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Partial Differential Equations

Two remarks on liftings of maps with values into S^1

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Abstract

Given a map $u \in L^1_{\mathrm{loc}}(\Omega, S^1)$ with some regularity: $|u|_X = R < \infty$, we consider the problem of finding a lifting φ of u (i.e. a measurable function satisfying $u = \mathrm{e}^{\mathrm{i}\varphi}$) with the same regularity and with an optimal control $|\varphi|_X \leqslant g(R)$. Two cases are treated here:

(i) $|\cdot|_X$ is a $W^{s,p}(0,1)$-seminorm, with $0 < s < 1 < p$ and $sp > 1$. We find a lifting φ such that $|\varphi|_{W^{s,p}(I)} \leqslant C(R + R^{1/s})$ and we show that the exponent $1/s$ cannot be improved.

(ii) $|\cdot|_X$ is the $BV(\Omega)$-seminorm where $\Omega \subset \mathbf{R}^d$ is a smooth open set. We give a simplified proof of a previous result [J. Dàvila, R. Ignat, Lifting of BV functions with values in S^1, C. R. Acad. Sci. Paris, Ser. I 337 (3) (2003) 159–164]: there exists $\varphi \in BV(\Omega)$ satisfying $|\varphi|_{BV} \leqslant 2R$. ***To cite this article: B. Merlet, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Résumé

Deux remarques sur les relèvements d'applications à valeurs dans S^1. Étant donnée une application $u \in L^1_{\mathrm{loc}}(\Omega, S^1)$ ayant une certaine régularité : $|u|_X = R < \infty$, nous cherchons un relèvement φ de u (i.e. une fonction mesurable telle que $u = \mathrm{e}^{\mathrm{i}\varphi}$) ayant la même régularité et avec le meilleur contrôle possible de $|\varphi|_X$ en fonction de R. On traite deux cas :

(i) $|\cdot|_X$ est une seminorme $W^{s,p}(0,1)$, avec $0 < s < 1 < p$ et $sp > 1$. Nous trouvons un relèvement φ satisfaisant $|\varphi|_{W^{s,p}(I)} \leqslant C(R + R^{1/s})$ et nous montrons que l'exposant $1/s$ ne peut être amélioré.

(ii) $|\cdot|_X$ est la seminorme $BV(\Omega)$ où $\Omega \subset \mathbf{R}^d$ est un ouvert régulier. Nous donnons une preuve simplifiée d'un résultat préexistant [J. Dàvila, R. Ignat, Lifting of BV functions with values in S^1, C. R. Acad. Sci. Paris, Ser. I 337 (3) (2003) 159–164] : il existe $\varphi \in BV(\Omega)$ telle que $|\varphi|_{BV} \leqslant 2R$. ***Pour citer cet article : B. Merlet, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Version française abrégée

Soit $\Omega \subset \mathbf{R}^d$ un ouvert régulier et $u : \Omega \to S^1$ une application mesurable. On appelle relèvement de u une fonction mesurable $\varphi : \Omega \to \mathbf{R}$ telle que $u = \mathrm{e}^{\mathrm{i}\varphi}$ presque partout dans Ω.

Nous abordons ici le problème de trouver un relèvement φ dont la régularité est contrôlée par la régularité de u. Ce problème a été traité par Bourgain, Brezis et Mironescu dans [2,4] pour des régularités $W^{s,p}$.

