Available online at www.sciencedirect.com
SCIENCE DIRECT®
C. R. Acad. Sci. Paris, Ser. I 342 (2006) 927–931
COMPTES RENDUS MATHEMATIQUE
http://france.elsevier.com/direct/CRASS1/

Harmonic Analysis

Universal sampling of band-limited signals

Alexander Olevskii [a,1], Alexander Ulanovskii [b]

[a] *School of Mathematical Sciences, Tel Aviv University, Ramat Aviv, Tel Aviv 69978, Israel*
[b] *Stavanger University, NO-4036 Stavanger, Norway*

Received 29 March 2006; accepted 30 March 2006

Available online 22 May 2006

Presented by Jean-Pierre Kahane

Abstract

We ask if there exist universal sampling sets of given density, which provide reconstruction or stable reconstruction of every band-limited signal whose spectrum has a small Lebesgue measure. For the stable reconstruction, we show that it is crucial whether the spectrum is compact or dense. On the other hand, the non-stable universal reconstruction is possible in general situation. ***To cite this article: A. Olevskii, A. Ulanovskii, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Échantillonnage universel de signaux à spectre borné. Nous posons le problème de l'existence d'ensembles discrets, de densité donnée, permettant par échantillonnage la reconstitution, ou la reconstitution stable, de tout signal dont le spectre a une mesure de Lebesgue assez petite. Pour la reconstitution stable, nous montrons que la réponse dépend de manière cruciale du fait que le spectre soit compact ou soit dense. La reconstitution simple, par contre, est toujours possible. ***Pour citer cet article : A. Olevskii, A. Ulanovskii, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Version française abrégée

Les signaux que nous considérons, notés f, sont des fonctions entières de type exponentiel dont le carré est intégrable sur la droite réelle $\mathbf{R}$, c'est-à-dire des transformées de Fourier de fonctions appartenant à $L^2(S)$, où S est une partie mesurable de $\mathbf{R}$. On dira que f a son spectre dans S. Soit Λ une partie discrète de $\mathbf{R}$, et $E(\Lambda)$ l'ensemble des exponentielles dont les fréquences appartiennent à Λ. Les deux problèmes de reconstitution s'expriment ainsi : (1) (reconstitution simple) $E(\Lambda)$ est-il total dans $L^2(S)$? (2) (reconstitution stable) $E(\Lambda)$ est-il un cadre (frame) pour $L^2(S)$ au sens de la formule (1) ? Les références principales sont Beurling et Malliavin [2], Beurling [1], et Landau [6,7], lorsque S est un intervalle ou une réunion finie d'intervalles.

Un exemple est donné d'une suite d'entiers Λ, de densité uniforme $\frac{1}{2}$, telle que pour tout S de mesure de Haar $\frac{1}{2}$ sur le cercle $\mathbf{T}$ la réponse à (1) soit positive (Théorème 1). Cependant (Théorème 2), pour tout Λ séparé (par exemple

E-mail addresses: olevskii@post.tau.ac.il (A. Olevskii), alexander.ulanovskii@uis.no (A. Ulanovskii).
1 The first author is partially supported by the Israel Science Foundation.

1631-073X/$ – see front matter
doi:10.1016/j.crma.2006.04.015

formé d'entiers) dont la densité uniforme inférieure de Beurling est < 1 (formules (2) et (3)), il existe un ouvert S, de mesure de Lebesgue arbitrairement petite, contenu dans un intervalle de longueur 1 (ou 2π, selon la normalisation), tel que la réponse à (2) soit négative. Par contre (Théorème 3), il existe un Λ arbitrairement proche de $\mathbf{Z}$ tel que la réponse à (2) soit positive pour tout S dont la mesure de Lebesgue est inférieure à un nombre donné. Le Théorème 4 étend le Théorème 1 au cas où S n'est pas borné, mais satisfait à la condition (5).

1. Introduction

A band-limited signal is an entire function f of finite exponential type square-integrable on the real axis. According to the classical Paley–Wiener theorem, f is the Fourier transform of an L^2 function with bounded support S, which is called the spectrum of f. We shall denote by PW_S the space of the Fourier transforms of functions from $L^2(S)$.

Given a set $S \subset \mathbf{R}$, let Λ be a real sequence of numbers. Two problems are of special interest:

(A) When a signal f can be recovered (= uniquely defined) by the samples $f(\lambda), \lambda \in \Lambda$?

