

Functional Analysis

Noncommutative index theory for mirror quantum spheres

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Abstract

We introduce and analyse a new type of quantum 2-spheres. Then we apply index theory for noncommutative line bundles over these spheres to conclude that quantum lens spaces are non-crossed-product examples of principal extensions of C^* -algebras. **To cite this article:** *P.M. Hajac et al., C. R. Acad. Sci. Paris, Ser. I 343 (2006).*

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Résumé

Théorie de l'indice non commutative pour des sphères quantiques miroirs. Nous introduisons et analysons un nouveau type de 2-sphères quantiques. Nous appliquons la théorie de l'indice pour les fibrés en droites non commutatifs sur ces sphères afin de déduire que les espaces lenticulaires quantiques sont des exemples d'extensions principales de C^* -algèbres qui ne sont pas des produits croisés. **Pour citer cet article :** *P.M. Hajac et al., C. R. Acad. Sci. Paris, Ser. I 343 (2006).*

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La C^* -algèbre $C(S_{pq+}^2)$ d'une sphère quantique de Podleś générique [18] peut être vu comme un produit fibré de deux algèbres de Toeplitz \mathcal{T} [19]. Le produit fibré est défini par la fonction symbole $\mathcal{T} \xrightarrow{\sigma} C(S^1)$ de la façon suivante : $C(S_{pq+}^2) \cong \{(x, y) \in \mathcal{T} \times \mathcal{T} \mid \sigma(x) = \sigma(y)\}$. Cela peut être vu comme un amalgame de deux disques quantiques dont on identifie les bords S^1 . Nous considérons un amalgame similaire de disques quantiques mais avec une identification différente des bords, plus précisément $C(S_{pq-}^2) := \{(x, y) \in \mathcal{T} \times \mathcal{T} \mid \sigma(x) = \bar{\sigma}(y)\}$ (où $\bar{\sigma}$ est le morphisme d'algèbre défini en envoyant l'isométrie génératrice de \mathcal{T} sur l'inverse du générateur unitaire de $C(S^1)$). Cela peut être vu comme le amalgame du bord d'un disque quantique avec celui son image-miroir. Du point de vue algébrique, nous commençons par $\mathcal{O}(D_p)$, la sous- $*$ -algèbre dense de \mathcal{T} définie par la $*$ -algèbre unitaire universelle pour la relation $x^*x - px x^* = 1 - p$, $p \in [0, 1)$, [15]. Nous considérons alors les versions algébriques de σ et de $\bar{\sigma}$ qui envoient x

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respectivement vers le générateur unitaire des polynômes de Laurent et vers son inverse. Nous étudions le produit fibré $\{(x, y) \in \mathcal{O}(D_p) \times \mathcal{O}(D_q) \mid \sigma(x) = \bar{\sigma}(y)\}$ qui se trouve être équivalent à la $*$ -algèbre unitaire universelle générée par C , E et F munis des relations $C^*C = 1 - pE - F$, $CC^* = 1 - E - qF$, $EC = pCE$, $CF = qFC$, $EF = 0$, $E = E^*$, $F = F^*$, $p, q \in [0, 1]$. Nous l'appelons l'algèbre des coordonnées d'une 2-sphère quantique miroir et nous la notons par $\mathcal{O}(S_{pq-}^2)$. Puisque l'algèbre de Toeplitz \mathcal{T} est la C^* -algèbre enveloppante de $\mathcal{O}(D_p)$ [15], nous avons que $C(S_{pq-}^2)$ est la C^* -algèbre enveloppante de $\mathcal{O}(S_{pq-}^2)$.