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On considère tout d'abord le cas $\Omega = I = (0,1)$ et $u \in W^{s,p}(I,S^1)$ avec $0 < s < 1 < p$ et $sp > 1$, muni de la seminorme :

$$|f|^p_{W^{s,p}(I)} := \int\limits_{I^2} \frac{|f(x)-f(y)|^p}{|x-y|^{1+sp}}\,\mathrm{d}x\,\mathrm{d}y.$$

Par l'inégalité de Morey, $W^{s,p}(I)$ s'injecte dans un espace de fonctions continues et pour $u \in W^{s,p}(I,S^1)$ on a l'existence d'un relèvement continu φ. On peut vérifier facilement que φ appartient à $W^{s,p}(I)$. Mais, à cause de la forme non locale de la seminorme $W^{s,p}$, on n'a pas un contrôle linéaire de $|\varphi|_{W^{s,p}(I)}$ en fonction de $|u|_{W^{s,p}(I)}$. Nous montrons :

Théorème 0.1. *Soit $u \in W^{s,p}(I,S^1)$, notons φ un relèvement continu de u (unique à une constante additive près), alors $\varphi \in W^{s,p}(I,\mathbf{R})$ et on a l'estimation*

$$|\varphi|_{W^{s,p}(I)} \leqslant C(s,p)\big(|u|_{W^{s,p}(I)} + |u|^{1/s}_{W^{s,p}(I)}\big).$$

De plus l'exposant $1/s$ est optimal.

Dans une seconde partie, nous nous intéressons au cas d'une application $u:\Omega \to S^1$, avec Ω domaine régulier de $\mathbf{R}^d$ et u à variation bornée : $|u|_{BV} := \int_\Omega |\nabla u| < \infty$ (il faut comprendre l'intégrale comme la masse totale de la mesure de Radon ∇u). Nous donnons une preuve alternative du résultat suivant dû à Dávila et Ignat [8] :

Théorème 0.2. *Soit $u \in L^1_{\mathrm{loc}}(\Omega,S^1)$ à variation bornée. Il existe $\varphi \in L^\infty(\Omega,\mathbf{R})$, relèvement de u, à variation bornée, satisfaisant l'estimation : $|\varphi|_{BV} \leqslant 2|u|_{BV}$.*

Notre preuve utilise une définition équivalente de la seminorme BV donnée par Dávila dans [7]. Cette définition est basée sur une caractérisation des espaces de Sobolev $W^{1,p}$ due à Bourgain, Brezis et Mironescu [3,5]. Cette nouvelle définition permet d'éviter d'effectuer la décomposition de Vol'pert des mesures ∇u, $\nabla\varphi$ et l'utilisation de la formule de composition pour les fonctions BV — voir [1,9].

1. Introduction

Let $\Omega \subset \mathbf{R}^d$ be an open set and $u:\Omega \to S^1 := \{z \in \mathbb{C}\colon |z| = 1\}$ a measurable function. A lifting of u is a measurable function $\varphi:\Omega \to \mathbf{R}$ such that $u = \mathrm{e}^{\mathrm{i}\varphi}$ a.e. in Ω.

This Note concerns the following question: let u be given with some regularity; can we find a lifting φ with the same regularity and such that the regularity of u controls the regularity of φ? This problem has been extensively treated by Bourgain, Brezis and Mironescu in [2,4] for $W^{s,p}$ data.

1.1. Case 1: $u \in W^{s,p}(I,S^1)$, $0 < s < 1 < p$ and $sp > 1$

First, we consider the case of $\Omega = I = (0,1)$ and $u \in W^{s,p}(I,S^1)$ with $0 < s < 1 < p$ and $sp > 1$. We define the $W^{s,p}(I)$-seminorm of a measurable map $f \in L^1_{\mathrm{loc}}(I,\mathbf{R}^q)$ by

$$|f|^p_{W^{s,p}(I)} := \int\limits_{I^2} \frac{|f(x)-f(y)|^p}{|x-y|^{1+sp}}\,\mathrm{d}x\,\mathrm{d}y,$$

where $|\cdot|$ is the Euclidean norm in $\mathbf{R}^q$ or in $\mathbf{R}$. If $|f|_{W^{s,p}(I)} < \infty$, then, from the Sobolev embedding theorem ($sp > 1$), f is almost everywhere equal to a $C^{0,\alpha}$ function $\tilde f$, with $\alpha = s - 1/p$ — in the sequel, we always consider $f = \tilde f$. Moreover, there exists $c_0 > 0$ depending only on s and p, such that for every x, y in I, $x \neq y$,

$$\frac{|f(x)-f(y)|}{|x-y|^{s-1/p}} \leqslant c_0 |f|_{W^{s,p}([x,y])}, \quad \text{where } |f|^p_{W^{s,p}([x,y])} := \int\limits_{[x,y]^2} \frac{|f(x')-f(y')|^p}{|x'-y'|^{1+sp}}\,\mathrm{d}x'\,\mathrm{d}y'. \tag{1}$$