(B) When this sampling provides a *stable reconstruction*, i.e. the condition

$$A\|f\|_{L^2(\mathbf{R})} \leqslant \left(\sum_{\lambda\in\Lambda}\left|f(\lambda)\right|^2\right)^{1/2} \leqslant B\|f\|_{L^2(\mathbf{R})} \tag{1}$$

holds with some positive constants A and B, for every function $f \in PW_S$?

The right inequality in (1) follows from the separation condition:

$$\inf_{\lambda,\gamma\in\Lambda,\ \lambda\neq\gamma} |\lambda-\gamma| > 0,$$

which we always assume below.

Problem (A) is equivalent to the problem of completeness of the exponential system $E(\Lambda) := \{\exp(\mathrm{i}\lambda t), \lambda \in \Lambda\}$ in the space $L^2(S)$, while restriction (1) means that $E(\Lambda)$ is a frame in this space.

The case $S = [a, b]$ is classical. Beurling and Malliavin [2,1] established that in this case the problems can be essentially solved in terms of appropriate densities of the sampling set Λ. Beurling discovered the role of the *lower uniform density*

$$D^-(\Lambda) := \lim_{r\to\infty}\inf_{t\in\mathbf{R}} \frac{\#(\Lambda \cap (t, r+t))}{r} \tag{2}$$

for the problem of stable reconstruction: If $D^-(\Lambda) > 1$, then $E(\Lambda)$ is a frame in $L^2(-\pi,\pi)$. If

$$D^-(\Lambda) < 1, \tag{3}$$

then $E(\Lambda)$ is not a frame in $L^2(-\pi,\pi)$.

Much attention has been paid to signals which are band-limited to several intervals or general measurable sets. Landau [6] discovered that signals with some specific structure of spectra can be recovered by sparse sequence of samples. On the other hand, he proved [7] that the stable reconstruction always requires the condition $D^-(\Lambda) \geqslant \mathrm{mes}(S)/2\pi$.

These fundamental results have initiated an intensive research in the area, see [5] and [10] for survey and references. We mention also the paper [4], where certain stochastic procedure is suggested showing that for every set $S \subset [-\pi,\pi]$, $\mathrm{mes}(S) < 2\pi$, there exists $\Lambda \subset \mathbf{Z}$, with upper density $D^*(L) < 1$, such that $E(\Lambda)$ is a frame in $L^2(S)$.

In the present note we are interested in the following version of problems (A) and (B): Is it possible to find a 'universal' sampling sequence Λ, which provides recovering, or stable reconstruction of every signal with sufficiently small spectrum?

Our main results show that for the problem of stable reconstruction of signals it is crucial whether the spectra are compact or dense. On the other hand, a non-stable reconstruction is possible in general situation.

2. Universal reconstruction by integer sampling

We start with the following simple, but maybe a bit surprising theorem:

Theorem 1. *There is a sequence Λ of integers with uniform density $D(\Lambda) = \frac{1}{2}$, such that the exponential system $E(\Lambda)$ is complete in $L^2(S)$, for every measurable set S on the circle $\mathbf{T}$ with (normalized) Lebesgue measure $|S| \leqslant \frac{1}{2}$.*

One says that the uniform density $D(\Lambda)$ is equal to d, if $D^+(\Lambda) = D^-(\Lambda) = d$, where D^+ is defined as in (2), but with inf replaced by sup.

Clearly, Theorem 1 is sharp: $E(\Lambda)$ cannot be complete even in L^2 on an arc of the (normalized) length $> 1/2$.

For a weaker result of such sort see [8].

We mention that in Theorem 1 one can take

$$\Lambda = (-2\mathbf{N}) \cup (2\mathbf{N} - 1).$$

A more general form of Theorem 1 is true:

Theorem 1*. *For every $n \in \mathbf{N}$ there is a sequence $\Lambda \subset \mathbf{Z}$, $D(\Lambda) = \frac{n}{n+1}$, such that there is no non-trivial finite Borel measure μ on the circle satisfying $|\operatorname{supp}(\mu)| \leqslant \frac{n}{n+1}$ and $\hat{\mu}(\lambda) = 0$, for $\lambda \in \Lambda$.*

Here and below the sign $\hat{(\cdot)}$ stands for the Fourier transform.