La C^* -algèbre $C(S_{pq-}^2)$ a, à équivalence unitaire près, deux $*$ -représentations bornées, irréductibles, infini-dimensionnelles ρ_+ et ρ_- et aussi une famille ρ_λ , $\lambda \in S^1$ de représentations uni-dimensionnelles. Cette classification des représentations et beaucoup de propriétés de $C(S_{pq-}^2)$ sont en accord avec celles de Podleś. Néanmoins, en considérant les comportements différents des K_0 -classes de deux projections minimales non-équivalentes dans $C(S_{pq-}^2)$ et dans $C(S_{pq+}^2)$, cela permet de montrer que les C^* -algèbres $C(S_{pq+}^2)$ et $C(S_{pq-}^2)$ ne sont pas Morita-équivalentes. La différence entre ces deux algèbres est aussi visible dans leur structure de module de Fredholm. Contrairement à $C(S_{pq+}^2)$, la paire de représentations irréductibles infini-dimensionnelles (ρ_+, ρ_-) n'est pas un module de Fredholm. À la place, nous obtenons :

Lemme 0.1. *Soit Tr la trace. La paire $(\rho_+ \oplus \rho_-, \int_{S^1} \rho_\lambda d\lambda)$ est un module de Fredholm 1-sommable sur $\mathcal{O}(S_{pq-}^2)$, i.e. $\forall x \in \mathcal{O}(S_{pq-}^2) : \text{Tr}|\rho_+(x) \oplus \rho_-(x) - \int_{S^1} \rho_\lambda(x) d\lambda| < \infty$.*

Soit $\mathcal{O}(S_{pq\theta}^3)$ l'algèbre des coordonnées de la 3-sphère quantique de type Heegaard [1]. On peut montrer que $\mathcal{O}(S_{pq-}^2) \subseteq \mathcal{O}(S_{pq\theta}^3)$ est une extension principale, dans le sens de [2], pour l'action naturelle $U(1)$ sur $\mathcal{O}(S_{pq\theta}^3)$ (de cette façon, nous pouvons voir cette extension comme une déformation non commutative de la fibration de Hopf). Les $\mathcal{O}(S_{pq-}^2)$ -modules associés à cette extension, via des représentations uni-dimensionnelles (fibrés en droites non commutatifs) peuvent être définis comme $\mathcal{O}(S_{pq\theta}^3)_\mu := \{x \in \mathcal{O}(S_{pq\theta}^3) \mid \alpha_g(x) = g^{-\mu}x, \forall g \in U(1)\}$, $\mu \in \mathbb{Z}$. En appliquant [2, Theorem 3.1], on peut montrer que les modules $\mathcal{O}(S_{pq\theta}^3)_\mu$ sont projectifs, finiment engendrés, et en outre on peut déterminer explicitement les idempotents E_μ qui les représentent. En remplaçant $\mathcal{O}(S_{pq\theta}^3)$ par la C^* -algèbre enveloppante $C(S_{pq\theta}^3)$ et en étendant par continuité l'action de $U(1)$, nous obtenons une extension principale $C(S_{pq-}^2) \subseteq C(S_{pq\theta}^3)$ dans le sens de [7], et la famille $C(S_{pq\theta}^3)_\mu$, $\mu \in \mathbb{Z}$ de $C(S_{pq-}^2)$ -modules. On peut montrer que ces modules peuvent être représentés par les mêmes idempotents E_μ . Maintenant, nous pouvons représenter le module de Fredholm du Lemme 0.1 par une trace tr sur $\mathcal{O}(S_{pq-}^2)$ et, en se servant de [8], nous calculons les couplages de ces idempotents avec le module de Fredholm qui sont les indices d'opérateurs de Fredholm.

Théorème 0.2. *Soit $\langle \cdot, \cdot \rangle$ le couplage entre cohomologie cyclique et K -théorie. Alors $\langle \text{tr}, [E_\mu] \rangle = \mu$, $\forall \mu \in \mathbb{Z}$.*