Let $u \in W^{s,p}(I, S^1)$. Since u is continuous, there exists a continuous lifting φ of u — which is unique up to an additive constant. But, due to the non local character of the $W^{s,p}$-seminorm, we do not have a linear control of $|\varphi|_{W^{s,p}(I)}$ by $|u|_{W^{s,p}(I)}$. We show:

Theorem 1.1. *Let u in $W^{s,p}(I, S^1)$, let φ be a continuous lifting of u, then*

$$|\varphi|_{W^{s,p}(I)} \leqslant C(s, p)\big(|u|_{W^{s,p}(I)} + |u|^{1/s}_{W^{s,p}(I)}\big).$$

Moreover, the power $1/s$ in this estimate is optimal.

1.2. Case 2: $u \in L^1_{\mathrm{loc}}(\Omega, S^1)$, $\int_\Omega |\nabla u| < \infty$

Let $\Omega \subset \mathbf{R}^d$ be an open set with a smooth boundary, we say that $f \in L^1_{\mathrm{loc}}(\Omega, \mathbf{R}^q)$ has a bounded variation and we write $f \in BV(\Omega)$ if the gradient of f in the sense of distributions is a Radon measure of finite total mass. In this case, we define the BV-seminorm of f by $|f|_{BV} := \int_\Omega |\nabla f|$.

We give an alternative proof of the following result of Dávila and Ignat [8]:

Theorem 1.2. *Let $u \in BV(\Omega, S^1)$. There exists a lifting $\varphi \in BV(\Omega, \mathbf{R})$ of u, $\varphi \in BV(\Omega, \mathbf{R})$ satisfying $|\varphi|_{BV} \leqslant 2|u|_{BV}$.*

Remark 1. The constant 2 cannot be improved, it is due to a topological obstruction. For example (see [8]), let Ω be the unit disc in $\mathbf{C}$ and let us define the map $u(re^{i\theta}) := e^{i\theta}$, $0 \leqslant r < 1$, $0 \leqslant \theta < 2\pi$. The function $\varphi(re^{i\theta}) := \theta$ is a lifting of u of minimal BV-seminorm and we have $|\varphi|_{BV} = 4\pi = 2|u|_{BV}$.

Our proof uses an equivalent definition of the BV-seminorm (Dávila [7]). This definition is based on the characterization of the Sobolev spaces $W^{1,p}$ introduced by Bourgain, Brezis and Mironescu [3,5]. With this new definition, we avoid applying Vol'pert's decomposition to the measures ∇u, $\nabla\varphi$ and using the chain rule formula for BV functions — see [1,9].

2. Lifting of $W^{s,p}(I, S^1)$-maps, $sp > 1$ – proof of Theorem 1.1

In this section, the reals $0 < s < 1$ and $p > 1$ are such that $sp > 1$, c_0 is the constant introduced in (1). The letter C denotes various positive constants only depending on s and p.

Counter-example. We show that the power $1/s$ is optimal by exhibiting an unbounded sequence (u_n) in $W^{s,p}(I, S^1)$ with associated liftings (φ_n) in $W^{s,p}$ such that $|u_n|^{1/s}_{W^{s,p}(I)} \leqslant C|\varphi_n|_{W^{s,p}(I)}$, $n \geqslant 0$.