3. No universal stable reconstruction for dense spectra

We show that, in general, the signals with dense spectra do not admit universal stable reconstruction:

Theorem 2. *Let Λ be a (separated) sequence of real numbers, satisfying (3). Then for every $d > 0$ there is an open set $S \subset [-\pi, \pi]$ such that $\operatorname{mes}(S) < d$, and Λ is not a set of stable sampling for the Paley–Wiener space PW_S. Equivalently, the exponential system $E(\Lambda)$ is not a frame in $L^2(S)$.*

We sketch below the proof. In what follows we denote by c positive constants depending only on Λ, and by $\|\cdot\|$ the $L^2(\mathbf{R})$-norm.

Observe that to proof Theorem 2 it is enough to show that for every $\epsilon > 0$ there is a set $S = S(\epsilon) \subset [-\pi, \pi]$ and a function $f \in PW_S$, such that $\operatorname{mes}(S) < \epsilon$ and $\|f\|_{l^2(\Lambda)} < c\epsilon\|f\|$.

(i) Let P be a smooth function on the circle $\mathbf{T}$, $\|P\|_{L^2(\mathbf{T})} = 1$, supported by the interval $(-\epsilon, \epsilon)$. Fix a number M such that

$$\sum_{|n|>M} |\widehat{P}(n)| < \epsilon.$$

(ii) Let N be a large number. Take an integer $q > MN/\epsilon$ and approximate Λ by a sequence $\Lambda(q) \subset (1/q)\mathbf{Z}$ with an l^∞-error less than $1/q$.

Split $\mathbf{R}^+$ up into segments of length $2N$. The intersection of $\Lambda(q)$ with any segment B can be identified with a set $V \subset (1/q)\mathbf{Z} \cap [-N, N]$. Since there are only finitely many different V, at least one is realized with positive upper density. By (3), we can assume that the number of elements in this set satisfies $\#V \leqslant 2rN$, with some $0 < r < 1$. The Szemeredy theorem [11] allows to choose $2M + 1$ segments B_j, $|j| \leqslant M$, situated in an arithmetic progression, which correspond to this V. In other words, there are natural numbers w and L such that:

$$\Lambda(q) \cap B_j = V + w + jL \quad (|j| \leqslant M).$$

(iii) Since $\#V \leqslant 2rN$, $r < 1$, one can show that there exists a function $h \in PW_{[-\pi,\pi]}$ such that $h(n) = 0, n \in V$, and

$$\|h\| = 1, \qquad \|h\|_{L^2(|x| \geqslant N)} < \frac{\epsilon}{M},$$

provided N is large enough.

(iv) Denote by H the inverse Fourier transform of h. Set

$$F(t) := H(t)P(Lt)\,\mathrm{e}^{\mathrm{i}wt}.$$

We may assume that L is sufficiently large, so factors H and P here are 'almost' independent, which imply $\|F\| > c$. Observe also that F is supported by a set of measure 2ϵ, which lies on $(-\pi, \pi)$.

(v) Denote $f := \widehat{F}$. Then

$$f(x) = \sum_{j \in \mathbf{Z}} \widehat{P}(j) h(x - w - Lj).$$

We have to show that $\|f\|_{l^2(\Lambda)} < c\epsilon$. The main part of this norm is the $l_2(\Lambda)$-norm of

$$\sum_{|j| \leqslant M} \widehat{P}(j) h_1(x - w - Lj),$$

where we set $h_1 := h \cdot \mathbf{1}_{[-N,N]}$. The square Q^2 of this norm is equal to:

$$Q^2 = \sum_{|j| \leqslant M} \left|\widehat{P}(j)\right|^2 \sum_{x+w+jL \in \Lambda(q) \cap B(j)} \left|h(x)\right|^2.$$

Due to (ii), each x in the second sum has distance $< 1/q$ from a point of V (where h is zero). By Bernstein's inequality, $|h(x)| < c/q$, which gives $Q < c\epsilon$.

The rest can be estimated by using the inequalities in (i) and (iii).

4. Stable reconstruction of signals with compact spectra

It is well known that if a set S has the following structure:

(C) *S is a finite union of intervals, the lengths of these intervals and of the gaps between them are commensurable*,

then there exists Λ (a finite union of arithmetic progressions) such that $E(\Lambda)$ is an exponential frame, and even a Riesz basis in $L^2(S)$ (see [3] and [5]).