Nous appliquons ce résultat aux espaces lenticulaires quantiques. Puisque $Z_n := \mathbb{Z}/n\mathbb{Z}$ est un sous-groupe de $U(1)$, nous devons définir de nouvelles sous-algèbres constituées de points fixes $C(L_{pq\theta}^n) := C(S_{pq\theta}^3)^{Z_n}$. Elles jouent le rôle de l'algèbre des fonctions continues sur les espaces lenticulaires (cf. [17,13]). Comme dans le cas classique, il y a toujours une action naturelle de $U(1)$ sur $C(L_{pq\theta}^n)$. On peut montrer que, pour tout $n \in \mathbb{N} \setminus \{0\}$, cette action est principale (libre) dans le sens de [7], et que les sous-algèbres constituées de points fixes $C(L_{pq\theta}^n)^{U(1)}$ coïncident avec $C(S_{pq-}^2)$. En utilisant ces nouvelles actions $U(1)$, nous définissons les $C(S_{pq-}^2)$ -modules $C(L_{pq\theta}^n)_\mu$, et on montre que $C(L_{pq\theta}^n)_\mu \cong C(S_{pq\theta}^3)_{n\mu}$, $\forall n \in \mathbb{N}$. D'un autre côté, nous montrons que les modules associés avec les produits croisés par \mathbb{Z} sont toujours libres. En combinant ces résultats avec le Théorème 0.2, nous obtenons notre résultat principal :

Théorème 0.3. *Soient $n \in \mathbb{N}$, $n > 0$, $p, q, \theta \in [0, 1]$. La C^* -algèbre $C(L_{pq\theta}^n)$ est une $U(1)$ -extension principale de $C(S_{pq-}^2)$, mais $C(L_{pq\theta}^n) \not\cong C(S_{pq-}^2) \rtimes \mathbb{Z}$ comme $U(1)$ - C^* -algèbres pour toutes actions de \mathbb{Z} .*

1. Introduction

The purpose of this article is twofold. First, we present and study quantum two-spheres of a new topological type (cf. [4]). They are like twin siblings of the generic Podleś quantum spheres [18], but of opposite gender. These new

spheres were encountered while trying to understand why the piecewise trivial quantum Hopf fibration built over a generic Podleś sphere comes from an action of $U(1)$ on S^3 that is equivalent to the standard Hopf-fibration action by an orientation-reversing homeomorphism [11]. Taking the natural action leads to a new fixed-point subalgebra that coincides with the fibre product of Toeplitz algebras over $C(S^1)$ given by the symbol map and its composition with the automorphism inverting the unitary generator of $C(S^1)$. In contrast, the C^* -algebra of a generic Podleś sphere is a fibre product of two Toeplitz algebras via two symbol maps (no inversion) [19]. Geometrically, this means that the new sphere arises by gluing over the boundary a quantum disc with its mirror image. It is precisely this type of gluing of *oriented* discs that yields an *oriented* sphere. While in the classical case the orientation issue does not show up in the C^* -algebra of continuous functions on S^2 , in the noncommutative setting it leads to two C^* -algebras that are not even Morita equivalent. Since orientation is vital for the Dirac operator, our new example may be a useful testing ground for spectral triples (cf. [5]). Another curious aspect of the new spheres is that, unlike the aforementioned Podleś C^* -algebra, their C^* -algebra is not a graph algebra. However, it is very close to graph algebras [12] and points at a more general concept of a graph algebra.

Our other main goal is to construct non-crossed-product examples of principal extensions of C^* -algebras [7]. Much as there is more to principal bundles than Cartesian products of groups with spaces, we argue that there is more to principal extensions of C^* -algebras than crossed products. In the algebraic context of Hopf–Galois extensions, one might prove that a given extension is not a crossed product by arguing that algebraic crossed products have many nontrivial invertible elements, whereas a given polynomial algebra might lack such elements (e.g., see [9, Appendix]). This argument does not apply on the C^* -level because it is easy to construct nontrivial invertible elements in a C^* -algebra. It is difficult to decide whether a given C^* -algebra is a crossed product with its fixed-point subalgebra because, contrary to the more restricted setting of von Neumann algebras, there appears to be no theory to manage such problems. Von Neumann algebras are at an opposite extreme to polynomial algebras. For them theory tells us when an extension of algebras has a crossed-product structure. (Think of measurable global sections of nontrivial topological bundles.) Therefore, we find the C^* -context most interesting to analyse this crossed-product problem. Our method follows the classical pattern of algebraic topology: to prove that a principal bundle is nontrivial, one computes an invariant of an associated vector bundle. Herein we use the result of an index computation for finitely generated projective modules over the C^* -algebra of a mirror quantum sphere to prove that the C^* -algebras of noncommutative lens spaces equipped with a natural $U(1)$ -action cannot be \mathbb{Z} -crossed products with the sphere C^* -algebra. This makes them essentially different from the standard example of the noncommutative torus: $C(S^1) \cong C(T_\theta^2)^{U(1)} \subseteq C(T_\theta^2) \cong C(S^1) \rtimes_\theta \mathbb{Z}$.

