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Differential Geometry

Floer homology for almost Hamiltonian isotopies

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Abstract

Seidel introduced a homomorphism from the fundamental group $\pi_1(\mathrm{Ham}(M))$ of the group of Hamiltonian diffeomorphisms of certain compact symplectic manifolds (M, ω) to a quotient of the automorphism group $\mathrm{Aut}(HF_*(M, \omega))$ of the Floer homology $HF_*(M, \omega)$. We prove a rigidity property: if two Hamiltonian loops represent the same element in $\pi_1(\mathrm{Diff}(M))$, then the image under the Seidel homomorphism of their classes in $\pi_1(\mathrm{Ham}(M))$ coincide. The proof consists in showing that Floer homology can be defined by using 'almost Hamiltonian' isotopies, i.e. isotopies that are homotopic relatively to endpoints to Hamiltonian isotopies. ***To cite this article: A. Banyaga, C. Saunders, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Homologie de Floer pour les isotopies presques hamiltonniennes. Seidel a introduit un homomorphisme du groupe fondamental $\pi_1(\mathrm{Ham}(M))$ du groupe des difféomorphismes Hamiltoniennes de certaines variétés symplectiques compactes (M, ω) dans un quotient du groupe $\mathrm{Aut}(HF_*(M, \omega))$ des automorphismes de l'homologie de Floer $HF_*(M, \omega)$. Nous démontrons que si deux lacets Hamiltoniennes representent le même élément dans $\pi_1(\mathrm{Diff}(M))$, alors les images par l'homomorphisme de Seidel de leurs classes dans $\pi_1(\mathrm{Ham}(M))$ coïncident (un phénomène de rigidité). La preuve consiste à montrer que l'homologie de Floer peut être définie en utilisant des isotopies presques Hamiltoniennes, c'est-à-dire des isotopies qui sont homotopes, relativement aux extrémités à des isotopies Hamiltoniennes. ***Pour citer cet article : A. Banyaga, C. Saunders, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

1. Introduction

A symplectic manifold is a manifold M which is equipped with a closed, non-degenerate 2-form ω. Non-degeneracy means that the interior product $i_X\omega$ of a vector field X with ω induces a bundle isomorphism from TM to T^*M. Given $H \in C^\infty(M \times \mathbb{R})$, this isomorphism defines a family of vector fields X_{H_t} on M by $i_{X_{H_t}}\omega = \mathrm{d}H_t$, where $H_t(x) = H(x, t)$. Integration of this family of vector fields leads to an isotopy $\theta^H = (\theta_t^H)$ of M. We say that θ^H

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is the Hamiltonian isotopy generated by H. The set of all time one maps of Hamiltonian isotopies forms a subgroup of $\mathrm{Diff}(M)$, called the group of Hamiltonian diffeomorphisms, and it will be denoted $\mathrm{Ham}(M)$ herein.

We are in general interested in the inclusion $i:\mathrm{Ham}(M)\to\mathrm{Diff}(M)$ and the induced map $i_*:\pi_1(\mathrm{Ham}(M))\to\pi_1(\mathrm{Diff}(M))$. In [6], Seidel introduced an extension of $\pi_1(\mathrm{Ham}(M))$, denoted $\widetilde{G}$, and a homomorphism $\tilde{\sigma}:\widetilde{G}\to\mathrm{Aut}(HF_*(M,\omega))$, where $HF_*(M,\omega)$ is the Floer homology of (M,ω). (At this point, $\tilde{\sigma}$ is only well defined under certain conditions on (M,ω).) There is thus an induced homomorphism

$$\sigma:\pi_1\big(\mathrm{Ham}(M)\big)\to\mathrm{Aut}\big(HF_*(M,\omega)\big)/\Theta.$$

The main point of this Note is to sketch a proof of the following theorem (under the conditions necessary to define σ.)

Theorem 1.1. *Let g_0 and g_1 be smooth loops at* id *in* $\mathrm{Ham}(M)$*, and let $[g_0]$ and $[g_1]$ be their respective classes in* $\pi_1(\mathrm{Ham}(M))$*. Then, if g_0 and g_1 define the same element of* $\pi_1(\mathrm{Diff}(M))$*, then* $\sigma([g_0])=\sigma([g_1])$.

In other words, this theorem says that $\ker i_*\subset\ker\sigma$. This gives more evidence for what has been called the 'topological rigidity' of the group $\mathrm{Ham}(M)$ [4].