Namely, for $n \geqslant 1$, we set $\varphi_n(x) := nx$, and $u_n(x) := e^{inx}$, for $0 \leqslant x \leqslant 1$. Clearly, φ_n belongs to $W^{s,p}(I, \mathbf{R})$ and is a lifting of $u_n \in W^{s,p}(I, S^1)$. Then we compute

$$|\varphi_n|^p_{W^{s,p}(I)} = n^p \int\limits_{I^2} |x - y|^{-1+(1-s)p}\, \mathrm{d}x\, \mathrm{d}y = Cn^p.$$

Next, considering the integral on $(x, y) \in I^2$ in the definition of $|u_n|^p_{W^{s,p}(I)}$, we use the estimate $|u_n(x) - u_n(y)| \leqslant n|x - y|$ in the case $|x - y| < 1/n$ and $|u_n(x) - u_n(y)| \leqslant 2$ in the other case. We obtain,

$$|u_n|^p_{W^{s,p}(I)} \leqslant Cn^p \int\limits_0^{1/n} r^{-1+(1-s)p}\, \mathrm{d}r + C \int\limits_{1/n}^1 r^{-1-sp}\, \mathrm{d}r \leqslant Cn^{sp} + Cn^{sp} \leqslant C|\varphi_n|^p_{W^{s,p}(I)}.$$

Proof of Theorem 1.1. Let $u \in W^{s,p}(I, S^1)$. From the Sobolev embedding theorem, u is continuous and there exists a continuous lifting of u denoted by φ. Next, define

$$E_0 := \big\{(x, y) \in I^2\colon\ a(x, y) \leqslant c_0^{-p}\big\} \quad \text{where } a(x, y) := |u|^p_{W^{s,p}([x,y])}|x - y|^{sp-1}, \text{ for } x, y \in I.$$

Let $(x, y) \in E_0$. From (1), for every $x', y' \in [x, y]$, we have $|u(x') - u(y')| \leqslant 1$ and the set $u([x, y])$ is contained in a half circle of S^1. By continuity, we have $|\varphi(x) - \varphi(y)| \leqslant (\pi/2)|u(x) - u(y)|$. Thus

$$\int_{E_0} \frac{|\varphi(x) - \varphi(y)|^p}{|x - y|^{1+sp}} \,\mathrm{d}x\,\mathrm{d}y \leqslant \left(\frac{\pi}{2}\right)^p \int_{E_0} \frac{|u(x) - u(y)|^p}{|x - y|^{1+sp}} \,\mathrm{d}x\,\mathrm{d}y \leqslant C|u|^p_{W^{s,p}(I)}. \tag{2}$$

Now let $(x, y) \in I^2$, we set

$$k_{x,y} := \inf\left\{2^q \colon \exists x = x_0, x_1, \ldots, x_{2^q} = y \in I \text{ with } (x_{i-1}, x_i) \in E_0 \text{ for } 1 \leqslant i \leqslant 2^q\right\}.$$

Clearly, $|\varphi(x) - \varphi(y)| \leqslant (\pi/2)k_{x,y}$, and

$$\int_{I^2 \setminus E_0} \frac{|\varphi(x) - \varphi(y)|^p}{|x - y|^{1+sp}} \,\mathrm{d}x\,\mathrm{d}y \leqslant \left(\frac{\pi}{2}\right)^p \int_{I^2 \setminus E_0} \frac{k^p_{x,y}}{|x - y|^{1+sp}} \,\mathrm{d}x\,\mathrm{d}y. \tag{3}$$

Lemma 2.1. *Let* $(x, y) \in I^2$. *There exists* $z \in (x, y)$ *such that* $\max(a(x, z), a(z, y)) \leqslant 2^{-sp} a(x, y)$.