2. The comparison of Podleś and mirror quantum spheres

Definition 2.1. Let $p, q \in [0, 1]$. We call the universal unital $*$ -algebra generated by C, E and F satisfying

$$C^*C = 1 - pE - F, \quad CC^* = 1 - E - qF, \quad EC = pCE, \quad CF = qFC, \quad EF = 0, \quad E = E^*, \quad F = F^*,$$

the coordinate algebra of a *mirror quantum 2-sphere*, and denote it by $\mathcal{O}(S_{pq-}^2)$.

The first two relations play the role of the sphere equation. Here E should be thought of as the coordinate measuring the square of the distance of a point on the upper hemisphere from the equator plane, while F measures the square of the distance of a point on the lower hemisphere from the equator plane. Indeed, if $p = 1 = q$, then we have the classical sphere, E and F are functions with disjoint support, and the relationship with the Cartesian coordinates is $C = x + iy, \sqrt{E} - \sqrt{F} = z$. These new spheres were found when examining a natural $U(1)$ -action on Heegaard-type quantum 3-spheres $S_{pq\theta}^3$ [1]. Recall that the polynomial algebra $\mathcal{O}(S_{pq\theta}^3)$ is the universal unital $*$ -algebra generated by elements a and b satisfying the relations:

$$(1 - aa^*)(1 - bb^*) = 0, \quad a^*a = paa^* + 1 - p, \quad b^*b = qbb^* + 1 - q, \quad ab = e^{2\pi i\theta} ba, \quad ab^* = e^{-2\pi i\theta} b^*a.$$

Here p, q and θ are parameters with values in $[0, 1)$. Rescaling the generators by a unitary complex number, i.e. $a \mapsto e^{i\phi}a, b \mapsto e^{i\phi}b$, gives an action $\alpha : U(1) \rightarrow \text{Aut } \mathcal{O}(S_{pq\theta}^3)$. Moreover, one can show that the subalgebra of all α -invariants $\mathcal{O}(S_{pq\theta}^3)^{U(1)}$ is independent of θ and isomorphic with the unital $*$ -algebra $\mathcal{O}(S_{pq-}^2)$.

The above-defined $U(1)$ -action extends by continuity to the C^* -completion $C(S_{pq\theta}^3)$ of $\mathcal{O}(S_{pq\theta}^3)$. The fixed-point C^* -algebra $C(S_{pq\theta}^3)^{U(1)}$ turns out to be the universal (enveloping) C^* -algebra $C(S_{pq-}^2)$ of the coordinate algebra

of S_{pq-}^2 . Since $C(S_{pq\theta}^3)$ and $C(S_{00\theta}^3)$ are isomorphic via a $U(1)$ -equivariant map [1, Theorem 2.8], we can conclude that $C(S_{pq-}^2) \cong C(S_{00-}^2)$ as C^* -algebras. Another way to think of this C^* -algebra is in terms of the fibre product of two Toeplitz algebras given by the symbol map $\mathcal{T} \xrightarrow{\sigma} C(S^1)$ and its composition with the automorphism of $C(S^1)$ sending the generating unitary to its inverse: $C(S_{pq-}^2) \cong \{(x, y) \in \mathcal{T} \times \mathcal{T} \mid \sigma(x) = \bar{\sigma}(y)\}$.

