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Probability Theory

The harmonic measure and a Reuter-type result for a process with Bessel components

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Abstract

Let $B_{p,q}$ be the R^2-valued process (B_p, B_q) with independent Bessel components B_p and B_q with indices p and q strictly positive. In this paper we compute explicitly the law of the hitting time and place of a circle, centered at the origin, when $B_{p,q}$ starts from the center and deduce a Reuter-type independence result. We use mainly analytical tools from PDE theory. ***To cite this article: A. Ziadi, A. Bencherif-Madani, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Résumé

La mesure harmonique et un résultat du type Reuter pour un processus à composantes besséliennes. Soit $B_{p,q}$ le processus (B_p, B_q) dans R^2 ayant deux composantes de Bessel indépendantes B_p et B_q d'indices, respectivement, p et q strictement positifs. Dans cet article, nous trouvons explicitement la loi du temps et place de sortie d'un cercle, centré à l'origine, quand $B_{p,q}$ démarre du centre et on déduit un théorème d'indépendance apparenté à celui de Reuter. On utilise principalement des outils analytiques de la théorie des EDP. ***Pour citer cet article : A. Ziadi, A. Bencherif-Madani, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Version française abrégée

Un processus de Bessel B_p de dimension réelle p (dans ce papier $p > 0$) est un processus de Markov réel non-négatif fort et continu de générateur $\mathcal{A}_p$ donné par

$$\mathcal{A}_p(\cdot) = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}(\cdot) + \frac{1}{2}\frac{p-1}{x}\frac{\mathrm{d}}{\mathrm{d}x}(\cdot)$$

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agissant, à l'intérieur de son espace d'états, sur les fonctions deux fois continûment différentiables. Rappelons que si $p \leqslant 0$ l'origine est une barrière de sortie mais n'est pas une barrière d'entrée. Quand $0 < p < 2$ l'origine devient une barrière régulière (il est nécessaire de spécifier ici la condition au bord $\lim_{x\to 0+} x^{p-1}\frac{\mathrm{d}f}{\mathrm{d}x}(x) = 0$, f appartenant au domaine de $\mathcal{A}_p$) et quand $p \geqslant 2$ celle-ci devient une barrière inaccessible. Par contre dans tous les cas, le point à l'infini reste inaccessible.

Soit Γ l'arc de cercle unité complexe $0 \leqslant \theta \leqslant \frac{\pi}{2}$ et soit $B_{p,q}$ le processus plan (B_p, B_q) dans lequel B_p et B_q sont deux Bessel indépendants, p et q strictement positifs, partant de z avec $|z| \leqslant 1$. Nous trouvons explicitement la mesure harmonique en z et nous montrons que lorsque $z = 0$ le processus $B_{p,q}$ exhibe un comportement de sortie de Γ qui rappelle un vieux théorème de Harry Reuter. Nous utilisons un théorème de représentation de Feynman–Kac ; et étant donné la compacité de nos variables aléatoires, nous réduisons les fonctions continues à un système de polynômes orthogonaux convenables (polynômes de Jacobi). Il suffit alors d'utiliser la technique bien connue de la séparation des variables.

1. Introduction

A Bessel process B_p with real dimension p, here $p > 0$, is a linear continuous strong Markov process defined on some probability space $(\Omega, \mathcal{F}, P)$ (when not stated otherwise B_p starts from zero), whose boundary behaviour depends on p. For the convenience of the reader, we remind that when $p \leqslant 0$ the origin is an exit boundary but not an entrance one, when $0 < p < 2$ it becomes a regular boundary and when $p \geqslant 2$ it is inaccessible; whereas in all cases the point at infinity remains inaccessible, see [1,14]. Locally, it has the generator $\mathcal{A}_p$ given by

$$\mathcal{A}_p(\cdot) = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}(\cdot) + \frac{1}{2}\frac{p-1}{x}\frac{\mathrm{d}}{\mathrm{d}x}(\cdot)$$

acting on twice continuously differentiable functions. That is, the process is kept alive by means of the main part of the well known *Bessel differential equation* before any killing takes place, whence the name *Bessel process*. When p is in $(0, 2)$, we need in addition the following boundary condition for functions f in the domain $\mathcal{D}_p$ of $\mathcal{A}_p$

$$\lim_{x\to 0+} x^{p-1}\frac{\mathrm{d}f}{\mathrm{d}x}(x) = 0$$

which makes zero a reflecting barrier.