Proof of Lemma 2.1. We assume that $a(x, y) > 0$ (otherwise there is nothing to prove) and we define $z, f : [0, 1] \to [0, 1]$ by

$$z(\lambda) := (1 - \lambda)x + \lambda y, \qquad f(\lambda) := |u|^p_{W^{s,p}([x,z(\lambda)])} / |u|^p_{W^{s,p}([x,y])}.$$

We have $a(x, z(\lambda)) = f(\lambda)\lambda^{sp-1}a(x, y)$ and $a(z(\lambda), y) \leqslant (1 - f(\lambda))(1 - \lambda)^{sp-1}a(x, y)$. Setting $g(\lambda) := f(\lambda)\lambda^{sp-1}$ and $h(\lambda) := (1 - f(\lambda))(1 - \lambda)^{sp-1}$, it is sufficient to prove that there exists $\lambda_\star \in (0, 1)$ such that $g(\lambda_\star) \leqslant 2^{-sp}$ and $h(\lambda_\star) \leqslant 2^{-sp}$.

The functions g and h are continuous and satisfy $g(0) = h(1) = 0$, $g(1) = h(0) = 1$. Therefore, there exists $\lambda_\star \in (0, 1)$ such that $g(\lambda_\star) = h(\lambda_\star)$. This identity reads $f(\lambda_\star) = (1 - \lambda_\star)^{sp-1}/(\lambda_\star^{sp-1} + (1 - \lambda_\star)^{sp-1})$, thus $g(\lambda_\star) = \lambda_\star^{sp-1}(1 - \lambda_\star)^{sp-1} / (\lambda_\star^{sp-1} + (1 - \lambda_\star)^{sp-1})$. And it is not difficult to check that the right hand side of the previous identity is smaller than 2^{-sp} for every $\lambda_\star \in I$. □

We use Lemma 2.1 to bound the quantities $k_{x,y}$. Let $(x, y) \in I^2 \setminus E_0$, so that $a(x, y) \geqslant c_0^{-p}$ and let $q \geqslant 0$, such that $(2^{q-1})^{sp} \leqslant c_0^p a(x, y) \leqslant (2^q)^{sp}$. Applying recursively the lemma, there exists $x = x_0, \ldots, x_{2^q} = y \in I$ such that for $1 \leqslant i \leqslant 2^q$, $a(x_{i-1}, x_i) \leqslant (2^{-sp})^q a(x, y) \leqslant c_0^{-p}$, thus $(x_{i-1}, x_i) \in E_0$ and $k_{x,y} \leqslant 2^q$.

We then compute $k_{x,y} \leqslant 2 \cdot 2^{q-1} \leqslant 2c_0^{1/s} a(x, y)^{1/(sp)}$ and plugging this estimate in (3), we get

$$\int_{I^2 \setminus E_0} \frac{|\varphi(x) - \varphi(y)|^p}{|x - y|^{1+sp}} \,\mathrm{d}x\,\mathrm{d}y \leqslant C \int_{I^2} \frac{|u|^{p/s}_{W^{s,p}([x,y])}}{|x - y|^m} \,\mathrm{d}x\,\mathrm{d}y,$$

where $m := 1 + 1/s - (1 - s)p$. For $(x, y) \in I^2$, we write

$$|u|^p_{W^{s,p}([x,y])} = |u|^p_{W^{s,p}(I)} \int_{[x,y]^2} \frac{1}{|u|^p_{W^{s,p}(I)}} \frac{|u(x') - u(y')|^p}{|x' - y'|^{1+sp}} \,\mathrm{d}x'\,\mathrm{d}y' \leqslant |u|^p_{W^{s,p}(I)} \mu\big([x, y]\big),$$

where μ is the probability measure on I with density

$$\rho(y') := \frac{1}{|u|^p_{W^{s,p}(I)}} \int_I \frac{|u(x') - u(y')|^p}{|x' - y'|^{1+sp}} \,\mathrm{d}x'.$$