An induction process involving infinitely many steps with small perturbation on each step, allows us to construct sets of universal stable reconstruction with optimal value of sampling density:

Theorem 3. *Given a number $d > 0$, there exists a sequence $\Lambda \subset \mathbf{R}$ which is arbitrarily close to $(1/d)\mathbf{Z}$ in l^∞-norm, such that the system $E(\Lambda)$ is a frame in $L^2(S)$ for every compact set $S \subset \mathbf{R}$, $\operatorname{mes}(S) < 2\pi d$.*

In fact we do more than that: we construct a universal exponential Riesz basis in $L^2(S)$, for all (C)-sets S having a given measure:

Theorem 3*. *There exists a sequence $\Lambda \subset \mathbf{R}$ (arbitrarily close to $\mathbf{Z}$ in l^∞-norm), such that $E(\Lambda)$ is a Riesz basis in $L^2(S)$ for every set $S \subset \mathbf{R}$ of measure 2π satisfying property* (C).

5. Reconstruction of signals with unbounded spectra

Here we come back to non-stable reconstructions, and show that by small perturbation of integers one may extend Theorem 1 to some classes of unbounded spectra:

Theorem 4. *Given a number $d > 0$, set*

$$\Lambda_d := \left\{ \frac{n}{d} + 2^{-|n|}, n \in \mathbf{Z} \right\}. \tag{4}$$

Then the exponential system $E(\Lambda_d)$ is complete in $L^2(S)$ for any measurable set $S \subset \mathbf{R}$ satisfying $\operatorname{mes}(S) < 2\pi d$ *and*

$$\int_S e^{|t|} \, dt < \infty. \tag{5}$$

The proof is similar to the proof of Theorem 4 in our paper [9].

The last assumption shows that the measure of the portion of S outside the interval $(-r, r)$ tends to zero fast enough as $r \to \infty$. It cannot be dropped: it can be shown that for every $\delta > 0$ there exists $S \subset \mathbf{R}$, $\mathrm{mes}(S) < \delta$, such that the system $E(\Lambda_1)$ is not complete in $L^2(S)$, where Λ_1 is defined in (4).

Theorem 4 can be generalized in several directions. In particular, the statement remains true for all sequences Λ which are 'quasi-analytically small' perturbations of $(1/d)\mathbf{Z}$, and the restriction $\mathrm{mes}(S) < 2\pi d$ can be replaced by $\mathrm{mes}(P_{\pi d}(S)) < 2\pi d$, where $P_{\pi d}(S)$ is the *projection* of S onto $[-\pi d, \pi d]$:

$$P_{\pi d}(S) := \left(\bigcup_{k \in \mathbf{Z}} (S + 2\pi k d)\right) \cap [-\pi d, \pi d].$$

Such sets S may have arbitrarily large measure.

References

[1] A. Beurling, Balayage of Fourier–Stietjes transforms, in: Collected Works of Arne Beurling, vol. 2, Harmonic Analysis, Birkhäuser, Boston, 1989.

[2] A. Beurling, P. Malliavin, On the closure of characters and the zeros of entire functions, Acta Math. 118 (1967) 79–93.

[3] L. Bezuglaya, V. Katsnelson, The sampling theorem for functions with limited multi-band spectrum, Z. Anal. Anwendungen 12 (3) (1993) 511–534.

[4] J. Bourgain, L. Tzafriri, Invertibility of "large" submatrices with applications to the geometry of Banach spaces and harmonic analysis, Israel J. Math. 57 (2) (1987) 137–224.

[5] J.R. Higgins, Sampling Theory in Fourier and Signal Analysis. Foundations, Clarendon Press, Oxford, 1996.

[6] H.J. Landau, A sparse regular sequence of exponentials closed on large sets, Bull. Amer. Math. Soc. 70 (1964) 566–569.

[7] H.J. Landau, Necessary density conditions for sampling and interpolation of certain entire functions, Acta Math. 117 (1967) 37–52.

[8] Yu. Lyubarskii, I. Spitkovsky, Sampling and interpolation for a lacunary spectrum, Proc. Roy. Soc. Edinburgh Sect. A 126 (1) (1996) 77–87.

[9] A. Olevskii, A. Ulanovskii, Almost integer translates. Do nice generators exist?, J. Fourier Anal. Appl. 10 (1) (2004) 93–104.

[10] K. Seip, Interpolation and Sampling in Spaces of Analytic Functions, University Lecture Series, vol. 33, American Mathematical Society, Providence, RI, 2004.

[11] E. Szemerédi, On sets of integers containing no k elements in arithmetic progression, Acta Arith. 27 (1975) 199–245.