Inspired by Kent [5] or Hendricks [4], we might ask about the hitting properties of the process $B_{p,q} = (B_p, B_q)$ in which B_p and B_q are independent Bessel processes with positive dimensions p and q respectively. In this Note, we run $B_{p,q}$ from a point z within a circle C of radius one centered at the origin and find the joint law of the couple $(\tau^z, B_{p,q}(\tau^z))$, where τ^z stands for the hitting time of C starting from z. It is remarkable that a well known theorem of Reuter concerning independence properties of Brownian motion with drift exiting from inside a ball in R^d, $d \geqslant 1$ (unpublished manuscript derived by a direct analytical computation – see [7], p. 84, and [5]) is valid here. Note that the first Reuter-type independence phenomenon appears in [9] for a linear Brownian motion with constant drift. This result was extended to R^d by Kent [5] and improved in [13], see also [12]. In fact, there is also an older Markov chains context, see for example [3,8]; where the first paper deals with a Markov chain with continuous time in which independence properties of exit time and space from a stable state are considered and the second one studies exit properties of a possibly asymmetric Bernoulli random walk on the integers, which is more in line with our situation here.

2. Preliminaries

We need first to discuss the semi-group theory for the process $B_{p,q}$ and in particular its boundary behaviour. If we denote by $P_t^p(x_1, y_1)$ and $P_t^q(x_2, y_2)$ the transition densities of B_p and B_q respectively, we can immediately manufacture a new transition density $P_t((x_1, x_2), (y_1, y_2)) = P_t^p(x_1, y_1)P_t^q(x_2, y_2)$. It is straightforward, thanks to standard procedures, to see that this transition density gives a strongly continuous contraction semi-group which corresponds to $B_{p,q}$ and to find a core for the generator $\mathcal{A}_{p,q}$ of $B_{p,q}$ with domain $\mathcal{D}_{p,q}$. Indeed, it suffices to put

$$A_{p,q} = \frac{1}{2}\left(\Delta + \frac{p-1}{x_1}\frac{\mathrm{d}}{\mathrm{d}x_1} + \frac{q-1}{x_2}\frac{\mathrm{d}}{\mathrm{d}x_2}\right),$$

where Δ is the standard Laplacian. We have

Lemma 2.1. *Let $f(x_1, x_2)$ be a twice continuously differentiable function in R^2 with compact support and even both in x_1 and x_2, i.e. for all x_1 and x_2, we have $f(x_1, x_2) = f(-x_1, x_2)$ and $f(x_1, x_2) = f(x_1, -x_2)$.*

Then the restriction of f to R_+^2, still noted f, lies in $\mathcal{D}_{p,q}$ and $\mathcal{A}_{p,q} f = A_{p,q} f$.

Proof. It is clear that the lemma holds for elementary products

$$f(x_1, x_2) = f_1(x_1) f_2(x_2),$$

where f_1 is in $\mathcal{D}_p$ and f_2 in $\mathcal{D}_q$. In order to capture general functions $f(x_1, x_2)$ satisfying our hypothesis, we use a standard uniform convergence result using the Stone–Weierstrass theorem, see for example [10], p. 409, to find an array of even twice continuously differentiable functions with compact supports $f_{1n}^j(x_1)$ and $f_{2n}^j(x_2)$, $j = 1, \ldots, n$, s.t. $f_n(x_1, x_2) = \sum_j^n f_{1n}^j(x_1) f_{2n}^j(x_2)$ converges, together with its derivatives up to the order two, uniformly to $f(x_1, x_2)$ and its derivatives respectively. It is easy to see that the functions $f_{1n}^j(x_1)$ and $f_{2n}^j(x_2)$, restricted to R_+, lie respectively in $\mathcal{D}_p$ and $\mathcal{D}_q$; from where it follows that f_n restricted to R_+^2 lies in $\mathcal{D}_{p,q}$ and that $\mathcal{A}_{p,q} f_n = A_{p,q} f_n$. Reaching f is now obvious. □