With this notation, we have:

$$\int_{I^2 \setminus E_0} \frac{|\varphi(x) - \varphi(y)|^p}{|x - y|^{1+sp}} \,\mathrm{d}x\,\mathrm{d}y \leqslant C|u|^{p/s}_{W^{s,p}(I)} \int_{I^2} \frac{\mu([x, y])^{1/s}}{|x - y|^m} \,\mathrm{d}x\,\mathrm{d}y \leqslant C|u|^{p/s}_{W^{s,p}(I)} \int_{I^2} \frac{\mu([x, y])}{|x - y|^m} \,\mathrm{d}x\,\mathrm{d}y.$$

Writing $\mu([x,y]) = \int_I \mathbf{1}_{[x,y]}(x')\,\mathrm{d}\mu(x')$, we get:

$$\int\limits_{I^2\setminus E_0} \frac{|\varphi(x)-\varphi(y)|^p}{|x-y|^{1+sp}}\,\mathrm{d}x\,\mathrm{d}y \leqslant C|u|^{p/s}_{W^{s,p}(I)} \int\limits_I \left(\int\limits_{I^2} \frac{\mathbf{1}_{[x,y]}(x')}{|x-y|^m}\,\mathrm{d}x\,\mathrm{d}y\right)\mathrm{d}\mu(x').$$

Let us bound the term in brackets. Let $x' \in I$; using the change of variables $(z,w) = ((x+y)/2 - x', (x-y)/2)$, we have $x' \in [x,y]$ if and only if $|z| \leqslant |w|$ and we compute

$$\int\limits_{I^2} \frac{\mathbf{1}_{[x,y]}(x')}{|x-y|^m}\,\mathrm{d}x\,\mathrm{d}y \leqslant 8 \int\limits_{-1/2}^{1/2} \frac{1}{w^m} \int\limits_{-w}^{w} \mathrm{d}z\,\mathrm{d}w \leqslant 16 \int\limits_{-1/2}^{1/2} \frac{1}{w^{m-1}}\,\mathrm{d}w \leqslant C,$$

where we have used $m - 1 = 1 - (1/s - 1)(sp - 1) < 1$. Finally,

$$\int\limits_{I^2\setminus E_0} \frac{|\varphi(x)-\varphi(y)|^p}{|x-y|^{1+sp}}\,\mathrm{d}x\,\mathrm{d}y \leqslant C|u|^{p/s}_{W^{s,p}(I)}, \tag{4}$$

and summing (2) and (4), we obtain Theorem 1.1. □

3. Lifting of BV-maps

In this section, $\Omega \subset \mathbf{R}^d$ is an open set with a smooth boundary and $|\cdot|$ denotes the classical Euclidean norm in $\mathbf{R}^q$.

Equivalent definition for the BV-seminorm. The key of the simplification of the proof of [8] is Theorem 1 in [7]. In [7], it is stated for functions $u \in BV(\Omega, \mathbf{R})$ but it is also valid, with the same proof, for maps in $BV(\Omega, \mathbf{R}^q)$. The result for maps in $BV(\Omega, \mathbf{R}^q)$ is also a particular case of Theorem 2 in [6] concerning maps with values into a metric space.

Theorem 3.1. *There exists a positive constant $K_{1,d}$ such that for every family of non negative radial mollifiers $(\rho_\varepsilon)_{1\geqslant\varepsilon>0} \subset L^1_{\mathrm{loc}}((0,\infty), \mathbf{R}_+)$ satisfying*:

$$\int\limits_0^\infty \rho_\varepsilon(r) r^{d-1}\,\mathrm{d}r = 1, \quad 1 \geqslant \rho_\varepsilon > 0 \quad \textit{and} \quad \lim_{\varepsilon\to 0} \int\limits_\delta^\infty \rho_\varepsilon(r) r^{d-1}\,\mathrm{d}r = 0, \quad \textit{for every } \delta > 0,$$

we have

$$\lim_{\varepsilon\to 0} \int\limits_{\Omega^2} \frac{|f(x)-f(y)|}{|x-y|} \rho_\varepsilon\big(|x-y|\big)\,\mathrm{d}x\,\mathrm{d}y = K_{1,d}|f|_{BV}, \quad \textit{for every } f \in L^1_{\mathrm{loc}}(\Omega, \mathbf{R}^q).$$

In particular, this equality means that the limit always exists in $\mathbf{R}_+ \cup \{+\infty\}$ and is finite if and only if ∇f is a Radon measure with a finite total mass.