In order to prove this theorem, we observe that σ is induced by a map $\hat{\sigma}$ with domain G consisting of smooth loops at id in $\mathrm{Ham}(M)$ (rather than $\pi_0(G)=\pi_1(\mathrm{Ham}(M))$). We extend the domain of $\hat{\sigma}$ to the possibly larger group $D:=\{\text{smooth loops in } \mathrm{Diff}(M) \text{ at id which are homotopic to a Hamiltonian loop}\}$.

Elements of D will be called *almost Hamiltonian* loops.

Theorem 1.1 is proved by showing that the extended map is well-defined on $\pi_0(D)=i_*(\pi_1(\mathrm{Ham}(M))$.

2. Floer homology and Seidel's homomorphism [5,6]

Let $\mathcal{J}(M,S^1)$ denote the space of smooth time dependent 1-periodic almost complex structures on M. Let $\mathcal{J}(M,\omega,S^1)\subset\mathcal{J}(M,S^1)$ be the subspace of ω-compatible almost complex structures. There are Chern classes $c_i\in H^{2i}(M,\mathbb{Z})$ associated to the symplectic manifold (M^{2n},ω), and throughout this note, we will assume the semi-positive condition W^+ given in [6]. While the new ideas in this note do not depend on this assumption, it is thus far necessary in defining Seidel's homomorphism.

Let $\mathcal{L}M$ denote the space of smooth, contractible loops in M, and let $p:\widetilde{\mathcal{L}M}\to\mathcal{L}M$ be the covering space given in [3]. For $H\in C^\infty(M\times S^1)$, let $\mathcal{P}(H)\subset\mathcal{L}M$ consist of 1-periodic contractible orbits of the Hamiltonian isotopy θ^H generated by the function H.

Let $(H,J)\in C^\infty(M\times S^1)\times\mathcal{J}(M,\omega,S^1)$ be a regular pair [2,3,6]. We denote by $HF_*(M,\omega,H,J)$ the Floer homology groups defined using the pair (H,J), which is essentially constructed using the set $\widetilde{\mathcal{P}(H)}:=p^{-1}(\mathcal{P}(H))$.

Any $g\in G$ gives a map $g:\mathcal{L}M\to\mathcal{L}M$ by $(g\cdot x)(t)=g_t(x(t))$. Let $\widetilde{G}$ be the group of all lifts of elements of G to homeomorphisms of $\widetilde{\mathcal{L}M}$. This gives an exact sequence of topological groups $1\to\Gamma\to\widetilde{G}\to G\to 1$, where Γ is the group of deck transformations of the covering map p.

For $g\in G$, a new regular pair (H^g,J^g) is defined in such a way that

$$g\cdot\mathcal{P}\big(H^g\big)=\mathcal{P}(H).\tag{1}$$

After lifting to $\tilde{g}\in\widetilde{G}$, this induces a map between the Floer chain complexes $CF_*(M,\omega,H^g)$ to $CF_*(M,\omega,H)$. The almost complex structure J^g is defined so that this is a chain map, and this gives an isomorphism on the level of homology, which we still call $\tilde{g}$.

There is a continuation isomorphism Φ between the Floer homology groups defined using any two regular pairs. Let Φ denote the continuation isomorphism between $HF_*(M,\omega,H^g,J^g)$ and $HF_*(M,\omega,H,J)$. Then we get an automorphism of $HF_*(M,\omega,H,J)$ by

$$HF_*(M,\omega,H,J)\xrightarrow{\Phi}HF_*\big(M,\omega,H^g,J^g\big)\xrightarrow{\tilde{g}}HF_*(M,\omega,H,J).\tag{2}$$

This automorphism commutes with the continuation isomorphism to the Floer homology defined using any other regular pair, and if $g_0,g_1\in G$ are connected through smooth Hamiltonian loops, then the corresponding isomorphisms agree. This describes a map

$$\tilde{\sigma}:\pi_0\big(\widetilde{G}\big)\to\mathrm{Aut}\big(HF_*(M,\omega)\big).$$

After quotienting by the image of Γ in $\mathrm{Aut}(HF_*(M,\omega))$, we obtain the map σ.

3. Sketch of the proof of the main result

First notice that it is a near triviality to see that if two functions H^0 and H^1 generate the same Hamiltonian isotopy, then $HF_*(M,\omega,H^0,J) = HF_*(M,\omega,H^1,J)$ (as long as both (H^0,J) and (H^1,J) are regular pairs). This means that Floer homology can be rephrased in terms of Hamiltonian isotopies.