Proposition 2.2. *Let $\{\xi_k\}_{k \in \mathbb{N}}$ be an orthonormal basis of a Hilbert space, and $\lambda \in U(1)$. Any irreducible bounded $*$ -representation of $\mathcal{O}(S_{pq-}^2)$ is unitarily equivalent to one of the following representations:*

$$\rho_+(C)\xi_k = \sqrt{1 - p^{k+1}} \xi_{k+1}, \quad \rho_+(E)\xi_k = p^k \xi_k, \quad \rho_+(F)\xi_k = 0, \quad (1)$$

$$\rho_-(C)\xi_k = \sqrt{1 - q^k} \xi_{k-1}, \quad \rho_-(E)\xi_k = 0, \quad \rho_-(F)\xi_k = q^k \xi_k, \quad (2)$$

$$\rho_\lambda(C) = \lambda, \quad \rho_\lambda(E) = 0, \quad \rho_\lambda(F) = 0. \quad (3)$$

We use the same symbols ρ_+ , ρ_- and ρ_λ to denote the extensions of these representations to the enveloping C^* -algebra $C(S_{pq-}^2)$. It turns out that the kernels of ρ_+ and ρ_- have zero intersection and are both isomorphic to the algebra \mathcal{K} of compact operators on a separable Hilbert space. The quotient of $C(S_{pq-}^2)$ by $\ker \rho_+ + \ker \rho_-$ is isomorphic to the algebra of continuous functions on the circle. Thus, the C^* -algebra $C(S_{pq-}^2)$ admits an exact sequence $0 \rightarrow \mathcal{K} \oplus \mathcal{K} \rightarrow C(S_{pq-}^2) \rightarrow C(S^1) \rightarrow 0$. Using this sequence or the fibre-product formula, one can compute: $K_0(C(S_{pq-}^2)) \cong \mathbb{Z} \oplus \mathbb{Z}$ and $K_1(C(S_{pq-}^2)) = 0$.

The properties of $C(S_{pq-}^2)$ described so far match the respective properties of the C^* -algebra of a generic Podleś sphere. One can show that the latter is isomorphic with $C(S_{pq+}^2)$, that is the universal unital C^* -algebra for the relations

$$\begin{aligned} D^*D &= 1 - pP - qQ, & DD^* &= 1 - P - Q, & PD &= pDP, \\ QD &= qDQ, & PQ &= 0, & P &= P^*, & Q &= Q^*, \end{aligned}$$

where $p, q \in [0, 1)$. These relations appear almost identical to the defining relations for $C(S_{pq-}^2)$. Furthermore, for the action given by $a \mapsto e^{i\phi}a$, $b \mapsto e^{-i\phi}b$, we have $C(S_{pq+}^2) \cong C(S_{pq\theta}^3)^{U(1)}$. The fibre-product formula for $C(S_{pq+}^2)$ also differs only slightly from its counterpart for $C(S_{pq-}^2)$, namely $C(S_{pq+}^2) \cong \{(x, y) \in \mathcal{T} \times \mathcal{T} \mid \sigma(x) = \sigma(y)\}$. On the other hand, consider two inequivalent minimal projections e_1^\pm and e_2^\pm in $C(S_{pq\pm}^2)$: it can be shown that their classes are non-zero in the respective K_0 -groups. However, $[e_1^+] = -[e_2^+]$ in $K_0(C(S_{pq+}^2))$, while $[e_1^-] = [e_2^-]$ in $K_0(C(S_{pq-}^2))$. This leads to:

Proposition 2.3. *The C^* -algebras $C(S_{pq+}^2)$ and $C(S_{pq-}^2)$ are not stably isomorphic.*

Finally, recall that $C(S_{pq+}^2)$ is a graph C^* -algebra [12], whose primitive ideal space consists of a pair of points (which is dense in the entire space) and a circle. The primitive ideal space of $C(S_{pq-}^2)$ turns out to be homeomorphic to that of $C(S_{pq+}^2)$, but a careful analysis of all graph algebras whose primitive ideal space is a circle plus two points (cf. [14]) shows that no such an algebra can be isomorphic to $C(S_{pq-}^2)$.