Proof of Theorem 1.2. Let $u \in BV(\Omega, S^1)$ and let (ρ_ε) be a family of radial mollifiers as above. As in [8], for $0 \leqslant \alpha < 2\pi$, we define the lifting φ_α, by $u(x) = \mathrm{e}^{\mathrm{i}\varphi_\alpha(x)}$, $\alpha \leqslant \varphi_\alpha(x) < 2\pi + \alpha$, for every $x \in \Omega$ and we prove

$$\int\limits_0^{2\pi} |\varphi_\alpha|_{BV}\,\mathrm{d}\alpha \leqslant 4\pi|u|_{BV}. \tag{5}$$

Let $1 \geqslant \varepsilon > 0$. From Fubini's theorem, we have

$$\int\limits_0^{2\pi}\int\limits_{\Omega^2} \frac{|\varphi_\alpha(x)-\varphi_\alpha(y)|}{|x-y|}\rho_\varepsilon\big(|x-y|\big)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\alpha = \int\limits_{\Omega^2} \frac{\rho_\varepsilon(|x-y|)}{|x-y|}\left(\int\limits_0^{2\pi} \big|\varphi_\alpha(x)-\varphi_\alpha(y)\big|\,\mathrm{d}\alpha\right)\mathrm{d}x\,\mathrm{d}y.$$

For $x, y \in \Omega$, there exist $\psi_1, \psi_2 \in \mathbf{R}$, such that $u(x) = \mathrm{e}^{\mathrm{i}\psi_1}$, $u(y) = \mathrm{e}^{\mathrm{i}\psi_2}$ and $|\psi_1 - \psi_2| \leqslant \pi$. It is not difficult to see that

$$\int_0^{2\pi} \left|\varphi_\alpha(x) - \varphi_\alpha(y)\right| \mathrm{d}\alpha = 2|\psi_1 - \psi_2|\bigl(2\pi - |\psi_1 - \psi_2|\bigr).$$

Now, by derivating it is easy to prove that the inequality $\sin\beta \geqslant \beta(1 - \beta/\pi)$ holds for every $0 \leqslant \beta \leqslant \pi/2$. Since $|\psi_1 - \psi_2| \leqslant \pi$, we have $|u(x) - u(y)| = 2|\sin\frac{\psi_1 - \psi_2}{2}| \geqslant |\psi_1 - \psi_2|(1 - \frac{|\psi_1 - \psi_2|}{2\pi})$. Thus,

$$\int_0^{2\pi} \left|\varphi_\alpha(x) - \varphi_\alpha(y)\right| \mathrm{d}\alpha \leqslant 4\pi \left|u(x) - u(y)\right|.$$

Plugging this estimate in the integral above, we get

$$\int_0^{2\pi}\int_{\Omega^2} \frac{|\varphi_\alpha(x) - \varphi_\alpha(y)|}{|x - y|} \rho_\varepsilon\bigl(|x - y|\bigr)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\alpha \leqslant 4\pi \int_{\Omega^2} \frac{|u(x) - u(y)|}{|x - y|} \rho_\varepsilon\bigl(|x - y|\bigr)\,\mathrm{d}x\,\mathrm{d}y.$$

Letting ε going to 0, we obtain (5) from Theorem 3.1 and Fatou's lemma. Finally, (5) implies that there exists $\alpha \in [0, 2\pi)$ such that φ_α is a (bounded) lifting of u satisfying $|\varphi_\alpha|_{BV} \leqslant 2|u|_{BV}$. □

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