2.1. A Feynman–Kac formula

The following Lemma is well known

Lemma 2.2. *Let $h(z)$ be a function defined on the arc Γ given by $z = x_1^2 + x_2^2 = 1$ and $f(x_1, x_2)$ be a twice continuously differentiable function in R^2 and even both in x_1 and x_2 (its restriction to R_+^2 is still noted f). If for $\lambda > 0$, and all z in $(0, \infty)^2$*

$$A_{p,q} f - \lambda f = 0, \tag{1}$$

then

$$f(z) = E^z\left[h\left(B_{p,q}(\tau)\right) \exp -\lambda\tau\right]$$

solves the boundary value problem $f(z) = h(z)$ on Γ corresponding to the elliptic operator $A_{p,q} - \lambda I$.

3. The joint law of $(\tau^z, B_{p,q}(\tau^z))$

Set $z = \rho\, \mathrm{e}^{\mathrm{i}\theta}$, $\rho \leqslant 1$, $\nu = (p-2)/2$ and $\mu = (q-2)/2$. Let us recall that the system of complete and orthonormal Jacobi polynomials are defined on the interval [0, 1] by the Rodrigues formula

$$P_n^{\nu,\mu}(x) = \frac{(-1)^n}{n!2^n} \cdot \frac{1}{(1-x)^\nu (1+x)^\mu} \cdot \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[(1-x)^{\nu+n}(1+x)^{\mu+n}\right],$$

where n is an integer and ν and μ are > -1. They satisfy the differential equation

$$(1-x^2)\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + \left[(\mu-\nu) - (\mu+\nu+2)x\right]\frac{\mathrm{d}f}{\mathrm{d}x} + n(\mu+\nu+n+1) f = 0,$$

see [6], p. 209.

Our main result is the following:

Theorem 3.1. *The function*

$$f(z)=\rho^{-(\nu+\mu+1)}\frac{I_{(\nu+\mu+2n+1)}(\rho\sqrt{2\lambda})}{I_{(\nu+\mu+2n+1)}(\sqrt{2\lambda})}P_n^{\mu,\nu}(\cos 2\theta),$$

where $I_{(\cdot)}(x)$ stands for the modified Bessel function of the first kind, solves Eq. (1).

Proof. It suffices to use polar coordinates and separation of variables. Eq. (1) becomes

$$\frac{\mathrm{d}^2 f}{\mathrm{d}\rho^2}+\frac{p+q-1}{\rho}\frac{\mathrm{d}f}{\mathrm{d}\rho}+\frac{1}{\rho^2}\left[\frac{\mathrm{d}^2 f}{\mathrm{d}\theta^2}+\left[(1-p)\tan\theta+(q-1)\cot\theta\right]\frac{\mathrm{d}f}{\mathrm{d}\theta}\right]-2\lambda f=0.$$

Let us put $\varsigma_n=2n(p+q-2)+4n^2$, then we have

$$\frac{d^2 f}{d\rho^2}+\frac{p+q-1}{\rho}\frac{\mathrm{d}f}{\mathrm{d}\rho}+\frac{1}{\rho^2}\left\{\frac{\mathrm{d}^2 f}{\mathrm{d}\theta^2}+\left[(1-p)\tan\theta+(q-1)\cot\theta\right]\frac{\mathrm{d}f}{\mathrm{d}\theta}+\varsigma_n f\right\}-\left(\frac{1}{\rho^2}\varsigma_n+2\lambda\right)f=0,$$

for all n. Consider first the angular equation

$$\frac{\mathrm{d}^2 g}{\mathrm{d}\theta^2}+\left[(1-p)\tan\theta+(q-1)\cot\theta\right]\frac{\mathrm{d}g}{\mathrm{d}\theta}+\varsigma_n g=0. \tag{2}$$

It has the particular solution

$$g(\theta)=P_n^{\mu,\nu}(\cos 2\theta).$$

Indeed, we shall look for a function $u(\cdot)$ s.t.

$$g(\theta)=(\sin\theta)^{-(q-1)/2}(\cos\theta)^{-(p-1)/2}u(2\theta)$$

solves (2). Letting $\alpha=2\theta$ Eq. (2) becomes

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\alpha^2}+\left[\frac{1-4((p-2)/2)^2}{16}\cdot\frac{1}{\cos^2\alpha/2}+\frac{1-4((q-2)/2)^2}{16}\cdot\frac{1}{\sin^2\alpha/2}+\left(n+\frac{p+q-2}{4}\right)^2\right]u(\alpha)=0.$$

A solution of this equation is given by, see [6], p. 214,

$$u(\alpha)=\left(\sin\frac{\alpha}{2}\right)^{-(q-1)/2}\left(\cos\frac{\alpha}{2}\right)^{-(p-1)/2}P_n^{\mu,\nu}(\cos\alpha),$$

which gives the required expression for $g(\theta)$.

