x)=\mathrm{e}^{\mathrm{i}u(x)}$, where $u(x)$ is a real additive function, $u(\xi)$ is a Gaussian random variable with mean 0. Moreover, a $S\alpha S$ with $\alpha=2$ need not be Gaussian (as in $(\mathbb{R}_+,\max)$ example).

Theorem 3.5. *Under conditions of Theorem* 3.4 *and the second distributivity law, the quadratic form q vanishes unless $\alpha=1$ or $\alpha=2$; if $\mathbb{K}$ is a group, q vanishes unless $\alpha=2$; and if the involution is identical, then q vanishes unless $\alpha=1$.*

The following result shows that in general, the LePage representation lives on the second dual space $\mathbb{K}^\sharp$:

Theorem 3.6. *If ξ is proper involution-symmetric $S\alpha S$ without Gaussian component, then*

$$\exp\big\{-\phi(\chi)\big\}=\exp\Bigg\{-\int\limits_{\mathbb{K}^\sharp\setminus\{\mathbf{1}\}}\big(1-\rho(\chi)\big)\lambda(\mathrm{d}\rho)\Bigg\}=\mathbf{E}\Bigg(\prod_{\rho\in\operatorname{supp}\Pi^\sharp}\rho\Bigg)(\chi),\quad \chi\in\widetilde{\mathbb{K}},$$

where $\Pi^\sharp$ is a Poisson process on $\mathbb{K}^\sharp$ with intensity measure λ. Hence one has the LePage representation for $\imath(\xi)$ on $\mathbb{K}^\sharp$: $\imath(\xi)=\prod_{\rho\in\operatorname{supp}\Pi^\sharp}\rho$.

Under certain conditions of the reflexivity type, the measure λ can be transported from $\mathbb{K}^\sharp$ onto $\mathbb{K}$ to give the canonical representation

$$\phi(\chi)=\mathrm{i}\ell(\chi)+q(\chi)+\int\limits_{\mathbb{K}\setminus\{\mathbf{e}\}}\big(1-\chi(x)+\mathrm{i}L(\chi,x)\big)\Lambda(\mathrm{d}x),\quad \chi\in\widetilde{\mathbb{K}}.\tag{8}$$

In this case Λ coincides with $\theta_\alpha\times\sigma$. So existence of the spectral measure and thus LePage representation in $\mathbb{K}$ are related to possibility to represent λ as an image of a Λ under evaluation map $\imath$. In particular, it is the case when $\mathbb{K}$ is a reflexive Banach space or a semigroup with compact unit sphere and a separating family of series-continuous characters $\widetilde{\mathbb{K}}$.

Models covered by our settings include examples with the second distributivity (Banach spaces, positive half-line, compact convex sets with Minkowski addition, upper semicontinuous functions, finite random measures, locally finite random measures) as well as without it (extreme values and $(\max,+)$ algebras, continuous functions with product and argument rescaling, compact sets with union as addition and scaling, integrable random probability measures with convolution operation, random measures with bounded supports and phase space scaling as multiplication). Full details can be found in [3].

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