3. Index computation for associated noncommutative line bundles

A first step in an index computation for finitely generated projective modules is to find appropriate summable Fredholm modules. Although the index computations for noncommutative line bundles over the generic Podleś and mirror quantum spheres proceed along the same lines (cf. [10]), it is in the structure of Fredholm modules where differences between these spheres are crucial. Indeed, one can take appropriate infinite-dimensional irreducible representations ρ_1 and ρ_2 of $C(S_{pq+}^2)$, and they yield a 1-summable Fredholm module over $\mathcal{O}(S_{pq+}^2)$, the universal $*$ -algebra for the defining relations of $C(S_{pq+}^2)$ [16,10]. This is in contrast with the situation for mirror quantum spheres, where (ρ_+, ρ_-) is not a Fredholm module. Instead, making an appropriate identification of the Hilbert space of the direct

sum of our infinite-dimensional representations with the Hilbert space of the direct integral of our one-dimensional representations,¹ we obtain:

Lemma 3.1. *Let Tr be the operator trace. The pair $(\rho_+ \oplus \rho_-, \int_{S^1} \rho_\lambda \, d\lambda)$ is a 1-summable Fredholm module over $\mathcal{O}(S_{pq-}^2)$, i.e. $\forall x \in \mathcal{O}(S_{pq-}^2): \text{Tr}|\rho_+(x) \oplus \rho_-(x) - \int_{S^1} \rho_\lambda(x) \, d\lambda| < \infty$.*

On the other hand, it is straightforward to show that $\mathcal{O}(S_{pq-}^2) \subseteq \mathcal{O}(S_{pq\theta}^3)$ is a principal extension in the sense of [2, Definition 2.1]. The $\mathcal{O}(S_{pq-}^2)$ -modules associated to this extension via 1-dimensional representations (noncommutative line bundles) can be defined as

$$\mathcal{O}(S_{pq\theta}^3)_\mu := \{x \in \mathcal{O}(S_{pq\theta}^3) \mid \alpha_g(x) = g^{-\mu}x, \forall g \in U(1)\}, \quad \mu \in \mathbb{Z}. \tag{4}$$

Applying [2, Theorem 3.1], one can prove that the modules $\mathcal{O}(S_{pq\theta}^3)_\mu$ are finitely generated projective and can be explicitly given by the idempotents $E_\mu = R_\mu^T L_\mu$. Here, respectively for $\mu < 0$ and $\mu > 0$,

$$R_\mu = (b^{|\mu|}, ab^{|\mu|-1}, \dots, a^{|\mu|}), \quad L_\mu = (p^{|\mu|} A^{|\mu|} b^{*|\mu|}, \binom{|\mu|}{1}_p p^{|\mu|-1} A^{|\mu|-1} b^{*|\mu|-1} a^*, \dots, a^{*|\mu|}), \tag{5}$$

$$R_\mu = (b^{*\mu}, a^* b^{*\mu-1}, \dots, a^{*\mu}), \quad L_\mu = (b^\mu, \binom{\mu}{1}_q b^{\mu-1} B a, \dots, B^\mu a^\mu), \tag{6}$$

and $A := 1 - aa^*, B := 1 - bb^*$. The parentheses with subscripts denote the standard deformed binomial coefficients. (Note that the entries of E_μ are indeed in $\mathcal{O}(S_{pq-}^2)$.) Replacing $\mathcal{O}(S_{pq\theta}^3)$ by $C(S_{pq\theta}^3)$ in (4), one obtains a definition of the $C(S_{pq-}^2)$ -modules $C(S_{pq\theta}^3)_\mu$ (continuous sections of noncommutative line bundles). One can argue that these are isomorphic to the finitely generated projective modules $C(S_{pq-}^2)^{|\mu|+1} E_\mu$ (row times matrix), with the E_μ 's given above.