To finish off, the equation

$$\rho^2\frac{\mathrm{d}^2 h}{\mathrm{d}\rho^2}+\rho(p+q-1)\frac{\mathrm{d}h}{\mathrm{d}\rho}+[-2\lambda\rho^2-\varsigma_n]h=0,$$

has, see [11], p. 83, the solution

$$h(\rho)=\rho^{-(\nu+\mu+1)}I_{(\nu+\mu+2n+1)}(\rho\sqrt{2\lambda}). \qquad \square$$

Corollary 3.2. *When $z=0$, τ and $B_{p,q}(\tau)$ are independent.*

Proof. Let us first derive the joint law of τ^z and $B_p(\tau^z)$ for $z=\rho\,\mathrm{e}^{\mathrm{i}\theta}$, $|z|<1$. By Lemma 2.2, we have

$$E^z\left[P_n^{\mu,\nu}\left(2B_p^2(\tau)-1\right)\exp-\lambda\tau\right]=\rho^{-(\nu+\mu+1)}\frac{I_{(\nu+\mu+2n+1)}(\rho\sqrt{2\lambda})}{I_{(\nu+\mu+2n+1)}(\sqrt{2\lambda})}P_n^{\mu,\nu}(\cos 2\theta).$$

Since the interval $[0,1]$ is compact, this formula does determine uniquely the law of the couple $(\tau^z, B_{p,q}(\tau^z))$, whence the harmonic measure at z.

For $z = 0$, we can write for all μ and $\nu > 0$

$$E^0\big[P_n^{\mu,\nu}\big(2B_p^2(\tau) - 1\big)\big] = \delta_{0,n},$$

where $\delta_{i,j}$ is the Kronecker symbol. Hence, by well known facts concerning orthogonality of the Jacobi polynomials we have

$$\int_0^1 P_n^{\mu,\nu}(2x_1^2 - 1) H_p^0(dx_1) = \int_0^1 P_n^{\mu,\nu}(2x_1^2 - 1)\big[c2^{(\nu+\mu+2)}(1 - x_1^2)^{\mu} x_1^{2\nu+1}\big]\,\mathrm{d}x_1,$$

where

$$c = \sqrt{\frac{\Gamma(\nu+\mu+2)}{2^{(\nu+\mu+1)}\Gamma(\nu+1)\Gamma(\mu+1)}},$$

and this clearly determines the law of $B_p(\tau^0)$, denoted $H_p^0(\mathrm{d}x_1)$. In the same way we find

$$H_q^0(\mathrm{d}x_2) = 2^{(\nu+\mu+3)/2}\frac{\Gamma(\nu+\mu+2)}{\Gamma(\nu+1)\Gamma(\mu+1)}(1 - x_2^2)^{\nu} x_2^{2\mu+1}\,\mathrm{d}x_2.$$

Now let us designate by c_0 the Laplace transform of τ^0. We clearly have

$$c_0 = \frac{(\lambda/2)^{\nu+\mu+1}}{\Gamma(\nu+\mu+2) I_{\nu+\mu+1}(\sqrt{2\lambda})},$$

so that we easily deduce from this

$$\int_\Omega \mathrm{e}^{-\lambda\tau} P_n^{\mu,\nu}\big(2B_p^2(\tau) - 1\big)P(\mathrm{d}\omega) = \int_\Omega \mathrm{e}^{-\lambda\tau} P(\mathrm{d}\omega)\int_\Omega P_n^{\mu,\nu}\big(2B_p^2(\tau) - 1\big)P(\mathrm{d}\omega).$$

Since we can do the same for $H_q^0(\mathrm{d}x_2)$ and since Γ is compact, the Corollary 3.2 is proved. □

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