The isotopy generated by the function H^g is given by $\theta_t^{H^g} = g_t^{-1}\theta_t^H$. We define an action of D on the set of all isotopies of M by $(g*\theta)_t := g_t^{-1}\theta_t$, and in this language, $\theta^{H^g} = g*\theta^H$.

Thus, we can rephrase (2) as

$$HF_*(M,\omega,\theta,J) \xrightarrow{\Phi} HF_*(M,\omega,g*\theta,J^g) \xrightarrow{\tilde{g}} HF_*(M,\omega,\theta,J). \tag{3}$$

This suggests a method of extending the domain of $\hat{\sigma}$ to the group D: we extend the space of available choices in defining $HF_*(M,\omega)$ to include pairs $(g*\theta,J^g)$ for $g \in D$ and Hamiltonian isotopies θ.

Notice that for $g \in D$ and a Hamiltonian isotopy θ, $g*\theta$ is homotopic, relative endpoints, to a Hamiltonian isotopy. Such isotopies will be called *almost Hamiltonian*, and we denote the group of all almost Hamiltonian isotopies by $\mathcal{I}$. There is an analogous extension $1 \to \Gamma \to \widetilde{D} \to D \to 1$, where $\widetilde{D}$ consists of lifts of the action of D on $\mathcal{L}M$ to homeomorphisms of $\widetilde{\mathcal{L}M}$.

Such isotopies may not preserve ω, so we need to adjust the compatibility requirements on J.

For $\psi \in \mathcal{I}$, we let $\mathcal{J}^\psi(M,\omega,S^1) = \{J \in \mathcal{J}(M,S^1) \mid J_t \in \mathcal{J}(M,(\psi_t^{-1})^*\omega)\}$. We then let $\mathcal{F}$ denote all pairs (ψ,J) with $\psi \in \mathcal{I}$ and $J \in \mathcal{J}^\psi(M,\omega,S^1)$. It is best to think of $\mathcal{F}$ as a bundle over $\mathcal{I}$, with fiber over $\psi \in \mathcal{I}$ given by $\mathcal{J}^\psi(M,\omega,S^1)$.

Theorem 3.1. *There is a dense subset of $\mathcal{F}$, denoted $\mathcal{F}_{\mathrm{reg}}$, such that for all pairs $(\psi,J) \in \mathcal{F}_{\mathrm{reg}}$, we can define Floer homology groups $HF_*(M,\omega,\psi,J)$. These groups are naturally independent of the choice of pair used to define them, and they recover the standard Floer homology groups $HF_*(M,\omega)$.*

The Floer homology groups $HF_*(M,\omega,\psi,J)$ in this theorem are constructed as in the Hamiltonian case, but we replace the Hamiltonian function by the generated isotopy. More precisely, for $\psi \in \mathcal{I}$, let X_t^ψ be the corresponding family of vector fields on M obtained by differentiation (see [1]), and set

$$\mathcal{P}(\psi) := \{x \in \mathcal{L}M \mid \dot{x}(t) = X_t^\psi(x(t))\}, \qquad \widetilde{\mathcal{P}(\psi)} := p^{-1}(\mathcal{P}(\psi)).$$

For certain (non-degenerate) isotopies, there is an index map $\mu_\psi : \widetilde{\mathcal{P}(\psi)} \to \mathbb{Z}$, which reduces to the Conley–Zehnder index in the Hamiltonian case. Notice that any homotopy from ψ to a Hamiltonian isotopy θ provides an identification $b : \widetilde{\mathcal{P}(\psi)} \to \widetilde{\mathcal{P}(\theta)}$. The index in the almost Hamiltonian case is well defined by setting $\mu_\psi(c) = \mu_\theta(b(c))$.

This graded set is used to create a chain complex, with a boundary operator defined as follows. For smooth $u : \mathbb{R} \times S^1 \to M$, define $\bar{\partial}_{\psi,J}(u) \in C^\infty(u^*TM)$ by

$$\bar{\partial}_{\psi,J}(u) = \frac{\partial u}{\partial s} - J_t\left(\frac{\partial u}{\partial s} + X_t^\psi\right). \tag{4}$$

For $c_\pm \in \widetilde{\mathcal{P}(\psi)}$, let $\mathcal{M}(c_-,c_+,\psi,J)$ consist of all smooth maps $u : \mathbb{R} \times S^1 \to M$ which satisfy $\bar{\partial}_{\psi,J}(u) = 0$, and which lift to a map $\tilde{u} : \mathbb{R} \to \widetilde{\mathcal{L}M}$ with $\lim_{s\to\pm\infty} \tilde{u}(s) = c_\pm$. There is an $\mathbb{R}$-action on $\mathcal{M}(c_-,c_+,\psi,J)$ by translation in the s-variable, and we denote the quotient space by $\mathcal{M}(c_-,c_+,\psi,J)/\mathbb{R}$.