Now, any summable Fredholm module yields a cyclic cocycle that can be paired with a K -group [3]. In particular, Lemma 3.1 gives us a trace on $\mathcal{O}(S_{pq-}^2)$ defined by $\text{tr}(x) = \text{Tr}((\rho_+ \oplus \rho_-)(x) - \int_{S^1} \rho_\lambda(x) \, d\lambda)$. The pairing of the cyclic 0-cocycle tr and the K_0 -class $[E_\mu]$, $\mu \in \mathbb{Z}$, yields complicated-looking rational functions that have already been encountered in [8]. Therein, with the help of the noncommutative index formula [3], they were proved to be the desired integers. This brings us to the main technical result:

Theorem 3.2. *Let $\langle \cdot, \cdot \rangle$ be the pairing of cyclic cohomology and K -theory. Then $\langle \text{tr}, [E_\mu] \rangle = \mu, \forall \mu \in \mathbb{Z}$.*

As in [8, Corollary 2.4] and [10, Corollary 3.4], we can use Theorem 3.2 to estimate the positive cone $K_0^+(\mathcal{O}(S_{pq-}^2))$.

4. Quantum lens spaces as non-crossed-product examples of extensions of C^* -algebras

Since $Z_n := \mathbb{Z}/n\mathbb{Z}$ is a subgroup of $U(1)$, we can define new fixed-point subalgebras $C(L_{pq\theta}^n) := C(S_{pq\theta}^3)^{Z_n}$. They play the role of the algebras of continuous functions on quantum lens spaces (cf. [17,13]). Much as in the classical case, there is still a natural $U(1)$ -action on $C(L_{pq\theta}^n)$. It can be checked that, for any $n \in \mathbb{N} \setminus \{0\}$, this action is principal (free) in the sense of [7], and that the fixed-point subalgebras $C(L_{pq\theta}^n)^{U(1)}$ always coincide with $C(S_{pq-}^2)$ (cf. [20]). Furthermore, we can replace $\mathcal{O}(S_{pq\theta}^3)$ by $C(L_{pq\theta}^n)$ in (4) to obtain the $C(S_{pq-}^2)$ -modules $C(L_{pq\theta}^n)_\mu$ and verify that they are isomorphic with the previously defined modules:

$$\forall n \in \mathbb{N} \setminus \{0\}: \quad C(L_{pq\theta}^n)_\mu \cong C(S_{pq\theta}^3)_{n\mu} \quad \text{as left } C(S_{pq-}^2)\text{-modules.} \tag{7}$$

Thus it follows from Theorem 3.2 that the modules $C(L_{pq\theta}^n)_\mu$ are *not* free unless $\mu = 0$. On the other hand, using Fourier analysis with coefficients in a C^* -algebra A [6, pp. 222–223] and plugging $A \rtimes \mathbb{Z}$ into (4) in place of $\mathcal{O}(S_{pq\theta}^3)$, one can prove:

¹ We are grateful to Elmar Wagner for determining this identification.

Lemma 4.1. *Let A be a unital C^* -algebra and $A \rtimes \mathbb{Z}$ an arbitrary crossed product of A with \mathbb{Z} . Then, for the natural action of $U(1)$ on $A \rtimes \mathbb{Z}$ and any $\mu \in \mathbb{Z}$, we have $(A \rtimes \mathbb{Z})_\mu \cong A$ as left A -modules.*

This yields the desired contradiction, proving our main result:

Theorem 4.2. *Let $n \in \mathbb{N}$, $n > 0$, $p, q, \theta \in [0, 1)$. The C^* -algebra $C(L_{pq\theta}^n)$ is a principal $U(1)$ -extension of $C(S_{pq-}^2)$, but $C(L_{pq\theta}^n) \not\cong C(S_{pq-}^2) \rtimes \mathbb{Z}$ as $U(1)$ - C^* -algebras for any action of \mathbb{Z} on $C(S_{pq-}^2)$.*

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