The boundary operator in classical Floer homology consisted of counting the number of elements of $\mathcal{M}(c_-,c_+,\psi,J)/\mathbb{R}$. We would like to do the same, so we show that if $\mu_\psi(c_-) - \mu_\psi(c_+) = 1$, then $\mathcal{M}(c_-,c_+,\psi,J)/\mathbb{R}$ is finite.

In the Hamiltonian case (when ψ is a Hamiltonian isotopy), there are two basic reasons which allow us to count $\mathcal{M}(c_-,c_+,\psi,J)/\mathbb{R}$. The first is the ellipticity of (4), and the second is a uniform bound on the energy $E(u) = \int_{\mathbb{R}\times S^1} |\frac{\partial u}{\partial s}(s,t)|^2$. The ellipticity gives that $\mathcal{M}(c_-,c_+,\psi,J)$ is a manifold, and the energy bound gives compactness (in the 0-dimensional case). The energy bound is proved by noticing that solutions to $\bar{\partial}_{\theta^H,J}(u) = 0$ satisfy the equation $\nabla a_H(\tilde{u}) = \frac{\mathrm{d}\tilde{u}}{\mathrm{d}s}$ for any lift $\tilde{u} : \mathbb{R} \to \widetilde{\mathcal{L}M}$, where a_H is the action functional. This implies that the energy of $u \in \mathcal{M}(c_-,c_+,\theta^H,J)$ is given by $E(u) = a_H(c_-) - a_H(c_+)$.

In the almost Hamiltonian case, we can establish the same facts: the term involving the vector field does not affect the ellipticity because it is of lower order. To give a bound on the energy, we decompose $(\psi, J) \in \mathcal{F}$ as $g * \theta^H$ for some $g \in D$ and Hamiltonian isotopy θ^H, and choose any lift $(g, \tilde{g}) \in \widetilde{D}$. The pair $(\psi, J) \in \mathcal{F}$ induces a metric $\tilde{h}$ on $\widetilde{\mathcal{L}M}$. Then lifts $\tilde{u}$ of solutions to $\bar{\partial}_{\psi,J}(u) = 0$ satisfy $\nabla \tilde{g}^* a_H(\tilde{u}) = \frac{\mathrm{d}\tilde{u}}{\mathrm{d}s}$, because for all $\xi \in T_{\tilde{u}(s)}\widetilde{\mathcal{L}M}$,

$$\tilde{h}\big(\nabla \tilde{g}^* a_H(\tilde{u})(s), \xi\big) = \tilde{g}^*(da_H)\big(\tilde{u}(s)\big)(\xi) = \int_0^1 \psi_t^* \omega\big(\dot{x}(t) - X_t^{g*\theta^H}\big(x(t)\big), J_t Dp\xi(t)\big)\,\mathrm{d}t,$$

where $x(t) = u(s,t)$. But this is exactly $\tilde{h}(\frac{\mathrm{d}\tilde{u}}{\mathrm{d}s}(s), \xi)$. Thus, for $u \in \mathcal{M}(c_-, c_+, \psi, J)$, $E(u) = \tilde{g}^* a_H(c_-) - \tilde{g}^* a_H(c_+)$.

We then define a boundary operator via counting exactly as in the Hamiltonian case, and show that $\partial^2 = 0$.

To prove the independence of the choice of pair, we define a *homotopy of regular pairs* as a map $\Phi : \mathbb{R} \to \mathcal{F}$ which is fixed outside of $[-1, 1]$. A map between the corresponding chain complexes can be defined by counting cylinders which satisfy a two-parameter version of (4). These sets can be counted for the same reasons as described above.

Theorem 3.1 can now be used to extend the domain of $\tilde{\sigma}$ to $\widetilde{D}$ by (3). The final step is proving that this map is well defined on $\pi_0(\widetilde{D})$. This is accomplished by combining Seidel's proof with the above ideas.

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